J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
37 (1990), 511-525.

On smooth exceptional curves in threefolds

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Introduction.

Let X be a 3-dimensional complex manifold containing a compact smooth curve C. C is said to be an *exceptional curve* in X, if there exists a proper modification $\varphi: X \longrightarrow Z$ such that φ is isomorphic outside of C and $\varphi(C)$ is a point. Typical examples of exceptional curves are the zero sections of negative vector bundles of rank two over C. But in general, there exists an example where the normal bundle is not negative. The purpose of this paper is to study several properties of exceptional curves in threefolds, after the work of Reid [R], Laufer [L] and Ando [An]. In §1, we shall prove the following criterion:

THEOREM (1.4). *For a curve C in a threefold X, C is exceptional in X if and only if there is an $\mathcal{O}_X$-ideal $\mathcal{J}$ such that* $\mathrm{Supp}(\mathcal{O}_X/\mathcal{J}) \supset C$, $\dim \mathrm{Supp}(\mathcal{O}_X/\mathcal{J})=1$ *and* $\mathcal{J}\otimes\mathcal{O}_C/(\text{torsion})$ *is an ample vector bundle.*

By using this criterion and a result of Ando [An], we obtain:

THEOREM (1.10). *If $C \simeq \boldsymbol{P}^1 \subset X$ is an exceptional curve with the conormal bundle $I_C/I_C^2 \simeq \mathcal{O}(a) \oplus \mathcal{O}(b)$ for integers $a \geq b$, then $a+2b \geq 1$.*

In §2, we shall consider the blowing up $\mu_1: X_1 \longrightarrow X$ along C. Then the exceptional divisor $E_1 = \mu_1^{-1}(C)$ is a ruled surface over C. There exists at most one section C_1 of the ruling $E_1 \longrightarrow C$ with $(C_1)^2_{E_1} < 0$. We call this section by the *negative section*. If E_1 has a negative section C_1, then let us consider the blowing-up $\mu_2: X_2 \longrightarrow X_1$ along C_1. In this way, we have a sequence of blowing-ups

$$(B_k): X_k \xrightarrow{\mu_k} X_{k-1} \xrightarrow{\mu_{k-1}} \cdots \longrightarrow X_1 \xrightarrow{\mu_1} X,$$

the exceptional ruled surfaces E_i on X_i $(1 \leq i \leq k)$ and the negative sections C_i on E_i $(1 \leq i \leq k)$ such that the μ_j is just the blowing-up of X_{j-1} along C_{j-1} and $E_j = \mu_j^{-1}(C_{j-1})$ for $1 \leq j \leq k$.

Then we have:

THEOREM (2.6). *If $C \subset X$ is an exceptional curve, then the normal bundle N_{C_k/X_k} is semi-stable for some k, namely, the sequence* (B.) *terminates.*

In the case $C \simeq \boldsymbol{P}^1$ with $N_{C/X} \simeq \mathcal{O} \oplus \mathcal{O}(-2)$, Reid [R, §5] has proved this and constructed the flip along C.

In §3, we shall construct $\boldsymbol{C}^2$-bundles X over $\boldsymbol{P}^1$, where X has a section $C \simeq \boldsymbol{P}^1$ which is exceptional in X. By this construction, we obtain:

THEOREM (3.2). *If $a \geq b$ and $a+2b \geq 2$, then there exists an exceptional curve $C \simeq \boldsymbol{P}^1 \subset X$ such that $I_C/I_C^2 \simeq \mathcal{O}(a) \oplus \mathcal{O}(b)$.*

Compairing with (1.10), there remains only the existence of exceptional curves with $a+2b=1$, in problem. For $a=3$, $b=-1$, there exist such curves which are studied in [R], [P], [An], etc.

Further we have:

THEOREM (3.4). *If $a \geq b > 0$ and $a-2b \geq 2$, then there is an exceptional curve $C \simeq \boldsymbol{P}^1 \subset X$ such that $I_C/I_C^2 \simeq \mathcal{O}(a) \oplus \mathcal{O}(b)$ and that any open neighborhood of C in X is not isomorphic to an open neighborhood of the zero section of the normal bundle $\mathcal{O}(-a) \oplus \mathcal{O}(-b)$.*

On the other hand, by Kawamata-Viehweg's vanishing theorem ([K] and [V], see also [KMM, §1-2] or [N, §3]), we obtain:

THEOREM (3.5). *If $C \simeq \boldsymbol{P}^1 \subset X$ has the conormal bundle $I_C/I_C^2 \simeq \mathcal{O}(a) \oplus \mathcal{O}(b)$, where $a \geq b > 0$ and $a-2b \leq 1$, then an open neighborhood of C in X is isomorphic to a neighborhood of the zero section of the normal bundle $\mathcal{O}(-a) \oplus \mathcal{O}(-b)$.*

The author is grateful to the Max-Planck-Institut für Mathematik at Bonn for the support and the hospitality. A part of §1 and §2 are written, during the stay at the institute, as a preprint "Termination of succesive blowing-ups along exceptional curves in threefolds" (1987). The author also thanks to the Fūjukai foundation for the support.

§1. Preliminaries.

Let $\mathcal{E}$ be a locally free sheaf of rank two on a smooth compact curve C.

LEMMA (1.1). (1) *If $\mathcal{E}$ is a semi-stable vector bundle, then there exist no curves Γ on the ruled surface $P_C(\mathcal{E})$ with $\Gamma^2<0$.*

(2) *If $\mathcal{E}$ is unstable, then there exists a unique (up to isomorphisms) exact sequence*

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{M} \longrightarrow 0,$$

which satisfies the following two conditions:

(i) *$\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on C,*

(ii) $\deg_C \mathcal{L} > \deg_C \mathcal{M}$.

PROOF. (1) Let $\mathcal{O}_{\mathcal{E}}(1)$ be the tautological line bundle on $P_C(\mathcal{E})$ with respect to the $\mathcal{E}$. Then $\mathcal{E}$ is semi-stable if and only if the line bundle $\mathcal{O}_{\mathcal{E}}(2)\otimes\pi^*(\det\mathcal{E})^{-1}$ is nef on $P_C(\mathcal{E})$, where π is the ruling $P_C(\mathcal{E})\longrightarrow C$ (cf. [M, (3.1)]). (1) is an easy consequence of this fact.

(2) Since $\mathcal{E}$ is unstable, there exists an exact sequence satisfying (i) and (ii). Assume that there is another sequence

$$0 \longrightarrow \mathcal{L}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{M}' \longrightarrow 0$$

satisfying (i) and (ii). Since $2\deg\mathcal{M}'<\deg(\det\mathcal{E})$, the composition $\mathcal{L}\longrightarrow\mathcal{E}\longrightarrow\mathcal{M}'$ must be zero. Therefore $\mathcal{L}'\simeq\mathcal{L}$ and $\mathcal{M}'\simeq\mathcal{M}$. Q.E.D.

DEFINITION (1.2). When $\mathcal{E}$ is unstable, we call the exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{M} \longrightarrow 0$$

satisfying the above conditions (i) and (ii), the *characteristic exact sequence* of $\mathcal{E}$. Here we define $d^+(\mathcal{E}):=\deg_C\mathcal{L}$, $d^-(\mathcal{E}):=\deg_C\mathcal{M}$ and $\delta(\mathcal{E}):=d^+(\mathcal{E})-d^-(\mathcal{E})$. If $\mathcal{E}$ is the conormal bundle $N_{C/X}^{\vee}\simeq I_C/I_C^2$ of a curve $C\subset X$, (where I_C is the defining ideal of C in X), we simply denote $d^+(\mathcal{E})$, $d^-(\mathcal{E})$, and $\delta(\mathcal{E})$ by $d^+(C)$, $d^-(C)$, and $\delta(C)$, respectively.

DEFINITION (1.3). A compact smooth curve C in a smooth threefold X is called an *exceptional curve*, if there exists a proper bimeromorphic morphism $\varphi: X\longrightarrow Z$ such that $\varphi(C)$ is a point and that φ is isomorphic outside of C.

We have the following criterion:

THEOREM (1.4). *Let $C\subset X$ be a compact smooth curve in a smooth threefold. Then C is an exceptional curve if and only if there exists a coherent $\mathcal{O}_X$-*ideal *$\mathcal{J}$ on a neighborhood of C satisfying the following condition*

(α): dim Supp$(\mathcal{O}_X/\mathcal{J})=1$, Supp$(\mathcal{O}_X/\mathcal{J})\supset C$, *and* $(\mathcal{J}\otimes_{\mathcal{O}_X}\mathcal{O}_C)$/torsion *is an ample vector bundle on* C.

PROOF. First we assume that C is an exceptional curve. Let $\varphi: X\longrightarrow Z$ be the contraction of the exceptional curve C. Then by the exponential exact sequence:

$$R^1\varphi_*\mathcal{O}_X\longrightarrow R^1\varphi_*\mathcal{O}_X^*\longrightarrow R^2\varphi_*\mathbf{Z}_X\simeq H^2(C,\mathbf{Z})\longrightarrow 0$$

there exists a line bundle $\mathcal{A}$ on X such that $(\mathcal{A}\cdot C)>0$. Since $H^0(X,\mathcal{A}^{-1})\simeq H^0(Z,\varphi_*(\mathcal{A}^{-1}))$, if we replace Z by an open neighborhood of the point $\varphi(C)$, we find two sections s_1, $s_2\in H^0(X,\mathcal{A}^{-1})$ such that $\dim(\mathrm{div}(s_1)\cap\mathrm{div}(s_2))=1$. Let $S_i:=\mathrm{div}(s_i)$ for $i=1, 2$, and let $\mathcal{J}$ be the ideal $\mathcal{O}_X(-S_1)+\mathcal{O}_X(-S_2)$. Then we have

$$\mathcal{J}\otimes\mathcal{O}_C\simeq(\mathcal{O}_C\otimes\mathcal{O}_X(-S_1))\oplus(\mathcal{O}_C\otimes\mathcal{O}_X(-S_2)).$$

Thus $\mathcal{J}$ satisfies the condition (α). Next we assume that there is an $\mathcal{O}_X$-ideal $\mathcal{J}$ satisfying the condition (α). By considering the primary decomposition of $\mathcal{J}$, we have an $\mathcal{O}_X$-ideal $\mathcal{J}_0\supset\mathcal{J}$ such that Supp$(\mathcal{O}_X/\mathcal{J}_0)=C$ and Supp$(\mathcal{J}_0/\mathcal{J})$ does not contain C. Hence there is an injection $(\mathcal{J}\otimes\mathcal{O}_C/\text{torsion})\subset(\mathcal{J}_0\otimes\mathcal{O}_C/\text{torsion})$, whose cokernel is a torsion sheaf. Therefore $\mathcal{J}_0$ also satisfies the condition (α). Applying the contraction criterion (cf. [Ar], [F]) to $\mathcal{J}_0$, we have the contraction $\varphi: X\longrightarrow Z$ of C. Q.E.D.

LEMMA (1.5). *Let* $C\subset X$ *be an exceptional curve.*

(1) *If the conormal bundle* $N_{C/X}^{\vee}\simeq I_C/I_C^2$ *is semi-stable, then* I_C/I_C^2 *is an ample vector bundle.*

(2) *If* I_C/I_C^2 *is unstable, then* $d^+(C)>0$.

PROOF. Take an ideal $\mathcal{J}$ satisfying (α) and the maximal integer k such that $\mathcal{J}\subset I_C^k$. Then we have an injection

$$\mathcal{J}/(\mathcal{J}\cap I_C^{k+1})\hookrightarrow I_C^k/I_C^{k+1}\simeq\mathrm{Sym}^k(I_C/I_C^2).$$

By the condition (α), $\mathcal{J}/(\mathcal{J}\cap I_C^{k+1})$ is an ample vector bundle. Therefore we have proved (1) and (2). Q.E.D.

REMARK. In the case of $C\simeq\mathbf{P}^1$, Laufer [L, (3.1)] proves $\deg(I_C/I_C^2)\geq 2$.

Let $C\subset X$ be an exceptional curve such that I_C/I_C^2 is unstable. Let us consider the blowing-up $\mu_1: X_1\longrightarrow X$, $E_1=\mu_1^{-1}(C)$, and the negative section C_1 corresponding to the characteristic exact sequence of I_C/I_C^2.

PROPOSITION (1.6). *$C_1 \subset X_1$ is also an exceptional curve.*

PROOF. Let $0 \longrightarrow \mathcal{L} \longrightarrow I_C/I_C^2 \longrightarrow \mathcal{M} \longrightarrow 0$ be the characteristic exact sequence. Assume that I_C/I_C^2 is ample. Then from the natural exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{O}_{C_1} \otimes \mathcal{O}_{X_1}(-E_1) & \longrightarrow & I_{C_1}/I_{C_1}^2 & \longrightarrow & \mathcal{O}_{C_1} \otimes \mathcal{O}_{E_1}(-C_1) & \longrightarrow 0, \\ & \wr\wr & & & & \wr\wr & \\ & \mathcal{M} & & & & \mathcal{L} \otimes \mathcal{M}^{-1} & \end{array}$$

and the condition $\deg \mathcal{L} > \deg \mathcal{M} > 0$, we see that $I_{C_1}/I_{C_1}^2$ is also ample. Next assume that I_C/I_C^2 is not ample. Then $\deg \mathcal{M} \leq 0$. Take an $\mathcal{O}_X$-ideal $\mathcal{J}$ satisfying the condition (α) for $C \hookrightarrow X$. Let us consider the $\mathcal{O}_{X_1}$-ideal $\mathcal{J}' := \mathrm{Image}(\mu_1^* \mathcal{J} \longrightarrow \mathcal{O}_{X_1})$. Since $\mathcal{J} \subset I_C$, we have $\mathcal{J}' \subset \mathcal{O}_{X_1}(-E_1)$. Take the maximal integer l such that $\mathcal{J}' \subset \mathcal{O}_{X_1}(-lE_1)$ and let $\mathcal{J}_1 := \mathcal{J}' \otimes \mathcal{O}_{X_1}(lE_1) \subset \mathcal{O}_{X_1}$. We shall prove that the $\mathcal{J}_1$ satisfies the condition (α) for $C_1 \subset X_1$. Since ($\mathcal{J} \otimes \mathcal{O}_C$/torsion) is ample on C, ($\mathcal{J}' \otimes \mathcal{O}_{C_1}$/torsion) is also ample on C_1. Now we have a natural homomorphism $\mathcal{J}' \otimes \mathcal{O}_{C_1} \longrightarrow \mathcal{O}_{X_1}(-lE_1) \otimes \mathcal{O}_{C_1} \simeq \mathcal{M}^l$. Since $\deg \mathcal{M} \leq 0$, this homomorphism must be zero. Therefore $\mathcal{J}_1 \subset I_{C_1}$. On the other hand, ($\mathcal{J}_1 \otimes \mathcal{O}_{C_1}$/torsion) is ample, because

$$\mathcal{J}_1 \otimes \mathcal{O}_{C_1} \simeq (\mathcal{J}' \otimes \mathcal{O}_{C_1}) \otimes (\mathcal{O}_{X_1}(lE_1) \otimes \mathcal{O}_{C_1}) \simeq (\mathcal{J}' \otimes \mathcal{O}_{C_1}) \otimes \mathcal{M}^{-l}.$$

Therefore $\mathcal{J}_1$ satisfies the condition (α). Q.E.D.

LEMMA (1.7). *Let $C \subset X$ be an exceptional curve. Then it is impossible to construct an infinitely descending filtration $I^{(k)}$ ($k \geq 1$) of the defining ideal $I_C =: I^{(1)}$ which satisfies the following condition*

(β): *$I^{(k)}$ is a coherent $\mathcal{O}_X$-ideal, $I^{(k)}/I^{(k+1)}$ is a non-zero semi-negative locally free $\mathcal{O}_C$-module for all k.*

PROOF. Let $\mathcal{J}$ be an $\mathcal{O}_X$-coherent sheaf satisfying the condition (α) for $C \subset X$. Then we have

(γ): $\mathcal{J} \subset \bigcap_{k \geq 1} I^{(k)}$.

Because if there is an integer k such that $\mathcal{J} \subset I^{(k)}$ and $\mathcal{J} \not\subset I^{(k+1)}$, then we have an injection $\mathcal{J}/(\mathcal{J} \cap I^{(k+1)}) \hookrightarrow I^{(k)}/I^{(k+1)}$. By the conditions ($\alpha$) and ($\beta$), we have a contradiction. Take a general point $x \in C$ and a general smooth divisor S on X such that $S \cap C = \{x\}$ and this intersection is transversal. Then from (β) and (γ), we have $\lim_{k \to \infty} \mathrm{length}(\mathcal{O}_{S,x}/(I^{(k)} \cdot \mathcal{O}_S)_x) = \infty$, and $\text{length}(\mathcal{O}_{S,x}/(\mathcal{J}\cdot\mathcal{O}_S)_x)\geq\text{length}(\mathcal{O}_{S,x}/(I^{(k)}\cdot\mathcal{O}_S)_x)$. This is a contradiction. Q.E.D.

REMARK (1.8). For an exceptional curve $C\subset X$, there exists a unique maximal ideal $\mathcal{J}_{\max}$ satisfying the condition (α). By the proof of (1.4), we know that $\sqrt{\mathcal{J}_{\max}}=I_C$.

DEFINITION (1.9). The *width* $w(C)=w(C\subset X)$ of the exceptional curve $C\subset X$ is defined by $\text{length}(\mathcal{O}_{S,x}/(\mathcal{J}_{\max}\cdot\mathcal{O}_S)_x)$, where $x\in C$ is a general point and $S\subset X$ is also a general smooth divisor such that the intersection $S\cap C=\{x\}$ is transversal.

REMARK. In the case $C\simeq\boldsymbol{P}^1$ with $I_C/I_C^2\simeq\mathcal{O}\oplus\mathcal{O}(2)$, the width $w(C)$ is the same one as in [R, §5].

From (1.7) and results of Ando [An], we have:

THEOREM (1.10). *If $C\simeq\boldsymbol{P}^1\subset X$ is an exceptional curve with the conormal bundle $I_C/I_C^2\simeq\mathcal{O}(a)\oplus\mathcal{O}(b)$ for integers $a\geq b$, then $a+2b\geq 1$.*

PROOF. Ando [An, (3.2)] says that $a+2b\geq 0$. Assume that $a+2b=0$. Then $b<0$ by (1.5). Let us consider the ideal $J:=\text{Ker}(I_C\longrightarrow I_C/I_C^2\longrightarrow\mathcal{O}(b))$. Then we have an extension $0\longrightarrow\mathcal{O}(2b)\longrightarrow J/I_CJ\longrightarrow\mathcal{O}(a)\longrightarrow 0$. If $J/I_CJ\simeq\mathcal{O}(c)\oplus\mathcal{O}(d)$ with $c\geq d$, then Ando [An, (3.4)] proves $(1/2)d+2b+a\geq 0$. Therefore $c=d=0$. Let $\mathcal{J}$ be an ideal satisfying the condition (α) of (1.4). Since $\mathcal{J}\otimes\mathcal{O}_C/(\text{torsion})$ is ample, $\mathcal{J}$ must be contained in I_CJ. In general, we obtain $I_CJ^k/J^{k+1}\simeq(I_C/J)\otimes(J^k/I_CJ^k)$ for all $k\geq 1$. Since $I_C/J\simeq\mathcal{O}(b)$ is negative and $J^k\otimes\mathcal{O}_C$ is trivial, $\mathcal{J}$ can be contained in J^k for all $k\geq 1$. This contradicts to (1.7). Q.E.D.

§2. Termination.

Let $C\subset X$ be an exceptional curve such that I_C/I_C^2 is unstable. Then we have the characteristic exact sequence

$$\text{(e.1)}:\qquad 0\longrightarrow\mathcal{L}\longrightarrow I_C/I_C^2\longrightarrow\mathcal{M}\longrightarrow 0.$$

Let us consider $\mu_1: X_1\longrightarrow X$, E_1 and C_1 of (B.). (See the Introduction.) Then we have an exact sequence

$$\text{(e.2)}:\quad 0\longrightarrow\underset{\displaystyle\mathcal{M}}{\underset{\wr\wr}{\mathcal{O}_{C_1}\otimes\mathcal{O}_{X_1}(-E_1)}}\longrightarrow I_{C_1}/I_{C_1}^2\longrightarrow\underset{\displaystyle\mathcal{L}\otimes\mathcal{M}^{-1}}{\underset{\wr\wr}{\mathcal{O}_{C_1}\otimes\mathcal{O}_{E_1}(-C_1)}}\longrightarrow 0.$$

Assume that $I_{C_1}/I_{C_1}^2$ is also unstable. Then we have the characteristic exact sequence

$$\text{(e.3)}: \qquad 0 \longrightarrow \mathcal{L}_1 \longrightarrow I_{C_1}/I_{C_1}^2 \longrightarrow \mathcal{M}_1 \longrightarrow 0.$$

The following lemma is easily proved.

LEMMA (2.1). (1) *If* $\deg \mathcal{L} < 2 \deg \mathcal{M}$, *then the exact sequence* (e.2) *is isomorphic to* (e.3).

(2) *If* $\deg \mathcal{L} \geq 2 \deg \mathcal{M}$, *then* $\deg \mathcal{M} \leq \deg \mathcal{M}_1$ *and* $\deg \mathcal{L}_1 \leq \deg \mathcal{L} - \deg \mathcal{M}$. *Here the equality* $\deg \mathcal{M} = \deg \mathcal{M}_1$ *holds* (*or equivalently*, $\deg \mathcal{L}_1 = \deg \mathcal{L} - \deg \mathcal{M}$), *if and only if the exact sequence* (e.2) *is split.*

DEFINITION (2.2). Let $C \subset X$ be an exceptional curve. C is called of *type S*, if I_C/I_C^2 is a semi-stable vector bundle. C is called of *type P* (resp. *type N*), if I_C/I_C^2 is unstable and ample (resp. not ample). C is called of *type I*, if there exist two prime divisors S_1 and S_2 on a neighborhood of C such that C is just the scheme-theoretic intersection $S_1 \cap S_2$.

LEMMA (2.3). *If C is of type P, then one of the following conditions are satisfied*:

(1) *C_1 is of type S,*

(2) *C_1 is of type P and C_2 is of type I,*

(3) *C_1 is of type P and* $0 < \delta(C_1) < \delta(C)$.

PROOF. Assume that C_1 is not of type S. Then by (e.2), C_1 is of type P. If $d^+(C) < 2d^-(C)$, then by Lemma (2.1)-(1), C_2 is just the intersection of E_2 and the proper transform E_1' of E_1 on X_2. Therefore the condition (2) is satisfied. If $d^+(C) \geq 2d^-(C)$, then by Lemma (2.1)-(2), we have

$$d^-(C) \leq d^-(C_1) < d^+(C_1) \leq \delta(C) < d^+(C).$$

Therefore the condition (3) is satisfied. Q.E.D.

PROPOSITION (2.4). *If C is of type P and of type I, then there is a positive integer k such that C_k is of type S.*

PROOF. Let S_1 and S_2 be prime divisors with $S_1 \cap S_2 = C$. Then S_1 and S_2 are smooth surface near C, and $I_C/I_C^2 \simeq \mathcal{O}_C(-S_1) \oplus \mathcal{O}_C(-S_2)$.

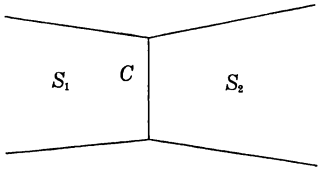

Figure 1.

Assume that $(S_1\cdot C)>(S_2\cdot C)$. Then we have $d^+(C)=-(S_2\cdot C)=-(C)^2_{S_1}>d^-(C)=-(S_1\cdot C)=-(C)^2_{S_2}$. Let us consider the $\mu_1: X_1\longrightarrow X$ and let S'_i be the proper transform of S_i on X_1 for $i=1, 2$. Then C_1 is just the complete intersection $S'_2\cap E_1$, and $I_{C_1}/I^2_{C_1}\simeq\mathcal{O}_{C_1}(-E_1)\oplus\mathcal{O}_{C_1}(-S'_2)$.

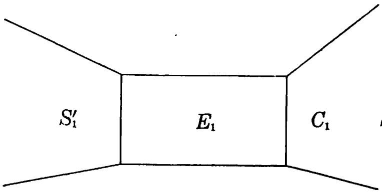

Figure 2.

Here we have $d^+(C_1)=\max(\delta(C), d^-(C))$ and $d^-(C_1)=\min(\delta(C), d^-(C))$. Therefore C_k is of type S for some k. Q.E.D.

PROPOSITION (2.5). *If C is of type N, then C_k is of type P or type S for some $k\leq w(C)$.*

PROOF. Assume that C_l is of type N for some l. Then by (2.3), C_j is of type N for all $j\leq l$. Let D_j be the effective divisor

$$E_j+\mu_j^*E_{j-1}+\mu_j^*\mu_{j-1}^*E_{j-2}+\cdots+(\mu_j^*\mu_{j-1}^*\cdots\mu_2^*)E_1$$

on X_j for $j\leq l$. Then we have

$$(*)_j \qquad K_{X_j}=\rho_j^*K_X+D_j,$$

and

$$(**)_j \qquad (-D_j)\cdot C_j=-(E_j\cdot C_j)+(-D_{j-1})\cdot C_{j-1}\leq 0,$$

where ρ_j is the composition $\mu_1\circ\cdots\circ\mu_j$. Let $I^{(j)}$ be the ideal $\rho_{j*}\mathcal{O}_{X_j}(-D_j)$.

Then we have a sequence of descending filtration $I^{(j)}$ of $I_C=I^{(1)}$. From the natural exact sequence:

$$0\longrightarrow I_{C_j}\mathcal{O}_{X_j}(-D_j)\longrightarrow\mathcal{O}_{X_j}(-D_j)\longrightarrow\mathcal{O}_{C_j}(-D_j)\longrightarrow 0,$$

we see that $I^{(j)}/I^{(j+1)}$ is contained in the non-ample line bundle $\mathcal{O}_{C_j}(-D_j)$ by $(**)_j$. On the other hand, by $(*)_j$, we have

$$\begin{aligned}R^1\rho_{j*}(I_{C_j}\mathcal{O}_{X_j}(-D_j)) &\simeq R^1\rho_{j+1*}\mathcal{O}_{X_{j+1}}(-D_{j+1})\\ &\simeq R^1\rho_{j+1*}\mathcal{O}_{X_{j+1}}(-K_{X_{j+1}})\otimes\mathcal{O}_X(K_X).\end{aligned}$$

By the construction, we see that $-K_{X_{j+1}}$ is relatively nef over some Zariski open subset U of X with $U\cap C\neq\emptyset$. Therefore by Kawamata-Viehweg's vanishing theorem (cf. [K], [V], [KMM, §1-2] or [N, §3]), $R^1\rho_{j+1*}\mathcal{O}_{X_{j+1}}(-K_{X_{j+1}})$ is supported in some points of C. This implies that $I^{(j)}/I^{(j+1)}$ is a line bundle. Therefore by (1.7) and (1.8), we obtain $j\leq l<w(C)$. Q.E.D.

Combining (2.3), (2.4) and (2.5), we obtain the following:

THEOREM (2.6). *The sequence of blowing-ups* (B.) *must terminate for all exceptional curves in threefolds.*

§ 3. On exceptional $\boldsymbol{P}^1$.

In this section, we treat only exceptional curves C which are isomorphic to $\boldsymbol{P}^1$. Let $(u:v)$ be a homogeneous coordinate of $\boldsymbol{P}^1$ and we set $U_0:=\{u\neq 0\}$, $U_1:=\{v\neq 0\}$ and $U:=U_0\cap U_1$. Then $t:=v/u$ (resp. $s:=u/v$) is a coordinate function on U_0 (resp. U_1). We shall consider the following $\boldsymbol{C}^2$-bundle $\pi: X\longrightarrow\boldsymbol{P}^1$:

For the trivializations $\phi_i:\pi^{-1}(U_i)\simeq U_i\times\boldsymbol{C}^2$ $(i=0,1)$ and for $\varphi:=\phi_1\circ\phi_0^{-1}$: $U\times\boldsymbol{C}^2\simeq U\times\boldsymbol{C}^2$, the transition is defined by

$$\begin{aligned}\varphi(x)&=t^{-a}x+\lambda(t)y^2\\ \varphi(y)&=t^{-b}y,\end{aligned}$$

where (x,y) is a cordinate system of $\boldsymbol{C}^2$, and $\lambda(t)$ is a holomorphic function on U. Since φ preserves $(x,y)=(0,0)$, X has a zero section C of π. Then the conormal bundle I_C/I_C^2 is determined by the transition matrix $\begin{pmatrix}t^{-a}, & 0\\ 0\ , & t^{-b}\end{pmatrix}$. Thus $I_C/I_C^2\simeq\mathcal{O}(a)\oplus\mathcal{O}(b)$. Let $J:=\ker(I_C\longrightarrow I_C/I_C^2\longrightarrow\mathcal{O}(b))$.

Then $J/I_C^2 \simeq \mathcal{O}(a)$ and $I_C/J \simeq \mathcal{O}(b)$. Since J is generated by x and y^2 over U_0 and U_1, we have an extension:

$$\begin{array}{ccccccccc} 0 \longrightarrow & I_C^2/I_CJ & \longrightarrow & J/I_CJ & \longrightarrow & J/I_C^2 & \longrightarrow 0, \\ & \| & & & & \| & \\ & \mathcal{O}(2b) & & & & \mathcal{O}(a) & \end{array}$$

which is given by the matrix $\begin{pmatrix} t^{-a}, & 0 \\ \lambda(t), & t^{-2b} \end{pmatrix}$. Therefore the function $\lambda(t)$ determines the vector bundle $J \otimes \mathcal{O}_C \simeq J/I_CJ$. Conversely, for any extension $0 \longrightarrow \mathcal{O}(2b) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(a) \longrightarrow 0$, we can find a function $\lambda(t)$ which induces the same extension. Thus we have:

PROPOSITION (3.1). *For integers a, b and for any extension $0 \longrightarrow \mathcal{O}(2b) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(a) \longrightarrow 0$ over $\boldsymbol{P}^1$, there exists a threefold $X \supset C$ such that $I_C/I_C^2 \simeq \mathcal{O}(a) \oplus \mathcal{O}(b)$ and $J \otimes \mathcal{O}_C \simeq \mathcal{E}$, where $J = \mathrm{Ker}(I_C \longrightarrow I_C/I_C^2 \longrightarrow \mathcal{O}(b))$.*

THEOREM (3.2). *If $a \geq b$ and $a+2b \geq 2$, then there exists a threefold $X \supset C \simeq \boldsymbol{P}^1$ such that C is an exceptional curve in X and $I_C/I_C^2 \simeq \mathcal{O}(a) \oplus \mathcal{O}(b)$.*

PROOF. If $b>0$, then the zero section of the vector bundle $\mathcal{O}(-a) \oplus \mathcal{O}(-b)$ is an exceptional curve. Thus we assume $b \leq 0$. By (3.1) and (1.4), we have only to construct an extension $0 \longrightarrow \mathcal{O}(2b) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(a) \longrightarrow 0$, where $\mathcal{E}$ is an ample vector bundle. Since $a+2b \geq 2$, we can write $a+2b=c+d$ for some positive integers c and d. Then $a-c>0$ and $a-d>0$. Therefore we obtain general sections $P \in H^0(\boldsymbol{P}^1, \mathcal{O}(a-c))$ and $Q \in H^0(\boldsymbol{P}^1, \mathcal{O}(a-d))$ such that $\mathrm{div}(P) \cap \mathrm{div}(Q) = \varnothing$. Hence P and Q give a surjection $\mathcal{O}(c) \oplus \mathcal{O}(d) \longrightarrow \mathcal{O}(a)$. Q.E.D.

REMARK. In [L, §2], Laufer constructed some examples of exceptional $\boldsymbol{P}^1$'s in $\boldsymbol{C}^2$-bundles over $\boldsymbol{P}^1$ whose patching relations are more complicated than our cases.

REMARK (3.3). If $\boldsymbol{P}^1 \simeq C \subset X$ is an exceptional curve with $I_C/I_C^2 \simeq \mathcal{O}(a) \oplus \mathcal{O}(b)$, where $a \geq b$ and $b \leq -2$, then the singularity obtained by contracting C is not rational. In fact, we have $H^1(X, \mathcal{O}_X/I_C^2) \simeq H^1(C, I_C/I_C^2) \neq 0$.

By comparing with (1.10) and (3.2), we are interested in the following:

Problem. If $a+2b=1$, $a \geq b$, then does there exist an exceptional curve $C \simeq \boldsymbol{P}^1$ with $I_C/I_C^2 \simeq \mathcal{O}(a) \oplus \mathcal{O}(b)$?

If $(a, b)=(1, 0)$, then this curve can be deformed, thus it is not exceptional. For $(a, b)=(3, -1)$, there exist such curves which are studied in [R], [L], [P], [An], etc.

Next we shall consider the case that the conormal bundle is ample. If $a-2b\geq 2$, then $\mathrm{Ext}^1_{P^1}(\mathcal{O}(a), \mathcal{O}(2b))\neq 0$. Therefore by (3.1), we obtain:

THEOREM (3.4). *If $a-2b\geq 2$, then there exists a threefold $X\supset C\simeq \boldsymbol{P}^1$ such that $I_C/I_C^2\simeq\mathcal{O}(a)\oplus\mathcal{O}(b)$ and that any open neighborhood of C in X is not isomorphic to any open neighborhood of the zero section of the normal bundle $\mathcal{O}(-a)\oplus\mathcal{O}(-b)$.*

On the other hand, by using Kawamata-Viehweg's vanishing, we have:

THEOREM (3.5). *If $\boldsymbol{P}^1\simeq C\subset X$ is an exceptional curve with $I_C/I_C^2\simeq \mathcal{O}(a)\oplus\mathcal{O}(b)$, where $a\geq b>0$, and $a-2b\leq 1$, then an open neighborhood of C in X is isomorphic to an open neighborhood of the zero section of the normal bundle $\mathcal{O}(-a)\oplus\mathcal{O}(-b)$.*

The rest of the paper is devoted to the proof of (3.5). First of all, we need the following:

LEMMA (3.6). *Let X be a 3-fold containing $C\simeq\boldsymbol{P}^1$. Suppose that there exist two smooth divisors S_1 and S_2 of X such that $S_1\cdot C<0$, $S_2\cdot C<0$, and C is the scheme-theoretic intersection $S_1\cap S_2$. Then an open neighborhood of C in X is isomorphic to an open neighborhood of the zero section of the normal bundle $\mathcal{O}_C(S_1)\oplus\mathcal{O}_C(S_2)$.*

PROOF. Since $I_C/I_C^2\simeq\mathcal{O}_C(-S_1)\oplus\mathcal{O}(-S_2)$ is ample, there exists the contraction $\varphi: X\longrightarrow Z$ of C, where we have $R^1\varphi_*\mathcal{O}_X=0$. Therefore if we replace X by an open neighborhood of C, then there exist two smooth divisors T_1 and T_2 on X such that $T_1\cdot C=T_2\cdot C=1$, $\mathcal{O}_X(T_1)\simeq\mathcal{O}_X(T_2)$, $T_1\cap T_2=\emptyset$, and that T_i intersects C transversally for $i=1, 2$. Thus we obtain the morphism $\pi: X\longrightarrow\boldsymbol{P}^1$ given by the linear system generated by T_1 and T_2. Here we see that $\pi_{|C}: C\longrightarrow\boldsymbol{P}^1$ is an isomorphism. Let $V\longrightarrow\boldsymbol{P}^1$ be the total space of the vector bundle $\mathcal{O}_C(S_1)\oplus\mathcal{O}_C(S_2)$, where we identify $C\simeq\boldsymbol{P}^1$ by π. Since $\mathcal{O}_X(S_i)\simeq\pi^*\mathcal{O}_C(S_i)$ for $i=1, 2$, we get a section of $\pi^*(\mathcal{O}_C(S_1)\oplus\mathcal{O}_C(S_2))$. Thus we have a morphism $\sigma: X\longrightarrow V$ over $\boldsymbol{P}^1$. It is easy to see that σ gives an isomorphism between a neighborhood of C in X and a neighborhood of the zero section of V. Q.E.D.

PROOF OF (3.5). We have only to construct two smooth divisors S_1 and S_2 as in (3.6). We devide the proof into the following 4 steps.

Step 1. *Construction of* S_2 *in the case* $a>b$.

Let $\mu_1: X_1 \longrightarrow X$ be the blowing-up of X along C, $E_1=\mu_1^{-1}(C)$ the exceptional divisor and let C_1 be the negative section of E_1. Let H be a divisor of X such that $H\cdot C=1$. We shall consider the divisor $L_b := -E_1-b\mu_1^*H$, and the following exact sequence:

$$\varphi_{1*}\mathcal{O}_{X_1}(L_b)\longrightarrow H^0(E_1, \mathcal{O}_{E_1}(L_b))\longrightarrow R^1\varphi_{1*}\mathcal{O}_{X_1}(-E_1+L_b),$$

where $\varphi_1 := \varphi\circ\mu_1$. For a fiber f_1 of the ruling $E_1\longrightarrow C$, we have $L_{b|E_1}\sim C_1+(a-b)f_1$. Therefore if $R^1\varphi_{1*}\mathcal{O}_{X_1}(-E_1+L_b)=0$, then we find a smooth divisor $S_2'\sim L_b$ such that $S_2'\cap C_1=\emptyset$ and the transversal intersection $S_2'\cap E_1$ is also a section of the ruling $E_1\longrightarrow C$. Thus $S_2 := \mu_1(S_2')$ is a smooth divisor of X with $S_2\cdot C=-b$. Therefore to obtain such S_2, we have only to show that $R^1\varphi_{1*}\mathcal{O}_{X_1}(-E_1+L_b)=0$. Let Δ be a $\boldsymbol{Q}$-divisor $L_b-K_{X_1}-(\delta+1)E_1\sim-(3+\delta)E_1+(-a-2b+2)\mu_1^*H$, where δ is a rational number with $0<\delta<1$. Then

$$\Delta_{|E_1}\sim_{\boldsymbol{Q}}(3+\delta)C_1+((2+\delta)a-2b+2)f_1,$$

and

$$\Delta\cdot C_1=(1+\delta)b-a+2.$$

Thus if $1-\delta$ is small enough, then Δ is φ_1-nef. Therefore by Kawamata-Viehweg's vanishing, $R^1\varphi_{1*}\mathcal{O}_{X_1}(-E_1+L_b)=R^1\varphi_{1*}\mathcal{O}_{X_1}(K_{X_1}+\ulcorner\Delta\urcorner)=0$.

Step 2. *Construction of* S_1 *and* S_2 *in the case* $a=b$.

As in the argument of *Step* 1, we consider the blowing-up $\mu_1: X_1\longrightarrow X$ along C, the exceptional divisor $E_1=\mu_1^{-1}(C)\simeq\boldsymbol{P}^1\times\boldsymbol{P}^1$, and the divisor $L_b=-E_1-b\mu_1^*H$. Then $\mathcal{O}_{E_1}(L_b)\simeq p_2^*\mathcal{O}(1)$, where $p_2: E_1\longrightarrow\boldsymbol{P}^1$ is the other ruling. Since $L_b-K_{X_1}-E_1$ is φ_1-nef in this case, by Kawamata-Viehweg's vanishing, we get $R^1\varphi_{1*}\mathcal{O}_{X_1}(-E_1+L_b)=0$. Therefore there exist two divisors S_1', $S_2'\in|L_b|$ such that $S_1'\cap S_2'=\emptyset$ and the transversal intersection $S_i'\cap E_1$ is a fiber of p_2 for $i=1,2$. Thus $S_1 := \mu_1(S_1')$ and $S_2 := \mu_1(S_2')$ are the desired divisors.

Step 3. *Construction of* S_1 *in the case* $a-2b\leq 0$.

Let us consider the blowing-up $\mu_2: X_2\longrightarrow X_1$ along C_1. If $a=2b$, then the new exceptional divisor $E_2=\mu_2^{-1}(C_1)$ is isomorphic to $\boldsymbol{P}^1\times\boldsymbol{P}^1$, and if $a<2b$, then the negative section of the ruling $E_2\longrightarrow C_1$ is nothing but the intersection $E_2\cap E_1'$, where $E_1'\simeq E_1$ is the proper transform of E_1. Here the curve $E_2\cap E_1'$ can be identified to C_1 under the isomorphism $E_1'\simeq E_1$. So we denote this curve $E_2\cap E_1'$ by C_1. Let $L_a:=-E_1-a\mu_1^*H$ be a divisor on X_1. Then we have the following exact sequence:

$$\varphi_{2*}\mathcal{O}_{X_2}(-E_2+\mu_2^*L_a)\longrightarrow H^0(E_2,\mathcal{O}_{E_2}(-E_2+\mu_2^*L_a))\longrightarrow R^1\varphi_{2*}\mathcal{O}_{X_2}(-2E_2+\mu_2^*L_a),$$

where $\varphi_2:=\varphi_1\circ\mu_2$. For a fiber f_2 of the ruling $E_2\longrightarrow C_1$, we have $(-E_2+\mu_2^*L_a)_{|E_2}\sim C_1+(2b-a)f_2$, and $(-E_2+\mu_2^*L_a)_{|E_1'}\sim 0$. Thus if the vanishing $R^1\varphi_{2*}\mathcal{O}_{X_2}(-2E_2+\mu_2^*L_a)=0$ holds, then we find a smooth divisor $S_1'\sim -E_2+\mu_2^*L_a$ such that $S_1'\cap E_1'=\emptyset$ and the transversal intersection $S_1'\cap E_2$ is a section of $E_2\longrightarrow C_1$. Therefore $S_1:=\mu_1\circ\mu_2(S_1')$ is a smooth divisor on X such that $S_1\cdot C=-a$ and $S_1\cap S_2=C$. Hence it is enough to prove the vanishing $R^1\varphi_{2*}\mathcal{O}_{X_2}(-2E_2+\mu_2^*L_a)=0$. Let Δ_2 be a $\boldsymbol{Q}$-divisor $-2E_2+\mu_2^*L_a-K_{X_2}-\varepsilon E_1'\sim -5E_2-(2+\varepsilon)E_1'-(2a+b-2)\mu_2^*\mu_1^*H$. Then $\Delta_{2|E_2}\sim_{\boldsymbol{Q}}(3-\varepsilon)C_1+(4b-2a+2)f_2$, $\Delta_2\cdot C_1=(1-\varepsilon)a+(2\varepsilon-2)b+2$, and $\Delta_{2|E_1'}\sim_{\boldsymbol{Q}}(2\varepsilon-1)C_1+(\varepsilon a-b+2)f_1$. Hence if $1-\varepsilon$ is small enough, then Δ_2 is φ_2-nef. Therefore $R^1\varphi_{2*}\mathcal{O}_{X_2}(-2E_2+\mu_2^*L_a)=R^1\varphi_{2*}\mathcal{O}_{X_2}(K_{X_2}+\ulcorner\Delta_2\urcorner)=0$.

Step 4. *Construction of* S_1 *in the case* $a-2b=1$.

Let us consider the blowing-up $\mu_2: X_2\longrightarrow X_1$, $E_2=\mu_2^{-1}(C_1)$, and the negative section C_2 on E_2 as in the *Step* 3. In this case, $C_2\cap E_1'=\emptyset$, and $I_{C_2}/I_{C_2}^2\simeq\mathcal{O}(b)\oplus\mathcal{O}(1)$. Further let us consider the blowing-up $\mu_3: X_3\longrightarrow X_2$ along C_2 and the new exceptional divisor E_3. If $b=1$, then $E_3\simeq\boldsymbol{P}^1\times\boldsymbol{P}^1$, and if $b>1$, then E_3 has a negative section $E_3\cap E_2'$, where $E_2'\simeq E_2$ is the proper transform of E_2. Here the curve $E_3\cap E_2'$ can be identified to C_2 under the isomorphism $E_2'\simeq E_2$. So we denote this curve $E_3\cap E_2'$ by C_2. We have the following exact sequence:

$$\begin{aligned}\varphi_{3*}\mathcal{O}_{X_3}(-E_3+\mu_3^*(-E_2+\mu_2^*L_a))&\longrightarrow H^0(E_3,\mathcal{O}_{E_3}(-E_3+\mu_3^*(-E_2+\mu_2^*L_a))\\&\longrightarrow R^1\varphi_{3*}\mathcal{O}_{X_3}(-2E_3+\mu_3^*(-E_2+\mu_2^*L_a)),\end{aligned}$$

where $\varphi_3:=\varphi_2\circ\mu_3$. Then for a fiber f_3 of the ruling $E_3\longrightarrow C_2$, we obtain $-E_3+\mu_3^*(-E_2+\mu_2^*L_a)_{|E_3}\sim C_2+(b-1)f_3$, $-E_3+\mu_3^*(-E_2+\mu_2^*L_a)_{|E_2'}\sim 0$ and $-E_3+\mu_3^*(-E_2+\mu_2^*L_a)_{|E_1'}\sim 0$. Let Δ_3 be a $\boldsymbol{Q}$-divisor $-2E_3+\mu_3^*(-E_2+\mu_2^*L_a)-K_{X_3}-\beta E_2'-\gamma E_1'$, for $0<\beta$, $\gamma<1$. Then we have

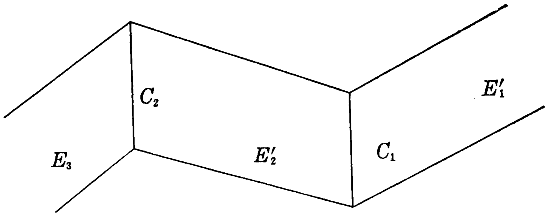

Figure 3.

$$\Delta_{3|E_3} \sim_{\mathbf{Q}} (3-\beta)C_2 + 2bf_3,$$
$$\Delta_{3|E_2'} \sim_{\mathbf{Q}} (2\beta-\gamma-1)C_2 + ((\beta-1)b + (\beta-\gamma)+2)f_2,$$
$$\Delta_{3|E_1'} \sim_{\mathbf{Q}} (2\gamma-\beta)C_1 + ((2\gamma-1)b+2+\gamma)f_1,$$
$$\Delta_3 \cdot C_2 = (\beta-1)b + (3-\beta),$$

and

$$\Delta_3 \cdot C_1 = (\beta-1)b + (\beta-\gamma)+2.$$

Therefore, if $2\beta \geq \gamma+1$, $2\gamma \geq \beta$, and $1-\beta$ is small enough, then Δ_3 is φ_3-nef. Thus by Kawamata-Viehweg's vanishing, $R^1\varphi_{3*}\mathcal{O}_{X_3}(K_{X_3} + \ulcorner \Delta_3 \urcorner) = R^1\varphi_{3*}\mathcal{O}_{X_3}(-2E_3 + \mu_3^*(-E_2 + \mu_2^* L_a)) = 0$. Hence we obtain a smooth divisor $S_1' \sim -E_3 + \mu_3^*(-E_2 + \mu_2^* L_a)$ such that $S_1' \cap E_2' = \emptyset$, $S_1' \cap E_1' = \emptyset$, and the transversal intersection $S_1' \cap E_3$ is a section of the ruling $E_3 \longrightarrow C_2$. Therefore the smooth divisors $S_1 := \mu_1 \circ \mu_2 \circ \mu_3(S_1')$ and the S_2 constructed in *Step* 1 satisfy the condition of Lemma (3.6). Q.E.D.

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(Received December 15, 1989)
(Revised February 15, 1990)

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