

On blow-up of positive solutions of semilinear parabolic equations

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§ 1. Introduction.

This paper is concerned with the blow-up of solutions of initial-boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + (b(t, x) \cdot \nabla u) + f(t, x, u) & \text{in } (0, T) \times \Omega, \\ u(0, x) = \varphi(x) & \text{on } \Omega, \\ u(t, x) = 0 & \text{on } (0, T) \times \partial\Omega \end{cases} \quad (1.1)$$

where Ω is a bounded domain in \mathbf{R}^n with C^2 boundary. We assume that the nonlinear term $f(t, x, u)$ satisfies:

$$f(t, x, u) \geq h(x)f(u) \quad \text{on } (0, T) \times \bar{\Omega} \times \mathbf{R} \quad (1.2)$$

where $h(x)$ is continuous, non-negative and not identically zero on $\bar{\Omega}$, and $f(u)$ satisfies certain growth condition (stated in §2). Under this assumption, we shall prove that the solution of (1.1) blows up in finite time provided $\max_{x \in \bar{\Omega}} \varphi(x)$ is sufficiently large. We note that our $f(\cdot, x, \cdot)$ is allowed to be identically zero somewhere in Ω .

The author [8] proved the blow-up of solutions of (1.1) in the case where $f(u) = |u|^p$ with $p > 1$ in (1.2) by means of comparison principle. Afterward the similar problems have been investigated by many scholars. There are several methods to show blow-up of solutions other than comparison principle in [8]. A typical one is to study the component of the solution u to the direction of the principal eigenfunction; see e.g. [2]. Energy method by J. Ball [1] is another typical one. Also H. Fujita [4] discussed the blow-up and non blow-up cases for the equation $u_t = \Delta u + u^{1+\alpha}$ in \mathbf{R}^n by a different method. Recently several remarkable results concerning the behavior of blowing-up solutions have been obtained; e.g.

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F. Weissler [10], A. Friedman-B. McLeod [3], C. Meuller-F. Weissler [9], Y. Giga-R. Kohn [5, 6, 7] and others—see also references given in [2, 6, 7]. In most of those papers, except [8], blow-up properties are discussed in the case where the nonlinear term in (1.1) is of the form $f=f(u)$, namely the growth order of f in u is independent of x . It seems to be somewhat interesting that, as is shown in the present paper, $h(x)$ in (1.2) is not necessarily positive everywhere in Ω for the solution to blow up.

The proof of blow-up in the present paper (§§ 3 and 4) is based on essentially the same idea as in the author's previous paper [8]. But the paper [8] has not been well-circulated as it was written in Japanese. So, in the present paper, the author will mention in detail the proofs of theorems which are generalizations of the result of [8].

§ 2. Assumptions and results.

Let Ω be a bounded domain in R^n with C^2 boundary, $b(t, x)$ be an n -vector valued function with components $b^j(t, x) \in C^1([0, \infty) \times \bar{\Omega})$ ($j=1, \dots, n$) and $f(t, x, u)$ be a C^1 function on $[0, \infty) \times \bar{\Omega} \times R$. Consider the initial-boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + (b(t, x) \cdot \nabla u) + f(t, x, u) & \text{in } (0, T) \times \Omega, \\ u(0, x) = a\varphi(x) & \text{on } \Omega, \\ u(t, x) = 0 & \text{on } (0, T) \times \partial\Omega \end{cases} \quad (2.1)$$

where $\varphi(x)$ is a bounded continuous function on $\bar{\Omega}$, non-negative and not identically zero on Ω , and a is a positive constant. $f(t, x, u)$ is assumed to be Hölder-continuous on $(0, \infty) \times \bar{\Omega} \times R$ and to satisfy

$$f(t, x, u) \geq h(x)f(u) \quad (2.2)$$

where $h(x)$ is a non-negative continuous function on $\bar{\Omega}$ not identically zero on Ω , and $f(u)$ satisfies the condition to be given in Theorem 1 or that in Theorem 2 below.

THEOREM 1. *Assume that the function $f(u)$ in (2.2) is of class C^1 on $[0, \infty)$ and that there exist $u_0 \geq 0$ and a function $G(u)$ satisfying the following conditions:*

$G(u)$ is C^1 and convex on $[u_0, \infty)$ and satisfies that

$$G > 0, G' \geq 0 \text{ on } (u_0, \infty) \text{ and that } \int_{u_0}^{\infty} \frac{du}{G(u)} < \infty, \quad (2.3)$$

$$\lim_{u \rightarrow \infty} \frac{f(u)}{G(u)} = \infty. \quad (2.4)$$

Then the solution of (2.1) blows up in finite time if a is sufficiently large.

COROLLARY. Assume that the function $G(u)$ in Theorem 1 satisfies (2.3) and the following condition:

$$\begin{aligned} &\text{there exist } c > 0 \text{ and } u_1 > u_0 \text{ such that} \\ &f'(u)G(u) - f(u)G'(u) \geq cG(u)G'(u) \text{ for any } u \geq u_1. \end{aligned} \quad (2.5)$$

Then the same conclusion as in Theorem 1 holds.

THEOREM 2. Assume that the function $f(u)$ in (2.2) is of class C^2 on $[0, \infty)$ and satisfies the following conditions:

$$f, f' \text{ and } f'' \geq 0 \text{ on } [0, \infty), \quad (2.6)$$

$$\int_0^{\infty} \frac{du}{\sqrt{F(u)}} < \infty \text{ holds for any primitive function } F \text{ of } f. \quad (2.7)$$

Then the solution of (2.1) blows up in finite time if a is sufficiently large.

Proofs of these theorems will be given in § 4.

Examples. i) The functions $f_1(u) = u(\log^+ u)^p$ with $p > 2$, $f_2(u) = u^p$ with $p > 1$ and $f_3(u) = e^{pu}$ with $p > 0$ satisfy the assumption in Theorem 2 and that in Corollary to Theorem 1, *a fortiori* that in Theorem 1.

ii) If $1 < p \leq 2$, the function $f(u) = u(\log^+ u)^p$ does not satisfy the assumption in Theorem 2 but it satisfies the assumption in Theorem 1; we may take $G(u) = u(\log^+ u)^q$ with $1 < q < p$.

REMARK 1. It may easily be seen from the argument in § 3 that Laplacian Δ in the parabolic equation in (2.1) may be replaced by Laplace-Beltrami operator with sufficiently smooth variable coefficients.

REMARK 2. We have assumed the boundedness of the domain Ω to avoid unessential complication in the argument in §§ 3 and 4. Even if Ω is not bounded, there exists a local solution of (2.2) (though the uniqueness is not assured) and the solution can be continued step-by-step in time provided it does not blow up. If we consider a bounded sub-

domain $\tilde{\Omega}$ of Ω and the solution $\tilde{u}(t, x)$ of (2.1) in $[0, T) \times \tilde{\Omega}$, we have $u(t, x) \geq \tilde{u}(t, x) \geq 0$ (whenever both solutions exist) by means of comparison principle. Hence, in the case of unbounded domain, we may obtain the same result as mentioned above.

§ 3. Preliminaries and the construction of subsolution.

It is well known that the (local) solution $u(t, x)$ of (2.1) is positive for $x \in \Omega$. Therefore, in proofs of theorems mentioned in § 2, we may assume without loss of generality that

$$\varphi(x) > 0 \quad \text{in } \Omega. \quad (3.1)$$

Throughout this paper, we always assume (3.1).

Let G be the function mentioned in Theorem 1. Hereinafter we fix $\xi_0 > u_0$ and $r > 0$ such that

$$G(\xi_0) > G(u_0) \quad (\geq 0) \quad \text{and} \quad \frac{1}{r} = 1 - \frac{G(u_0)}{G(\xi_0)}.$$

Then

LEMMA 1. $kG(\xi) \leq rG(k\xi)$ for any $k \geq 1$ and any $\xi \geq \xi_0$.

PROOF. Since G is convex in $\xi \geq \xi_0$, we have

$$\frac{G(\xi) - G(u_0)}{\xi - u_0} \leq \frac{G(k\xi) - G(u_0)}{k\xi - u_0} \leq \frac{G(k\xi)}{k\xi - ku_0},$$

which implies $k\{G(\xi) - G(u_0)\} \leq G(k\xi)$. Hence, for any $\xi \geq \xi_0$, we have $\frac{1}{r} \leq 1 - \frac{G(u_0)}{G(\xi)}$ and accordingly

$$\frac{k}{r}G(\xi) \leq k\{G(\xi) - G(u_0)\} \leq G(k\xi),$$

namely $kG(\xi) \leq rG(k\xi)$.

q.e.d.

Next we fix a positive number c_0 and a subdomain D of Ω such that $\sup_{x \in D} h(x) > c_0 > 0$ and that $\bar{D} \subset \{x \in \Omega \mid h(x) > c_0\}$.

Let $V \equiv V(x)$ be a nonnegative C^2 function on $\bar{\Omega}$ such that

$$V(x)=0 \quad \text{outside } D \quad \text{and} \quad \max_{x \in \bar{\Omega}} V(x)=1, \quad (3.2)$$

and $U \equiv U(t, x)$ be the solution of the initial-boundary value problem:

$$\begin{cases} \frac{\partial U}{\partial t} = \Delta U + (b \cdot \nabla U) & \text{in } (0, \infty) \times \Omega, \\ U(0, x) = \varphi(x) & \text{on } \Omega, \\ U(t, x) = 0 & \text{on } (0, \infty) \times \partial\Omega. \end{cases} \quad (3.3)$$

Then it follows from (3.1) that $U(t, x) > 0$ on $[0, \infty) \times \Omega$, and accordingly, by virtue of (3.2), the following function $\Phi(t, x)$ is well defined and continuous on $[0, \infty) \times \bar{\Omega}$:

$$\Phi(t, x) = V(x) - t \left\{ \frac{2(\nabla U(t, x) \cdot \nabla V(x))}{U(t, x)} + \Delta V(x) + (b(t, x) \cdot \nabla V(x)) \right\}. \quad (3.4)$$

We put $T^* = \int_{\xi_0}^{\infty} \frac{du}{G(u)}$, which is finite by (2.3). Then

$$\delta \equiv \inf_{0 < t \leq T^*, x \in D} U(t, x) > 0 \quad (3.5)$$

and accordingly

$$M \equiv \max \left\{ 1, \frac{\gamma}{c_0} \left[\sup_{0 \leq t \leq T^*, x \in \bar{D}} |\Phi(t, x)| \right] \right\} \quad (3.6)$$

is finite. Since $\Phi = 0$ for $x \notin D$ (by (3.2)), we have

$$\gamma |\Phi(t, x)| \leq M h(x) \quad \text{on } [0, T^*] \times \Omega. \quad (3.7)$$

Since $G(u)$ satisfies (2.4), there exists ξ_1 such that

$$f(u) \geq MG(u) \quad \text{for any } u > \xi_1; \quad (3.8)$$

here we may assume $\xi_1 \geq \xi_0$. Consider the function $\lambda = \Psi(\xi)$ ($\xi \geq \xi_1$) defined by $\Psi(\xi) = \int_{\xi_1}^{\xi} \frac{du}{G(u)}$, and put $\lambda_{\infty} = \int_{\xi_1}^{\infty} \frac{du}{G(u)}$. Then $\lambda_{\infty} \leq T^* < \infty$, and $\Psi'(\xi) = \frac{1}{G(\xi)} > 0$ for any $\xi > \xi_1$. Hence the inverse function $\xi = \phi(\lambda)$ of Ψ is well defined in $0 \leq \lambda < \lambda_{\infty}$ and satisfies

$$\phi(0) = \xi_1, \quad \lim_{\lambda \uparrow \lambda_{\infty}} \phi(\lambda) = \infty \quad (3.9)$$

and

$$\phi'(\lambda) = G(\phi(\lambda)) > 0, \quad \phi''(\lambda) = G'(\phi(\lambda))\phi'(\lambda) \geq 0. \quad (3.10)$$

Now we define the function $v(t, x)$ by

$$v(t, x) = \phi(tV(x))U(t, x) \quad \text{on } (0, \lambda_\infty) \times \bar{\Omega}, \quad (3.11)$$

which will play essentially the role of a subsolution of (2.1) in Proof of Theorem 1 in §4.

LEMMA 2. *The function $v \equiv v(t, x)$ satisfies:*

$$v(0, x) = \xi_1 \varphi(x) \quad \text{on } \Omega \quad (3.12)$$

and

$$\frac{\partial v}{\partial t} - \Delta v - (b \cdot \nabla v) \leq G(\phi(tV)) \cdot U\Phi \quad \text{in } (0, \lambda_\infty) \times \Omega. \quad (3.13)$$

PROOF. (3.12) is clear from (3.9) and (3.3). (3.13) is proved as follows. By elementary computation, we have

$$\begin{cases} \frac{\partial v}{\partial t} = \phi(tV) \frac{\partial U}{\partial t} + \phi'(tV)UV, \\ \nabla v = \phi(tV)\nabla U + t\phi'(tV)U\nabla V, \\ \Delta v = \phi(tV)\Delta U + 2t\phi'(tV)(\nabla U \cdot \nabla V) + t^2\phi''(tV)U(\nabla V \cdot \nabla V) + t\phi'(tV)U\Delta V. \end{cases}$$

Since U satisfies $\frac{\partial U}{\partial t} = \Delta U + (b \cdot \nabla U)$ and since

$$t^2\phi''(tV)U(\nabla V \cdot \nabla V) \geq 0 \quad (\text{by (3.10)}),$$

we get

$$\begin{aligned} & \frac{\partial v}{\partial t} - \Delta v - (b \cdot \nabla v) \\ & \leq \phi'(tV)U \left\{ V - \frac{2t(\nabla U \cdot \nabla V)}{U} - t\Delta V - t(b \cdot \nabla V) \right\} \\ & = G(\phi(tV)) \cdot U\Phi \quad (\text{by (3.10) and (3.4)}). \end{aligned} \quad \text{q.e.d.}$$

For any $t \in [0, \lambda_\infty)$ and any $x \in \bar{D}$, we have $\delta^{-1}U(t, x) \geq 1$ by (3.5), and accordingly $\delta^{-1}U\phi(tV) \geq \xi_1 \geq \xi_0$ by (3.9) and (3.10). Hence, by means of Lemma 1, (3.7) and (3.8), we get

$$\begin{aligned}\delta^{-1}UG(\phi(tV))|\Phi| &\leq \gamma G(\delta^{-1}U\phi(tV))|\Phi| \\ &\leq Mh \cdot G(\delta^{-1}U\phi(tV)) \leq h \cdot f(\delta^{-1}U\phi(tV)).\end{aligned}\quad (3.14)$$

On the other hand, if we define the function $v_\delta \equiv v_\delta(t, x)$ by

$$v_\delta(t, x) = \delta^{-1}v(t, x) = \phi(tV(x))\delta^{-1}U(t, x) \quad (\text{cf. (3.11)}) \quad (3.15)$$

and replace the function v in Lemma 2 by v_δ , we obtain that

$$\frac{\partial v_\delta}{\partial t} - \Delta v_\delta - (b \cdot \nabla v_\delta) \leq G(\phi(tV))\delta^{-1}U\Phi. \quad (3.16)$$

From (3.14), (3.15) and (3.16) follows that

$$\begin{cases} \frac{\partial v_\delta}{\partial t} \leq \Delta v_\delta + (b \cdot \nabla v_\delta) + h \cdot f(v_\delta) & \text{in } (0, \lambda_\infty) \times \Omega, \\ v_\delta(0, x) = \delta^{-1}\xi_1\varphi(x) & \text{on } \Omega, \\ v_\delta(t, x) = 0 & \text{on } (0, \lambda_\infty) \times \partial\Omega. \end{cases} \quad (3.17)$$

Furthermore we may see from (3.2), (3.9) and (3.15) that

$$\lim_{t \uparrow \lambda_\infty} \sup_{x \in \Omega} v_\delta(t, x) = \infty. \quad (3.18)$$

§ 4. Proof of Theorems.

PROOF OF THEOREM 1. Let $v_\delta \equiv v_\delta(t, x)$ be the function defined by (3.15). Then v_δ satisfies (3.17) and (3.18). We shall prove that the solution of (2.1) blows up in finite time whenever $a > \delta^{-1}\xi_1$.

By means of (2.2), the solution u of (2.1) satisfies

$$\begin{cases} \frac{\partial u}{\partial t} \geq \Delta u + (b \cdot \nabla u) + hf(u) & \text{in } (0, T) \times \Omega, \\ u(0, x) = a\varphi(x) & \text{on } \Omega, \\ u(t, x) = 0 & \text{on } (0, T) \times \partial\Omega \end{cases} \quad (4.1)$$

for suitable $T > 0$. From (3.17) and (4.1) follows that the function $w(t, x) = u(t, x) - v_\delta(t, x)$ satisfies

$$\frac{\partial w}{\partial t} \geq \Delta w + (b \cdot \nabla w) + h\{f(u) - f(v_\delta)\}$$

whenever $0 < t < T_\infty \equiv \min\{T, \lambda_\infty\}$. Since f is of class C^1 , there exists a

continuous function $f_1(\xi, \eta)$ such that

$$f(u) - f(v_\delta) = f_1(u, v_\delta)(u - v_\delta) = f_1(u, v_\delta)w.$$

Hence

$$\frac{\partial w}{\partial t} \geq \Delta w + (b \cdot \nabla w) + h \cdot f_1(u, v_\delta)w.$$

On the other hand, w satisfies that $w(0, x) = (a - \delta^{-1}\xi_1)\varphi(x) > 0$ for $x \in \Omega$ and $w(t, x) = 0$ for $x \in \partial\Omega$. Therefore $w(t, x) > 0$ namely $u(t, x) > v_\delta(t, x)$ on $(0, T_\infty) \times \Omega$. Since v_δ satisfies (3.18), T cannot exceed λ_∞ ; this means that the solution $u(t, x)$ of (2.1) blows up at certain $T \leq \lambda_\infty$. Theorem 1 is thus proved.

PROOF OF COROLLARY TO THEOREM 1. It suffices to show that (2.4) follows from (2.3) and (2.5). For any $u \geq u_1$, we have

$$\left(\frac{f}{G}\right)' = \frac{f'G - fG'}{G^2} \geq \frac{cG'}{G} \quad (\text{by (2.5)});$$

accordingly

$$\frac{f(u)}{G(u)} - \frac{f(u_1)}{G(u_1)} \geq c \log \frac{G(u)}{G(u_1)}. \quad (4.2)$$

Since G is convex and $\int_{u_0}^{\infty} \frac{du}{G(u)} < \infty$, it holds that $\lim_{u \rightarrow \infty} G(u) = \infty$. Hence (4.2) implies (2.4).

PROOF OF THEOREM 2. Let F be a primitive function of f . By means of (2.6) and (2.7), there exists $u_2 > 0$ such that $F(u) > 0$ on $[u_2, \infty)$ and $\int_{u_2}^{\infty} \frac{du}{\sqrt{F(u)}} < \infty$. We shall prove that the function $G(u) = \sqrt{F(u)}$ satisfies (2.3) and (2.4) if u_0 in (2.3) is replaced by suitable u^* ($> u_2$).

It is clear that G is of class C^3 , $G > 0$ and $G' \geq 0$ in (u_2, ∞) ; in particular we notice that

$$G'(u) = \frac{f(u)}{2G(u)} \quad \text{in } (u_2, \infty). \quad (4.3)$$

Furthermore, as we shall prove later,

$$G'(u) \text{ is unbounded in } (u_2, \infty) \quad (4.4)$$

and

$$G''(u) > 0 \text{ in } [u^*, \infty) \text{ for some } u^* > u_2. \quad (4.5)$$

Hence G is convex on $[u^*, \infty)$ and $\int_{u^*}^{\infty} \frac{du}{G(u)} < \infty$. It follows from (4.4) and (4.5) that $\lim_{u \rightarrow \infty} G'(u) = \infty$. Hence $G(u)$ satisfies (2.4) by means of (4.3). Theorem 2 is thus proved.

PROOF OF (4.4). Suppose that $G'(u) \leq M$ in (u_2, ∞) for some $M < \infty$. Then

$$\int_{u_2}^{\infty} \frac{du}{\sqrt{F(u)}} = \int_{u_2}^{\infty} \frac{du}{G(u)} \geq \int_{u_2}^{\infty} \frac{du}{M(u - u_2) + G(u_2)} = \infty$$

contrary to our assumption. Hence G' is unbounded in (u_2, ∞) .

PROOF OF (4.5). By means of (4.4), there exists $u^* > u_2$ such that

$$G''(u^*) > 0. \quad (4.6)$$

Since $\frac{1}{2}f = GG'$ by (4.3), we have

$$\frac{1}{2}G^2f'' = G^2(GG')'' = 3G^2G'G'' + G^3G''' = (G^3G'')'.$$

Hence, by (2.6) and (4.6), we get

$$G(u)^3G''(u) = G(u^*)^3G''(u^*) + \frac{1}{2} \int_{u^*}^u G(u)^2f''(u)du > 0 \text{ for any } u \geq u^*,$$

which implies (4.5).

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