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Linear plane divisors of homology planes

Dedicated to Professor Akio Hattori on his sixtieth birthday

By Tammo TOM DIECK

1. Introduction and statement of results.

Let V be a non-singular affine surface over the complex numbers. The surface V is called *R-homology plane* if its reduced homology $\tilde{H}_*(V;R)$ is zero; here R is a subring of the rational numbers. For $R=\boldsymbol{Z}$ we simply talk about homology planes. It has been shown by Gurjar and Shastri [6] that a homology plane V is isomorphic to a complement $X\backslash D$ where X is a non-singular rational projective surface and D a reduced effective divisor of smooth rational curves with normal crossings. One can choose X such that there exists a contraction $p: X\rightarrow \boldsymbol{P}^2$ onto the projective plane $\boldsymbol{P}^2$. The image $p(D)$ is then a union of irreducible rational curves in $\boldsymbol{P}^2$ and is called a plane divisor of V. In this paper we determine all plane divisors of $\boldsymbol{Q}$- and $\boldsymbol{Z}$-homology planes which are unions of lines. We call such arrangements of lines *linear plane divisors.* The results of this paper are part of a general project of Ted Petrie and the author to classify homology planes. We conjecture that most homology planes have linear plane divisors. In [4] an algorithm is described which yields all homology planes which result from the arrangements of lines produced in this essay.

Suppose $L=\{L_1, \cdots, L_n\}$ is an arrangement of n lines in $\boldsymbol{P}^2$. A point x which is contained in exactly r lines L_i is called an *r-point* of L or a point of *valence* $m(x)=r$. Let t_r denote the number of r-points. Then we call $n(t_2, t_3, \cdots)$ the combinatorial data of L. Often we write n as roman numeral, e.g. VI (3, 4). Let $N=\{x|m(x)\geqq 3\}$. A function $d: N\rightarrow\{0,1\}$ is called a *selection function.* We blow up $\boldsymbol{P}^2$ in all points of N. Let $E(x)$ be the exceptional curve belonging to $x\in N$ and let L_1' be the proper transform of L_i. Let $S(L,d)$ be the set of curves

$$\{L_i' \mid i=1, \cdots, n\}\cup\{E(x) \mid x\in N,\ d(x)=1\}$$

and $T(L,d)$ its dual graph. It is shown in [3], [4] that a pair (L,d) can lead to a $\boldsymbol{Q}$-homology plane only if the following conditions hold:

(1.1) $$\sum_{x\in N}(1-d(x))=:\mu(d)\leqq n-1.$$

(1.2) $T(L,d)$ is connected and has at most $n-1-\mu(d)$ independent cycles.

We say that L has a tree resolution if there exists a selection function d such that (1.1) and (1.2) hold. The following is derived in [3].

(1.3) PROPOSITION. *Suppose the arrangement possesses a tree resolution. Then*:

(i) $$2n-2\geqq\sum_{r\geqq 2}t_r=:f_0.$$

(ii) $$3n-3\geqq 2f_0-t_2.$$

Actually (1.2) implies the following, see [3].

(1.4) $$\sum_{x\in N}d(x)(m(x)-2)\leqq 2n-2-f_0.$$

We call a selection function *admissible* if it satisfies (1.4).

Arrangements of lines in general satisfy certain combinatorial inequalities. First there is the elementary identity

(1.5) $$\binom{n}{2}=\sum_{r\geqq 2}\binom{r}{2}t_r.$$

Moreover we have the important inequality of Hirzebruch [1].

(1.6) PROPOSITION. *Suppose the arrangement satisfies* $t_n=t_{n-1}=t_{n-2}=0$. *Then*

$$t_2+\frac{3}{4}t_3\geqq n+\sum_{r\geqq 5}(2r-9)t_r.$$

Using these results it was shown in [3] that the following holds.

(1.7) PROPOSITION. *Suppose an arrangement of n lines with* $t_n=t_{n-1}=t_{n-2}$ *possesses a tree resolution. Then* $n\leqq 13$.

We start from these results. It is a trivial matter to list all solutions of (1.3), (1.5), (1.6) such that $6\leqq n\leqq 13$ and $t_n=t_{n-1}=t_{n-2}$. But apart from these basic inequalities there are more combinatorial conditions which arrangements of lines have to satisfy, as exemplified by (1.8)-(1.11). We collect a few observations which are proved by simple counting arguments (left to the reader).

We consider arrangements of the following type: Two points x resp. y of valence k resp. l with $k \leqq l$ and s additional lines not going through x or y. We call x and y *connected* if they both lie on some line of the arrangement. We denote arrangements of this type by

$$((k, l))+s\,, \quad \text{if } x \text{ and } y \text{ are connected,}$$

$$[[k, l]]+s\,, \quad \text{if } x \text{ and } y \text{ are not connected.}$$

We let t'_r be the number of points of valence r which are different from x and y.

(1.8) LEMMA. *The arrangement* $[[k, l]]+0$ *has* $f_0=2+kl$ *and* $t'_2=kl$. □

This lemma is used to show that in certain (hypothetical) arrangements two points of heighest possible valences k and l have to be connected because $2+kl>f_0$.

(1.9) LEMMA. $((k, l))+1$ *and* $[[k, l]]+1$ *have* $t'_r=0$ *for* $r>3$. *Moreover*:

(i) $((k, l))+1$ *has* $t'_3 \leqq k-1$.

(ii) $[[k, l]]+1$ *has* $t'_3 \leqq k$. □

(1.10) LEMMA. *Consider* $((k, l))+2$. *Then*:

(i) $t'_r=0$ *for* $r>4$.

(ii) $t'_4 \leqq 1$.

(iii) $t'_3 \leqq 2k-1$.

(iv) *If* $t'_4=1$, *then* $t'_3 \leqq 2k-4$. □

(1.11) LEMMA. *Consider* $((k, l))+3$. *Then either* (i) *or* (ii) *holds.*

(i) $t'_5=1, \quad t'_4=0$.

(ii) $t'_r=0$ *for* $r>4$, $t'_4 \leqq 3$. □

It is now a straightforward computation to list all solution vectors $(t_2, t_3, \cdots)$ of the inequalities (1.3), (1.5), (1.6) which are not ruled out by (1.8)-(1.11).

These solutions are covered by Tables 1 and 2 below. These tables are set up in order to suit the following statements.

(1.12) THEOREM A. *The data* $n(t_2, t_3, \cdots)$ *which occur for arrangements of lines satisfying* (i)-(iv) *are listed in Table* 1.

(i) $6 \leqq n \leqq 13$.

(ii) $t_n = t_{n-1} = t_{n-2} = 0$.

(iii) $f_0 \leqq 2n-2$.

(iv) $2f_0 - t_2 \leqq 3n-3$.

(1.13) THEOREM B. *The combinatorial data of Table* 2 *cannot occur for arrangements of lines.*

The tables contain more information to be explained below. Also we have to deal with the data not covered by (1.7) and (1.12). We also point out that the combinatorial data do not, in general, determine the arrangement uniquely up to projective automorphisms. There exist more combinatorial data then just the cardinalities of r-points. This becomes clear later. The reader is refered to the body of the paper for the actual classification and description of the arrangements.

A pair (L, d) is called *redundant* if it has the following property: There exist three non collinear points x, y, z such that the following holds:

(1.14) (i) Each line of the arrangement goes through at least one of these points.

(ii) If k is the number of lines in L connecting two of the points, then

$$k > d(x) + d(y) + d(z).$$

We call (L, d) *minimal* if it is not redundant. The significance of these notions is explained in [4]: If a pair is redundant, then one can apply a plane Cremona transformation with centers x, y, z in order to obtain an arrangement with fewer lines which leads to the same family of $\boldsymbol{Q}$-homology planes. Thus for the purpose of constructing acyclic surfaces it is only necessary to list the minimal arrangements. Therefore we do not check whether redundant arrangements yield homology planes or not.

There are three further properties why certain (L, d) need not be considered for the construction of homology planes. For use in our tables we indicate the following occurrences.

(1.15) (c): $T(L, d)$ is not connected.

($\boldsymbol{Q}$): (L, d) leads only to $\boldsymbol{Q}$-homology planes but never to $\boldsymbol{Z}$-homology planes.

(0): (L, d) leads only to compactification trees with determinant zero.

(r): (L, d) is redundant.

If *all* (L, d) which belong to a datum $n(t_2, t_3, \cdots)$ have one or several of the properties (1.15), then we have listed the cases which occur in column (#) of Table 1. Therefore only the data with empty (#)-row have to be considered for the construction of $\boldsymbol{Z}$-homology planes.

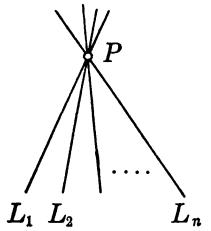
We now deal with the data which are not covered by Tables 1 and 2. These are the cases in which t_n, t_{n-1}, t_{n-2} are not all zero. We list the possibilities.

(1.16) (i) Arrangements of type $L_0(n)$:
n lines $L_1, \cdots, L_n$ intersect in a single point ($n \geqq 1$).

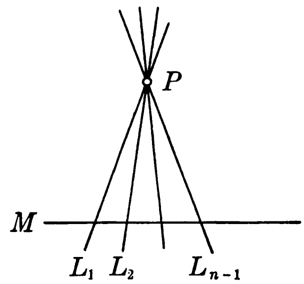
(ii) Arrangements of type $L_1(n)$:
$n-1$ lines $L_1, \cdots, L_{n-1}$ intersect in a single point P; an additional line M not containing P. ($n \geqq 3$).

(iii) Arrangements of type $L_2(n)$:
$n-2$ lines $L_1, \cdots, L_{n-2}$ intersect in a single point P; two additional lines M_1, M_2 not containing P and intersecting in Q. The point Q is contained in $L_1 \cup \cdots \cup L_{n-2}$. Here $n \geqq 5$.

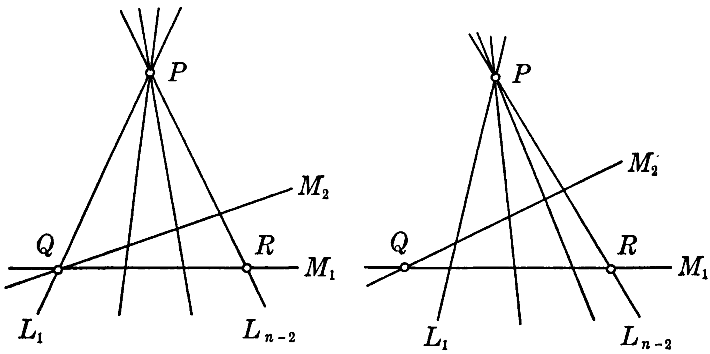
(iv) Arrangements of type $L_3(n)$:
Same as in (iii) but Q is not contained in $L_1 \cup \cdots \cup L_{n-2}$. Here $n \geqq 4$.



(i)



(ii)



(iii) (iv)

(1.17) PROPOSITION. *The arrangements of type $L_2(n)$, $n\geqq 6$ and $L_3(n)$, $n\geqq 5$ are redundant. They are plane divisors of homology planes.*

PROOF. Consider $L_2(n)$. We have $t_2=2(n-3)$, $t_3=1$, $t_{n-2}=1$ and therefore $f_0=2n-4$.

An admissible selection function must satisfy $d(Q)+d(P)(n-4)\leqq 2$, see (1.4). This requires $d(P)=0$ if $n\geqq 7$. We apply a Cremona transformation with centers P, Q, R, see the figure for (iii). The exceptional curve corresponding to P has to be removed. One checks that the result is an arrangement of type $L_1(n-1)$. For $n=6$ one could also make the choice $d(Q)=0$, $d(P)=1$. The same Cremona transformation leads to $L_2(5)$.

Now consider $L_3(n)$. One has $t_2=2(n-2)+1$, $t_{n-2}=1$ and hence $f_0=2n-2$. An admissible selection function must satisfy $d(P)=0$. One applies again a Cremona transformation with centers P, Q, R. The line connecting P, Q, has to be added. The resulting arrangement is $L_1(n)$. Thus $L_3(n)$ is not redundant in the strict sense that one can reduce the number of lines by a Cremona transformation. But one can transform into an arrangement of a different type.

Since the arrangements $L_1(n)$ lead to homology planes (see Gurjar-Miyanishi [5], tom Dieck-Petrie [2]), also the $L_2(n)$ and $L_3(n)$. □

The two remaining arrangements $L_2(5)$ and $L_3(4)$ are very important for the construction of homology planes. They are studied in detail in [4]. They are uniquely determined by the combinatorial data up to projective isomorphism. We have already refered to the $L_1(n)$. According to Gurjar-Miyanishi loc. cit. they can be used for the construction of all homology planes of logarithmic Kodaira dimension one.

The arrangements $L_0(n)$ finally appear as plane divisors of the standard affine plane $\boldsymbol{C}^2$. But they are also plane divisors of interesting $\boldsymbol{Q}$-homology planes.

Table 1.

n	2	3	4	5	6	Δ	(♯)
VI	6	3				1	r
	3	4				3	r
VII	6	5				1	r, $\boldsymbol{Q}$
	3	6				3	
	9	2	1			0	r
	6	3	1			2	r
	9	0	2			1	c
VIII	7	7				0	$r, c, 0$
	4	8				2	c
	7	5	1			1	r, c
	4	6	1			3	r
	10	2	2			0	r
	7	3	2			2	r
	9	3	0	1		1	r
	12	0	1	1		0	c
IX	0	12				4	$0, \boldsymbol{Q}$
	6	8	1			1	$\boldsymbol{Q}$
	9	5	2			0	$r, c, 0$
	6	6	2			2	
	9	3	3			1	r
	6	4	3			3	r
	8	6	0	1		1	r
	11	3	1	1		0	r
	12	3	0	0	1	0	r
X	3	10	2			3	c
	9	6	3			0	$c, \boldsymbol{Q}$
	6	7	3			2	$c, \boldsymbol{Q}$
	9	4	4			1	$\boldsymbol{Q}$
	5	10	0	1		2	$0, \boldsymbol{Q}$
	8	7	1	1		1	
	11	4	2	1		0	r

Table 1 (Continued)

n	2	3	4	5	6	Δ	(#)
XI	10	5	5			0	c
	10	3	6			1	c
	6	9	2	1		2	r
	11	6	1	2		0	r
XII	12	0	9			1	c
	9	9	0	3		1	r

r=redundant (not minimal)
c=graph disconnected
$\boldsymbol{Q}=\boldsymbol{Q}$-homology planes
0=determinant 0
$\Delta=2n-2-f_0$

Table 2.

n	2	3	4	5	6
IX	3	9	1		
X	12	1	5		
	9	2	5		
XI	7	6	5		
	9	6	3	1	
	12	3	4	1	
	9	4	4	1	
	8	9	0	2	
XII	9	5	7		
	12	2	8		
	11	5	5	1	
	11	3	6	1	
	14	0	7	1	
	10	6	3	2	
	9	8	3	0	1
	12	5	4	0	1
	15	0	6	0	1
XIII	12	2	10		
	14	0	9	1	
	13	3	6	2	
	12	6	3	3	
	14	4	2	4	

2. Classification of arrangements: simple cases.

We use the following notation: A line of an arrangement which contains the points of valence $r_1, \cdots, r_t$ is denoted by $L(r_1, \cdots, r_t)$. Note that

$$\sum_{i=1}^{t}(r_i-1)=n-1 \tag{2.1}$$

in an arrangement of n lines.

If there exists an arrangement with given data in the real projective plane we draw a figure of lines in the plane. Often the line at infinity L_∞ will be part of the arrangement. This fact is indicated by the phrase "with ∞". Note that parallel lines intersect at infinity. Lines in a figure which look parallel (no intersection point drawn) are meant to be parallel. If the arrangement is redundant in the sense of section one we draw three fat points which prove this. If such a point appears on the infinite line we draw it related to the corresponding pencil of parallels. Note that a selection function d must satisfy (1.4). This usually forces most of the $d(x)$ to be zero. (In the cases at hand the right hand side of (1.4) is at most 4). On the other hand the requirement that the graph $T(L, d)$ has to be connected forces some of the $d(x)$ to be one. If these two requirements lead to a contradiction we conclude that L has no tree resolution. (This is the case marked (c) in column (♯) of Table 1.) In dealing with redundancy we only treat the case where equality holds in (1.4). If these are redundant the others are a fortiori.

To begin with we dispose of the simple cases $((k, l))+0$ and $((k, l))+1$. We only consider arrangements which satisfy the conditions of Theorem A. Consider arrangements of type $((k+1, l+1))+0$. Then $3 \leqq k \leqq l$, $n=k+l+1$, $kl+2=f_0 \leqq 2n-2=2k+2l$. We obtain

$$\frac{2l-2}{l-2} \geqq k \geqq 3\,, \qquad 2l-2 \geqq 3l-6\,, \qquad 4 \geqq l\,.$$

Hence the only solutions are $(k, l)=(3, 3)$ and $(3, 4)$. They correspond to VII $(9, 0, 2)$ and VIII $(12, 0, 1, 1)$ in Table 1. Every arrangement with these data is of the type described because the two points of highest valence have to be connected. Since $d(x)=0$ for the points of highest valence, by (1.4), the corresponding graph is disconnected.

Consider arrangements of type $((k+1, l+1))+1$. Suppose the additional line passes through r of the points of $((k+1, l+1))$. Then $r \leqq k$. We obtain $k+l+1-2r$ new points. The condition $f_0 \leqq 2n-2$ now reads

$$(k-1)(l-1) \leqq 2r\,.$$

The solutions and the associated combinatorial data are

(2.2)

k	l	r	data
2	2	1, 2	VI (6, 3), VI (3, 4)
2	3	1, 2	VII (9, 2, 1), VII (6, 3, 1)
2	4	2	VIII (9, 3, 0, 1)
2	5	2	IX (12, 3, 0, 0, 1)
3	3	2, 3	VIII (10, 2, 2), VIII (7, 3, 2)
3	4	3	IX (11, 3, 1, 1)

The two points of highest valence in arrangements with these data have to be connected. Therefore we have

(2.3) PROPOSITION. *The arrangements of type* $((k+1, l+1))+1$ *are those listed in* (2.2). *Any arrangement with the data of* (2.2) *is of type* $((k+1, l+1))+1$. □

One should observe that the arrangements in (2.3) are in general not unique up to automorphisms of $\boldsymbol{P}^2$; they can contain lines which can be moved within a given pencil.

If one looks at the two centers of the pencils and a suitable point of valence at least 3 on the additional line one sees easily

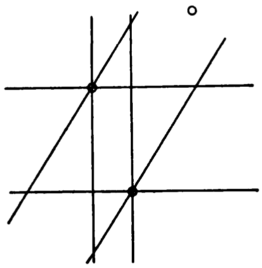
(2.4) PROPOSITION. *The arrangements of* (2.2) *are redundant.* □

The remaining data of Table 1 will be considered individually in the order in which they are listed.

3. Classification of arrangements: individual class cases.

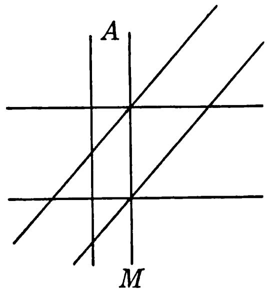
VII (6, 5)

Suppose there exist two unconnected 3-points. They use up 6 lines. The additional line must then produce three 3-points. If any two 3-points are connected but never three on one line, then the five 3-points would be in general position and would have ten connecting lines. Hence there exists always a line with three 3-points which we take as the infinite line. We can normalize by a projective transformation and assume that $z_1=0, 1$ and $z_2=0, 1$ are contained in the arrangement (z_1, z_2 complex coordinates). If there are unconnected 3-points we can assume that these are $(0,1)$ and $(1,0)$. In the other case we can assume that $(0,1)$ and $(1,1)$ are 3-points.



with ∞

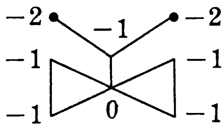
(1)



with ∞

(2)

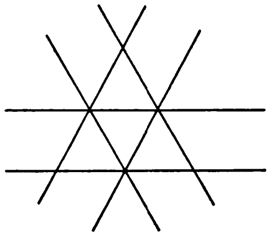
Case (1) is redundant (1.14), because $k=2$ and $d(x)+d(y)+d(z)\leqq 1$ for any admissible selection function d. In case (2) a selection function must have value 1 at a single 3-point. In order to obtain a connected graph this has to be a point on L_∞ and M, hence the point A. This selection function leads to the following weighted dual graph.



The $(-2,-1,-2)$-head of this butterfly shows that the determinant of any tree obtained by cutting cycles is divisible by 2^2. (See tom Dieck-Petrie [4] for the relevant notions.) Thus one never obtains $\boldsymbol{Z}$-homology planes. By symmetry of the figure there are three possibilities to cut cycles.

VII (3, 6)

Any two 3-points are connected. Any arrangement of this type must contain four lines in general position. Such lines produce six 2-points. Among them there are three unconnected pairs and the pairs are disjoint. Thus we have to connect the pairs and obtain a VII (3, 6). This conceptual description also shows that the data determine the arrangement uniquely up to projective transformations.



with ∞

This arrangement has many symmetries. In particular the three 2-points are equivalent. The arrangement is minimal and leads to several infinite families of homology planes [4].

VIII (7, 7)

The lines of an arrangement have the form

(i) $L(3, 3, 3, 2)$

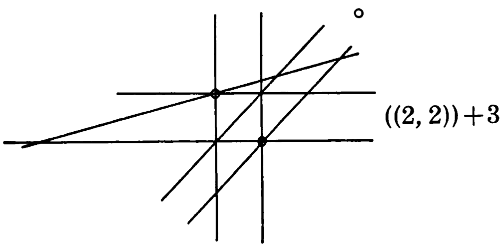
(ii) $L(3, 3, 2, 2, 2)$

(iii) $L(3, 2, 2, 2, 2, 2)$.

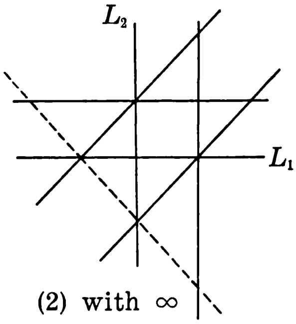
Not all lines can have the form (i), since otherwise we could pair them off according to the containing 2-point and we would have only four 2-points.

If we remove a line of type (iii), there remains a VII (3, 6). This arrangement was projectively unique. Because of symmetry its three 2-points are equivalent. We fix a 2-point and add a line which is not parallel to the 3 pairs of parallels. We obtain case (1).

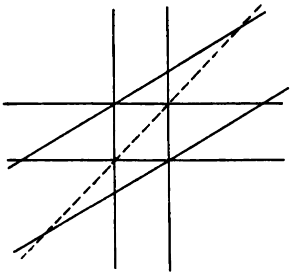
There remains the case that only lines of types (i) and (ii) exist and at least one of type (ii) which, when removed, yields a VII (6, 5). The cases (2) and (3) are the structurally different enlargements of VII $(6, 5)_1$, the cases (4) and (5) those of VII $(6, 5)_2$.



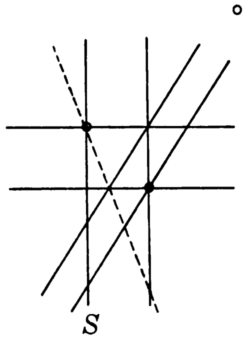
(1) with ∞



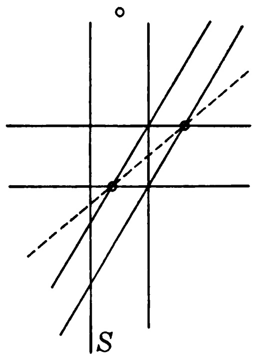
(2) with ∞



(3) with ∞



(4) with ∞



(5) with ∞

The broken line is the one added. We consider in these figures the 3-points and count the number of lines with three 3-points through them. We get

(2)	3 2 2 2 2 2 2
(3)	3 3 2 2 2 2 1
(4)	3 3 3 2 2 2 1
(5)	3 3 3 3 2 2 2

This shows that these arrangements have different combinatorial data and are hence not isomorphic.

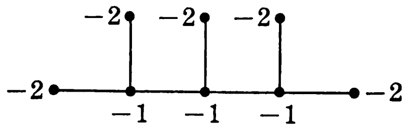
The added line in (2) and (3) has different position with respect to pairs of parallel lines. The added line in (4) and (5) has different position with respect to the line S with four 2-points in VII $(6, 5)_2$.

A selection function must have value zero on all 3-points. The fat points show that cases (1), (4), and (5) are redundant.

In case (2) there exist two lines L_1, L_2 which intersect in a 2-point.

They become separated when we blow up. Thus the graph is not connected.

In case (3) the weighted dual graph is

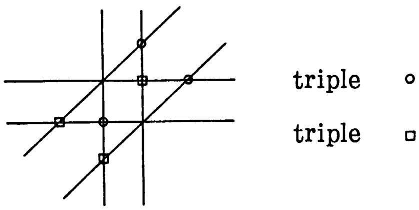


and has determinant zero.

VIII (4, 8)

A line can have the form $L(3,3,3,2)$ or $L(3,3,2,2,2)$. Removing a line of the second type leads to VII (3, 6). In VII (3, 6) any two 2-points are connected. Therefore it is impossible to add a line of the required type. Hence all lines must have type $L(3,3,3,2)$. Removing a line then leads to VII (6, 5). In VII $(6,5)_2$ any set of three 2-points contains two on a line. Therefore it is impossible to enlarge this arrangement.

In case VII $(6,5)_1$ there exist two triples of 2-points in which no two lie on a line:



The situation is symmetric with respect to these two triples.

Projective normalization leads to a triple

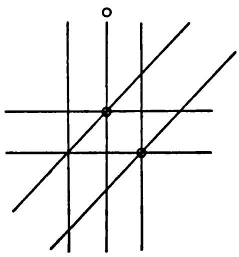
$$(0,0),(1,a),\left(\frac{a}{a-1},1\right) \qquad a\neq 1\,.$$

We have to investigate when these points lie on a line. The line has the form $bz_1=z_2$ since it passes through (0, 0).

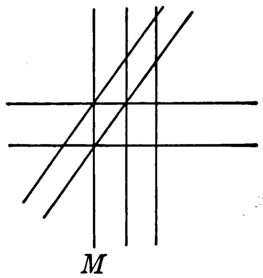
We obtain the conditions $b=a$, $b=a^{-1}(a-1)$, hence $a^2=a-1$. We write $a=1+\omega$ and see $\omega^3=1$, $\omega\neq 1$. Interchanging z_1 and z_2 transforms the two cases into each other. Therefore the arrangement is projectively unique. It arises from the famous IX (0, 12) by removing any one of its lines.

A selection function d has value one on two 3-points. All lines have type $L(3,3,3,2)$. They are paired according to the 2-point they contain. In order not to disconnect the graph a point x with $d(x)=1$ has to lie on each pair. One checks that this is impossible.

VIII (7, 5, 1)



with ∞
(1)



with ∞
(2)

The 4-point has to be connected with each 3-point since otherwise there would exist at least 14 points. There exists a line through the 4-point which contains at least two 3-points. Such a line is chosen as L_∞.

The arrangement contains a ((2, 3)). The two additional lines are parallel. They have to go through three 2-points of ((2, 3)). One of them must go through exactly two 2-points. This line and ((2, 3)) can projectively be normalized to

$$z_1 = 0, 1, a\,; \quad z_2 = 0, 1\,; \quad z_1 = z_2\,; \quad L_\infty\,.$$

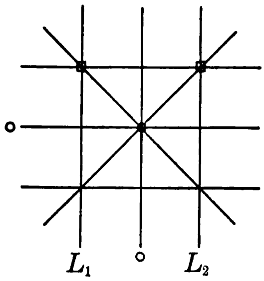
The additional line $z_1 = z_2 + c$, $c \neq 0$, goes through one of the points (1, 0), (0, 1), $(a, 0)$, $(a, 1)$. If it goes through $(a, 0)$ or $(a, 1)$ the lines through the 4-points have the form $L(4, 3, 3)$, $L(4, 3, 2, 2)$ three times. If it goes through (1, 0) or (0, 1), these lines have the form $L(4, 3, 3)$ twice, $L(4, 3, 2, 2)$ and $L(4, 2, 2, 2, 2)$. The two cases (1) and (2) are therefore combinatorially distinct.

Case (1) is redundant. A selection function d has value one on a single 3-point x. In order not to disconnect the graph in case (2) x has to lie on M and L_∞. This is impossible since $M \cap L_\infty$ is a 4-point.

VIII (4, 6, 1)

The 4-point is connected to each 3-point and we can therefore choose as L_∞ a line $L(4, 3, 3)$. The additional lines to ((2, 3)) are parallel. In order to produce six 3-points each of the two additional lines has to go through two 2-points of ((2, 3)).

The configuration is projectively unique.

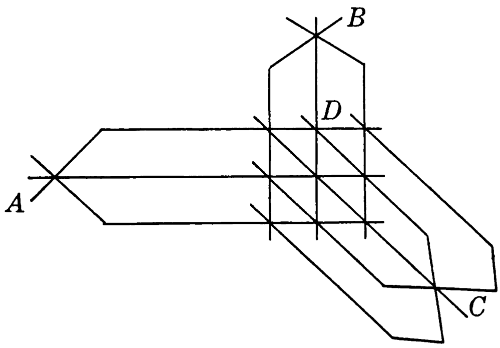


A selection function d has value one on two 3-points. In order not to disconnect the graph these points have to lie on L_1 and L_2. Up to symmetry these are the points □. The fat points then show that the arrangement is redundant.

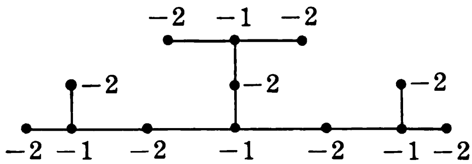
We could also choose a selection function with value one on the 4-point. Again we see that the situation is redundant.

IX (0, 12)

An arrangement with these data exists and is projectively unique. It is called the dual Hesse arrangement in [1], p. 76. This arrangement cannot be realized in the real projective plane. We draw a figure which indicates the intersection pattern of the lines.

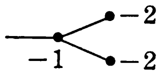


A selection function must have value one on four 3-points. Suppose there exist two unconnected 3-points with value one, say A and B by symmetry. Then, in order not to disconnect, there has to be a point with value one on each line through C, hence C must have value one. The final point with value one can be any one in the center square, say D by smmetry. The weighted dual graph is



The center is D, blown up. The determinant is 4^3.

There remains the possibility that all points on a single line have value one. The resulting tree has a center with -3 and three branches



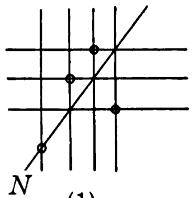
attached to it. The determinant is zero.

IX $(6, 8, 1)$

Suppose there exists a 3-point which is not connected to the 4-point. The lines through these two points use up 7 lines. The two additional lines have to produce seven more 3-points. Thus they have to intersect on a line S through the 4-point and they have to go through three 2-points lying on the other three lines through the 4-point. Therefore the line S is of type $L(4, 3, 2, 2, 2)$. Removing it leads to VIII $(4, 8)$. This latter arrangement is projectively unique and there is a unique way of enlarging it to IX $(6, 8, 1)_1$. This leads to (1).

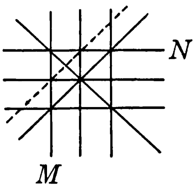
There remains the situation that no line of type $L(4, 3, 2, 2, 2)$ exists. Then all lines through the 4-point must be of type $L(4, 3, 3, 2)$.

Suppose there exists a line of type $L(3, 3, 2, 2, 2, 2)$. Removing it leads to VIII $(4, 6, 1)$. The possible connections of two 2-points in VIII $(4, 6, 1)$ are equivalent, because of symmetry. Thus we obtain a second realization (2) with no line of type $L(4, 3, 2, 2, 2)$ and two lines of type $L(3, 3, 3, 3)$. Finally, suppose also no line of type $L(3, 3, 2, 2, 2, 2)$ exists. Then there must exist a line of type $L(3, 3, 3, 2, 2)$. Removing it leads to VIII $(7, 5, 1)$. The second case cannot be augmented appropriately. The first case has a unique augmentation leading to type (3). (It fixes the free parameter in VIII $(7, 5, 1)_1$. There exist augmentations of other types.)

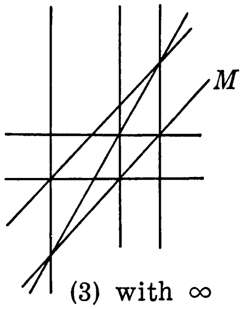


(1)

line M through ∘



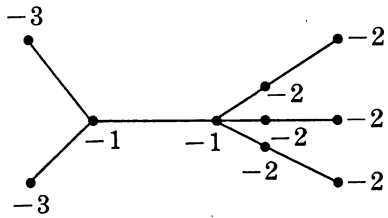
(2)



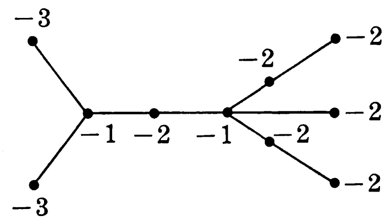
(3) with ∞

We show that in all three cases the weighted dual graph $T(L, d)$ is already a tree and its determinant is not ± 1. Thus one obtains $\boldsymbol{Q}$-homology planes but not integral ones.

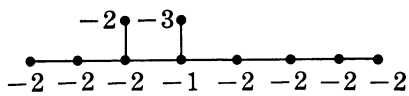
A selection function must have value one on a single 3-point. In cases (1) and (2) it has to be the point $M \cap N$ in order to obtain a connected graph. In case three it has to be a point on M. One computes the following weighted dual graph and their determinants.



(i) $2^2 \cdot 3^4$



(ii) 3^2

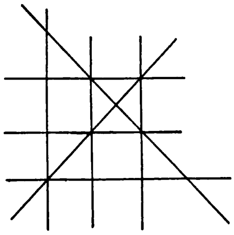


(iii) 10^2

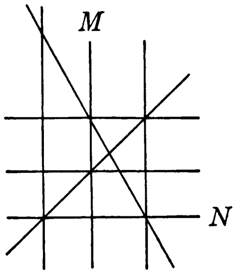
IX (9, 5, 2)

The two 4-points have to be connected. The connecting line of the 4-points has the form $L(4, 4, 3)$ or $L(4, 4, 2, 2)$. A selection function has value zero on all points x of valence $m(x) \geqq 3$. Hence a line $L(4, 4, 3)$ will be separated in the graph. So we need not consider these cases. (There are several as the reader may find out.)

We consider the $((3, 3))+2$ arrangement where $((3, 3))$ is given by the lines through the 4-points. If the two additional lines intersect on a line through a 4-point, then the arrangement is redundant. Therefore the two additional lines have to intersect in a 2-point of the complete arrangement. One has to be of type $L(3, 3, 3, 2, 2)$ the other of type $L(3, 3, 2, 2, 2, 2)$ in order to obtain five 3-points. There are two different intersection patterns as drawn in the following figure.

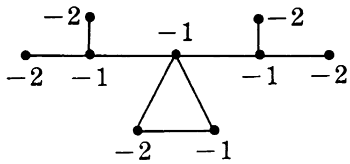


(1) with ∞

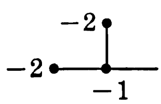


(2) with ∞

In case (2) the lines $M \cup N$ become separated in the graph. The remaining case (1) has the dual graph $T(L, d)$



Cutting cycles in this graph leads always to a tree with determinant zero. This is due to the appearance of the two



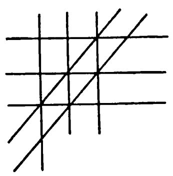
pieces.

IX (6, 6, 2)

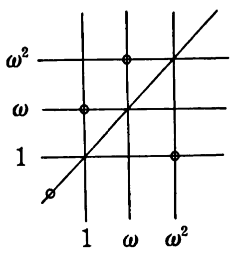
The two 4-points have to be connected. Therefore we have a ((3, 3))+2 arrangement. The connecting line of the 4-points has ths form $L(4, 4, 3)$ or $L(4, 4, 2, 2)$.

First case. The two additional parallel lines have to go through 3 resp. 2 of the 2-points of ((3, 3)). The resulting arrangement is projectively unique. There exists a line of type $L(4, 3, 2, 2, 2)$. Removing it leads to VIII (4, 6, 1). It was unique and the augmentations are unique, by symmetry.

Second case. We remove $L(4, 4, 2, 2)$ and obtain the unique VIII (4, 8). The augmentation is again unique, by symmetry.



(1) with ∞



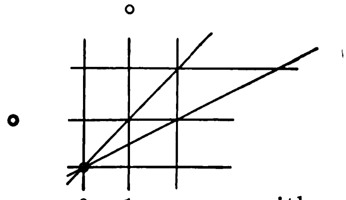
(2) with ∞

line through ∘

Both cases lead to several families of homology planes as shown in [4].

IX (9, 3, 3)

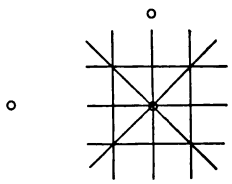
Again we must have a ((3, 3))+2 arrangement. The two additional lines have to intersect in a 2-point of ((3, 3)) and must go through two resp. one further 2-points. We see that arrangements of this type are redundant.



with ∞

IX (6, 4, 3)

This is similar to the previous case. The line at infinity has type $L(4, 4, 2, 2)$. Removing it leads to VIII (4, 6, 1) which was unique. A selection function d has value one on three 3-points or on a 3-point and a 4-point. Both cases are redundant.



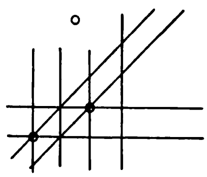
with ∞

IX (8, 6, 0, 1)

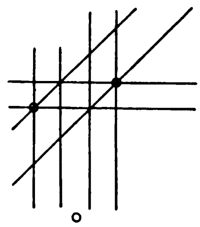
The 5-point has to be connected with all 3-points. Thus we have a $((2, 4))+2$. There exists a line through the 5-point which contains two 3-points; it is chosen as L_∞. The two additional lines are therefore parallel. In order to obtain six 3-points the two additional lines must go through two 2-points of $((2, 4))$ each.

Case (1): A line through the 5-point does not contain a 3-point. Removing this line leads to VIII (4, 6, 1).

A selection function has $d(x)=1$ for a single 3-point. Therefore both cases are redundant.



(1) with ∞



(2) with ∞

X (3, 10, 2)

If the 4-points were unconnected, then the remaining two lines could produce at most nine 3-points. The line connecting the 4-points can have the form (i) $L(4, 4, 2, 2, 2)$ or (ii) $L(4, 4, 3, 2)$.

In case (i), removing it leads to IX (0, 12). All augmentations of IX (0, 12) are equivalent by symmetry.

In case (ii) the other line through the 2-point of $L(4, 4, 3, 2)$ must have the form $L(3, 3, 3, 3, 2)$ since the removal of lines of other type contradicts Hirzebruch's inequality (1.6). Removing $L(3, 3, 3, 3, 2)$ leads to IX (6, 6, 2) and in arrangements of this type there are never four 2-points on one line. Thus case (ii) does not occur.

Since we have three 2-points on a single line there exist six lines of type $L(4, 3, 3, 3)$. A selection function has $d(x)=1$ on three 3-points or on a 3-point and a 4-point. There has to be a point with $d(x)=1$ on each of the six lines in order to obtain a connected graph. From this information one checks easily that any choice of a selection function leads to a disconnected graph.

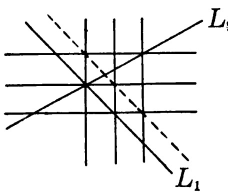
X (9, 6, 3)

Suppose the three 4-points lie on a line L_∞. Since a selection function has $d(x)=0$ on all points x of valence $m(x)\geqq 3$ the line $L_\infty=L(4, 4, 4)$ becomes separated. (The reader is invited to check that there exist different arrangements of this type.)

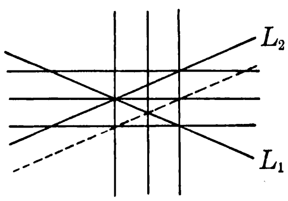
Now suppose that all 4-points are connected but they do not lie on a line. A connecting line has the form (i) $L(4, 4, 3, 2)$ or (ii) $L(4, 4, 2, 2, 2)$. Not all three of them have type (ii). For, otherwise, they would contain all 2-points. Then there must exist a line with only 3-points. This is impossible in an arrangement with 10 lines. Therefore we choose a line of type (i) as L_∞. We have a $((3, 3))+3$ arrangement. Two of the additional lines are parallel, two intersect in a 2-point of $((3, 3))$ to produce a 4-point.

Thus there exists exactly one line without 4-point. It must have one of the following types (a) $L(3,3,3,3,2)$, (b) $L(3,3,3,2,2,2)$, or (c) $L(3,3,2,2,2,2,2)$ since removing a line of another type leads to a contradiction with Hirzebruch's inequality (1.6).

Case (a). We remove the line and obtain IX (12, 2, 3). There are two intersection patterns: The 3-points on a line or not.

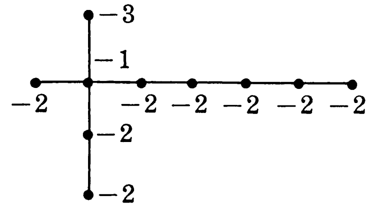


(1) with ∞



(2) with ∞

We have to add a line through four 2-points which is parallel to one of the two additional lines L_1, L_2 (the dotted line). Case (2) leads to a disconnected graph. Case (1) leads to a tree with determinant 12^2. It has the following structure.



It is remarkable that the tree is star-shaped.

Case (b). We remove the line and obtain a IX (9, 3, 3). This again leads to case (1) of (a).

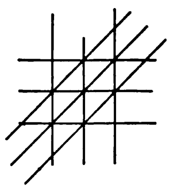
Case (c). We remove and obtain IX (6, 4, 3). An appropriate augmentation is impossible.

Finally we have the case that there exist two unconnected 4-points. The two additional lines must have the form $L(4, 3, 3, 3)$. Therefore any such arrangement leads to a disconnected graph.

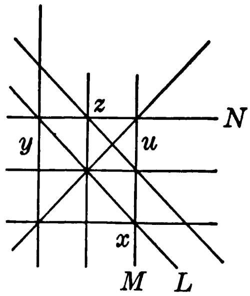
X (6, 7, 3)

Any two 4-points are connected. Suppose they lie on one line. Then we are in case (1). A selection function has $d(x)=1$ on two 3-points or on one 4-point. Both cases lead to a disconnected graph.

If the 4-points do not lie on one line, then there exists a line without 4-point. It must have the form (i) $L(3, 3, 3, 3, 2)$ or (ii) $L(3, 3, 3, 2, 2, 2)$. In the first case we are led to an augmentation of IX (9, 3, 3) which cannot have the required form. In the second case we have an augmentation of IX (6, 4, 3) which is unique, by symmetry of the figure.



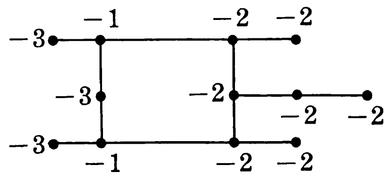
(1) with ∞



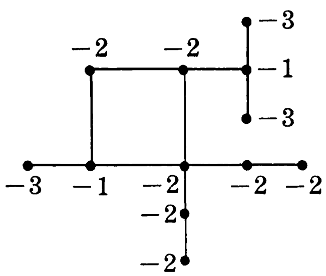
(2) with ∞

A selection function has value 1 on two 3-points or on a single 4-point. In order not to separate one has to have value one on a point on L, M, and N. Hence the second case of a selection function is impossible. Up to symmetry of the figure there are three choices of pairs of 3-points which can have value 1, namely $\{x, u\}$, $\{x, y\}$, $\{x, z\}$.

Case $\{x, y\}$. It leads to the following weighted dual graph.



The case $\{x, u\}$ leads to the same graph. Case $\{u, z\}$ and case $\{x, z\}$ lead to the following graph.



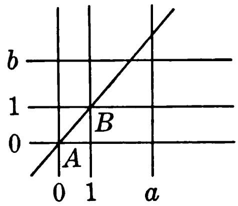
The $\overset{-3}{\bullet}\!-\!\!-\!\!-\overset{-1}{\bullet}\!-\!\!-\!\!-\overset{-3}{\bullet}$ piece in the second graph shows that one can never obtain integral homology planes, but of course $\boldsymbol{Q}$-planes. It can also be shown that the three possibilities to cut cycles in the first graph lead to $\boldsymbol{Q}$-homology planes but never to integral ones.

X (9, 4, 4)

Any two 4-points are connected. No three can lie on one line because such a situation would already use up 10 lines and there could not exist a fourth 4-point.

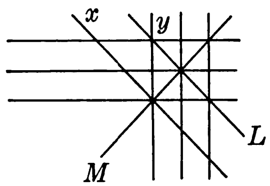
There exists a line of type $L(4, 4, 3, 2)$. For if not, then all 6 connecting lines of the 4-points would be of type $L(4, 4, 2, 2, 2)$. Therefore there would exist through each 4-point a line of type $L(4, 3, 3, 3)$. Two lines of this type would already yield at least five 3-points.

We take a line of type $L(4, 4, 3, 2)$ as L_∞. We normalize the four 4-points and obtain



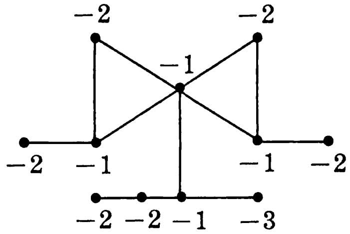
$a=b$ possibly

with two additional parallel lines through A and B. In case $a=b$ they have to produce two finite 3-points. The only possibility is



If $a \neq b$, there are (up to interchange of coordinates) two solutions with $1=ab$, namely $a=2$ and $a=-\omega$ with $\omega^3=1$. The case $a=2$ is isomorphic to the previous one, the case $a=-\omega$ leads to a disconnected graph. Therefore we study only the first case.

A selection function has value one on a single 3-point. In order not to separate this has to be a point on L. Up to symmetry there are two cases: x and y. Case y leads to a separation of $M \cup L_\infty$. Case x leads to the graph

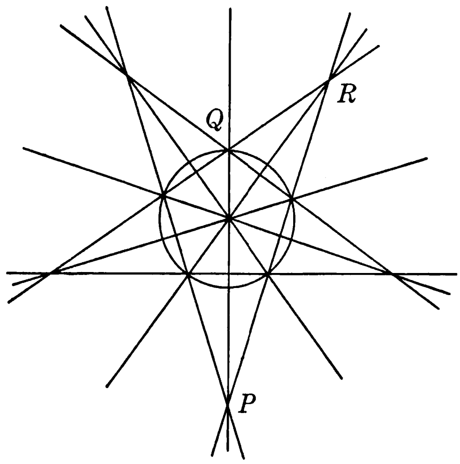


The socle of this monument shows that one can never obtain integral homology planes, but of course $\boldsymbol{Q}$-planes.

X (5, 10, 0, 1)

A line through the 5-point can have the form (i) $L(5,3,3,2)$. (ii) $L(5,3,2,2,2)$, or (iii) $L(5,2,2,2,2,2)$.

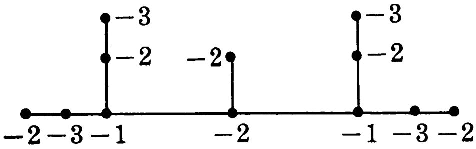
Removing a line (iii) leads to IX (0, 10, 1) and this contradicts (1.6). Removing (ii) leads to IX (3, 9, 1) and this is shown to be impossible in Section 4. Removing $L(5,3,3,2)$ leads to IX (6, 8, 1). Only IX $(6,8,1)_3$ has a (unique) augmentation of the required type:



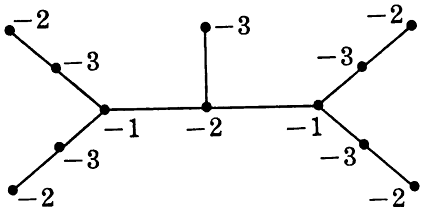
A regular five-gone together with the connection of the vertices with the center.

A selection function has value one on two 3-points. In order to obtain a connected graph these are, up to symmetry, the cases $\{P,R\}$ and $\{P,Q\}$.

The case $\{P,R\}$ leads to the graph



which has determined zero. The case $\{P, Q\}$ leads to the graph

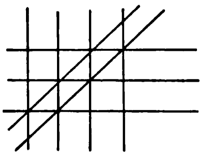


with determinant $(25)^2$.

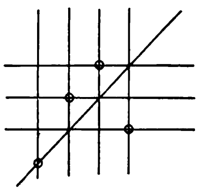
X (8, 7, 1, 1)

The 5-point and the 4-point are connected. Therefore we have a ((3, 4))+2 arrangement. If there exists a line $L_\infty = L(5, 4, 3)$, then we obtain necessarily case (1), which is projectively unique.

If $L_\infty = L(5, 4, 2, 2)$, then the two additional lines have to intersect on a line of ((3, 4)), but not in a 2-point. The line L of intersection must go through the 5-point, since otherwise there could not exist seven 3-points. Removing L leads to IX $(6, 6, 2)_2$. Hence case (2).



(1) with ∞



(2) with ∞ line through ∘

Case (2) is redundant. Case (1) leads to integral homology planes [4].

X (11, 4, 2, 1)

The 5-point is connected to the 4-points. The 4-points have to be connected, since otherwise there could not exist a 5-point. Arrangements exist. Necessarily all lines go through a 5-point or a 4-point. This shows the arrangements to be redundant.

XI (10, 5, 5)

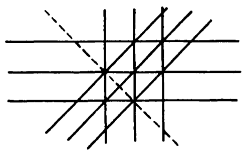
Suppose the five 4-points are in general position. If any two 4-points are connected this yields a X (15, 0, 5). The added line must be of type $L(3, 3, 3, 3, 3)$. Such arrangements exist: Take a regular 5-gon, add all connections of vertices and the line at infinity. Since a selection function has value zero on all points of valence greater two the line $L(3, 3, 3, 3, 3)$ will become separated. Unconnected 4-points cannot exist: Consider two such and the eight lines through them. Two additional lines L_1, L_2 intersect in a 2-point of the eight lines to produce a 4-point. Also they have to produce at least five 3-points in order that after adding the final line we have five 3-points. Therefore L_1 or L_2 will be an $L(4, 4, 3, 3)$ in the final arrangement and becomes separated.

Now consider the other case:

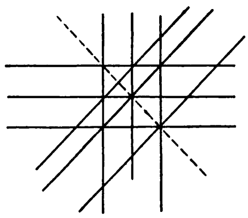
Three 4-points lie on one line L_∞. This uses 10 lines. The remaining line must produce two 4-points, hence go through two 3-points of the 10 lines.

(i) There exist already seven 3-points. Then we have an augmentation of X (6, 7, 3) with three 4-points on one line, hence of X $(6, 7, 3)_1$. This gives (1).

(ii) There exist only six 3-points. Then we leave an augmentation of X (9, 6, 3) with three 4-points on one line. This gives (2).



(1) with ∞



(2) with ∞

(2) is different from (1), since in (2) each connecting line of 4-points contains a 2-point. In both cases the graph is disconnected.

XI (10, 3, 6)

Suppose two 4-points are unconnected. Then we can produce with the additional 3 lines at most five 4-points.

Suppose six 4-points are in general position. Then there would exist 15 connecting lines. Therefore there exists a line of type $L(4, 4, 4, 2)$. We choose one as L_∞.

The lines through the 4-points at L_∞ use 10 lines. The additional line must go through three 3-points of the 10 lines. Hence there exists a second line of type $L(4, 4, 4, 2)$.

A selection function must have value zero on 4-points. Hence the two lines of type $L(4, 4, 4, 2)$ become seperated in the graph.

The existence of an XI (10, 3, 6) is shown by removing a line from XII (12, 0, 9).

XI (6, 9, 2, 1)

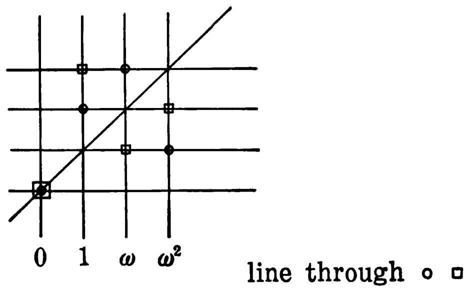
The 5-point is connected with the 4-points.

CLAIM. *The 5-point and the two 4-points do not lie on a line.*

PROOF. There must exist a line of type $L(4, 3, 3, 3, 2)$ since all other cases with a single 4-point contradict (1.6). An augmentation of X (8, 7, 1, 1) of the required type does not exist.

The 4-points are not connected. Otherwise the lines through the 5-point and the 4-points would give a X-arrangement. The eleventh line must be of type $L(3, 3, 3, 3, 3)$ since all other cases lead to impossible X-data. Therefore we would have a X (11, 4, 2, 1) which does not contain five 2-points on a line.

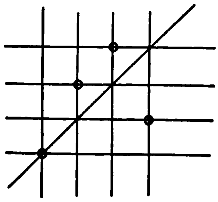
There remains an arrangement of the type which arises from XII (9, 9, 0, 3) by removing a connection of 5-points. It is redundant.



XI (11, 6, 1, 2)

Each 5-point is connected to the 4-point and the 5-points are connected. We therefore have a $((4, 4))+2$ arrangement. The two additional lines have to produce six 3-points. Since all lines go through the 4- and 5-

points the arrangement is seen to be redundant.



XII (12, 0, 9)

These data determine an arrangement which is projectively unique. It is called the Hesse arrangement in [1], p. 71. The resulting graph is disconnected.

XII (9, 9, 0, 3)

Any two 5-points are connected. The connections have necessarily the form $L(5, 5, 2, 2, 2)$. Removing them leads to a IX (0, 12). The arrangement is redundant.

4. Exclusion of combinatorial data.

We consider the data of Table 2 individually. The strategy to disprove the existence of arrangements is as follows: By counting arguments we show that there must exist a line L of specific type. We omit this line and can reach a contradiction in one of several ways:

(i) We arrive at data which were already shown to be impossible.

(ii) We arrive at data for which the existing arrangements cannot be augmented suitably.

(iii) We arrive at data which contradict Hirzebruch's inequality (1.6).

In case (ii) we refer to our classification and ask the reader to verify the impossibility of an augmentation.

We generally assume that there exists an arrangement with the given data and deduce a contradiction.

IX (3, 9, 1)

There exists a line L through the 4-point without 2-points, since there exist only three 2-points. A line of type $L(4, 3, \cdots, 3)$ is impossible for an arrangement of 9 lines by (2.1).

X (12, 1, 5) and X (9, 2, 5)

Suppose there are two unconnected 4-points. Then we can have at most three 4-points. Therefore any pair of 4-points is connected.

Suppose three 4-points lie on a line. Then the lines through these three 4-points give the whole arrangement. The other points produced by these lines can only be 3-points.

Therefore the five 4-points are in general position and any pair of them is connected by a line. But then we should have at least 15 lines.

XI (7, 6, 5)

Suppose a line L of type $L(4, 4, 4, 2)$ exists. Then the two remaining 2-points must lie on the other line L' through the 2-point of L. Therefore $L'=L(4, 4, 3, 2, 2)$. Removing L' leads to X (6, 7, 3) which cannot be augmented.

Suppose there are two unconnected 4-points. Then the lines through these 4-points produce already 18 points. Hence also two connected 4-points exist. The remaining case of a line $L(4, 4, 2, 2, 2)$ leads on removal to a contradiction with (1.6).

XI (13, 0, 7)

The lines can have the following structure:

(i) $L(4, 4, 4, 2)$.

(ii) $L(4, 4, 2, 2, 2, 2)$.

(iii) $L(4, 2, \cdots, 2)$.

(iv) $L(2, \cdots, 2)$.

Omitting a line of this type leads to:

(i) X (12, 3, 4).

(ii) X (9, 2, 5).

(iii) X (6, 1, 6).

(iv) X (3, 0, 7).

We have ruled out (ii) and (iii), (iv) contradict (1.6). There remains the case that all lines have type (i). Then we can pair off the lines according to containing the same 2-point and this is obviously impossible.

XI (9, 6, 3, 1)

There exist at least two lines through the 5-point without 4-point. These lines have the form:

(i) $L(5, 3, 3, 3)$.

(ii) $L(5, 3, 3, 2, 2)$.

(iii) $L(5, 3, 2, \cdots, 2)$.

(iv) $L(5, 2, \cdots, 2)$.

Omitting (iii), (iv) leads to X (6, 5, 4), X (3, 6, 4) contradicting (1.6). Omitting (ii) leads to X (9, 4, 4). By our description of this arrangement we see that there does not exist a line through exactly one 4-point and two 2-points. If there are two lines of type (i) they use up all 3-points. Hence there must exist a line of type $L(5, 4, 2, 2, 2)$ which when omitted leads to X (6, 7, 3). By our classification of these arrangements we see that 4- and 3-points are already connected.

X (12, 3, 4, 1)

Removing a line without 4-point through the 5-point lead to a case which we have already ruled out or contradicts (1.6) unless this line is $L(5, 3, 3, 3)$. If there exists such a line, then there must exist a line of type $L(5, 4, 2, 2, 2)$. Its omission leads to X (9, 4, 4) which cannot be augmented appropriately.

XI (9, 4, 4, 1)

Each 4-point has to be connected to the 5-point. Therefore there exists a line of type $L(5, 4, 3, 2)$ or $L(5, 4, 2, 2, 2)$. The first case leads to X (9, 4, 4) which cannot be augmented and the second case leads to X (6, 5, 4) which contradicts (1.6).

XI (8, 9, 0, 2)

There exists a line through a 5-point of type:

(i) $L(5, 3, 3, 2, 2)$.

(ii) $L(5, 3, 2, \cdots, 2)$.

(iii) $L(5, 2, \cdots, 2)$.

Omitting a line of type (ii) or (iii) leads to a situation which was already shown to be impossible. Omitting (i) leads to X (8, 7, 1, 1) which, by our classification, cannot be augmented.

We postpone the case XII (9, 5, 7) since it is somewhat involved.

XII (12, 2, 8)

There exists a line containing only 2-points and 4-points. These lines are:

(i) $L(4, 4, 4, 3, 3)$.

(ii) $L(4, 4, 2, \cdots, 2)$.

(iii) $L(4, 2, \cdots, 2)$.

(iv) $L(2, \cdots, 2)$.

Removing in cases (ii)—(iv) contradicts (1.6). In case (i) we are led to XI (10, 5, 5) which cannot be augmented.

XII (11, 5, 5, 1)

A 4-point is connected to a 5-point. The lines through the 5-point containing a 4-point have the form:

(i) $L(5, 4, 4, 2)$.

(ii) $L(5, 4, 3, 2, 2)$.

(iii) $L(5, 4, 2, 2, 2, 2)$.

Not all such lines can have type (i). Removing (iii) leads to XI (7, 6, 5) which was excluded. Removing (ii) leads to XI (10, 5, 5) which cannot be augmented.

XII (11, 3, 6, 1)

Suppose there is a line through the 5-point without 4-point. Removing $L(5, 3, 3, 3, 2)$ leads to XI (13, 0, 7) which was already excluded, and removal of the other possibility contradicts (1.6). Therefore all lines through the five point must contain 4-points. There can be at most one line of type $L(5, 4, 4, 2)$ since we have only six 4-points. But removing a line of type $L(5, 4, 2, 2, 2, 2)$ again contradicts (1.6).

XII (14, 0, 7, 1)

There must exist a line through the 5-point of type $L(5, 4, 2, 2, 2, 2)$ or $L(5, 2, \cdots, 2)$. Removal contradicts (1.6).

XII (10, 6, 3, 2)

The line connecting the 5-points cannot have the form $L(5, 5, 4)$ since the lines through these points could produce at most 3-points not on this line. Therefore the connecting line has type

(i) $L(5, 5, 3, 2)$ or

(ii) $L(5, 5, 2, 2, 2)$.

Removal of (ii) leads to XI (7, 6, 5) which was excluded. Removal of (i) leads to XI (10, 5, 5) which cannot be augmented.

XII (9, 8, 3, 0, 1), XII (12, 5, 4, 0, 1)

There must exist a line of type

(i) $L(6, 3, 3, 2, 2)$.

(ii) $L(6, 3, 2, \cdots, 2)$.

(iii) $L(6, 2, \cdots, 2)$.

Removal leads to a case which was already excluded or contradicts (1.6).

XII (15, 0, 6, 0, 1).

There must exist a line of type $L(6, 4, 2, 2, 2)$ or $L(6, 2, \cdots, 2)$. Removing it leads to XI (12, 1, 5, 1) or XI (9, 0, 6, 1) which is impossible.

XIII (12, 2, 10)

There exists a line which contains only 2-points or 4-points. The only possibility for such a line which, when removed, does not lead to an excluded case or to a contradiction with (1.6) is $L(4, 4, 4, 4)$. But if such a line exists the lines through its four 4-points could produce at most three more 4-points.

XIII (14, 0, 9, 1)

There exists a line through the 5-point with at most one 4-point. Removing this line contradicts (1.6).

XIII (13, 3, 6, 2)

There exists a line through a 5-point without further 5-points and 3-points. Remove it!

XIII (12, 6, 3, 3).

Two 5-points have to be connected. A line of type $L(5, 5, 5)$ cannot exist, since then the lines could only produce 2- and 3-points outside this line. For a similar reason lines of type $L(5, 5, 4, 2)$ and $L(5, 5, 3, 3)$ cannot exist. Thus a connection of two 5-points must be of type $L(5, 5, 3, 2, 2)$ or $L(5, 5, 2, 2, 2, 2)$. In both cases removal leads to a contradiction.

XIII (14, 4, 2, 4)

Any two 5-points are connected and no three can lie on one line. Therefore there are six connecting lines and eight further lines. A contradiction.

XII (9, 5, 7)

Removing a line $L(3, 3, 3, 3, 3, 2)$ leads to XI (13, 0, 7) which was already excluded. Removing another type of line without 4-point contradicts (1.6). Therefore each line contains at least one 4-point. The lines which when removed do not lead to an arrangement datum already ruled out or contradicting (1.6) are

(i)	$L(4, 3, 3, 3, 3)$
(ii)	$L(4, 3, 3, 3, 2, 2)$
(iii)	$L(4, 4, 3, 3, 2)$
(iv)	$L(4, 4, 4, 3)$
(v)	$L(4, 4, 4, 2, 2)$

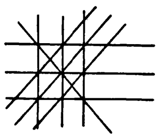
CLAIM. *There exists a line of type* (v).

PROOF. Consider all lines through a 4-point. At most two lines can be of type (i), (ii) or (iii) since there are only five 3-points. If there are two, then one of the remaining lines can only be of type (iv) and the other must be of type (v). If there is one of type (i) or (ii), then only two can have type (iv) and at least one must have type (v). If there is none of type (i)—(iii), then there would be more than seven 4-points. If there is one of type (iii), then the remaining ones cannot all be of type (iv) and (v) because again this would yield too many 4-points.

By removing a line of type (v) one arrives at XI (7, 8, 4). Unfortunately these arrangements are not contained in our classification. So we have to classify them and show that they cannot be augmented suitably. We give only an outline.

First one shows that in an XI (7, 8, 4) any two 4-points have to be connected. A line of type $L(4, 4, 3, 2, 2)$ leads to X (6, 9, 2) and one shows that this does not exist. A line of type $L(4, 4, 2, 2, 2, 2)$ leads to X (3, 10, 2). This leads to an XI (7, 8, 4) which, from our classification of X (3, 10, 2), cannot be augmented.

If no three 4-points lie on a line and a connecting line of 4-points always has the form $L(4, 4, 3, 3)$, then there exists a line of type $L(4, 3, 3, 2, 2, 2)$, which has a unique augmentation to XI (7, 8, 4) which



with ∞

cannot be augmented to XII (9, 5, 7). The other cases contradict (1.6).

If there are three 4-points on a line consider the lines through the remaining 4-point and see that one is led to a case already considered.

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