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Principal zeta-function of non-degenerate complete intersection singularity

Dedicated to Professor A. Hattori on his sixtieth birthday

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Abstract. Let $\boldsymbol{f}=(f_1, \cdots, f_k):(\boldsymbol{C}^{n+k}, 0)\to(\boldsymbol{C}^k, 0)$ be a germ of an analytic mapping such that $V=\{z\in\boldsymbol{C}^{n+k}; f_1(z)=\cdots=f_k(z)=0\}$ is non-degenerate complete intersection variety with an isolated singularity at the origin. We give a formula for the principal zeta-function of the monodromy of the Milnor fibration. As a corollary, we obtain a formula for the zeta-function of iterated hyperplane sections of a Milnor fibration of a non-degenerate analytic function.

§ 1. Introduction.

Let $f(z)$ be a germ of an analytic function of $(n+k)$-variables with $f(\vec{0})=0$ and we assume that f is non-degenerate in the sense of the Newton boundary. Let $l_1(z), \cdots, l_{k-1}(z)$ be general linear forms and we consider the iterated hyperplane sections of f by $L=\{z\in\boldsymbol{C}^{n+k}; l_1(z)=\cdots=l_{k-1}(0)=0\}$. Namely let f' be the restriction of f to L. It is known that the Milnor fibration of f' is uniquely determined by $\Gamma(f)$ and the integer $k-1$ by a result of Teissier ([T]). However, as far as I know, we do not have any concrete formula for the zeta-function of the monodromy of f' in term of the Newton boundary $\Gamma(f)$ as the formula of Varchenko in the case of $k=1$. The case that $n=1$, $k=2$ is treated by Mima ([Mi]). The main difficulty comes from the fact that f' may be in general degenerate on some faces of the Newton boundary as a function of $(n+1)$-variables if we eliminate $(k-1)$-variables using $l_1=\cdots=l_{k-1}=0$. However the variety $f'^{-1}(\vec{0})$ is non-degenerate as a complete intersection variety $\{z\in\boldsymbol{C}^{n+k}; l_1(z)=\cdots=l_{k-1}(z)=f(z)=0\}$. This paper starts from this observation.

Let $\boldsymbol{f}=(f_1, \cdots, f_k):(\boldsymbol{C}^{n+k}, \vec{0})\to(\boldsymbol{C}^k, \vec{0})$ be a germ of an analytic mapping at $\vec{0}$ with $\boldsymbol{f}(\vec{0})=\vec{0}$. We assume that the fiber of the origin $V=\{z\in\boldsymbol{C}^{n+k}; f_1(z)=\cdots=f_k(z)=0\}$ is non-degenerate in the sense of the Newton boundary. (See § 4 for the definition.) We consider the Milnor fibration of $\boldsymbol{f}$ at the origin. Then our original problem is reduced to the determination of the zeta-function of the monodromy along the path $f_1=\cdots=f_{k-1}=0$, $|f_k|=\varepsilon$ i.e., the k-th principal zeta-function. (See § 3.) A key step for the determination of the principal zeta-function is to calculate the topological Euler-Poincaré characteristic of the non-degenerate complete intersection variety in the affine torus

$$Z=\{\boldsymbol{y}=(y_1, \cdots, y_m)\in \boldsymbol{C}^{*m}\,;\ h_1(\boldsymbol{y})=\cdots=h_k(\boldsymbol{y})=0\}$$

in the words of the respective Newton polyhedra $\{\Delta(h_i)\}_{i=1,\cdots,k}$. This is done by Khovanskii in [Kh 2]. Now using a toroidal resolution of V and a theorem of A'Campo [A], we obtain our main result Theorem (6.8) which gives a formula about the principal zeta-function $\zeta_k(t)$ as follows.

$$\zeta_k(t)=\prod_{|I|\geq k}\ \prod_{Q\in\mathcal{S}_I}(1-t^{d(Q;f_k^I)})^{-\chi(Q)}.$$

Here $\mathcal{S}_I$ is a finite set which is called the I-data set of V and it depends only on the restriction of $\boldsymbol{f}$ to $\boldsymbol{C}^I$. $\chi(Q)$ is an integer which depends only on $\Delta(Q\,;f_1^I), \cdots, \Delta(Q\,;f_k^I)$. This formula is a generalization of the result of Varchenko in the case of non-degenerate hypersurfaces ([V]). We also gives a formula about the Milnor number as an immediate corollary. The zeta-function of the iterated hyperplane sections is obtained simultaneously (See § 8.). This paper is composed as follows:

§ 2. Minkowski's mixed volume and a theorem of Bernshtein.

We first recall the definition of Minkowski's mixed volume. For details, see Busemann [Bs]. Let $S_1, \cdots, S_m$ be compact convex polyhedra in $\boldsymbol{R}^m$ and let $\lambda_1, \cdots, \lambda_m$ be positive real numbers. It is known that $\mathrm{Vol}_m(\lambda_1 S_1+\cdots+\lambda_m S_m)$ is a homogeneous polynomial of the variables $\lambda_1, \cdots, \lambda_m$ where Vol_m is the m-dimensional Euclidean volume. Minkowski's mixed volume $V_m(S_1, \cdots, S_m)$ is defined by (the coefficient of $\lambda_1\cdots\lambda_m)/m!$ in the polynomial $\mathrm{Vol}_m(\lambda_1 S_1+\cdots+\lambda_m S_m)$. It is also known that $V_m(S_1, \cdots, S_m)$ is a symmetric multilinear function of the 'variables' $S_1, \cdots, S_m$. $V_m(S_1, \cdots, S_m)$ is invariant under the parallel translation of each variable $S_i\mapsto S_i+\boldsymbol{a}_i$ for any vector $\boldsymbol{a}_i$. It satisfies the following equality ([B]).

$$m!\,V_m(S_1, \cdots, S_m) = \sum_{I \subset \{1,\cdots,m\}} (-1)^{m-|I|} \mathrm{Vol}_m\Big(\sum_{i \in I} S_i\Big). \tag{2.1}$$

As a special case, we have

$$V_m(S, \cdots, S) = \mathrm{Vol}_m(S). \tag{2.2}$$

The case that $S_1 = \cdots = S_s = S$ and $S_{s+1} = \cdots = S_m = K$ is particularly important. We denote the corresponding Minkowski's mixed volume by $V_{m,m-s}(S, K)$. Then we have the following equality.

$$\mathrm{Vol}_m(\lambda S + \mu K) = \sum_{s=0}^{m} \binom{m}{s} \lambda^{m-s} \mu^s V_{m,s}(S, K). \tag{2.3}$$

More generally, let $S_1, \cdots, S_k$ be compact convex polyhedra such that each S_i is generated by integral vertices and $\dim(S_1 + \cdots + S_k) = k$. By taking a parallel translation if necessary, we may assume that the affine subspace of minimal dimension which includes S_i contains the origin for $i = 1, \cdots, k$. Take a unimodular matrix σ such that the respective images $\sigma(S_1), \cdots, \sigma(S_k)$ are included in $\boldsymbol{R}^k = \{\boldsymbol{x} \in \boldsymbol{R}^m ; x_j = 0 \ (j > k)\}$. We define *the generalized Minkowski's mixed volume* $V_k(S_1, \cdots, S_k)$ by the Minkowski's mixed volume $V_k(\sigma(S_1), \cdots, \sigma(S_k))$.

Let $h(\boldsymbol{y}) = \sum_{i=1}^N a_i \boldsymbol{y}^{\nu_i}$ be a Laurent polynomial. The Newton polyhedron $\Delta(h)$ of h is, by definition, the minimal convex polyhedron which contains $\nu_1, \cdots, \nu_N$. Here we have assumed that $a_i \neq 0$ for each i. Let P be any covector. Let $d(P; h) = \min\{P(\nu_1), \cdots, P(\nu_N)\}$. The face function $h_P(\boldsymbol{y})$ is defined as usual by the partial sum $\sum_{P(\nu_i) = d(P;h)} a_i \boldsymbol{y}^{\nu_i}$. Now we recall a beautiful theorem of Bernshtein which plays an important role in this paper. Let $h_1(\boldsymbol{y}), \cdots, h_m(\boldsymbol{y})$ be given Laurent polynomials and let $Z = \{\boldsymbol{y} \in \boldsymbol{C}^{*m} ; h_1(\boldsymbol{y}) = \cdots = h_m(\boldsymbol{y}) = 0\}$. We say that Z is *non-degenerate in the weak sense* if the following condition is satisfied. (a) For any non-zero covector P, the system of equations $\{h_{1P}(\boldsymbol{y}) = \cdots = h_{mP}(\boldsymbol{y}) = 0\}$ has no solution in $\boldsymbol{C}^{*m}$. We say that Z is *non-degenerate* if the following condition is also satisfied. (b) The holomorphic m-form $dh_1 \wedge \cdots \wedge dh_m$ is nowhere vanishing on Z. This condition is equivalent to the simplicity of the solutions of Z.

THEOREM (2.4) ([B]). *Assume that Z is non-degenerate in the weak sense. Then the number of Z (counting the multiplicity) is equal to $m!\,V_m(\Delta(h_1), \cdots, \Delta(h_m))$. If Z is non-degenerate, Z consists of $m!\,V_m(\Delta(h_1), \cdots, \Delta(h_m))$ simple points.*

By the definition, it is obvious that Minkowski's mixed volume is non-negative and $V_m(S_1, \cdots, S_m)=0$ if $\dim(S_1+\cdots+S_m)<m$. However this is not a necessary condition for $V_m(S_1, \cdots, S_m)=0$. In fact, a necessary and sufficient condition for the positivity is the following.

PROPOSITION (2.5). *$V_m(S_1, \cdots, S_m)>0$ if and only if the following condition is satisfied.*

(A_0) *For any subset I of $\{1, \cdots, m\}$, $\dim \sum_{i\in I} S_i \geqq |I|$. (Here $|I|$ is the cardinality of I.)*

PROOF. First assume that $\dim \sum_{i\in I} S_i<|I|$ for some I. To show that $V_m(S_1, \cdots, S_m)=0$, we way assume that each S_i is generated by rational vertices as $V_m(S_1, \cdots, S_m)$ is continuous in $S_1, \cdots, S_m$. Multiplying a suitable integers and using the linearity of $V_m(S_1, \cdots, S_m)$ in S_i, we may even assume that each S_i is generated by integral vertices. Let $h_1(\boldsymbol{y}), \cdots, h_m(\boldsymbol{y})$ be Laurent polynomials in $\boldsymbol{y}=(y_1, \cdots, y_m)$ such that $\Delta(h_i)=S_i$ and the system of equation $h_1(\boldsymbol{y})=\cdots=h_m(\boldsymbol{y})=0$ is non-degenerate. Let $s=\dim \sum_{i\in I} S_i$. Using a Laurent coordinate change if necessary (see § 5 for definition), we may assume that $h_i(\boldsymbol{y})$ contains only the variables $y_1, \cdots, y_s$ for $i\in I$. By Theorem (2.4) and the assumption $s<|I|$, the subsystem of equation $\{\boldsymbol{y}\in \boldsymbol{C}^{*s}\,;\, h_i(\boldsymbol{y})=0,\ i\in I\}$ has no solution. Thus $h_1(\boldsymbol{y})=\cdots=h_m(\boldsymbol{y})=0$ has no solution. Again by Theorem (2.4), this implies that $V_m(S_1, \cdots, S_m)=0$. Now we show the opposite direction. Assume that the (A_0)-condition is satisfied. First note that $\dim S_i\geqq 1$ by the (A_0)-condition.

ASSERTION (2.5.1). *Assume that (A_0)-condition is satisfied. If $\dim S_i>1$, there exists a codimension one boundary face Δ_i of S_i such that $\{S_1, \cdots, \Delta_i, \cdots, S_m\}$ satisfies the (A_0)-condition.*

PROOF OF ASSERTION (2.5.1). We assume that the assertion is false. Let Δ_i and Δ_i' be codimension one boundary faces of S_i such that $\dim(\Delta_i+\Delta_i')=\dim S_i$. Then by the assumption there are subsets I and J of $\{1, \cdots, m\}$ such that $i\notin I\cup J$ and

$$\dim\Big(\Delta_i+\sum_{j\in I} S_j\Big)\leqq |I|\,, \qquad \dim\Big(\Delta_i'+\sum_{j\in J} S_j\Big)\leqq |J|\,.$$

As $\dim \sum_{i\in I\cap J} S_i\geqq |I\cap J|$ by the (A_0)-condition, we have that

$$\dim\Big(\Delta_i+\Delta_i'+\sum_{j\in I\cup J} S_j\Big)\leqq |I|+|J|-|I\cap J|=|I\cup J|\,.$$

This implies that $\dim(S_i+\sum_{j\in I\cup J} S_j)\leqq |I\cup J|$, contradicting the ($\mathrm{A}_0$)-condition on $S_1, \cdots, S_m$.

We apply the assertion successively to obtain one-dimensional faces L_i

of S_i respectively for $i=1, \cdots, m$ so that $\{L_1, \cdots, L_m\}$ satisfies (A_0)-condition. Let $L_i=\overline{P_iQ_i}$ and let $\vec{L}_i=P_i-Q_i$. Then it is easy to see that $m!\, V_m(L_1, \cdots, L_m) = |\det(\vec{L}_1, \cdots, \vec{L}_m)|$. The ($A_0$)-condition for $\{L_1, \cdots, L_m\}$ implies that $|\det(\vec{L}_1, \cdots, \vec{L}_m)|>0$. By the monotone property of Minkowski's mixed volume (see [Bs]), we have the inequality

$$V_m(S_1, \cdots, S_m) \geqq V_m(L_1, \cdots, L_m)>0\,.$$

This completes the proof of Proposition (2.5).

§ 3. Complete intersection variety and Milnor fibration.

Let $\boldsymbol{f}=(f_1, \cdots, f_k) : (\boldsymbol{C}^{n+k}, \vec{0}) \to (\boldsymbol{C}^k, \vec{0})$ be a germ of an analytic mapping such that $\boldsymbol{f}(\vec{0})=\vec{0}$. We assume that $V=\boldsymbol{f}^{-1}(\vec{0})$ is a germ of a complete intersection variety with an isolated singularity at $\vec{0}$. By an abuse of language, we assume that $\boldsymbol{f}$ is defined in a neighborhood of the origin. Let ε_0 be a sufficiently small positive number so that any sphere S_ε ($\varepsilon \leqq \varepsilon_0$) meets transversely with V. Here S_ε is the sphere of radius ε i.e., $S_\varepsilon = \{\boldsymbol{z} \in \boldsymbol{C}^{n+k} ; \|\boldsymbol{z}\|=\varepsilon\}$. Let U be a sufficiently small neighborhood of the origin $\vec{0}$ of the target space $\boldsymbol{C}^k$ such that S_{ε_0} meets transversely with any fiber $\boldsymbol{f}^{-1}(s)$ with $s \in U$. Let $B_\varepsilon=\{\boldsymbol{z} ; \|\boldsymbol{z}\| \leqq \varepsilon\}$ and let $D_{\boldsymbol{f}}$ be the set of the critical values of the restriction $\boldsymbol{f}|\boldsymbol{f}^{-1}(U)\cap B_{\varepsilon_0}$. Let $X^*=\boldsymbol{f}^{-1}(U)\cap B_{\varepsilon_0}-\boldsymbol{f}^{-1}(D_{\boldsymbol{f}})$. Then by the fibration theorem of Ehresmann ([W]), $\boldsymbol{f}: X^* \to U-D_{\boldsymbol{f}}$ is a C^∞-fibration. This fibration is called the Milnor fibration of $\boldsymbol{f}$ at the origin. Let F be the fiber. It is known that F is homotopically a bouquet of n-dimensional spheres by Milnor for $k=1$ ([M]) and by Hamm for the general case ([H1]). See also [L], [Di 2] and [Lo]. The Milnor number is defined by the n-th Betti number of F and we denote the Milnor number by μ. Let $\rho_i : \pi_1(U-D_{\boldsymbol{f}}) \to \mathrm{Aut}(H_i(F ; Q))$ be the monodromy representation homomorphism. For each $g \in \pi_1(U-D_{\boldsymbol{f}})$, we associate the corresponding zeta-function $\zeta_g(t)$ by the formula $\zeta_g(t)=P_0(t ; g)^{-1}P_n(t ; g)^{(-1)^{n-1}}$ where $P_i(t ; g)= \det(I-t\rho_i(g))$. Note that the Milnor number μ can be computed from the equality:

$$\text{(3.1)} \qquad \operatorname{degree} \zeta_g(t)=-1+(-1)^{n-1}\mu\,.$$

Let $V_{k-1}=\{\boldsymbol{z} \in B_{\varepsilon_0} ; f_1(\boldsymbol{z})= \cdots =f_{k-1}(\boldsymbol{z})=0\}$. We assume that V_{k-1} is also a complete intersection variety with an isolated singularity at the origin. We are particularly interested in $\zeta_g(t)$ where g is represented by the loop $f_j=0$ ($j \neq i$) and $|f_i|=\varepsilon$ where ε is small enough. The corresponding monodromy is called the i-th *principal monodromy* ([Da]). For brevity's sake, we denote the corresponding zeta-function by $\zeta_i(t)$ and we call $\zeta_i(t)$ *the*

i-th principal zeta-function. If $\{f_j\}_{j=1,\cdots,k-1}$ are generic linear forms, $\zeta_k(t)$ is equal to the zeta-function of the Milnor fibration of the function f_k^L which is the restriction of f_k to the $(n+1)$-dimensional plane $L=\{f_1=\cdots=f_{k-1}=0\}$. Let $\pi:\tilde{V}_{k-1}\to V_{k-1}$ be a resolution of V_{k-1} and let f'_k be the composition $f_k\circ\pi$. We assume that the following conditions are satisfied. Let $\tilde{V}_k$ be the proper transform of V_k by π and let $f_k'^{-1}(0)=\tilde{V}_k\cup D_1\cup\cdots\cup D_s$ where D_i is assumed to be irreducible. Then (1) $\pi:\tilde{V}_k\to V_k$ is a resolution of V_k. (2) Each exceptional divisor D_i is smooth and $f_k'^{-1}(0)$ has only normal crossing singularities. Let m_a be the order of zeros of the function f'_k along D_a and let $D'_a=D_a-(\cup_{i\neq a}D_i)\cup\tilde{V}_k$. The following theorem is due to A'Campo ([A]).

THEOREM (3.2). *Under the above assumption, we have the equality*:

$$\zeta_k(t)=\prod_{i=1}^{s}(1-t^{m_i})^{-\chi(D'_i)}$$

where $\chi(D'_i)$ is the topological Euler-Poincaré characteristic of D'_i.

§ 4. Non-degenerate complete intersection variety and its resolution.

Let $f(\boldsymbol{z})=\sum_\nu a_\nu \boldsymbol{z}^\nu$ be an analytic function of $(n+k)$-variables which is defined in a neighborhood of the origin. The Newton polyhedron $\Gamma_+(f)$ is the convex hull of the union of $\{\nu+\boldsymbol{R}_+^{n+k}\}$ for ν such that $a_\nu\neq 0$. The Newton boundary $\Gamma(f)$ is the union of the compact faces of the Newton polyhedron $\Gamma_+(f)$. The dual space of $\boldsymbol{R}^{n+k}$ can be canonically identified with $\boldsymbol{R}^{n+k}$ itself by the Euclidean inner product. Let N be the set of integral dual vectors under this identification and let N^+ be the set of positive integral dual vectors. A weight vector or covector is a synonym of an integral dual vector in this paper. We use the column vectors to show the dual vectors hereafter. Let $P={}^t(p_1,\cdots,p_{n+k})$. For each $\boldsymbol{x}\in\boldsymbol{R}^{n+k}$, $P(\boldsymbol{x})$ is defined by $\boldsymbol{x}P=\sum_{i=1}^{n+k}p_ix_i$. P is called a positive (respectively a strictly positive) dual vector if $p_i\geqq 0$ (respectively $p_i>0$) for $i=1,\cdots,n+k$. The notation $P\gg 0$ shows that P is strictly positive. For a positive integral dual vector $P\in N^+$, we define $d(P;f)$ as the minimal value of the restriction $P|\Gamma_+(f)$ i.e., $d(P;f)=\min\{P(\boldsymbol{x});\boldsymbol{x}\in\Gamma_+(f)\}$ and let $\Delta(P;f)=\{\boldsymbol{x}\in\Gamma_+(f);P(\boldsymbol{x})=d(P;f)\}$. We define $f_P(\boldsymbol{z})=f_{\Delta(P;f)}(\boldsymbol{z})$ where $f_\Delta(\boldsymbol{z})=\sum_{\nu\in\Delta}a_\nu\boldsymbol{z}^\nu$. We call $f_P(\boldsymbol{z})$ *the face function of f* with respect to P. We define the coordinate subspace $\boldsymbol{C}^I$ and $\boldsymbol{C}^{*I}$ by $\boldsymbol{C}^I=\{\boldsymbol{z}=(z_1,\cdots,z_{n+k});z_j=0,\ j\notin I\}$ and $\boldsymbol{C}^{*I}=\{\boldsymbol{z}\in\boldsymbol{C}^{n+k};z_j=0\Leftrightarrow j\notin I\}$ respectively. We also use the notations $f^I=f|\boldsymbol{C}^I$ and $N_I=\{P\in N;p_i=0 \text{ if } i\notin I\}$.

Let $f(z)=(f_1(z),\cdots,f_k(z))$ be an analytic mapping from a neighborhood U of the origin of C^{n+k} to C^k such that $f(\vec{0})=\vec{0}$. We say that the variety $V=\{z\in U\,;f_1(z)=\cdots=f_k(z)=0\}$ is a *non-degenerate complete intersection variety* at $\vec{0}$ (with respect to the Newton boundary) if for any strictly positive weight vector $P={}^t(p_1,\cdots,p_{n+k})$, the k-form $df_{1P}\wedge\cdots\wedge df_{kP}$ does not vanish on $V^*(P)=\{z\in C^{*(n+k)}\,;f_{1P}(z)=\cdots=f_{kP}(z)=0\}$ ([Kh 1]). Here $f_{\nu P}$ is the face function of f_ν with respect to P. Hereafter we assume that V is a non-degenerate complete intersection variety and that each f_a is convenient. Namely $f_a^{\{i\}}$ is not identically zero for any $a=1,\cdots,k$ and $i=1,\cdots,n+k$. We define an equivalent relation $\sim$ on the space of the positive dual vectors N^+ by $P\sim Q$ if and only if $\Delta(P\,;f_i)=\Delta(Q\,;f_i)$ for each $i=1,\cdots,k$. This defines a conical polyhedral subdivision $\Gamma^*(f_1,\cdots,f_k)$ of N^+ which we call the dual Newton diagram of $f=(f_1,\cdots,f_k)$. If we define $f=f_1\cdots f_k$, the dual Newton diagram $\Gamma^*(f)$ in the sense of [O 2] is equal to $\Gamma^*(f_1,\cdots,f_k)$. Let Σ^* be a fixed unimodular simplicial subdivision of $\Gamma^*(f_1,\cdots,f_k)$ and let $\hat{\pi}: X\to C^{n+k}$ be the associated toroidal modification map ([E], [V], [O 2]). Let $\tilde{V}$ be the proper transform of V and let $\pi:\tilde{V}\to V$ be the restriction of $\hat{\pi}$ to $\tilde{V}$. It is well-known that $\pi:\tilde{V}\to V$ is a good resolution of V. We assume that the set of the vertices of Σ^* which are not strictly positive is equal to $\{R_1,\cdots,R_{n+k}\}$ where $R_i={}^t(0,\cdots,\overset{i}{1},\cdots,0)$. This implies, in particular, that $\hat{\pi}: X-\hat{\pi}^{-1}(0)\to C^{n+k}-\{\vec{0}\}$ is biholomorphic.

We briefly recall the construction of X. X is covered by affine spaces C_σ^{n+k} with coordinate $y_\sigma=(y_{\sigma,1},\cdots,y_{\sigma,n+k})$ where σ moves in $(n+k)$-simplices of Σ^*. An $(n+k)$-simplex is always identified with a unimodular $(n+k)\times(n+k)$ matrix. If $\sigma=(P_1,\cdots,P_{n+k})$ is an $(n+k)$-unimodular matrix, the corresponding cone in Σ^* is defined by $\{\sum_{i=1}^{n+k}t_iP_i\,;t_1,\cdots,t_{n+k}\geqq 0\}$. $P_1,\cdots,P_{n+k}$ are called vertices of the simplex σ. Let $\sigma=(p_{ij})$ be an $(n+k)$-simplex. Then $\hat{\pi}|C_\sigma^{n+k}$ is defined by $\hat{\pi}(y_\sigma)=z=(z_1,\cdots,z_{n+k})$ where $z_i=\prod_{j=1}^{n+k}y_{\sigma j}^{p_{ij}}$. Let P be a vertex of Σ^*. Then P defines a divisor $\hat{E}(P)$ of X as follows. Let $\sigma=(P_1,\cdots,P_{n+k})$ be an $(n+k)$-simplex of Σ^* such that $P=P_1$. Then $\hat{E}(P)\cap C_\sigma^{n+k}$ is defined by the divisor $y_{\sigma 1}=0$. For an $(n+k)$-simplex τ, $\hat{E}(P)\cap C_\tau^{n+k}\neq\emptyset$ iff P is a vertex of τ. If $P\gg 0$, the union of $\{\hat{E}(P)\cap C_\sigma^{n+k}\,;P\in\sigma\}$ for σ is a compact toric variety of dimension $n+k-1$. For the general properties of the toric varieties, see [K-K-M-S] and [Od]. If $P\gg 0$, $\hat{E}(P)$ is an exceptional divisor i. e., $\hat{\pi}(\hat{E}(P))=\{\vec{0}\}$. On the other hand, $\hat{E}(R_i)$ is isomorphic to the hyperplane $\{z_i=0\}$ in the base space C^{n+k} by the projection $\hat{\pi}$. Let $\hat{E}(P)^*=\hat{E}(P)-\bigcup_{Q\neq P}\hat{E}(Q)$. This is isomorphic to the affine torus $C^{*(n+k-1)}$. For finite vertices $Q_1,\cdots,Q_s$ of Σ^*, we define a subvariety $\hat{E}(Q_1,\cdots,Q_s)$ of X by $\hat{E}(Q_1)\cap\cdots\cap\hat{E}(Q_s)$ and let $\hat{E}(Q_1,\cdots,Q_s)^*=\hat{E}(Q_1,\cdots,Q_s)-\bigcup_{P\neq Q_1,\cdots,Q_s}\hat{E}(P)$. Note that $\hat{E}(Q_1,\cdots,Q_s)^*$ is non-empty if and only if

$Q_1, \cdots, Q_s$ are vertices of an $(n+k)$-simplex of Σ^*. In this case, we say that $\{Q_1, \cdots, Q_s\}$ are *compatible*. $\hat{E}(P)$ has the canonical toric stratification $\{\hat{E}(P, Q_1, \cdots, Q_s)^*; P, Q_1, \cdots, Q_s: \text{compatible}\}$. We also define $E(Q_1, \cdots, Q_s)$ by $\hat{E}(Q_1, \cdots, Q_s) \cap \tilde{V}$ and let $E(Q_1, \cdots, Q_s)^* = \hat{E}(Q_1, \cdots, Q_s)^* \cap \tilde{V}$. Let $\sigma = (P_1, \cdots, P_{n+k})$ be such an $(n+k)$-simplex with $P_i = Q_i$, $i = 1, \cdots, s$. Let $\Delta_a = \bigcap_{i=1}^s \Delta(Q_i; f_a)$ for $a = 1, \cdots, k$ and let

$$h_a(y_{\sigma,s+1}, \cdots, y_{\sigma,n+k}) = f_{a\Delta_a}(\pi_\sigma(\boldsymbol{y}_\sigma)) / \prod_{i=1}^{n+k} y_{\sigma i}^{d(P_i; f_a)}, \qquad a = 1, \cdots, k.$$

Then we can see easily that

$$E(Q_1, \cdots, Q_s)^* = \{\boldsymbol{y}'_\sigma \in \boldsymbol{C}^{*(n+k-s)}; h_1(y'_\sigma) = \cdots = h_k(y'_\sigma) = 0\} \tag{4.1}$$

where $\boldsymbol{y}'_\sigma = (y_{\sigma,s+1}, \cdots, y_{\sigma,n+k})$. In particular, $E(Q_1, \cdots, Q_s)^*$ is a non-degenerate complete intersection variety in the complex torus $\hat{E}(Q_1, \cdots, Q_s)^* \cong \boldsymbol{C}^{*(n+k-s)}$ by the non-degeneracy assumption of V. See Lemma (5.2). $E(P)$ has also the canonical smooth stratification $E(P) = \cup E(P, Q_1, \cdots, Q_s)^*$ where $P, Q_1, \cdots, Q_s$ are compatible and $s = 0, \cdots, n-2$. The necessary and sufficient condition for $E(Q_1, \cdots, Q_s)^*$ to be non-empty is that $\{\Delta_1, \cdots, \Delta_k\}$ satisfies the (A_0)-condition (See Proposition (5.4).).

REMARK (4.2). Let $P \in \Sigma^*$ be a strictly positive vertex and let $\sigma = (P_1, \cdots, P_{n+k})$ be a simplex of Σ^* with $P_1 = P$. Then by (4.1)

$$E(P)^* = \{\boldsymbol{y}'_\sigma \in \boldsymbol{C}^{*(n+k-1)}; h_1(\boldsymbol{y}'_\sigma) = \cdots = h_k(\boldsymbol{y}'_\sigma) = 0\}$$

where

$$h_a(\boldsymbol{y}'_\sigma) = f_{aP}(\pi_\sigma(\boldsymbol{y}_\sigma)) / \prod_{i=1}^{n+k} y_{\sigma i}^{d(P_i; f_a)}.$$

Let $\tau = (P, Q_2, \cdots, Q_{n+k})$ be an arbitrary unimodular matrix which is not necessarily a simplex of Σ^*. It is easy to see $\sigma^{-1}\tau$ can be written as

$$\begin{pmatrix} 1 & * \\ 0 & \Lambda \end{pmatrix}$$

where Λ is an $(n+k-1) \times (n+k-1)$-unimodular matrix. Thus $\phi = \pi_{\sigma^{-1}\tau}|_{y_{\tau 1}=0}$ gives an isomorphism $\pi_\Lambda: \{y_{\tau 1} = 0\} \to \{y_{\sigma 1} = 0\}$. The pullback $\phi^{-1}(E(P)^*)$ is defined by

$$\{\boldsymbol{y}_\tau \in \boldsymbol{C}^{*(n+k-1)}; \phi^* h_1(\boldsymbol{y}'_\tau) = \cdots = \phi^* h_k(\boldsymbol{y}'_\tau) = 0\}.$$

We claim that

$$\phi^* h_a(\boldsymbol{y}'_\tau) \equiv f_{aP}(\pi_\tau(\boldsymbol{y}_\tau)) / y_{\tau 1}^{d(P; f_a)} \tag{4.2.1}$$

modulo a multiplication of a monomial in $\boldsymbol{y}'_\sigma$. In fact, the assertion is easily derived from the composition property $\pi_{\sigma^{-1}\tau}=\pi_\sigma^{-1}\circ\pi_\tau$. In later sections, we have to calculate the topological Euler-Poincaré characteristic $\chi(E(P)^*)$ to obtain the principal zeta-function of the Milnor fibration of a complete intersection variety. For this purpose, (4.2.1) gives a practical way to calculate $\chi(E(P)^*)$ without carrying out the subdivision Σ^* of $\Gamma^*(f_1,\cdots,f_k)$.

§ 5. Euler-Poincaré characteristic of an affine complete intersection variety.

Let $h_1(\boldsymbol{y}),\cdots,h_k(\boldsymbol{y})$ be given Laurent polynomials of m-variables $\boldsymbol{y}=(y_1,\cdots,y_m)$ and let $Z=\{\boldsymbol{y}\in\boldsymbol{C}^{*m}\,;\,h_1(\boldsymbol{y})=\cdots=h_k(\boldsymbol{y})=0\}$. We say that Z is *non-degenerate* if for any weight vector P, the variety $Z(P)=\{\boldsymbol{y}\in\boldsymbol{C}^{*m}\,;\,h_{1P}(\boldsymbol{y})=\cdots=h_{kP}(\boldsymbol{y})=0\}$ is a smooth complete intersection variety i.e., if the k-form $dh_{1P}\wedge\cdots\wedge dh_{kP}$ is nowhere vanishing on $Z(P)$. Here P is not necessarily strictly positive. Taking $P=0$, we see that Z is also a complete intersection variety ([Kh1]). Let $A=(a_{ij})$ be a non-singular integral matrix i.e., an integral square matrix with non-zero determinant. We define a morphism $\pi_A:\boldsymbol{C}^{*m}\to\boldsymbol{C}^{*m}$ by $\pi_A(\boldsymbol{y})=(y_1^{a_{11}}\cdots y_m^{a_{1m}},\cdots,y_1^{a_{m1}}\cdots y_m^{a_{mm}})$. π_A gives a $|\det A|$-fold covering map. In the case of $\det A=\pm1$, we say that π_A is *a Laurent coordinate change.*

PROPOSITION (5.1). *Let $Z=\{\boldsymbol{y}\in\boldsymbol{C}^{*m}\,;\,h_1(\boldsymbol{y})=\cdots=h_k(\boldsymbol{y})=0\}$ be a non-degenerate complete intersection variety. Let A be a non-singular integral $m\times m$ matrix and let $L_1,\cdots,L_k$ be integral vectors. Let $Z^A=\{\boldsymbol{y}\in\boldsymbol{C}^{*m}\,;\,h_1(\pi_A(\boldsymbol{y}))=\cdots=h_k(\pi_A(\boldsymbol{y}))=0\}$ and $Z'=\{\boldsymbol{y}\in\boldsymbol{C}^{*m}\,;\,\boldsymbol{y}^{L_1}h_1(\boldsymbol{y})=\cdots=\boldsymbol{y}^{L_k}h_k(\boldsymbol{y})=0\}$. Then Z^A and Z' are non-degenerate complete intersection varieties.*

PROOF. Let P be a weight vector and let $\boldsymbol{y}^\nu$ be a monomial. Then it is easy to see that $\pi_A^*(\boldsymbol{y}^\nu)=\boldsymbol{y}^{\nu A}$ and $\mathrm{degree}_P(\pi_A^*(\boldsymbol{y}^\nu))=\mathrm{degree}_{AP}(\boldsymbol{y}^\nu)$. Thus we have that $(\pi_A^*h_\alpha)_P=\pi_A^*((h_\alpha)_{AP})$. Thus the non-degeneracy of Z^A results immediately from this equality. It is clear that $Z'(P)=Z(P)$ for any weight vector P as a set. Let $h'_\alpha(\boldsymbol{y})=\boldsymbol{y}^{L_\alpha}h_\alpha(\boldsymbol{y})$. Then $dh'_{1P}\wedge\cdots\wedge dh'_{kP}(\boldsymbol{y})=\boldsymbol{y}^L dh_{1P}\wedge\cdots\wedge dh_{kP}(\boldsymbol{y})\neq0$ for any $\boldsymbol{y}\in Z'(P)=Z(P)$ where $L=L_1+\cdots+L_k$. This proves the non-degeneracy of Z'.

LEMMA (5.2). (i) *Let $V=\{\boldsymbol{z}\in\boldsymbol{C}^{n+k}\,;\,f_1(z)=\cdots=f_k(z)=0\}$ be a germ of a non-degenerate complete intersection variety with an isolated singularity at the origin and let $\pi:\tilde{V}\to V$ be a toroidal resolution of V which is constructed as in § 4. Let P be a strictly positive vertex of Σ^*. Then*

$E(P)^* \subset \hat{E}(P)^* = \boldsymbol{C}^{*(n+k-1)}$ is a non-degenerate complete intersection variety.

(ii) *Let $Z=\{\boldsymbol{y}\in \boldsymbol{C}^{*m}\,;\, h_1(\boldsymbol{y})=\cdots=h_k(\boldsymbol{y})=0\}$ be a non-degenerate complete intersection variety. Then there is a germ of a non-degenerate complete intersection variety $V=\{\boldsymbol{z}\in\boldsymbol{C}^{m+1}\,;\, f_1(\boldsymbol{z})=\cdots=f_k(\boldsymbol{z})=0\}$ and a resolution $\pi:\tilde{V}\to V$ as in § 4 such that $Z\cong E(P)^*$ for $P={}^t(1,\cdots,1)$.*

PROOF. Let $\sigma=(P_1,\cdots,P_{m+k})$ be a simplex of Σ^* with $P=P_1$. Then we have

$$E(P)^*=\{\boldsymbol{y}'_\sigma\in\boldsymbol{C}^{*(n+k-1)}\,;\, h_1(\boldsymbol{y}'_\sigma)=\cdots=h_k(\boldsymbol{y}'_\sigma)=0\}$$

where $\boldsymbol{y}'_\sigma=(y_{\sigma 2},\cdots,y_{\sigma(m+k)})$ and $h_a(\boldsymbol{y}'_\sigma)=f_{aP}(\pi_\sigma(\boldsymbol{y}_\sigma))/\prod_{j=1}^{m+k} y_{\sigma j}^{d(P_j;f_a)}$. Thus the non-degeneracy of $E(P)^*$ follows from Proposition (5.1). Now we consider the assertion (ii). Consider homogeneous polynomials $\tilde{h}_a(\boldsymbol{z})=z_{m+1}^M h_a(z_1/z_{m+1},\cdots,z_m/z_{m+1})$ and $f_a(z)=\tilde{h}_a(z)+\sum_{j=1}^{m+1} b_{aj}z_j^N$. Here $M\geqq \max_a \text{degree}\, h_a$ and $N>M$ and the coefficients $\{b_{aj}\}$ are generically chosen so that $\{\boldsymbol{z}\in\boldsymbol{C}^{m+1}\,;\, f_1(\boldsymbol{z})=\cdots=f_k(\boldsymbol{z})=0\}$ is a non-degenerate complete intersection variety. We choose a unimodular simplicial subdivision Σ^* of $\Gamma^*(f_1,\cdots,f_k)$ and we consider the corresponding toroidal resolution $\pi:\tilde{V}\to V$. Let $P={}^t(1,\cdots,1)$ and let $\sigma=(R_1,\cdots,R_m,P)$. (Recall that $R_i={}^t(0,\cdots,1,\cdots,0)$.) Though P is a vertex of Σ^*, σ is not necessarily an $(n+k)$-simplex of Σ^*. Note that $f_{aP}=\tilde{h}_a$ and $\tilde{h}_a(\pi_\sigma(\boldsymbol{y}_\sigma))=h_a(y_{\sigma 1},\cdots,y_{\sigma m})\cdot y_{\sigma,m+1}^M$. Thus by Remark (4.2), $E(P)^*$ is isomorphic to the variety $\{\boldsymbol{y}'\in\boldsymbol{C}^{*m}\,;\, h_1(\boldsymbol{y}'_\sigma)=\cdots=h_k(\boldsymbol{y}'_\sigma)=0\}$ where $\boldsymbol{y}'_\sigma=(y_{\sigma 1},\cdots,y_{\sigma m})$. This completes the proof.

For a more direct toric compactification of the variety Z, see Khovanskii [Kh 1]. In this paper, we want to calculate the zeta-function $\zeta_k(t)$. For this purpose, it is necessary to know the Euler-Poincaré characteristic of a non-degenerate complete intersection variety $Z=\{\boldsymbol{y}\in\boldsymbol{C}^{*m}\,;\, h_1(\boldsymbol{y})=\cdots=h_k(\boldsymbol{y})=0\}$. Using the result of Ehlers on the Chern classes of the toric variety ([E]), Khovanskii has determined the Euler-Poincaré characteristic $\chi(Z)$ in [Kh2] as follows.

THEOREM (5.3). *Let $Z=\{\boldsymbol{y}\in\boldsymbol{C}^{*m}\,;\, h_1(\boldsymbol{y})=\cdots=h_k(\boldsymbol{y})=0\}$. Let $\Delta_i=\Delta(h_i)$. Then*

$$\chi(Z)=\left(\prod_{i=1}^{k}\frac{\Delta_i}{1+\Delta_i}\right)_m.$$

Here the right side is, by definition, the homogeneous component of degree m of $\prod_{i=1}^k \Delta_i(1+\Delta_i)^{-1}$ in the formal series of $\Delta_1,\cdots,\Delta_k$. Namely we have

$$\chi(Z)=(-1)^{m-k}\sum_{\substack{a_1,\cdots,a_k\geqq 1\\ a_1+\cdots+a_k=m}}\Delta_1^{a_1}\cdots\Delta_k^{a_k}$$

where $\Delta_1^{a_1}\cdots\Delta_k^{a_k}$ *is the integer defined by the Minkowski's mixed volume* $m!\,V_m(\underbrace{\Delta_1,\cdots,\Delta_1}_{a_1},\cdots,\underbrace{\Delta_k,\cdots,\Delta_k}_{a_k})$.

COROLLARY (5.3.1). *Assume that* $\dim\sum_{i=1}^{k}\Delta_i<m$. *Then* $\chi(Z)=0$.

A non-degenerate complete intersection variety Z may be empty. A criterion for the non-emptiness of Z is the following.

PROPOSITION (5.4). *Z is non-empty if and only if the following condition is satisfied.*

(A_0) *For any subset* $I\subset\{1,\cdots,k\}$, $\dim\sum_{i\in I}\Delta_i\geqq|I|$.

PROOF. This proposition is announced as a result of Bernshtein in [Kh 2]. The necessity of (A_0) is obvious by Theorem (2.4). (See the proof of Proposition (2.5).) Assume that (A_0) is satisfied. Let $h_j(\boldsymbol{y})$ $(j=k+1,\cdots,m)$ be generic polynomials such that $\dim\Delta(h_j)=m$ for $j=k+1,\cdots,m$ and $\{\boldsymbol{y}\in\boldsymbol{C}^{*m}\,;\,h_1(\boldsymbol{y})=\cdots=h_m(\boldsymbol{y})=0\}$ is non-degenerate. Then $\{\Delta(h_1),\cdots,\Delta(h_m)\}$ clearly satisfies the (A^0)-condition. By Proposition (2.5) and Theorem (2.4), we conclude that $\{\boldsymbol{y}\in\boldsymbol{C}^{*m}\,;\,h_1(\boldsymbol{y})=\cdots=h_m(\boldsymbol{y})=0\}$ is non-empty. In particular, Z is non-empty. This completes the proof.

Example (5.5) (Hypersurface). Let $Z=\{\boldsymbol{y}\in\boldsymbol{C}^{*m}\,;\,h(\boldsymbol{y})=0\}$ be a non-degenerate hyper-surface. Then $\chi(Z)=(-1)^{m-1}m!\,\mathrm{Vol}_m(\Delta(h))$. This is proved by Kouchnirenko ([K], [O 1]).

Example (5.6) (Similar complete intersection variety). Let $Z_k=\{\boldsymbol{y}\in\boldsymbol{C}^{*m}\,;\,h_1(\boldsymbol{y})=\cdots=h_k(\boldsymbol{y})=0\}$ and assume that there is a Laurent polynomial $h(\boldsymbol{y})$ such that $\Delta(h_i)=d_i\Delta(h)$ for some positive integers $d_1,\cdots,d_k$. Let $\Delta=\Delta(h)$. As $(1+d_i\Delta)^{-1}=\sum_{j=0}^{\infty}(-1)^j d_i^j\Delta^j$, we obtain by Theorem (5.3) that

$$\chi(Z_k)=(-1)^{m-k}\Delta^m\sum_{i_1+\cdots+i_k=m-k}d_1^{i_1+1}\cdots d_k^{i_k+1}. \tag{5.6.1}$$

For the later purpose, we define the homogeneous polynomial $F_k^m(d_1,\cdots,d_k)$ by

$$F_k^m(d_1,\cdots,d_k)=\sum_{i_1+\cdots+i_k=m-k}d_1^{i_1+1}\cdots d_k^{i_k+1}. \tag{5.6.2}$$

Note that we have the equality:

$$(F_{k-1}^m(d_1,\cdots,d_{k-1})+F_k^m(d_1,\cdots,d_k))d_k=F_k^{m+1}(d_1,\cdots,d_k). \tag{5.6.3}$$

Assume that $d_1=\cdots=d_k=1$. Then $F_k^m(1,\cdots,1)=\binom{m-1}{k-1}$ and

$$(5.6.4) \qquad \chi(Z_k)=(-1)^{m-k}\binom{m-1}{k-1}\Delta^m .$$

§ 6. Zeta-function and the Milnor number.

Let $V_j=\{z\in \boldsymbol{C}^{n+k}\,;\, f_1(z)=\cdots=f_j(z)=0\}$ $(j=k-1,k)$ be germs of non-degenerate complete intersection varieties with isolated singularities at the origin and let $\hat{\pi}: X\to \boldsymbol{C}^{n+k}$ be the toroidal modification map with respect to Σ^* which is compatible with $\Gamma^*(f_1,\cdots,f_k)$. See § 4 for the construction. Let $\tilde{V}_j$ be the proper transformation of V_j by $\hat{\pi}$ for $j=k-1,k$. We denote the restriction of $\hat{\pi}$ to $\tilde{V}_{k-1}$ by π_{k-1}. The purpose of this paper is to calculate the k-th principal zeta-function $\zeta_k(t)$ which corresponds to the restriction of the Milnor fibration of $\boldsymbol{f}=(f_1,\cdots,f_k)$ to V_{k-1}. Let B_ε be a small disc of radius ε and let $U_\delta=\{\eta\in \boldsymbol{C}\,;\,|\eta|<\delta\}$ and let $U_\delta^*=U_\delta-\{0\}$ where δ is sufficiently small comparing with ε. Let $X_{k-1}=V_{k-1}\cap B_\varepsilon\cap f_k^{-1}(U_\delta)$ and $X_{k-1}^*=V_{k-1}\cap B_\varepsilon\cap f_k^{-1}(U_\delta^*)$. We study the fibration $f_k: X_{k-1}^*\to U_\delta^*-\{0\}$. Let $\tilde{X}_{k-1}=\pi_{k-1}^{-1}(X_{k-1})$ and $\tilde{X}_{k-1}^*=\pi_{k-1}^{-1}(X_{k-1}^*)$. As $\pi_{k-1}: \tilde{X}_{k-1}^*\to X_{k-1}^*$ is biholomorphic, the above fibration is equivalent to $f_k': \tilde{X}_{k-1}^*\to U_\delta^*$ where $f_k'=\pi_{k-1}\circ f_k$. Now $\pi_{k-1}: \tilde{X}_{k-1}\to X_{k-1}$ is already a good resolution of X_{k-1} which satisfies the conditions of Theorem (3.2). Therefore we can apply Theorem (3.2) to this situation. The exceptional divisors of $f_k'\,|\tilde{X}_{k-1}$ are $\{E_{k-1}(P)\,;\, P\in\Sigma^*,\ P\gg 0\}$ where $E_{k-1}(P)$ is defined by $\hat{E}(P)\cap\tilde{V}_{k-1}$. We also define $E_k(P)=\hat{E}(P)\cap\tilde{V}_k$, $E_{k-1}(P)^*=E_{k-1}(P)-\bigcup_{Q\neq P}E_{k-1}(Q)$ and

$$E(P)'=E_{k-1}(P)-E_k(P)-\bigcup_{Q\neq P,\,Q\gg 0}E_{k-1}(Q)\,.$$

Note that $E_k(P)=E_{k-1}(P)\cap\tilde{V}_k$ and $E(P)'\supset E_{k-1}(P)^*-\tilde{V}_k$. The multiplicity of the function f_k' on $E(P)'$ is equal to $d(P\,;f_k)$ by the definition of $d(P\,;f_k)$. Thus by Theorem (3.2) we have

$$(6.1) \qquad \zeta_k(t)=\prod_{P\in\Sigma^*,\,P\gg 0}(1-t^{d(P;f_k)})^{-\chi(E(P)')}\,.$$

Now the main task in this section is to interpret the Euler number $\chi(E(P)')$ in the words of the respective faces $\{\Delta(P\,;f_i)\,;\, i=1,\cdots,k\}$.

(I) Assume first that $E_{k-1}(P)$ does not intersect with any non-compact divisors $E_{k-1}(R_j)$ $(j=1,\cdots,n+k)$. Then $E(P)'=E_{k-1}(P)^*-E_k(P)^*$. If $\dim(\Delta(P\,;f_1)+\cdots+\Delta(P\,;f_k))<n+k-1$, both of the Euler-Poincaré characteristics of $E_{k-1}(P)^*$ and $E_k(P)^*$ are zero by Corollary (5.3.1). Thus the

vertex P does not contribute to the zeta function $\zeta_k(t)$. Assume that $\dim(\Delta(P;f_1)+\cdots+\Delta(P;f_k))=n+k-1$. Then P is uniquely determined by $\Delta(P;f_1),\cdots,\Delta(P;f_k)$ and by the property that P is a positive primitive integral vector. The corresponding Euler-Poincaré characteristic can be computed by Theorem (5.3) using any unimodular matrix with the first column vector P by Remark (4.2). By the additivity of the Euler-Poincaré characteristics, we have $\chi(E_{k-1}(P)^*-E_k(P)^*)=\chi(E_{k-1}(P)^*)-\chi(E_k(P)^*)$. As $\chi(E(P)')$ depends only on P, we define $\chi(P)=\chi(E_{k-1}(P)^*-E_k(P)^*)$.

(II) Assume that $E_{k-1}(P)$ intersects with a non-compact divisor $E_{k-1}(R_i)$ for some i. Then the contributions from the lower dimensional coordinate plane sections may be non-trivial. Thus we cannot ignore the vertex P even if $\dim\sum_{i=1}^{k}\Delta(P;f_i)<n+k-1$. Let $I\subset\{1,\cdots,n+k\}$ and we define

$$E_j(P)^{*I}=E_j(P)\bigcap_{i\notin I}E_j(R_i)-\bigcup_{Q\neq P,Q\gg 0}E_j(Q)-\bigcup_{i\in I}E_j(R_i).$$

If $I=\{1,\cdots,n+k\}$, $E_j(P)^{*I}$ is simply equal to $E_j(P)^*$. The family $\{E_j(P)^{*I}\}$ gives a regular stratification of $E_j(P)-\bigcup_{Q\neq P,Q\gg 0}E_j(Q)$. Thus we have a canonical stratification of $E(P)'$

$$E(P)'=\bigcup_{I}(E_{k-1}(P)^{*I}-E_k(P)^{*I}). \tag{6.2}$$

Assume that $E_{k-1}(P)^{*I}\neq\emptyset$. This implies that there is an $(n+k)$-simplex σ which contains P and R_i $(i\notin I)$. For brevity's sake, let us assume that $I=\{s+1,\cdots,n+k\}$. Changing the ordering of the vertices if necessary, we can assume that $\sigma=(Q_1,\cdots,Q_{n+k})$ where $Q_i=R_i$ for $i=1,\cdots,s$ and $Q_{s+1}=P$. Then σ can be written as

$$\sigma=\begin{pmatrix} I_s & * \\ 0 & A \end{pmatrix}. \tag{6.3}$$

Here I_s is the $s\times s$ identity matrix. In particular, A is an $(n+k-s)\times(n+k-s)$-unimodular matrix. Let $\Delta_a=\bigcap_{1\le i\le s+1}\Delta(Q_i;f_a)$. It is easy to see that $\Delta_a\in\Gamma(f_a^I)$. Let $P_I\in N_I$ be the I-projection of P. (Under the above assumption, $P_I={}^t(p_{s+1},\cdots,p_{n+k})$ if $P={}^t(p_1,\cdots,p_{n+k})$.) Then we have $\Delta(P_I;f_a^I)=\Delta_a$. As A in (6.3) is a unimodular matrix and P_I is the first column vector of A, P_I must be a primitive weight vector. Let

$$h_a(y_{\sigma,s+2},\cdots,y_{\sigma,n+k})=f_{a\Delta_a}(\pi_\sigma(\boldsymbol{y}_\sigma))/\prod_{i=1}^{n+k}y_{\sigma i}^{d(Q_i;f_a)}. \tag{6.4}$$

As σ is a simplex of Σ^*, it must have the property that $\bigcap_{i=1}^{n+k}\Delta(Q_i;f_a)\neq\emptyset$ for $a=1,\cdots,k$. Therefore $\Delta(Q_i;f_a)\cap\Delta_a\neq\emptyset$. Thus we obtain the equality:

$$h_a(y_{\sigma,s+2}, \cdots, y_{\sigma,n+k}) = f^I_{\sigma P_I}(\pi_\Delta(\boldsymbol{y}'_\sigma)) / \prod_{i=s+1}^{n+k} y_{\sigma i}^{d(Q_{iI}; f^I_a)}.$$

Here we use the notation $\boldsymbol{y}'_\sigma = (y_{\sigma,s+2}, \cdots, y_{\sigma,n+k})$. Then by (4.1), we have:

$$E_{k-1}(P)^{*I} = \{\boldsymbol{y}'_\sigma \in \boldsymbol{C}^{*(n+k-s-1)}; h_1(\boldsymbol{y}'_\sigma) = \cdots = h_{k-1}(\boldsymbol{y}'_\sigma) = 0\},$$

$$E_k(P)^{*I} = \{\boldsymbol{y}'_\sigma \in \boldsymbol{C}^{*(n+k-s-1)}; h_1(\boldsymbol{y}'_\sigma) = \cdots = h_k(\boldsymbol{y}'_\sigma) = 0\}.$$

Note that $\dim(\Delta_1 + \cdots + \Delta_k) = \dim(\Delta(h_1) + \cdots + \Delta(h_k))$. Thus if $\dim(\Delta_1 + \cdots + \Delta_k) < n+k-s-1$, both of the Euler-Poincaré characteristics of $E_{k-1}(P)^{*I}$ and $E_k(P)^{*I}$ are zero by Corollary (5.3.1). Assume that $\dim(\Delta_1 + \cdots + \Delta_k) = n+k-s-1$. Then P_I is uniquely determined by the property that $\Delta(P_I; f^I_a) = \Delta_a$. Conversely we claim:

LEMMA (6.5). *Let Q be a primitive strictly positive weight vector in N_I such that* $\dim(\Delta(Q; f^I_1) + \cdots + \Delta(Q; f^I_k)) = |I| - 1$. *Then there is a unique vertex $P \in \Sigma^*$ whose I-projection is equal to Q.*

PROOF. We assume that $I = \{1, \cdots, s\}$ for simplicity. Let $W = \{P \in N; \Delta(P; f_a) \supset \Delta(Q; f^I_a),\ a = 1, \cdots, k\}$. As $\dim(\sum_{i=1}^k \Delta(Q; f^I_i)) = s-1$ by the assumption, W is a polyhedral cone of dimension $n+k-s+1$. Note that $R_i \in W$ for $i = s+1, \cdots, n+k$. Let $Q = {}^t(q_1, \cdots, q_s)$ and define $S = {}^t(s_1, \cdots, s_{n+k}) \in N^+$ by $s_i = q_i$ for $i \leqq s$ and $s_i = M$ otherwise. If M is a sufficiently large positive integer, it is easy to see that $S \in W$ and $\Delta(S; f_a) = \Delta(Q; f^I_a)$ for $a = 1, \cdots, k$. Therefore $W \cap N^+ \neq \emptyset$. Note that the interior of $W \cap N^+$ is an equivalent class in the dual Newton diagram $\Gamma^*(f_1, \cdots, f_k)$. See §4 for the definition. As Σ^* is a unimodular subdivision of $\Gamma^*(f_1, \cdots, f_k)$ and $\dim W = n+k-s+1$, the above observation implies that there is unique vertex P of $W \cap \Sigma^*$ such that $\{P, R_{s+1}, \cdots, R_{n+k}\}$ is an $(n+k-s+1)$-simplex in Σ^*. This completes the proof.

Let Q and P as in Lemma (6.5). Then the following equality results immediately from the inclusion $\Delta(P; f_k) \supset \Delta(Q; f^I_k)$.

ASSERTION. $d(P; f_k) = d(Q; f^I_k)$.

Now we can formulate our result as follows. Let $\mathcal{S}_I$ be the set of primitive strictly positive weight vectors Q in N_I such that (a) (Non-emptiness) $\{\Delta(Q; f^I_i); i = 1, \cdots, k-1\}$ satisfies the (A_0)-condition and (b) (Maximal dimension) $\dim(\Delta(Q; f^I_1) + \cdots + \Delta(Q; f^I_k)) = |I| - 1$. We call $\mathcal{S}_I$ *the I-data set of* $\boldsymbol{f}$. Let P be as in Lemma (6.5). The condition (a) implies that $E_{k-1}(P)^{*I} \neq \emptyset$. If (b) is not satisfied, we have $\chi(E_{k-1}(P)^{*I}) = \chi(E_k(P)^{*I}) = 0$. Let $f(z) = f_1(z) \cdots f_k(z)$ and $\mathcal{S}'_I = \{Q \in N_I; Q \gg 0,\ \dim \Delta(Q; f^I) = |I| - 1\}$.

As $\Delta(Q;f^I)=\Delta(Q;f_1^I)+\cdots+\Delta(Q;f_k^I)$, $\mathcal{S}_I$ is a subset of $\mathcal{S}_I'$. Therefore $\mathcal{S}_I$ is a finite set. Let $Q\in\mathcal{S}_I$ and let $\sigma=(Q_1,\cdots,Q_{|I|})$ be an arbitrary $|I|\times|I|$-unimodular matrix with $Q_1=Q$. Let $\Delta_i=\Delta(Q;f_i^I)$ for $i=1,\cdots,k$ and let $h_i(\boldsymbol{y}_\sigma')=f^I_{i\Delta_i}(\pi_\sigma(\boldsymbol{y}_\sigma))/y_{\sigma 1}^{d(Q;f_i^I)}$ where $\boldsymbol{y}_\sigma'=(y_{\sigma 2},\cdots,y_{\sigma|I|})$. Let us define an integer $\chi(Q)$ by

$$\chi(Q)=\left(\prod_{i=1}^{k-1}\frac{\Delta_i'}{(1+\Delta_i')}-\prod_{i=1}^{k}\frac{\Delta_i'}{(1+\Delta_i')}\right)_{|I|-1} \tag{6.6}$$

where $\Delta_i'=\Delta(h_i)$. By Remark (5.2), $\chi(Q)$ depends only on Q and $f_1^I,\cdots,f_k^I$. For notation's simplicity, we define

$$\Delta_1^{i_1}\cdots\Delta_k^{i_k}=(|I|-1)!\ V_{|I|-1}(\underbrace{\Delta_1,\cdots,\Delta_1}_{i_1},\cdots,\underbrace{\Delta_k,\cdots,\Delta_k}_{i_k})$$

where the right side is the generalized Minkowski's mixed volume defined in §2. Then $\Delta_1^{i_1}\cdots\Delta_k^{i_k}=(\Delta_1')^{i_1}\cdots(\Delta_k')^{i_k}$ by the definition itself. The equaliy (6.6) can be written as

$$\chi(Q)=\left(\prod_{i=1}^{k-1}\frac{\Delta_i}{(1+\Delta_i)}-\prod_{i=1}^{k}\frac{\Delta_i}{(1+\Delta_i)}\right)_{|I|-1}. \tag{6.7}$$

Note that the I-data set $\mathcal{S}_I$ is independent of the choice of Σ^* and that $\mathcal{S}_I=\emptyset$ if $|I|<k$.

Let $P\in\Sigma^*$ and let $P\gg 0$. By the additivity of the Euler-Poincaré characteristics and by the above argument, we have $(1-t^{d(P;f_k)})^{-\chi(E(P)')}=$ $=\prod_Q'(1-t^{d(Q;f_k^I)})^{-\chi(Q)}$ where the product is taken for $Q\in\mathcal{S}_I$, $|I|\geqq k$, such that $P_I=Q$ and $\{P,R_i\,;\,i\notin I\}$ is compatible in Σ^*. On the other hand, for a given $Q\in\mathcal{S}_I$ there exists a vertex $P\in\Sigma$ such that the term $(1-t^{d(Q;f_k^I)})^{-\chi(Q)}$ coincides with to a factor of $(1-t^{d(P;f_k)})^{-\chi(E(P)')}$ in the above product expression where P is uniquely characterized by Lemma (6.5). Thus by (6.1) we obtain the following theorem.

MAIN THEOREM (6.8). (i) *The k-th principal zeta-function* $\zeta_k(t)$ *is determined by*

$$\zeta_k(t)=\prod_{|I|\geqq k}\ \prod_{Q\in\mathcal{S}_I}(1-t^{d(Q;f_k^I)})^{-\chi(Q)}.$$

(ii) *The Milnor number* μ *of* $\boldsymbol{f}$ *is determined by*

$$1+(-1)^n\mu=\sum_{|I|\geqq k}\ \sum_{Q\in\mathcal{S}_I}d(Q;f_k^I)\chi(Q).$$

In the above formula, we can replace $\mathcal{S}_I$ by $\mathcal{S}_I'$ as every vertex $Q\in\mathcal{S}_I'-\mathcal{S}_I$ contributes trivially to $\zeta_k(t)$. Note that the above formula does not depend on the choice of a unimodular simplicial subdivision Σ^* and it is a generalization of Varchenko's formula for the zeta-function of a non-degenerate analytic function ([V]). In general, $\zeta_i(t)$ depends on i but μ can be computed through any $\zeta_i(t)$ using (ii). See Theorem (7.2) and Example (8.4). It should be mentioned here that Morales has determined the δ-genus of a non-degenerate complete intersection variety ([Mo]).

Now we give a practical method to compute the integer $\Delta_1^{i_1}\cdots\Delta_k^{i_k}$. Let us consider the polynomial $g(z)=\prod_{i=1}^k f_i(z)^{n_i}$ where $n_1,\cdots,n_k$ are variables in the positive integers. Then $g_Q^I(z_I)=\prod_{i=1}^k f_{iQ}^I(z_I)^{n_i}$. Let σ be as above. Then we have

$$g_Q^I(\pi_\sigma(\boldsymbol{y}_\sigma))=y_{\sigma 1}^{d(Q;g^I)}\prod_{i=1}^k h_i(\boldsymbol{y}'_\sigma)^{n_i}$$

where $d(Q;g^I)=\sum_{i=1}^k n_i d(Q;f_i^I)$. For a compact polyhedron Δ in $\boldsymbol{R}^{|I|}$, we denote the cone of Δ with the vertex at the origin by $C\Delta$. Note that $C(\sum_{i=1}^k n_i\Delta_i)=\sum_{i=1}^k n_i C\Delta_i$. Thus we have the following equality.

$$\begin{aligned}|I|!\,\mathrm{Vol}_{|I|}(C\Delta(g_Q^I))&=|I|!\,\mathrm{Vol}_{|I|}\Big(C\Big(\sum_{i=1}^k n_i\Delta_i\Big)\Big)\\
&=\Big(\sum_{i=1}^k n_i d(Q;f_i^I)\Big)(|I|-1)!\,\mathrm{Vol}_{|I|-1}\Big(\sum_{i=1}^k n_i\Delta'_i\Big)\\
&=\sum_{i=1}^k n_i d(Q;f_i^I)\Big(\sum_{i=1}^k n_i\Delta'_i\Big)^{|I|-1}\\
&=\sum_{i=1}^k n_i d(Q;f_i^I)\sum_{i_1+\cdots+i_k=|I|-1}\frac{(|I|-1)!}{i_1!\cdots i_k!}n_1^{i_1}\cdots n_k^{i_k}(\Delta'_1)^{i_1}\cdots(\Delta'_k)^{i_k}.\end{aligned}$$

Therefore replacing Δ'_i by Δ_i we obtain

(6.9) $$\begin{aligned}&|I|!\,\mathrm{Vol}_{|I|}\Big(\sum_{i=1}^k n_i C\Delta_i\Big)\\
&=\sum_{i=1}^k n_i d(Q;f_i^I)\sum_{i_1+\cdots+i_k=|I|-1}\frac{(|I|-1)!}{i_1!\cdots i_k!}n_1^{i_1}\cdots n_k^{i_k}\Delta_1^{i_1}\cdots\Delta_k^{i_k}.\end{aligned}$$

The above equality say that $|I|!\,\mathrm{Vol}_{|I|}(\sum_{i=1}^k n_i C\Delta_i)/\sum_{i=1}^k n_i d(Q;f_i^I)$ is a polynomial in $n_1,\cdots,n_k$ and the integer $\Delta_1^{i_1}\cdots\Delta_k^{i_k}$ is equal to $(i_1!\cdots i_k!/(|I|-1)!)$ $\times$(the coefficient of the monomial $n_1^{i_1}\cdots n_k^{i_k}$). For example, we have the following important formula:

(6.10) $$\Delta_i^{|I|-1}=|I|!\,\mathrm{Vol}_{|I|}(C\Delta_i)/d(Q;f_i^I).$$

The equalities (6.9) and (6.10) enable us to compute $\chi(Q)$ without calculating $h_1(\boldsymbol{y}'_\sigma),\cdots,h_k(\boldsymbol{y}'_\sigma)$.

§ 7. Similar complete intersection variety.

In this section, we consider a germ of a non-degenerate complete intersection variety $V=\{z\in \boldsymbol{C}^{n+k}\,;\, f_1(z)=\cdots=f_k(z)=0\}$ where the respective Newton boundaries $\{\Gamma(f_i)\}_{i=1,\cdots,k}$ are similar. Namely we assume that there is a polynomial $f(z)$ such that $\Gamma(f_i)=d_i\Gamma(f)$ for some positive integers d_i, $i=1,\cdots,k$. Then the I-data set $\mathcal{S}_I$ is simply the set of the primitive I-weight vectors which corresponds to the $(|I|-1)$-dimensional faces of $\Gamma(f^I)$. Let $Q\in\mathcal{S}_I$. The number $\chi(Q)$ can be explicitly computed as follows.

$$\begin{aligned}(7.1)\qquad \chi(Q)&=(-1)^{|I|-k}F_k^{|I|}(d_1,\cdots,d_k)d_k^{-1}\Delta(Q\,;f^I)^{|I|-1}\\ &=(-1)^{|I|-k}F_k^{|I|}(d_1,\cdots,d_k)d_k^{-1}\cdot|I|!\,\mathrm{Vol}_{|I|}(C\Delta(Q\,;f^I))/d(Q\,;f^I)\,.\end{aligned}$$

Here the first equality is derived from (5.6.1) and (5.6.3) and the second one is derived from (6.10). Thus we obtain the following formula.

THEOREM (7.2). *Under the above assumption, we have*

$$\text{(i)}\qquad \zeta_k(t)=\prod_{|I|\geq k}\prod_{Q\in\mathcal{S}_I}(1-t^{d_k d(Q;f^I)})^{-\chi(Q)}$$

$$\text{(ii)}\qquad \mu=\sum_{|I|\geq k}(-1)^{n+k-|I|}F_k^{|I|}(d_1,\cdots,d_k)\sum_{Q\in\mathcal{S}_I}|I|!\,\mathrm{Vol}_{|I|}(C\Delta(Q\,;f^I))+(-1)^{n-1}$$

where $\chi(Q)$ *is defined by* (7.1) *and* μ *is the Milnor number. In particular, assume that* $d_1=\cdots=d_k=1$. *Then*

$$\chi(Q)=(-1)^{|I|-k}\binom{|I|-1}{k-1}|I|!\,\mathrm{Vol}_{|I|}(C(\Delta(Q\,;f^I)))/d(Q\,;f^I)$$

and

$$\mu=\sum_{|I|\geq k}(-1)^{n+k-|I|}\binom{|I|-1}{k-1}\sum_{Q\in\mathcal{S}_I}|I|!\,\mathrm{Vol}_{|I|}(C\Delta(Q\,;f^I))+(-1)^{n-1}.$$

We can easily see that $\zeta_i(t)\neq\zeta_j(t)$ if $d_i\neq d_j$. However μ is certainly a symmetric polynomial in $d_1,\cdots,d_k$.

Example (7.3). Let us consider the case where $f(z)=z_1^{a_1}+\cdots+z_{n+k}^{a_{n+k}}$ and $\Gamma(f_a)=d_a\Gamma(f)$, $a=1,\cdots,k$. This case is studied by Hamm ([H 2]) and Greuel-Hamm ([G-H]). They have determined the χ_y-genus of the corresponding quasi-projective complete intersection variety. Dimca also studied the monodromy of the S^1-action ([Di 1]). In this case, $\mathcal{S}_I=\{P(I)\}$ where $P(I)$ is the weight vector of f^I. Namely let a_I be the least common multiple of $\{a_i ; i\in I\}$. Then $P(I)={}^t(p_i)_{i\in I}$ where $p_i=a_I/a_i$ $(i\in I)$. Thus the contribution of $P(I)$ to the principal zeta-function $\zeta_k(t)$ is $(1-t^{d_k a_I})^{-\chi(P(I))}$ where

$$\chi(P(I))=(-1)^{|I|-k}F_k^{|I|}(d_1,\cdots,d_k)d_k^{-1}a_I^{-1}\prod_{i\in I}a_i\,. \tag{7.3.1}$$

Let $\sigma_i(a_1,\cdots,a_{n+k})$ be the i-th elementary symmetric function of $a_1,\cdots,a_{n+k}$ defined by

$$\prod_{i=1}^{n+k}(t-a_i)=t^{n+k}-\sigma_1 t^{n+k-1}+\cdots+(-1)^{n+k}\sigma_{n+k}\,.$$

Then by Theorem (7.2), we have the equality:

$$\mu=\sum_{s\geq k}(-1)^{n+k-s}F_k^s(d_1,\cdots,d_k)\sigma_s(a_1,\cdots,a_{n+k})+(-1)^{n-1}\,. \tag{7.3.2}$$

Assume that $d_1=\cdots=d_k=1$. Then Milnor number is determined by the following simple formula which is a generalization of the formula of Orlik-Milnor ([Or-M]).

$$\mu=\sum_{s\geqq k}(-1)^{n+k-s}\binom{s-1}{k-1}\sigma_s(a_1,\cdots,a_{n+k})+(-1)^{n-1}\,. \tag{7.3.3}$$

§ 8. Generic hyperplane sections.

In this section, we consider the Milnor fibration of the generic hyperplane sections of a non-degenerate function. Let $f(z)=f_k(z)$ be a given germ of a non-degenerate analytic function. Let $f_i(z)=l_i(z)$ $(i=1,\cdots,k-1)$ be generic linear forms. Thus we may assume that $l_i(z)=a_{i,1}z_1+\cdots+a_{i,n+k}z_{n+k}$ with $a_{i,j}\neq 0$ for any $i=1,\cdots,k-1$ and $j=1,\cdots,n+k$. Let $Q\in N_I^+$ be a strictly positive I-weight vector. Let $Q={}^t(q_i)_{i\in I}$ and let $q_{\min}=\min\{q_i ; i\in I\}$ and $I(Q)=\{i ; q_i=q_{\min}\}$. Let $\varDelta=\varDelta(Q;l_1)$ and $\Xi=\varDelta(Q;f)$. Note that $\varDelta$ is generated by the vertices $e_i=(0,\cdots,\overset{i}{1},\cdots,0)$ for $i\in I(Q)$. Now it is easy to see that $Q\in\mathcal{S}_I$ if and only if (a) $|I(Q)|\geqq 2$ and (b) $\dim(\varDelta+)\Xi=|I|-1$. Let $Q\in\mathcal{S}_I$. Using the notations defined in § 6, we have

$$\chi(Q)=\left(\frac{\varDelta^{k-1}}{(1+\varDelta)^{k-1}}+\frac{\varDelta^{k-1}\Xi}{(1+\varDelta)^{k-1}(1+\Xi)}\right)_{|I|-1} \tag{8.1}$$

$$=(-1)^{|I|-k}\sum_{i=0}^{|I|-k}\binom{k+i-2}{k-2}\varDelta^{k+i-1}\Xi^{|I|-k-i}\,.$$

The integers $\{\Delta^i \Xi^j\}$ can be computed using (2.3) and (6.10). Let us examine several important cases.

(I) $k=2$. This case corresponds to a hyperplane section. Let $m+1=|I|$. Then (8.1) says that

$$\chi(Q)=(-1)^{m-1}(\Delta^m+\Delta^{m-1}\Xi+\cdots+\Delta\Xi^{m-1}). \tag{8.2}$$

For instance, assume that $n=1$. The case $n=1$, $k=2$ is studied by Mima ([Mi]). Then $|I|=3$ or 2 and we have

$$\chi(Q)=\begin{cases} -(\Delta^2+\Delta\Xi) & \text{if } |I|=3 \\ \Delta & \text{if } |I|=2. \end{cases}$$

(II) $k=3$. Let $m=|I|-1$ as above. Then

$$\chi(Q)=(-1)^m((m-1)\Delta^m+(m-2)\Delta^{m-1}\Xi+\cdots+\Delta^2\Xi^{m-2}). \tag{8.3}$$

For instance, assume that $n=1$ and $k=3$. Then $|I|=4$ or 3.

$$\chi(Q)=\begin{cases} -(2\Delta^3+\Delta^2\Xi) & \text{if } |I|=4 \\ \Delta^2 & \text{if } |I|=3. \end{cases}$$

Example (8.4). Let $l_1(z)=z_1+z_2+z_3$ and $f(z)=z_1^p+z_2^p+z_3^p+z_1^{s_1}z_2^{s_2}z_3^{s_3}$. We assume that $p>s_1+s_2+s_3$. Then $\mathcal{S}_{(1,2,3)}$ contains four vectors $P, P_1, \cdots, P_3$ where $P={}^t(1,1,1)$ and P_i is characterized as the weight vector such that $f_{P_i}(z)=\sum_{j\neq i} z_j^p+z_1^{s_1}z_2^{s_2}z_3^{s_3}$. For instance, we assume that $i=1$. Let r_1 be the greatest common divisor of $p-s_2-s_3$ and s_1. By an easy calculation, we can write $P_1={}^t(a_1, b_1, b_1)$ where $a_1=(p-s_2-s_3)/r_1$ and $b_1=s_1/r_1$. For I with $|I|=2$, $\mathcal{S}_I$ consists of a single weight vector P_I where P_I is the I-projection of P. The weight vector P contributes to $\zeta_2(t)$ by $(1-t^{s_1+s_2+s_3})$. Now we consider $\chi(P_1)$. Using (6.10), we have

$$\chi(P_1)=-\Delta\Xi=-((\Delta+\Xi)^2-\Delta^2-\Xi^2)/2=-s_1/b_1=-r_1.$$

Thus P_1 gives the term $(1-t^{ps_1/r_1})^{r_1}$. The weight vector P_I contributes by $(1-t^p)^{-1}$. Thus we obtain that

$$\zeta_2(t)=(1-t^{s_1+s_2+s_3})\left(\prod_{i=1}^{3}(1-t^{ps_i/r_i})^{r_i}\right)(1-t^p)^{-3}.$$

In particular, $\mu=(p+1)(s_1+s_2+s_3)-3p+1$.

If we eliminate z_3 in $f(z)$ using the equality $l_1=0$, we have

$$f|_{l_1=0}(z_1, z_2)=z_1^p+z_2^p+(-z_1-z_2)^p+z_1^{s_1}z_2^{s_2}(-z_1-z_2)_3^{s_3}.$$

Thus if $s_3>1$, f_P is degenerate (as a function of z_1 and z_2) for $P={}^t(1,1)$.

Now we consider $\zeta_1(t)$. The corresponding data are $\mathcal{S}_{\{1,2,3\}}=\{P_1, P_2, P_3\}$ and $\mathcal{S}_I=\{P_I\}$ for $|I|=2$. This time P_1 contribute to $\zeta_1(t)$ by $(1-t^{s_1/r_1})^{r_1(p+1)}$. If $|I|=2$, P_I gives the term $(1-t)^{-p}$. Thus

$$\zeta_1(t)=\left(\prod_{i=1}^{3}(1-t^{s_i/r_i})^{r_i(p+1)}\right)(1-t)^{-3p}.$$

Thus $\zeta_1(t)\neq\zeta_2(t)$ but they give the same Milnor number as degree $\zeta_1(t)=$ degree $\zeta_2(t)$.

§ 9. Irreducible components of a complete intersection space curve.

In this section, we consider the case $n=1$. Thus $V=\{z\in \boldsymbol{C}^{k+1}\,;\, f_1(z)=\cdots=f_k(z)=0\}$ is a curve. Let $\mathcal{S}=\mathcal{S}_{\{1,\cdots,k+1\}}$ and $\mathcal{S}_i=\{1,\cdots,k+1\}-\{i\}$. Then the zeta-function $\zeta_k(t)$ is determined as

$$\zeta_k(t)=\prod_{P\in\mathcal{S}}(1-t^{d(P;f_k)})^{-\chi(P)}\prod_{i=1}^{k+1}\prod_{Q\in\mathcal{S}_i}(1-t^{d(Q;f_k)})^{-\chi(Q)}. \tag{9.1}$$

Note that $\chi(Q)=(k-1)!\,V_{k-1}(\Delta(Q\,;f_1),\cdots,\Delta(Q\,;f_{k-1}))$. Now let us consider the number of the irreducible components $r(V)$ of V at the origin. We consider a toroidal resolution $\pi_k: \tilde{V}\to V$. Let $\mathcal{T}$ be the set of the positive vertex $P\in\Sigma^*$ such that $E_k(P)\neq\emptyset$ i.e., (a) $\{\Delta(P\,;f_1),\cdots,\Delta(P\,;f_k)\}$ satisfies the (A_0)-condition. Note that $E_k(P)$ consists of finite points. It is obvious that $\mathcal{T}\subset\mathcal{S}$. Let $V=\bigcup_{t=1}^{r(V)}C_t$ be the decomposition into irreducible components of V and let $\tilde{V}=\bigcup_{t=1}^{r(V)}\tilde{C}_t$ be the proper transform of V. For any $\tilde{C}_t$, there is a unique $P\in\mathcal{T}$ such that $\hat{E}_k(P)\cap\tilde{C}_t\neq\emptyset$. Therefore the irreducible components $\tilde{C}_t$ which intersect with $\hat{E}_k(P)$ correspond bijectively with the points of $E_k(P)$. The number of $E_k(P)$ is characterized by Theorem (2.4): $\chi(E_k(P))=k!\,V_k(\Delta(P\,;f_1),\cdots,\Delta(P\,;f_k))$. Thus we obtain the formula:

$$r(V)=\sum_{P\in\mathcal{T}}k!\,V_k(\Delta(P\,;f_1),\cdots,\Delta(P\,;f_k)). \tag{9.2}$$

Example (9.3). Assume that $\Gamma(f_i)=d_i\Gamma(f)$, $i=1,\cdots,k$ as in §7. Then $\mathcal{T}=\mathcal{S}$ and $\mathcal{S}$ corresponds bijectively to the maximal faces Δ of $\Gamma(f)$. Thus by (5.6.1) and (6.10) we obtain the formula:

$$r(V)=\sum_{P\in\mathcal{S}}d_1\cdots d_k(k+1)!\,\mathrm{Vol}_{k+1}(C(\Delta(P\,;f)))/d(P\,;f). \tag{9.3.1}$$

In particular, (9.3.1) implies that $r(V) \geqq |S|$.

(I) Assume that $k=1$ and $d_1=1$. Then V is a plane curve. By (9.3.1) we have

$$(9.3.2) \qquad r(V)=\sum_{P\in S} 2\,\mathrm{Vol}_2(C(\Delta(P;f)))/d(P;f).$$

Now it is easy to see that $2\,\mathrm{Vol}_2(C(\Delta(P;f)))/d(P;f)$ is equal to the number of the integral points on $\Delta(P;f)$ minus 1 by (6.10). *Thus* $r(V)$ *is equal to the number of integral points on* $\Gamma(f)$ *minus* 1. Here f is assumed to be convenient. I believe that this formula is well-known to specialists. See for example [O 2] or [Br-Kn].

(II) Assume that $f(z)=z_1^{a_1}+\cdots+z_{k+1}^{a_{k+1}}$. Let b be the least common multiple of $a_1,\cdots,a_{k+1}$. Then we have $r(V)=(\prod_{j=1}^{k} d_j \prod_{i=1}^{k+1} a_i)/b$. In particular, $r(V)=1$ if and only if $d_i=1$ for $i=1,\cdots,k$ and $b=a_1\cdots a_{k+1}$ i.e., $\{a_1,\cdots,a_{k+1}\}$ are mutually prime.

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