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Comments on Yoshida's manifolds

Dedicated to Professor Akio Hattori on his sixtieth birthday

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§1. Introduction

The purpose of this note is to prove

PROPOSITION. *Let M^n be a closed smooth manifold which has Stiefel-Whitney numbers satisfying*

$$w_\omega(\tau)[M^n]=0 \qquad if \quad \omega\neq(2^{i_1},2^{i_2},\cdots,2^{i_p}),$$

where for $\omega=(j_1,\cdots,j_r)$, $w_\omega=w_{j_1}\cdots w_{j_r}$. If $n\neq 2^{s+1}$, $s\geqq 0$, then M^n bounds. If $n=2^{s+1}$, $s\geqq 0$, then M^n is cobordant to a union of the manifolds

$$RP^{2^{s+1}},(RP^{2^s})^2,\cdots,(RP^2)^{2^s}.$$

Note. $w_{2^t}^{2^{s+1-t}}[(RP^{2^{s+1-t}})^{2^t}]=w_{2^{s+1}}[(RP^{2^{s+1-t}})^{2^t}]\neq 0$ for $0\leqq t\leqq s$, and all other Stiefel-Whitney numbers are zero. Thus any union of the manifolds $(RP^{2^{s+1-t}})^{2^t}$, $0\leqq t\leqq s$, has the desired property.

This result was suggested by the work of Toshio Yoshida [5], who considers manifolds M^n satisfying $w_i(M)=0$ for i not a power of 2. Such manifolds provided a nice place to apply his formulae for $\chi(Sq^i)$ to calculate Wu classes. He showed that in many special cases M^n was a boundary.

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§2. A Poincaré Algebra

The first step is to reduce the problem to a special case.

LEMMA 1. *Let M^n be a closed manifold satisfying $w_\omega[M^n]=0$ if $\omega\neq(2^{i_1},2^{i_2},\cdots,2^{i_p})$. If M^n does not bound then there is a $t\geqq 0$ so that M^n has some nonzero Stiefel-Whitney number involving w_{2^t}, and t is minimal with this property. Then $n=2^tn'$, and there is a Poincaré algebra $P^{n'}$ of* *dimension n' having $w_i(P)=0$ for i not a power of 2, and so that M^n is cobordant to* $P^{2^t}=\underbrace{P\times\cdots\times P}_{2^t}$.

Note. This means $w_\omega[M^n]=0$ if $\omega=(i_1,\cdots,i_p)$ and some $i_\alpha\not\equiv 0(2^t)$, and if $\omega=(2^t i_1',\cdots,2^t i_p')$, then $w_\omega[M^n]=w_{(i_p',\cdots,i_p')}[P^{n'}]$. In particular, $P^{n'}$ has some nonzero Stiefel-Whitney number involving w_1.

PROOF. Letting t be as stated, $w_\omega[M]=0$ unless $\omega=(i_1,\cdots,i_p)$ with each $i_\alpha=2^{t_\alpha}$ with $t_\alpha\geqq t$ and in particular unless each $i_\alpha\equiv 0(2^t)$. By Milnor [2], M^n is cobordant to a 2^t-th power, and so $M^n\cong(N^{n'})^{2^t}$ where $n=2^t n'$, and $w_\omega[N^{n'}]=0$ unless $\omega=(2^{i_1},2^{i_2},\cdots,2^{i_r})$. From [4], one may then find a Poincaré algebra $P^{n'}$ with $w_\omega[P^{n'}]=w_\omega[N^{n'}]$ so that any characteristic class which is always zero in Stiefel-Whitney number of $N^{n'}$ is actually zero as a class in $P^{n'}$. $\square$

Shifting notation in an obvious way, one has

HYPOTHESIS. *P^n is an n-dimensional Poincaré algebra satisfying $w_i(P)=0$ for i not a power of 2 so that*

$$w(P)=1+w_1+w_2+w_4+\cdots+w_{2^t}+\cdots$$

and P has some nonzero Stiefel-Whitney number divisible by w_1.

LEMMA 2 (Yoshida [5], (4.2)). *If $j\geqq 2$, $w_1w_{2^j}=0$.*

PROOF. $0=Sq^2w_{2^j-1}=w_{2^j-1}w_2+w_{2^j}w_1+w_{2^j+1}=w_{2^j}w_1$. $\square$

Observation. In the characteristic numbers of P involving w_1, the Stiefel-Whitney class of P acts as if $w(P)$ were $1+w_1+w_2$; i.e.

$$w_1\cdot w(P)=w_1(1+w_1+w_2)\,.$$

In particular, the only characteristic numbers of P involving w_1 which can possibly be nonzero are $w_1^{n-2p}w_2^p[P]$, $n>2p$.

LEMMA 3. *n is even, and P has a nonzero Stiefel-Whitney number of the form $w_1^{2a}w_2^{2b}[P^n]$ with $a>0$.*

PROOF. Suppose n is odd. Then

$$w_1^{2p+1}w_2^{2q}[P^n]=Sq^1(w_1^{2p}w_2^{2q})[P^n]=0$$

and

$$\begin{aligned} w_1^{2p+1}w_2^{2q+1}[P^n] &= v_3(w_1^{2p}w_2^{2q})[P^n] \\ &= Sq^3(w_1^{2p}w_2^{2q})[P^n] \\ &= 0\,, \end{aligned}$$

for one has the Wu classes $v_1=w_1$, $v_3=w_1w_2$ and $Sq^{2j+1}(x^2)=0$ always. Thus all numbers of P involving w_1 must be zero, which is a contradiction.

Now assuming n even, suppose $w_1^{2a}w_2^{2b+1}[P^n]\neq 0$, $a>0$. Then

$$\begin{aligned} 0\neq w_1^{2a}w_2^{2b+1}[P^n] &= (v_2+w_1^2)(w_1^{2a}w_2^{2b})[P^n] \\ &= Sq^2(w_1^{2a}w_2^{2b})[P^n]+w_1^{2a+2}w_2^{2b}[P^n] \\ &= \{Sq^1(w_1^a w_2^b)\}^2[P^n]+w_1^{2a+2}w_2^{2b}[P^n]\,. \end{aligned}$$

Since $Sq^1w_2=w_3+w_2w_1$, this gives

$$0\neq(a+b+1)w_1^{2a+2}w_2^{2b}[P^n]\,,$$

and so $w_1^{2a+2}w_2^{2b}[P^n]\neq 0$, i.e. there is a nonzero number of P involving w_1 for which w_1 and w_2 both occur to even powers. □

LEMMA 4. *There are integers $0\leqq r\leqq s$ with $n=2^{r+1}+2^{s+1}$, or $n=2$.*

PROOF. Letting $w_1^{2a}w_2^{2b}[P^n]\neq 0, a>0$, one has $v_{n/2}\cdot w_1^a w_2^b[P]=Sq^{n/2}(w_1^a w_2^b)[P]$ $=w_1^{2a}w_2^{2b}[P]\neq 0$ and so $w_1v_{n/2}\neq 0$. Now $w_1v=w_1Sq^{-1}w(P)=w_1Sq^{-1}(1+w_1+w_2)$. Using the splitting principle, one may let $1+w_1+w_2=(1+x)(1+y)$ and then

$$\begin{aligned} Sq^{-1}\{(1+x)(1+y)\} &= (1+x+x^2+x^4+\cdots+x^{2^i}+\cdots)(1+y+y^2+\cdots+y^{2^j}+\cdots) \\ &= \sum_{i,j} x^{2^i}y^{2^j}\,. \end{aligned}$$

Thus $w_1v_i\neq 0$ only if $i=2^p+2^q$ for some p, q or $i=2^0$. There are then integers $0\leqq r\leqq s$ with $n/2=2^r+2^s$ or $n/2=1$. □

Note. $1+w_1+w_2$ is not the Stiefel-Whitney class of a 2-plane bundle, but after multiplying by w_1 it satisfies the same formulae over the Steenrod algebra as if it were. The use of the splitting principle is then justifiable.

Note. One has $w_1v_{2^k}=w_1(w_1^{2^k}+w_2^{2^{k-1}})$ if $k\geqq 1$, for the term $x^{2^k}+y^{2^k}+(xy)^{2^{k-1}}$ is the part of $\sum_{i,j} x^{2^i}y^{2^j}$ of degree 2^k.

LEMMA 5. *There are integers α_j, independent of P, with*

$$w_1^{n-2j}w_2^j[P^n]=\alpha_j w_1^n[P^n]$$

for $n-2j>0$.

PROOF. Let $\bar{w}=1/w$ be the dual Stiefel-Whitney class. Then

$$w_1\bar{w}=w_1(1/(1+w_1+w_2))$$
$$=w_1\left(\sum_k\left\{\sum_{q=0}^{[k/2]}\binom{k-q}{q}w_1^{k-2q}w_2^q\right\}\right),$$

and for $n-2j>0$

$$w_1^{n-2j}\bar{w}_{2j}=\sum_{q=0}^{j}\binom{2j-q}{q}w_1^{n-2q}w_2^q,$$

or

$$w_1^{n-2j}w_2^j[P]=\binom{j}{j}w_1^{n-2j}w_2^j[P]$$
$$=\sum_{q=0}^{j-1}\binom{2j-q}{q}w_1^{n-2q}w_2^q[P]+w_1^{n-2j}\bar{w}_{2j}[P]$$
$$=\sum_{q=0}^{j-1}\binom{2j-q}{q}w_1^{n-2q}w_2^q[P]+\chi(Sq^{2j})w_1^{n-2j}[P]$$
$$=\sum_{q=0}^{j-1}\binom{2j-q}{q}w_1^{n-2q}w_2^q[P]+\binom{n+2j}{n}w_1^n[P].$$

Induction on j gives the integer α_j iteratively by

$$\alpha_j=\sum_{q=1}^{j-1}\binom{2j-q}{q}\alpha_q+\left\{1+\binom{n+2j}{n}\right\}. \quad \square$$

COROLLARY. *If* $n=2^{s+1}$, $s\geqq 0$, *and* M^n *is a closed smooth manifold satisfying* $w_\omega[M^n]=0$ *for* $\omega\neq(2^{i_1},\cdots,2^{i_r})$, *then* M *is cobordant to a union of the manifolds* $(RP^{2^{s+1-t}})^{2^t}$ *with* $0\leqq t\leqq s$.

PROOF. One may induct on s. For $n=2$, the only nonzero class in $\mathfrak{N}_2$ is the class of RP^2, with $w(RP^2)=1+w_1+w_2$. Then let $n=2^{s+1}$, $s>0$, and assume the result known for $n'=2^{s'+1}$, $s'\geqq 0$ if $s'<s$. If all numbers of M^n divisible by w_1 are zero, M is cobordant to $(M')^2$, with M' satisfying the same hypotheses, so M' is cobordant to a union of $RP^{2^s},\cdots,(RP^2)^{2^{s-1}}$, and squaring that union gives the result for M. If some number of M^n involving w_1 is nonzero, then by the Lemma, one must have $w_1^n[M^n]\neq 0$. Then $M^n\cup RP^{2^{s+1}}$ satisfies the hypotheses and has w_1^n zero, hence all numbers involving w_1, and so is a union of the $(RP^{2^{s+1-t}})^{2^t}$ with $0<t\leqq s$. (*Note.* The use of P is implicit in obtaining $w_1^n[M^n]=w_1^n[P^n]\neq 0$.) $\square$

Observation. One has $[M^{2^s+1}]=\sum_{t=0}^{s}\alpha_t[(RP^{2^{s+1-t}})^{2^t}]$ in $\mathfrak{N}_*$, where $\alpha_t=w_{2^t}^{2^{s+1-t}}[M^{2^{s+1}}]$. The only other Stiefel-Whitney number of M which can possibly be nonzero is $w_{2^{s+1}}[M^{2^{s+1}}]$ which is $\sum_{t=0}^{s}\alpha_t=\sum_{t=0}^{s}w_{2^t}^{2^{s+1-t}}[M^{2^{s+1}}]$.

§ 3. The case $n=2^{r+1}+2^{s+1}$

Having completely settled what happens for n a power of 2, one is now reduced to

ASSUMPTION. $n=2^{r+1}+2^{s+1}$ with $0\leqq r<s$.

LEMMA 6. *For j odd, $n-2j>0$, $w_1^{n-2j}w_2^j[P^n]=0$.*

PROOF. One proves this by induction on j. Since n is even, $n-2j>0$ gives $n-2j-1>0$ and

$$\begin{aligned} w_1^{n-2j-1}\bar{w}_{2j+1}[P^n]&=\chi(Sq^{2j+1})w_1^{n-2j-1}[P^n]\\ &=\chi(Sq^{2j})\chi(Sq^1)w_1^{n-2j-1}[P^n]\\ &=\chi(Sq^{2j})w_1^{n-2j}[P^n]\\ &=w_1^{n-2j}\bar{w}_{2j}[P^n]. \end{aligned}$$

Thus

$$\begin{aligned} 0&=\{w_1^{n-2j}\bar{w}_{2j}+w_1^{n-2j-1}\bar{w}_{2j+1}\}[P^n]\\ &=\sum_{q=0}^{j}\left\{\binom{2j-q}{q}+\binom{2j+1-q}{q}\right\}w_1^{n-2q}w_2^q[P^n]. \end{aligned}$$

For q even, $\binom{2j+1-q}{q}\equiv\binom{2j-q}{q}$ mod 2 and so

$$jw_1^{n-2j}w_2^j[P^n]=\sum_{\substack{0<q<j\\ q\text{ odd}}}\left\{\binom{2j-q}{q}+\binom{2j+1-q}{q}\right\}w_1^{n-2q}w_2^q[P^n]$$

which inductively is zero. □

LEMMA 7. *For $n-2j>0$, $j\not\equiv 0(2^{r+1})$, $w_1^{n-2j}w_2^j[P^n]=0$.*

PROOF. For $r=0$, Lemma 6 gives this, and hence one supposes $r>0$. Then suppose inductively that one has $w_1^{n-2q}w_2^q[P^n]=0$ if $n-2q>0$, $q\not\equiv 0(2^k)$ and $2^k<2^{r+1}$, which holds for $k=1$ by Lemma 6.

Consider $w_1^{n-2^{k+1}t}w_2^{2^k t}$ with $n-2^{k+1}t>0$. One has $n-2^{k+1}t-2^k>0$, since $n-2^{k+1}\equiv 0(2^{k+1})$ and is positive. Thus

$$
\begin{aligned}
0&=w_1^{n-2(2^{k-1}(2t+1))}\cdot w_2^{2^{k-1}(2t+1)}[P^n]\\
&=w_2^{2^{k-1}}(w_1^{n-2^k(2t+1)}w_2^{2^k t})[P^n]\\
&=(v_{2^k}+w_1^{2^k})(w_1^{n-2^k(2t+1)}w_2^{2^k t})[P^n]\\
&=\{Sq^{2^k}(w_1^{n-2^k(2t+1)}w_2^{2^k t})+w_1^{n-2^{k+1}t}w_2^{2^k t}\}[P^n]\\
&=[\{Sq^1(w_1^{2n/2^k-(2t+1)}w_2^t)\}^{2^k}+w_1^{n-2^{k+1}t}w_2^{2^k t}][P^n]\\
&=[(n/2^k-(2t+1)+t)\{w_1^{n/2^k-2t}w_2^t\}^{2^k}+w_1^{n-2^{k+1}t}w_2^{2^k t}][P^n]\\
&=(n/2^k-t)w_1^{n-2^{k+1}t}w_2^{2^k t}[P^n].
\end{aligned}
$$

For t odd, $n/2^k$ is even and one has $w_1^{n-2^{k+1}t}w_2^{2^k t}[P^n]$, i.e. $w_1^{n-2q}w_2^q[P^n]=0$ if $n-2q>0$ and $q=2^k t\equiv 2^k \bmod(2^{k+1})$. Thus $w_1^{n-2q}w_2^q[P^n]=0$ if $n-2q>0$ and $q\not\equiv 0(2^{k+1})$, completing the inductive step. □

LEMMA 8. $w_1^n[P^n]=0$.

PROOF. With $n=2^{r+1}+2^{s+1}$, $0\leqq r<s$, one uses the formula of Lemma 5, for $w_1^{n-2^{r+1}}w_2^{2^r}$, i.e.

$$
w_1^{n-2^{r+1}}w_2^{2^r}[P^n]=\sum_{q=0}^{2^r-1}\binom{2^{r+1}-q}{q}w_1^{n-2q}w_2^q[P^n]+\binom{n+2^{r+1}}{2^{r+1}}w_1^n[P^n],
$$

noting by Lemma 6 that $w_1^{n-2q}w_2^q[P^n]=0$ for $1\leqq q\leqq 2^r$ to obtain

$$
\left\{\binom{n+2^{r+1}}{2^{r+1}}+\binom{2^{r+1}}{0}\right\}w_1^n[P^n]=0\,.
$$

However, $\binom{2^{s+1}+2^{r+1}+2^{r+1}}{2^{r+1}}\equiv 0 \bmod 2$ and $\binom{2^{r+1}}{0}\equiv 1 \bmod 2$, so $w_1^n[P^n]=0$. □

This gives a contradiction. Specifically assuming P^n has a non-zero number involving w_1, one has $w_1^n[P^n]=0$, and by Lemma 5, $w_1^{n-2j}w_2^j[P]=\alpha_j w_1^n[P]=0$ if $n-2j>0$, and all numbers involving w_1 are zero. Thus, the assumption $n=2^{r+1}+2^{s+1}$ with $0\leqq r<s$ leads to a contradiction. Such a $P^{n'}$ had to exist beginning with a nonbounding M^n in Lemma 1, $n=2^t n'$. Thus, one has

Observation. If M^n is nonbounding, and $w_\omega[M]=0$ for $\omega\neq(2^{i_1},\cdots,2^{i_r})$ then $n=2^{r+1}+2^{s+1}$ with $r=s\geqq 0$, or $n=2$, i.e. n is a power of 2.

§ 4. Remarks

1) For manifolds satisfying Yoshida's property, one has additional restrictions arising due to Hopf invariant one phenomena. To illustrate this one has

PROPOSITION. *Let M^n be a manifold with $w_i(M)=0$ for $i\neq 2^s$. If there is a $t\geqq 4$ with $w_{2^t-1}(M)w_{2^t}(M)=0$, then $w_{2^t}^{n/2^t}[M^n]=0$ if either*

a) $n>2^{t+1}$, *or*

b) $n=2^{t+1}$ *and* $w_i(M)=0$ *for* $i<2^t$.

PROOF. It is tacitly assumed that 2^t divides n, so that $w_{2^t}^{n/2^t[M]}$ is a Stiefel-Whitney number. In order to be nonzero one must have $n=2^{r+t}$, for some r.

One has $Sq^i w_{2^t}=w_i w_{2^t}$ $1\leqq i\leqq 2^t$ since $w_{2^t+j}=0$ for $1\leqq j<2^t$, and by Yoshida ([5], (4.2)) $w_{2^i}w_{2^j}=0$ if $j>i+1$ (as in Lemma 2, by calculating $Sq^{2^{i+1}}w_{2^j-2^i}$). Thus $Sq^{2^i}w_{2^t}=0$ for $0\leqq i<t$, and $w_{2^t}^2=Sq^{2^t}w_{2^t}=\sum_\lambda P^\lambda(\Phi_\lambda w_{2^t})$ where P^λ are primary cohomology operations, $0<\deg P^\lambda<2^t$, and $\Phi_\lambda w_{2^t}$ is given by Adams' secondary operations modulo "indeterminancy" (Adams [1]).

Then

$$w_{2^t}^{2^r}[M]=w_{2^t}^2\cdot w_{2^t}^{2^r-2}[M]=\left\{\sum_\lambda P^\lambda(\Phi_\lambda w_{2^s})\cdot w_{2^t}^{2^r-2}\right\}[M]$$

$$=\sum_\lambda P^\lambda\{(\Phi_\lambda w_{2^t})\cdot w_{2^t}^{2^r-2}\}[M]=\sum_\lambda v_\lambda\cdot(\Phi_\lambda w_{2^t})\cdot w_{2^t}^{2^r-2}[P]$$

where $(v_\lambda x)[M]=(P^\lambda x)[M]$ giving a Wu-type class. Now v_λ is a polynomial in $w_1, w_2, \cdots, w_{2^t-1}$ and so is zero in the case $n=2^{t+1}$, and for $n>2^{t+1}$, $v_\lambda w_{2^t}^{2^r-2}=0$. □

Note. With the hypotheses for the case $n=2^{t+1}$, $w(M)=1+w_{2^t}+w_{2^{t+1}}$, and $w_{2^t}^2[M]=0$ implies that M bounds. The case $n>2^{t+1}$ is more interesting since the hypotheses are far less restrictive. If $w(M)=1+w_1+w_2+\cdots+w_{2^s}+\cdots$ with gaps in the powers of 2 appearing, some of the Stiefel-Whitney numbers of M which one anticipated are simply zero.

2) One should note that there are examples of manifolds having Yoshida's property. Specifically

a) $\boldsymbol{R}P^{2^s}$, $\boldsymbol{C}P^{2^s}$, $\boldsymbol{H}P^{2^s}$, Cayley P^2: $w=1+\alpha+\alpha^{2^s}$.

b) Connected sums of these: e.g. $\boldsymbol{R}P^{2^{s+1}}\#\boldsymbol{C}P^{2^s}$ with $w=1+w_1+w_2$ effectively cancelling out the top class.

c) Disjoint unions: e.g. $\boldsymbol{R}P^{2^{s+1}}\cup\boldsymbol{C}P^{2^s}$ with $w=1+w_1+w_2+w_{2^{s+1}}$, not cancelling at the top.

d) The space $FP(2^s-1, 2^s-1, 0)$ of F-lines in the fibers of the F vector bundle $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ over $FP^{2^s-1} \times FP^{2^s-1} \times FP^0$, with $F=\boldsymbol{R}$, $\boldsymbol{C}$, or $\boldsymbol{H}$ and $s \geqq 0$. This is nonbounding and has

$$w=\begin{cases} 1+w_1+w_2, & F=\boldsymbol{R}, \\ 1+w_2+w_4, & F=\boldsymbol{C}, \\ 1+w_4+w_8, & F=\boldsymbol{H}. \end{cases}$$

Notes. Yoshida (5.21) gives $\boldsymbol{R}P(2^s-1, 2^s-1, 0)$ for $s=1, 2, 3$, but all s work. In the $\boldsymbol{H}$-case, the usual projective bundle theorems are not available, but one can verify the above. No Cayley number analogue exists—even for $s=0$; the Cayley plane is not the space of lines in $R^{24}=(\text{Cayley})^3$. The case of $\boldsymbol{H}(2^s-1, 2^s-1, 0)$ is curious, since none of the previous examples have w of this form, except for $\boldsymbol{H}P^2$ and $\boldsymbol{H}P^4 \# \text{Cayley } P^2$.

3) It is immediate (Yoshida [5], (5.4) (iii)) that a manifold M^n with $w(M)=1+w_i$ bounds. For $n \not\equiv 0(i)$, one can form no nontrivial Stiefel-Whitney numbers. For $n \equiv 0(i)$, $(w_i)^{n/i}=\bar{w}_n=0$. In particular, if $w(M)=1+w_1$, M bounds. However, the possible classes of manifolds M^n with $w(M)=1+w_1+w_2$ were not previously characterized. One now knows that the only nonzero classes are the classes of $RP^{2^{s+1}} \# \boldsymbol{C}P^{2^s}$ or equivalently $RP(2^s-1, 2^s-1, 0)$. One would like to know the possible classes of manifolds M^n for which the tangent bundle is (stably) equivalent to a 2-plane bundle. For RP^2, $RP(1, 1, 0)$, $RP(3, 3, 0)$, and $RP(7, 7, 0)$ the tangent bundle actually reduces to a 2-plane bundle (the bundle along the fibers) since RP^0, RP^1, RP^3, and RP^7 are parallelizable.

The stable question asks for the image of the forgetful homomorphism $\pi_*^s(M(\gamma_2^\perp)) \to \pi_*^s(MO) \cong \mathfrak{N}_*$ from the stable homotopy of the Thom space of the complement $\gamma_2^\perp$ of the universal bundle over BO_2. If one begins with M^n having $\tau \cong \xi^2$ stably, the submanifold of M dual to ξ is framable, and that dual to 2ξ, P^{n-4}, immerses in codimension 2, with $\bar{w}_2^{(n-4)/2}[P^{n-4}]=w_2^{n/2}[M^n]$. Applying this construction to $RP(2^s-1, 2^s-1, 0)$ gives a manifold cobordant to $(RP^{2^s-2})^2$, which immerses in codimension 2 for $0 \leqq s \leqq 3$.

As a partial converse, notice that if $\xi \to N^{n-2}$ is a 2-plane bundle over a framed manifold with $w_2(\xi)^{(n-2)/2}[N^{n-2}] \neq 0$ then $RP(\xi \oplus 1)$ has tangent bundle $(n-2) \oplus \theta^2$, where θ^2 is the bundle along the fibers, and $w_2^{n/2}[RP(\xi \oplus 1)] \neq 0$. Thus beginning with M having τ stably a 2-plane bundle, one obtains an example with τ reducing to a 2-plane bundle. (Note. V. Snaith and J. Tornehave assert in [3] that this occurs if and only if there is a framed manifold P^{n-2} with nonzero Arf-Kervaire invariant.)

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