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Resilient leaves in transversely projective foliations

Dedicated to Professor Akio Hattori on his sixtieth birthday

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1. Introduction

A transformation of the real projective line $\boldsymbol{R}\cup\{\infty\}$ is *projective* if it is of the form $x\mapsto(ax+b)/(cx+d)$, $ad-bc=1$. A projective transformation whose domain is restricted to the real line $\boldsymbol{R}$ is *affine* if $c=0$. Let $\mathcal{F}$ be a codimension one foliation on a manifold M. We say that $\mathcal{F}$ is *transversely projective* (resp. *transversely affine*) if M is covered by a collection of $\mathcal{F}$-distinguished charts for which the coordinate transformations are projective (resp. affine) in the direction transverse to $\mathcal{F}$. A leaf of $\mathcal{F}$ is said to be *resilient* if it is nonproper (i.e., locally dense or exceptional) and with nontrivial holonomy.

In the previous paper [In], one of the authors has studied about resilient leaves in transversely affine foliations. The purpose of this paper is to extend the study to transversely projective foliations.

Let $PSL(2, \boldsymbol{R})$ be the group of projective transformations of the real projective line. A transversely projective foliation $\mathcal{F}$ on a manifold M induces a holonomy homomorphism $h:\pi_1(M)\to PSL(2, \boldsymbol{R})$. We call the image of h the *global holonomy group* of $\mathcal{F}$. Now the first result of this paper gives some sufficient conditions for the existence of resilient leaves in transversely projective foliations.

THEOREM 1.1. *Let $\mathcal{F}$ be a codimension one transversely projective foliation on a closed manifold. If $\mathcal{F}$ satisfies either of the following conditions, then $\mathcal{F}$ has resilient leaves.*

1) *$\mathcal{F}$ is not almost without holonomy.*

2) *The global holonomy group of $\mathcal{F}$ contains a non-abelian free subgroup.*

Here $\mathcal{F}$ is said to be *almost without holonomy* if every noncompact leaf of $\mathcal{F}$ has trivial holonomy. We remark that in transversely projective foliations, unlike in transversely affine foliations ([In, Theorem 1.2]), the existence of resilient leaves cannot be detected completely by informations about the global holonomy group. In fact, there exist two transversely projective foliations with the same global holonomy group, one with resilient leaves and the other without (see § 3).

The next theorem generalizes Levitt's recent result ([Lev, Th. III. 2], see also [Me, Prop. III. 2.10]) on the non-existence of resilient leaves of exceptional type in transversely affine foliations. In fact, almost all ideas of the proof have already been contained in [Lev].

THEOREM 1.2. *Let $\mathcal{F}$ be a codimension one transversely projective foliation on a (possibly open) manifold M. If $\pi_1(M)$ does not contain any non-abelian free subgroup, then $\mathcal{F}$ does not have an exceptional leaf.*

Let M be a closed manifold and $\mathcal{F}$ a transversely orientable C^2 codimension one foliation on M. A subset $\mathcal{M}$ of M is called a *minimal set* of $\mathcal{F}$ if it is nonempty, closed and saturated, and if it contains no proper subset of the same property. A minimal set which is neither a single compact leaf nor all of M is called *exceptional*. We say that an exceptional minimal set $\mathcal{M}$ is *Markov* if the holonomy pseudogroup on $\mathcal{M}$ is nearly generated by a one-sided subshift of finite type (for a precise definition, see § 5). The structure of Markov exceptional minimal sets has been clarified considerably (cf., [Ma], [CC2], [CC3]). A pair $(M, \mathcal{F})$ is called a *foliated circle bundle* if M is the total space of a circle bundle and the foliation $\mathcal{F}$ is transverse to the fibers.

Now the third result of this paper analyzes the structure of exceptional minimal sets of a certain class of transversely projective foliations.

THEOREM 1.3. *Let $(M, \mathcal{F})$ be a foliated circle bundle, M a closed manifold and $\mathcal{M}$ an exceptional minimal set of $\mathcal{F}$. Suppose that $\mathcal{F}$ is transversely projective. Then $\mathcal{M}$ is Markov.*

Some of the results of this paper were announced in [IM].

2. Subgroups of $PSL(2, \boldsymbol{R})$

Let $SL(2, \boldsymbol{R})$ be the group of all real 2×2 matrices with determinant 1. Let $PSL(2, \boldsymbol{R})=SL(2, \boldsymbol{R})/\{\pm I\}$. $PSL(2, \boldsymbol{R})$ acts on the real projective line $S^1_\infty=\boldsymbol{R}\cup\{\infty\}$ as linear fractional transformations: $x\mapsto(ax+b)/(cx+d)$. The stabilizer $\mathrm{Stab}(\infty)$ of ∞ is the subgroup of $PSL(2, \boldsymbol{R})$ consisting of all upper triangular matrices. It is possible to restrict the action of $\mathrm{Stab}(\infty)$ to $\boldsymbol{R}$. This yields the group $\mathrm{Aff}^+(\boldsymbol{R})$ of orientation preserving affine transformations. An element A of $PSL(2, \boldsymbol{R})$ is *hyperbolic, parabolic* or *elliptic* if A has 2, 1 or 0 fixed points in S^1_∞.

The following is well-known (see e.g. [dlH]).

PROPOSITION 2.1. *Let Γ be a subgroup of $PSL(2, \boldsymbol{R})$ which does not contain a non-abelian free subgroup. Then either of the following holds:*

1) *Γ is conjugate in $PSL(2, \boldsymbol{R})$ to a subgroup of the rotation group* $SO(2)$.

2) *Γ consists of hyperbolic elements with common fixed point set and elliptic elements which keep the fixed point set invariant.*

3) *Γ is conjugate in $PSL(2, \boldsymbol{R})$ to a subgroup of* $\mathrm{Stab}(\infty)$.

A group is *virtually abelian* if it has an abelian subgroup of finite index.

COROLLARY 2.2. *Let Γ be a virtually abelian subgroup of $PSL(2, \boldsymbol{R})$. Then Γ satisfies either* 1), 2) *in Proposition* 2.1 *or* 3)′ *Γ consists of parabolic elements with common fixed point.*

3. Proof of Theorem 1.1

Let $\mathcal{F}$ be a codimension one, transversely projective foliation on a closed manifold M. Let $p: \tilde{M} \to M$ be the universal covering of M and let $\tilde{\mathcal{F}} = p^{-1}(\mathcal{F})$. Fix a base point $\tilde{x}_0$ of $\tilde{M}$ and set $x_0 = p(\tilde{x}_0) \in M$. Take a distinguished neighborhood V_0 of x_0 and a distinguished submersion $f_0: V_0 \to S^1_\infty$. Denote by $\tilde{V}_0$ the lift of V_0 to $\tilde{M}$ which contains $\tilde{x}_0$. f_0 uniquely lifts to a submersion $\tilde{f}_0: \tilde{V}_0 \to S^1_\infty$. By analytic continuation, $\tilde{f}_0$ extends uniquely to a developing submersion $D: \tilde{M} \to S^1_\infty$. D yields a holonomy homomorphism $h: \pi_1(M, x_0) \to PSL(2, \boldsymbol{R})$ such that $D \circ \gamma = h(\gamma) \circ D$, for $\gamma \in \pi_1(M, x_0)$. We denote by Γ the global holonomy group of $\mathcal{F}$, i.e., the image of h. See e.g., [G, Chap. III. 3] for more background material.

First we observe the following simple fact.

PROPOSITION 3.1. *If Γ is virtually abelian, then $\mathcal{F}$ is almost without holonomy.*

PROOF. If Γ is virtually abelian, then by Corollary 2.2, the set $F = \{x \in S^1_\infty \mid \gamma(x) = x$ for some $\gamma \in \Gamma$, $\gamma \neq \mathrm{id}\}$ is finite. In fact, it either is empty or consists of one or two points. Hence $p \circ D^{-1}(F)$ is a union of compact leaves of $\mathcal{F}$. On the other hand, it is clear that $p \circ D^{-1}(F)$ contains the union of leaves of $\mathcal{F}$ with nontrivial holonomy. Proposition 3.1 is proved.

Now we will prove Theorem 1.1. First we recall the following fundamental fact on the structure of open saturated subsets without holonomy, which is easily proved by using Imanishi's theorem [Im]. See also [In, Lemma 3.3].

LEMMA 3.2. *Let U be an open saturated subset of M such that $\mathcal{F}|U$ is without holonomy. Suppose further that some (and hence every) border leaf of U has nontrivial holonomy. Let $\tilde{U}$ be a connected component of $p^{-1}(U)$. Then there exists a closed transversal in U, and for every closed transversal τ in U, each lift of τ in $\tilde{U}$ intersects every leaf of $\tilde{\mathcal{F}}|\tilde{U}$ exactly once.*

Let U be as in Lemma 3.2. Note that U is nothing but a maximal component such that $\mathcal{F}|U$ is without holonomy. See [In] for the definition of a maximal component.

Let $\alpha: I\to M$ be a path such that $\alpha(0)=x_0$ and $\alpha(1)\in U$, and let $\tilde{\alpha}: I\to\tilde{M}$ be the lift of α such that $\tilde{\alpha}(0)=\tilde{x}_0$. Denote by $\tilde{U}^{\alpha}$ the connected component of $p^{-1}(U)$ such that $\tilde{\alpha}(1)\in\tilde{U}^{\alpha}$. Let $\tilde{L}$ be any border leaf of $\tilde{U}^{\alpha}$, and let $\tilde{\beta}: I\to\tilde{M}$ be a path such that $\tilde{\beta}(0)=\tilde{\alpha}(1)$ and $\tilde{\beta}(1)\in\tilde{L}$. Put $L=p(\tilde{L})$ and $\beta=p\circ\tilde{\beta}$. Take a loop $\gamma: I\to L$ such that $\gamma(0)=\gamma(1)=\beta(1)$ and that the germinal holonomy along γ is nontrivial.

Notation. In what follows, for a loop a and a path b we denote the conjugation bab^{-1} by a^{b}.

LEMMA 3.3. *The developing image $D(\tilde{U}^{\alpha})$ of $\tilde{U}^{\alpha}$ coincides with a connected component of $S_{\infty}^{1}-\mathrm{Fix}\,h(\gamma^{\alpha\beta})$, where $\mathrm{Fix}\,h(\gamma^{\alpha\beta})$ is the set of fixed points of $h(\gamma^{\alpha\beta})$.*

PROOF. By the definitions of α and β, we easily see that $D(\tilde{L})$ is a fixed point of $h(\gamma^{\alpha\beta})$. Since $h(\gamma^{\alpha\beta})$ is nontrivial, $h(\gamma^{\alpha\beta})$ is either hyperbolic or parabolic. Take a small compact arc $\tilde{\sigma}$ transverse to $\tilde{\mathcal{F}}$ whose interior is contained in $\tilde{U}^{\alpha}$ and which has $\tilde{\beta}(1)$ as an endpoint. Let $\sigma=p\circ\tilde{\sigma}$. Since γ induces a nontrivial germinal holonomy, there are a subarc σ_1 of σ in U and a path γ_1 contained in a leaf and having the same endpoints with σ_1 such that the composite loop $\sigma_1\gamma_1$ is freely homotopic to γ. Let $\tilde{\sigma}_1$ be the lift of σ_1 such that $\tilde{\sigma}_1\subset\tilde{\sigma}$ and $\widetilde{\sigma_1\gamma_1}$ the lift of $\sigma_1\gamma_1$ such that $\tilde{\sigma}_1\subset\widetilde{\sigma_1\gamma_1}$. Then by the construction of $\sigma_1\gamma_1$, we see that the developing image of one of the endpoints of $\tilde{\sigma}_1$ is mapped by $h(\gamma^{\alpha\beta})$ to the developing image of the other endpoint of $\tilde{\sigma}_1$. From this follows that $D(\widetilde{\sigma_1\gamma_1})$ coincides with a connected component of $S_{\infty}^{1}-\mathrm{Fix}\,h(\gamma^{\alpha\beta})$. On the other hand, by a usual argument,

$\sigma_1\gamma_1$ can be deformed to a closed transversal in U. Thus by Lemma 3.2, we have that $D(\widetilde{\sigma_1\gamma_1})=D(\tilde{U}^\alpha)$. This proves Lemma 3.3.

Notice at this point that $D(\tilde{U}^\alpha)$ does not depend on $\tilde{L}$, $\tilde{\beta}$ and γ. Therefore, if we replace $\tilde{L}$, $\tilde{\beta}$ and γ in Lemma 3.3 by other ones satisfying the same properties, $\mathrm{Fix}\, h(\gamma^{\alpha\beta})$ is left invariant. Hereafter we denote it simply by F_U^α. $D(\tilde{U}^\alpha)$ is a proper subset of S_∞^1 and the developing image of every border leaf of $\tilde{U}^\alpha$ belongs to F_U^α.

Next, suppose that there exists a maximal component V without holonomy which is adjacent to U along L. Let $\alpha_1: I\to M$ be a path such that $\alpha_1(0)=\alpha(1)$ and that $\alpha_1([0, T))\subset U$, $\alpha_1(T)\in L$ and $\alpha_1((T, 1])\subset V$ for some T, $0<T<1$. Let $\tilde{\alpha}_1$ be the lift of α_1 such that $\tilde{\alpha}_1(0)=\tilde{\alpha}(1)$. By applying Lemma 3.3 to V, $\alpha\alpha_1$, $\tilde{\alpha}_1^{-1}\tilde{\beta}$ and γ, we obtain that $D(\tilde{V}^{\alpha\alpha_1})$ is a connected component of $S_\infty^1-\mathrm{Fix}\, h(\gamma^{(\alpha\alpha_1)(\alpha_1^{-1}\beta)})=S_\infty^1-\mathrm{Fix}\, h(\gamma^{\alpha\beta})$. In particular, we have $F_U^\alpha=F_V^{\alpha\alpha_1}$.

PROPOSITION 3.4. *If $\mathcal{F}$ is almost without holonomy, then Γ does not contain a non-abelian free subgroup.*

PROOF. If $\mathcal{F}$ is without holonomy, then the classical argument of Sacksteder (see e.g., [Im]) shows that Γ is abelian. So let us assume that $\mathcal{F}$ has a compact leaf with nontrivial holonomy. Here we note that since $\mathcal{F}$ is real analytic, the compact leaves of $\mathcal{F}$ are finite in number and each of them has nontrivial holonomy. Let K be the union of compact leaves of $\mathcal{F}$. Let U be a connected component of $M-K$. Then U is a maximal component of $\mathcal{F}$. Since the compact leaves are finite in number, every path that starts at x_0 can be perturbed to a path transverse to all the compact leaves. Therefore, it follows from the argument in the paragraph just above Proposition 3.4 that F_U^α depends neither on U nor on α. Denote this set by F. Then, for any connected component U of $M-K$ and for any lift $\tilde{U}$ of U, $D(\tilde{U})$ is a connected component of S_∞^1-F. In particular, $D(p^{-1}(K))=F\cap D(\tilde{M})$. Since $p^{-1}(K)$ is invariant under $\pi_1(M)$, $F\cap D(\tilde{M})$ is invariant under Γ. Thus Γ has one or two points as its invariant set. Hence Γ satisfies 2) or 3) of Proposition 2.1, which implies that Γ does not contain any non-abelian free subgroup. Proposition 3.4 is proved.

REMARK. It is obvious that if Γ leaves invariant a set consisting of two points, then Γ is virtually abelian. By this fact and the proof of Proposition 3.4, we get the following partial converse of Proposition 3.1: if $\mathcal{F}$ is almost without holonomy and if, in addition, Γ contains no parabolic elements, then Γ is virtually abelian.

Now, to complete the proof of Theorem 1.1, it suffices to show the following.

PROPOSITION 3.5. *If $\mathcal{F}$ has no resilient leaves, then $\mathcal{F}$ is almost without holonomy.*

PROOF. Since $\mathcal{F}$ is real analytic, $\mathcal{F}$ is of finite level [CC1]. Assume that $\mathcal{F}$ has no resilient leaves and that $\mathcal{F}$ is not almost without holonomy. Since $\mathcal{F}$ has no resilient leaves, M consists of finitely many maximal components without holonomy and finitely many proper leaves with nontrivial holonomy. Since $\mathcal{F}$ is not almost without holonomy, there exists a maximal component U with a noncompact border leaf L along which another maximal component V is adjacent to U. (V may coincide with U.) Let $\tilde{U}$ and $\tilde{V}$ be lifts of U and V respectively such that they are mutually adjacent. Then by Lemma 3.3, we obtain that $D(\overline{\tilde{U}\cup\tilde{V}})=S_\infty^1$.

Now let F be a leaf of $\mathcal{F}$ which is contained in the limit set of L. Let δ be a compact arc transverse to $\mathcal{F}$ such that one of the endpoints of δ is on F and that δ meets L (and hence also U and V) infinitely often. Then by the above observation, we see that the projective structure of $\operatorname{Int}\delta$ induced from the transverse projective structure of $\mathcal{F}$ must contain an infinite number of mutually disjoint intervals each of which is projectively equivalent to a fundamental domain of the universal covering map $\widetilde{S_\infty^1}\to S_\infty^1$. But then the projective structure of $\operatorname{Int}\delta$ cannot extend to the projective structure of the *compact* arc δ. This is a contradiction. Proposition 3.5 is proved. The proof of Theorem 1.1 is complete.

In the following, we will show, by examples, that the existence of resilient leaves cannot be characterized in terms of the global holonomy groups.

Define $p, q\in\operatorname{Stab}(\infty)$ by $p(x)=x+1$ and $q(x)=2x$. Denote by Γ the group generated by p and q. Let $(M_1, \mathcal{F}_1)$ be a foliated S_∞^1-bundle over a closed orientable surface Σ of genus greater than one which is determined by a homomorphism $\Phi:\pi_1(\Sigma)\to PSL(2, \boldsymbol{R})$ such that $\operatorname{Image}\Phi=\Gamma$ (such a Φ clearly exists).

Next, foliate $S^1\times[0, 1]\times S_\infty^1$ by $S^1\times[0, 1]\times\{x\}$, $x\in S_\infty^1$, and consider the quotient manifold $Q=S^1\times[0, 1]\times S_\infty^1/(s, 1, x)\sim(s, 0, p(x))$. Denote by $\mathcal{G}$ the induced foliation on Q. Let τ_1 and τ_2 be mutually disjoint closed transversals to $\mathcal{G}$ such that the projective structure of τ_1 (resp. τ_2) induced from $\mathcal{G}$ is $\boldsymbol{R}/\langle p\rangle$ (resp. $\boldsymbol{R}/\langle p^2\rangle$). Turbulize $\mathcal{G}$ along τ_1 in such a way that the obtained foliation $\mathcal{G}_1$ is still transversely projective. τ_1 is contained in the resulting Reeb component and is still a closed transversal. Let N_1

(resp. N_2) be a small open tubular neighborhood of τ_1 (resp. τ_2) such that $\mathcal{G}_1|N_1$ (resp. $\mathcal{G}_1|N_2$) is a product foliation by disks. Identify two boundary components of $Q-N_1-N_2$ by a foliation preserving diffeomorphism $\partial N_1 \to \partial N_2$ such that the induced map between their leaf spaces coincides with q. The existence of such a diffeomorphism is guaranteed by the fact that $q \circ p = p^2 \circ q$. We denote by $(M_2, \mathcal{F}_2)$ the resulting transversely projective foliation. Now the following is obvious.

PROPOSITION 3.6. 1) *The global holonomy groups of* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *both coincide with* Γ. 2) $\mathcal{F}_1$ *has resilient leaves, while* $\mathcal{F}_2$ *is almost without holonomy.*

4. Proof of Theorem 1.2

First we prove the following.

PROPOSITION 4.1. *Let* $\mathcal{F}$ *be a codimension one, transversely projective foliation on a (possibly open) manifold* M. *If the global holonomy group* Γ *of* $\mathcal{F}$ *does not contain any non-abelian free subgroup, then either of the following holds:*

1) $\mathcal{F}$ *is almost without holonomy.*

2) *There exists a discrete family* $\{L_n\}$ *of closed leaves in* M *such that in each connected component* U *of* $M-\bigcup_n L_n$, *the restricted foliation* $\mathcal{F}|U$ *is transversely affine.*

PROOF. By the assumption, Γ satisfies the condition of Proposition 2.1. If Γ satisfies 1) or 2) of Proposition 2.1, then by Proposition 3.1, $\mathcal{F}$ is almost without holonomy. If Γ satisfies 3) of Proposition 2.1, then we may assume that Γ is contained in Stab(∞). We see that $p \circ D^{-1}(\infty)$ is a discrete family of closed leaves and that the developing image of $\tilde{M}-D^{-1}(\infty)$ is contained in $\boldsymbol{R}$. Thus the global holonomy group of each connected component of $p(\tilde{M}-D^{-1}(\infty))$ is a subgroup of Stab(∞) whose action is restricted to $\boldsymbol{R}$. This means that $\mathcal{F}|p(\tilde{M}-D^{-1}(\infty))$ is transversely affine. Proposition **4.1** is proved.

DEFINITION 4.2 [Lev]. Let $\mathcal{F}$ be a codimension one foliation on a manifold M. Let $\mathcal{L}$ be the normal subgroup of $\pi_1(M)$ generated by all free homotopy classes of loops contained in leaves of $\mathcal{F}$ and with trivial holonomy. Then the *fundamental group* $\pi_1(M/\mathcal{F})$ *of the leaf space* $M/\mathcal{F}$ is the quotient of $\pi_1(M)$ by $\mathcal{L}$.

The following result on transversely affine foliations is obtained by Levitt [Lev, Th. III. 2]. (We notice that a similar result is obtained also by Meigniez [Me, Prop. III. 2. 10].)

THEOREM 4.3 [Lev]. *Let $\mathcal{F}$ be a codimension one, transversely affine foliation on a (possibly open) manifold M. Suppose that $\pi_1(M/\mathcal{F})$ does not contain any non-abelian free subgroup. Then $\mathcal{F}$ does not have any exceptional leaf.*

Theorem 1.2 is a generalization of this theorem.

PROOF OF THEOREM 1.2. Since $\pi_1(M)$ does not contain any non-abelian free subgroups, neither does the global holonomy group Γ of $\mathcal{F}$. Hence Proposition 4.1 applies. If $\mathcal{F}$ satisfies 1) of Proposition 4.1, we are done. So let $\mathcal{F}$ satisfy 2) of Proposition 4.1. Then we have a discrete family $\{L_n\}$ of closed leaves such that in each connected component U of $M-\bigcup_n L_n$, the restricted foliation $\mathcal{F}|U$ is transversely affine. By the same argument as in [Lev, Lemme III. 3], we see that the homomorphism $\pi_1(U/\mathcal{F})\to\pi_1(M/\mathcal{F})$ which is induced from the inclusion map $U\to M$ is injective, and hence that $\pi_1(U/\mathcal{F})$ does not contain any non-abelian free subgroup. Thus we can apply Theorem 4.3 to $\mathcal{F}|U$ and obtain that $\mathcal{F}|U$ has no exceptional leaves. This proves Theorem 1.2.

REMARK. As Levitt's theorem, Theorem 1.2 in fact holds if we replace the assumption on $\pi_1(M)$ by that on $\pi_1(M/\mathcal{F})$.

5. Markov exceptional minimal sets

First of all we give a precise definition of Markov exceptional minimal set.

DEFINITION 5.1. Let Γ be a pseudogroup of local C^2 diffeomorphisms of a compact 1-dimensional manifold T. Let C be an exceptional minimal set for Γ such that $\partial T\cap C=\emptyset$. C is said to be *Markov* for Γ if one can find elements $\gamma_1, \cdots, \gamma_m$ of Γ and closed intervals $I_1, \cdots, I_m$ of T such that

1) $\operatorname{Int} I_k$'s are pairwise disjoint,
2) $C\subset\bigcup_{k=1}^m I_k$,
3) $C\cap\operatorname{Int} I_k\neq\emptyset$ for each k,
4) the domain of γ_k contains I_k for each k,
5) $\gamma_k|I_k\cap C$'s generate $\Gamma|C$, and
6) if $\gamma_k(I_k)\cap\operatorname{Int} I_j\neq\emptyset$, then $\gamma_k(I_k)\supset I_j$.

We call the collection $\{\gamma_1, \cdots, \gamma_m; I_1, \cdots, I_m\}$ a *Markov system* for C.

DEFINITION 5.2. Let $\mathcal{F}$ be a transversely orientable, C^2 codimension one foliation on a closed manifold and $\mathcal{M}$ an exceptional minimal set of $\mathcal{F}$. We say that $\mathcal{M}$ is *Markov* if there exists a compact 1-manifold T transverse to $\mathcal{F}$ such that $\mathcal{M}\cap T$ is Markov for the holonomy pseudogroup of $\mathcal{F}$ induced on T.

A similar definition can be found in [CC2]. However their definition of Markov exceptional minimal set is more restrictive than ours. They impose the condition that any two of the intervals I_k be disjoint in Definition 5.1. There are a multitude of examples of exceptional minimal sets which fail to be Markov in this sense. On the other hand, one might even ask whether every exceptional minimal set is Markov in our sense.

The same terminology of different usages may cause some confusions. However we adopt the name Markov, since our definition seems to be more natural and parallel to the definition of Markov maps ([Bo 1], [BS]) and Markov partitions ([Bo 2]). We hope to investigate in future structures of Markov exceptional minimal sets by using symbolic dynamics. Compare [Ma], [CC2] and [CC3].

6. Fuchsian groups of the second kind and Markovness

We denote by G the group of orientation preserving isometries of the Poincaré disk $\Delta=\{z\in \boldsymbol{C} \mid |z|<1\}$. It is well-known that G is naturally isomorphic to $PSL(2, \boldsymbol{R})$. A discrete subgroup of G is called a *Fuchsian group*. Denote by L_Γ the limit set of a Fuchsian group Γ. It is a closed subset of the circle at infinity S^1_∞. The cardinality of L_Γ is 0, 1, 2 or ∞. Γ is called *elementary, of the first kind* or *of the second kind* if L_Γ is finite, $L_\Gamma=S^1_\infty$, or neither. In the last case, L_Γ is a Cantor sat. For further informations about Fuchsian groups, see [Be].

In this section we prove the following result which is the key to the proof of Theorem 1.3.

THEOREM 6.1. *Let Γ be a finitely generated Fuchsian group of the second kind. Then L_Γ is Markov for the pseudogroup generated by Γ.*

Given a Fuchsian group Γ of the first kind, Nielsen introduced a method that assigns a symbol sequence to each point of S^1_∞. This method is called a *Nielsen development* ([Bo 1], [BS]). The above theorem is regarded as a Nielsen development for Fuchsian groups of the second kind.

PROOF. Let Γ be a finitely generated Fuchsian group of the second kind. Put $\Sigma=\Delta/\Gamma$. Then the quotient map $\Delta\to\Sigma$ is a branched covering with a finite number of branch points, say, $e_1, \cdots, e_r$ of order $m_1, \cdots, m_r$. Σ is homeomorphic to a closed orientable surface of genus g with finitely many punctures $h_0, \cdots, h_s, p_1, \cdots, p_t$. The covering transformation corresponding to a small oriented loop surrounding h_i (resp. p_i) is hyperbolic (resp. parabolic).

As is well-known, there is a unique deleted neighborhood H_i of h_i which is homeomorphic to a cylinder and is bounded by a closed geodesic. We can choose as a deleted neighborhood of p_i a set P_i which lifts to a horodisk in Δ. By taking P_i sufficiently small, we may assume that P_i's are mutually disjoint and do not intersect H_i's and e_i's.

Now choose mutually disjoint curves α_i, β_i $(1\leqq i\leqq g)$, ε_j $(1\leqq j\leqq r)$, η_k $(1\leqq k\leqq s)$ and π_l $(1\leqq l\leqq t)$ so that

1) ∂H_0 intersects $\alpha_1, \beta_1, \alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \alpha_g, \beta_g, \varepsilon_1, \cdots, \varepsilon_r, \eta_1, \cdots, \eta_s, \pi_1, \cdots, \pi_t$ in counterclockwise order.

2) every connected component of the intersection of each curve and each H_k (resp. P_l) is a geodesic of infinite length orthogonal to ∂H_k (resp. ∂P_l),

3) ε_j joins H_0 to e_j,

4) η_k joins H_0 to H_k, and

5) π_l joins H_0 to P_l.

See Figure 1.

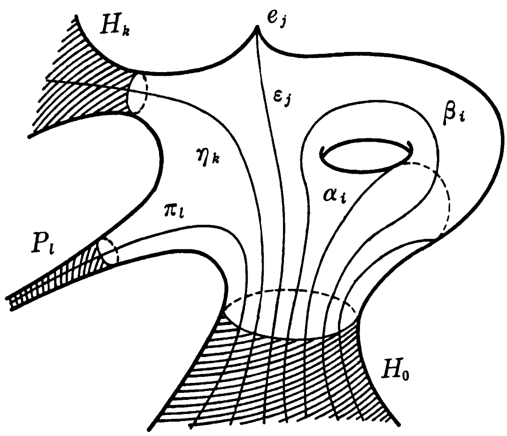

Figure 1.

Cut Σ along these curves and lift it to Δ. Then we obtain a compact polygon F in D^2 $(=\Delta\cup S_\infty^1)$ whose sides, which are not necessarily geodesic segments, appear in the following order (see Figure 2):

$$\left(\prod_{i=1}^{g}\alpha_i f\beta_i f\alpha_i^{-1}f\beta_i^{-1}f\right)\left(\prod_{j=1}^{r}\varepsilon_j\varepsilon_j^{-1}f\right)\left(\prod_{k=1}^{s}\eta_k f\eta_k^{-1}f\right)\left(\prod_{l=1}^{t}\pi_l\pi_l^{-1}f\right).$$

Here all free sides (i.e., sides lying on S_∞^1) are labeled f, and the side which is identified with s by some element of Γ is denoted by s^{-1}.

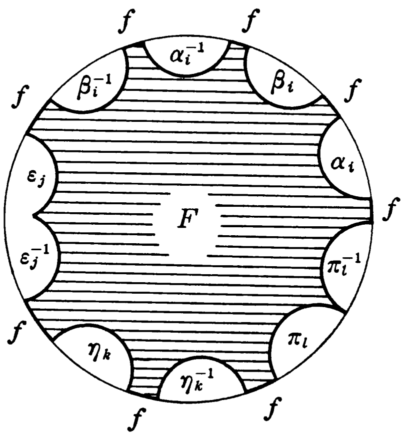

Figure 2.

Throughout this paragraph, we let X be one of the following: α_i, β_i, ε_j, η_k, π_l. Denote by $g(X)$ the element of Γ which maps X to X^{-1}. Then $g(X)$ is hyperbolic if $X=\alpha_i$, β_i or η_k, elliptic of order m_j if $X=\varepsilon_j$, and parabolic if $X=\pi_l$. Γ is the free product of cyclic groups generated by these elements. Let Γ_1 be the finite subset of Γ consisting of all the elements of the form $g(X)^\delta$ where $1\leqq\delta\leqq m_j-1$ if $X=\varepsilon_j$, and $\delta=\pm 1$ otherwise. For $g(X)^\delta\in\Gamma_1$ which is not elliptic, we define $I(g(X)^\delta)$ to be the closure of the connected component of S_∞^1-F which has the same endpoints with X^δ. For $g(\varepsilon_j)^\delta$, we denote the point $\varepsilon_j\cap S_\infty^1$ by c_j, and define $I(g(\varepsilon_j)^\delta)$ to be the closed interval in $\overline{S_\infty^1-F}$ having $g(\varepsilon_j)^{-\delta+1}c_j$ and $g(\varepsilon_j)^{-\delta}c_j$ as endpoints.

Then $\{\Gamma_1; \{I(g)\}_{g\in\Gamma_1}\}$ is a Markov system for L_Γ. In fact, the conditions 1), 2), 3) and 4) in Definition 5.1 are obvious. 5) easily follows from the fact that Γ_1 generates Γ. 6) is verified if we observe that

$$g(X)^\delta(I(g(X)^\delta))=\overline{S_\infty^1-I(g(X)^{-\delta})}$$

if $g(X)$ is not elliptic, and

$$g(\varepsilon_j)^{\delta}(I(g(\varepsilon_j)^{\delta}))=\overline{S^1_\infty-\bigcup_{\mu=1}^{m_j-1}I(g(\varepsilon_j)^{\mu})}.$$

The proof of Theorem 6.1 is complete.

7. Markov exceptional minimal sets in transversely projective foliations

PROOF OF THEOREM 1.3. Let $\pi: M\to B$ be an S^1-bundle and let $\mathcal{F}$ be a transversely projective foliation on M which is transverse to the fibers of π and which admits an exceptional minimal set $\mathcal{M}$. Fix a fiber C of π. C is transverse to $\mathcal{F}$. Recall that the class $[C]$ lies in the center of $\pi_1(M)$. This implies that the holonomy image $h[C]$ is the identity. In fact, otherwise, the global holonomy group Γ is contained in the centralizer of $h[C]$. That is, Γ is abelian. Then, by Proposition 3.1, $\mathcal{F}$ is almost without holonomy, contrary to our hypothesis.

Now, since $h[C]=\mathrm{id}$, the developing map $D|\tilde{C}$ projects down to yield a finite covering map $q: C\to S^1_\infty$, where $\tilde{C}$ is a connected component of the inverse image of C by the universal covering map $\tilde{M}\to M$. The image $\mathcal{N}=q(\mathcal{M}\cap C)$ is a Γ-minimal Cantor set. From this follows that Γ is a Fuchsian group of the second kind such that $L_\Gamma=\mathcal{N}$. Notice that the only point which needs proof is that Γ is a discrete subgroup. If not, however, $\mathcal{N}$ is kept invariant by the connected subgroup of the closure of Γ, which is clearly impossible.

By Theorem 6.1, one gets a Markov system for $\mathcal{N}$, which can easily be lifted to $\mathcal{M}\cap C$. This completes the proof of Theorem 1.3.

Bibliography

[Be] Beardon, A. F., The geometry of discrete groups, Graduate Texts in Math. vol. 91, Springer Verlag, New York-Berlin, 1983.

[Bo 1] Bowen, R., Hausdorff dimension of quasi-circles, Publ. Math. I. H. E. S. **50** (1979), 11-25.

[Bo 2] Bowen, R., Markov partitions for Axiom A diffeomorphisms, Amer. J. Math. **92** (1970), 725-747.

[BS] Bowen, R. and C. Series, Markov maps associated with Fuchsian groups, Publ. Math. I. H. E. S. **50** (1979), 153-170.

[CC1] Cantwell, J. and L. Conlon, Analytic foliations and the theory of levels, Math. Ann. **265** (1983), 253-261.

[CC2] Cantwell, J. and L. Conlon, Foliations and subshifts, Tôhoku Math. J. **40** (1988), 165-187.

[CC3] Cantwell, J. and L. Conlon, Leaves of Markov local minimal sets in foliations of codimension one, to appear in Publ. Sec. Mat. Univ. Autònoma Barcelona.

[dlH] de la Harpe, P., Free groups in linear groups, Enseign. Math. **29** (1983), 129-144.

[G] Godbillon, C., Feuilletages, Études géométriques I, II, Publ. I. R. M. A., Univ. Louis Pasteur, Strasbourg, 1986.

[Im] Imanishi, H., On the theorem of Denjoy-Sacksteder for codimension one foliations without holonomy, J. Math. Kyoto Univ. **14** (1974), 607-634.

[In] Inaba, T., Resilient leaves in transversely affine foliations, Tôhoku Math. J. **41** (1989), 625-631.

[IM] Inaba, T. and S. Matsumoto, Some qualitative aspects of transversely projective foliations, Proc. Japan Acad. **65** (1989), 116-118.

[Lev] Levitt, G., Groupe fondamental de l'espace des feuilles dans les feuilletages sans holonomie, Prepublicationes Univ. de Paris Sud, Orsay, 1988.

[Ma] Matsumoto, S., Measure of exceptional minimal sets of codimension one foliations, A Fête of Topology, Papers dedicated to Itiro Tamura (ed. Y. Matsumoto et al.) Academic Press, Orlando, Florida, 1988, 81-94.

[Me] Meigniez, G., Actions de groupes sur la droite et feuilletages de codimension 1, Thèse. Univ. Claude Bernard-Lyon I, Villeurbanne, 1988.

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