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Applications of the Malliavin calculus to McKean equations

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0. Introduction.

The stochastic calculus for the Wiener functionals initiated by P. Malliavin—so-called Malliavin calculus—has been applied to various problems of stochastic analysis so far while its mathematical foundations have been organized and made clear through the effort of many people.

In particular, one of the most popular applications is to the problem of regularity of the solutions to a certain class of second order partial differential equations. Kusuoka-Stroock [2] applied the Malliavin calculus to time-homogeneous stochastic differential equations (S. D. E. in abbreviation) and recovered (and also partly extended) Hörmander's basic theory. After that, Taniguchi [6] investigated the case that the equations have time-dependent coefficients with looser conditions of smoothness and gave sufficient conditions for the solutions to be smooth.

The aim of the present article is to extend the method developed in those articles to the stochastic integral equations of the form:

$$(0.1)\qquad X(t,\theta)=a(\theta)+\sum_{i=1}^{d}\int_0^t V_i[X(s,\theta),\pi(s)]\circ d\theta_i(s)+\int_0^t V_0[X(s,\theta),\pi(s)]ds\,.$$

The precise notation is given in the second section and here we only note that $\pi(t)$ stands for the probability distribution of the solution $X(t)$ itself on $\boldsymbol{R}^n$. The above equation was introduced by H. P. McKean [3] to describe the motion of a particle (molecule) in the bath of infinitely many particles interacting with one another. In this case also, the future motion of each particle depends only on the present state, but not only on its present location. The future is also influenced by the present distribution of all the particles, and this property is called 'Markovian in the sense of McKean'.

Our main problem here is to find the sufficient conditions for the solutions to McKean equations, whose coefficients depend on the distributions of the solutions, to have smooth density functions. It is true that McKean equations can be regarded as time-inhomogeneous S. D. E.'s, but our results contain no time parameter, and in this sense, differ from the results by Taniguchi [6], in which, as is mentioned above, more general types of equations are treated than in this paper. This may sound a bit strange, but our solutions intrinsically depend only on the initial distributions due to the structure of McKean equations, and so the results here make sense.

It is also true that our main result could be obtained by means of the powerful device called partial Malliavin calculus or partial hypoellipticity. (cf. [2], [5]) as is going to be referred to later. But another aim of this article is to show how the ordinary Malliavin calculus is carried out directly on the time-inhomogeneous equations without recourse to such a weapon.

Although no notation is shown yet, let us present the main theorem here. This theorem will be stated again in the third section as Theorem (3.26) under complete notation and formulation.

(0.2) THEOREM. *Suppose that* $\mathrm{span}\{(V_k)_{(\alpha)}[x, q];\ 1\leqq k\leqq d,\ \alpha\in\mathcal{A}\}=\boldsymbol{R}^n$ *for any* $x\in\mathrm{supp}\,q$.

(i) *Assume in addition that there exists an* $L\in\boldsymbol{N}$ *satisfying that* $\mathrm{span}\{(V_k)_{(\alpha)}[x, q];\ 1\leqq k\leqq d,\ \alpha\in\mathcal{A}$ *with* $\|\alpha\|<L\}=\boldsymbol{R}^n$ *for any* $x\in\mathrm{supp}\,q$ *and also that* $\inf_{x\in\mathrm{supp}\,q}\left\{\inf_{\eta\in S^{n-1}}\left(\sum_{k=1}^{d}\sum_{\|\alpha\|<L}((V_k)_{(\alpha)}[x, q], \eta)^2\right)\wedge 1\right\}>0$. *Then the unique solution to the McKean equation* (0.1) *has a smooth density function.*

(ii) *In particular, if* $\mathrm{supp}\,q$ *is compact, then the above L exists. Here,* $V_k[\cdot, \cdot]$*'s are the coefficients of the equation and* q *is the initial distribution.* $(V_k)_{(\alpha)}[\cdot]$ *is expressed as a combination of Lie brackets and a certain differential operator.*

In the first section, we recall some results of the Malliavin calculus we use later. And then, McKean equations with generalized coefficients are formulated in the Section 2. Section 3 is devoted to the L^p-estimation on the Malliavin covariance matrices of the solutions to McKean equations and our main theorem on the regularity of the solutions is shown there. In the last section, the main theorem is applied to the original form of McKean equations and the explicit calculation is executed.

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1. Malliavin calculus.

Let us make a quick review of the results of the Malliavin calculus before going into our topic. The terminology used in this article is from

Kusuoka-Stroock [1], [2] and Watanabe [7]. Precise definitions and proofs can be found there.

(1.1) *Notation.* $W=(\Theta, \mathcal{B}, \mu)$ is the Wiener space where $\Theta=\{\theta\in C([0, \infty)\to \boldsymbol{R}^d)\,;\, \theta(0)=0\}$, $\mathcal{B}$ is the Borel field over Θ and μ denotes the Wiener measure on $(\Theta, \mathcal{B})$. $H=\left\{h\in\Theta\,;\text{ absolutely continuous and } \int_0^\infty |h'(t)|^2dt<\infty\right\}$ with $\|h\|_H^2=\int_0^\infty |h'(t)|^2dt$. The Hilbert space H is often called the Cameron-Martin space. $\mathcal{M}(\boldsymbol{R}^n)$ is the space of all the finite signed measures on $\boldsymbol{R}^n$, while $\mathcal{P}(\boldsymbol{R}^n)$ denotes the space of the probability measures on $\boldsymbol{R}^n$.

(1.2) DEFINITION. Let E be a real separable Hilbert space and Φ be a measurable map from Θ to E. Then Φ is called to be *'differentiable in the sense of Malliavin'* if $D\Phi$ can be defined as a measurable map from Θ to $\mathcal{H}(E)=H.S.(H; E)$ (the space of the Hilbert-Schmidt operators from H to E) satisfying

$$\lim_{t\to 0}\mu\left(\left\{\theta\,;\,\left\|\frac{\Phi(\theta+th)-\Phi(\theta)}{t}-D\Phi(\theta)(h)\right\|_E\geqq\varepsilon\right\}\right)=0 \tag{1.3}$$

for any $h\in H$ and $\varepsilon>0$. Following S. Watanabe, we let $\boldsymbol{D}(E)$ denote the class of functionals Φ on which the above D can operate as many times as desired with $D^k\Phi\in L^p(\mu)$ for every $k\in\boldsymbol{N}$ and $1\leqq p<\infty$.

(1.4) REMARK. This class is called $\mathcal{G}(\mathcal{L}\,; E)$ in Kusuoka-Stroock [1] in connection with the Ornstein-Uhlenbeck operator $\mathcal{L}$, which is not used in this paper.

(1.5) DEFINITION. Let E be $\boldsymbol{R}^n$ and $\Phi=(\Phi^1, \Phi^2, \cdots, \Phi^n)$. Then $D\Phi^i(\theta)(\cdot)$ is regarded as an element of H for any $\theta\in\Theta$ and $i=1, 2, \cdots, n$ by Riesz' theorem and so an $n\times n$ matrix

$$A(\theta)=(D\Phi(\theta), D\Phi(\theta))_H=((D\Phi^i(\theta)(\cdot), D\Phi^j(\theta)(\cdot))_H)_{1\leq i,j\leq n} \tag{1.6}$$

can be defined and is called *'Malliavin covariance matrix'*.

(1.7) PROPOSITION (Malliavin). *Let $\Phi\in\boldsymbol{D}(\boldsymbol{R}^n)$ and $A=(D\Phi, D\Phi)_H$ be its Malliavin covariance matrix. If $\Delta=\det A$ is 'large' enough to satisfy*

$$1/\Delta\in\bigcap_{p=1}^{\infty} L^p(\mu), \tag{1.8}$$

then the induced distribution $p=\mu\circ\Phi^{-1}$ on $\boldsymbol{R}^n$ has a density function (also denoted by p) i.e. $p(dx)=p(x)dx$ and p belongs to $\mathcal{S}(\boldsymbol{R}^n)$ (the totality of the rapidly decreasing functions on $\boldsymbol{R}^n$).

Various estimations for the function p are known, but such precise arguments are not necessary for our problem.

2. McKean equations.

The stochastic differential equations H. P. McKean introduced to describe the motion of molecules are of the following type and are often called McKean equations.

$$(2.1)\qquad X(t,\theta)=a(\theta)+\sum_{i=1}^{d}\int_0^t V_i[X(s,\theta),\pi(s)]\circ d\theta_i(s)+\int_0^t V_0[X(s,\theta),\pi(s)]ds,$$

where (i) $a(\cdot)$ is an $\boldsymbol{R}^n$-valued random variable independent of the Brownian motion $b(\theta)=\theta\in\Theta$ and stands for the initial state,

(ii) $\pi(t)=\pi(t,dy)$ is the distribution of the solution $X(t)$ on $\boldsymbol{R}^n$ and

(iii) $V_i[x,p]=\int_{\boldsymbol{R}^{n\times m}}V_i(x,y_1,y_2,\cdots,y_m)p^{\otimes m}(dy_1,dy_2,\cdots,dy_m)$ for $x\in\boldsymbol{R}^n$, $p\in\mathcal{M}(\boldsymbol{R}^n)$, $i=0,1,\cdots,d$ and $m\in\boldsymbol{N}$ (common to all i's) with $V_i(\cdot,\cdots,\cdot)\in C_b^\infty(\boldsymbol{R}^{n\times(m+1)};\boldsymbol{R}^n)$, $i=0,1,\cdots,d$. In the above equation, '$\circ$' denotes Stratonovich integral.

Let us define a differential operator $L[p]$ for $p\in\mathcal{M}(\boldsymbol{R}^n)$ by

$$(2.2)\qquad L[p]=\frac{1}{2}\sum_{i=1}^{d}V_i[\cdot,p]^2+V_0[\cdot,p],$$

where $V_i, i=0,1,\cdots,d$ are regarded as vector fields in the usual way i.e. $V_i=\sum_{j=1}^{n}V_i^j\partial/\partial x^j$. Then, the infinitesimal generator associated to (2.1) is expressed by $L[\pi(t)]$ and $\pi(t)$ satisfies

$$(2.3)\qquad \frac{d}{dt}\langle f,\pi(t)\rangle=\langle L[\pi(t)]f,\pi(t)\rangle,$$

where $f\in C_b^\infty(\boldsymbol{R}^n,\boldsymbol{R})$ and $\langle f,p\rangle=\int_{\boldsymbol{R}^n}f(x)p(dx)$ for $p\in\mathcal{M}(\boldsymbol{R}^n)$.

Now we are ready to apply the method of the Malliavin calculus to the equation (2.1), but instead of the original form of McKean equations, we are going to deal with more general type of equations. To do so, a class of functions on $\mathcal{M}(\boldsymbol{R}^n)$ should be introduced.

(2.4) DEFINITION. Let $V[x,p]$ be an $\boldsymbol{R}$-valued function on $\boldsymbol{R}^k\times\mathcal{M}(\boldsymbol{R}^n)$ where $k\in\boldsymbol{N}\cup\{0\}$ and $n\in\boldsymbol{N}$. We define a class $C^1(\boldsymbol{R}^k\times\mathcal{M}(\boldsymbol{R}^n))$ of all the functions with following properties:

(i) for each $M>0$, there exists $C_M>0$ such that $|V[x,p]|\leq C_M$ for any $x\in\boldsymbol{R}^k$ and $p\in\mathcal{M}(\boldsymbol{R}^n)$ satisfying $\|p\|_{\text{var.}}\leq M$, where $\|\cdot\|_{\text{var.}}$ denotes the total variation of a signed measure,

(ii) $V[\cdot, p] \in C_b^\infty(\boldsymbol{R}^k ; \boldsymbol{R})$ and each derivative satisfies the same property as in (i) where C_M can be different for each index,

(iii) $V[x, \cdot]$ is Fréchet differentiable i.e. there exists a bounded linear operator $\mathcal{D}V[x, p]: \mathcal{M}(\boldsymbol{R}^n) \to \boldsymbol{R}$ such that

$$|V[x, p+p'] - V[x, p] - \mathcal{D}V[x, p](p')| / \|p'\|_{\text{var.}} \longrightarrow 0 \quad \text{as } \|p'\|_{\text{var.}} \to 0 \tag{2.5}$$

for all $x \in \boldsymbol{R}^k$ and $p \in \mathcal{M}(\boldsymbol{R}^n)$,

(iv) the convergence (2.5) is uniform in $x \in \boldsymbol{R}^k$ and

(v) there exists an $\boldsymbol{R}$-valued function $W[x, y, p]$ on $\boldsymbol{R}^k \times \boldsymbol{R}^n \times \mathcal{M}(\boldsymbol{R}^n)$ with the same boundedness as in (i) and its Fréchet derivative $\mathcal{D}V[x, p](\cdot)$ is expressed by

$$\mathcal{D}V[x, p](p') = \int_{\boldsymbol{R}^n} W[x, y, p] p'(dy). \tag{2.6}$$

We let $\mathcal{F}V[x, y, p]$ denote $W[x, y, p]$ here and thus define a linear operator $\mathcal{F}$ from $C^1(\boldsymbol{R}^k \times \mathcal{M}(\boldsymbol{R}^n))$ to the space of $\boldsymbol{R}$-valued functions on $\boldsymbol{R}^k \times \boldsymbol{R}^n \times \mathcal{M}(\boldsymbol{R}^n)$.

Suppose that $V \in C^1(\boldsymbol{R}^k \times \mathcal{M}(\boldsymbol{R}^n))$ and $\mathcal{F}V \in C^1(\boldsymbol{R}^{k+n} \times \mathcal{M}(\boldsymbol{R}^n))$. Then $\mathcal{F}$ can operate on $\mathcal{F}V$ again and $\mathcal{F}\mathcal{F}V = \mathcal{F}^2 V$ is a function on $\boldsymbol{R}^{k+2n} \times \mathcal{M}(\boldsymbol{R}^n)$. In this way, $\mathcal{F}^j V$ is defined inductively so long as $\mathcal{F}^{j-1}V$ belongs to $C^1(\boldsymbol{R}^{k+(j-1)n} \times \mathcal{M}(\boldsymbol{R}^n))$.

(2.7) *Notation.*

$$C^\infty(\boldsymbol{R}^k \times \mathcal{M}(\boldsymbol{R}^n)) = \{V \in C^1(\boldsymbol{R}^k \times \mathcal{M}(\boldsymbol{R}^n))\,;\, \mathcal{F}^j V \in C^1(\boldsymbol{R}^{k+jn} \times \mathcal{M}(\boldsymbol{R}^n)), j \in \boldsymbol{N}\}.$$

$$C^\infty(\boldsymbol{R}^k \times \mathcal{M}(\boldsymbol{R}^n)\,;\, \boldsymbol{R}^m) = \{V = (V_1, V_2, \cdots, V_m)\,;\, V_i \in C^\infty(\boldsymbol{R}^k \times \mathcal{M}(\boldsymbol{R}^n)),\ i = 1, 2, \cdots, m\}.$$

Under the above notation, more general type of McKean equations can be stated as below:

Generalized McKean equations.

$$X(t, \theta) = a(\theta) + \sum_{i=1}^d \int_0^t V_i[X(s, \theta), \pi(s)] \circ d\theta_i(s) + \int_0^t V_0[X(s, \theta), \pi(s)] ds, \tag{2.8}$$

where (i) $a(\cdot)$ is an $\boldsymbol{R}^n$-valued random variable independent of the Brownian motion $b(\cdot)$,

(ii) $\pi(t)$ is the distribution of the solution $X(t)$ on $\boldsymbol{R}^n$ and

(iii) $V_i[\cdot, \cdot] \in C^\infty(\boldsymbol{R}^n \times \mathcal{M}(\boldsymbol{R}^n)\,;\, \boldsymbol{R}^n), i = 0, 1, \cdots, d.$

(2.9) REMARK. McKean [3], [4] showed the existence and uniqueness of the solutions to the original type of McKean equations (2.1) by the iteration method and also proved that the distribution $\pi(t, dx)$ of the solution $X(t)$ has smooth density function $\pi(t, x)$ with respect to the Lebesgue measure when the system is strictly positive, namely, elliptic.

(2.10) PROPOSITION. *The equation* (2.8) *has the uuique solution.*

PROOF. We follow the iteration method like in McKean [3]. Let $X^0(t, \theta)\equiv a(\theta)$ and for $k=0, 1, \cdots$, define

$$X^{k+1}(t, \theta)=a(\theta)+\sum_{i=1}^{d}\int_0^t V_i[X^k(s, \theta), \pi^k(s)]\circ d\theta_i(s)+\int_0^t V_0[X^k(s, \theta), \pi^k(s)]ds$$

inductively, where $\pi^k(t)$ is the distribution of $X^k(t, \theta)$. First, note that for $V\in C^\infty(\mathcal{M}(\boldsymbol{R}^n))=C^\infty(\boldsymbol{R}^0\times\mathcal{M}(\boldsymbol{R}^n))$,

$$\begin{aligned}
&|V[\pi^k(t)]-V[\pi^{k-1}(t)]|^2\\
&=\left|\int_0^1\frac{\partial}{\partial u}V[(1-u)\pi^{k-1}(t)+u\pi^k(t)]du\right|^2\\
&=\left|\int_0^1\langle\mathcal{F}V[\cdot, (1-u)\pi^{k-1}(t)+u\pi^k(t)], \pi^k(t)-\pi^{k-1}(t)\rangle du\right|^2\\
&\leqq\int_0^1|E[\mathcal{F}V[X^k(t), (1-u)\pi^{k-1}(t)+u\pi^k(t)]\\
&\quad-\mathcal{F}V[X^{k-1}(t), (1-u)\pi^{k-1}(t)+u\pi^k(t)]]|^2du\\
&\leqq C\cdot E[|X^k(t)-X^{k-1}(t)|^2]
\end{aligned}$$

for some $C>0$ by the condition (v) of Definition (2.4). And using the above estimation, we have

$$\begin{aligned}
&E[\|X^{k+1}(t)-X^k(t)\|_{R^n}^2]\\
&=E\Big[\Big\|\sum_{i=1}^{d}\int_0^t\{V_i[X^k(s, \theta), \pi^k(s)]-V_i[X^{k-1}(s, \theta), \pi^{k-1}(s)]\}\circ d\theta_i(s)\\
&\quad+\int_0^t\{V_0[X^k(s, \theta), \pi^k(s)]-V_0[X^{k-1}(s, \theta), \pi^{k-1}(s)]\}ds\Big\|_{R^n}^2\Big]\\
&\leqq C_1\sum_{i=1}^{d}E\Big[\Big\|\int_0^t\{V_i[X^k(s, \theta), \pi^k(s)]-V_i[X^{k-1}(s, \theta), \pi^{k-1}(s)]\}\circ d\theta_i(s)\Big\|_{R^n}^2\Big]\\
&\quad+C_1E\Big[\Big\|\int_0^t\{V_0[X^k(s, \theta), \pi^k(s)]-V_0[X^{k-1}(s, \theta), \pi^{k-1}(s)]\}ds\Big\|_{R^n}^2\Big]
\end{aligned}$$

$$\leqq C_2(1+t)\int_0^t E[\|X^k(s)-X^{k-1}(s)\|^2_{R^n}]ds \qquad \text{for some} \quad C_1, C_2>0.$$

The rest of the proof is standard and we have a solution to the equation (2.8) as the limit

(2.11) $$X^\infty(t,\theta)=\lim_{k\to\infty} X^k(t,\theta) \qquad \text{a.s. and in } L^2(\mu).$$

The uniqueness is also proved in the same way. Q. E. D.

As is usual in investigating stochastic differential equations with random initial conditions, we introduce another equation here. Let q and $\pi(t)$ be the distributions of $a(\cdot)$ and $X(t,\cdot)$ in (2.8) respectively. Then the following equation has the unique solution $Y(t,x,\theta\,;q)$:

(2.12) $$Y(t,x,\theta\,;q)=x+\sum_{i=1}^d \int_0^t V_i[Y(s,x,\theta\,;q),\pi(s)]\circ d\theta_i(s)$$
$$+\int_0^t V_0[Y(s,x,\theta\,;q),\pi(s)]ds, \qquad x\in \boldsymbol{R}^n \text{ and } t\geqq 0.$$

Here, $\pi(t)$ is NOT the distribution of $Y(t,x,\theta\,;q)$ but that of $X(t,\theta)$ (the solution to (2.8)). Since the equations (2.8) and (2.12) are the same except for the initial states,

(2.13) $$X(t,\theta)=Y(t,a(\theta),\theta\,;q) \qquad \text{a.e. } \theta\,(\mu).$$

Moreover, the independence between $a(\theta)$ and $\{\theta_i\}$ implies

(2.14) $$\pi(t,dy)=\int_{R^n}\rho(t,x,dy)q(dx),$$

where $\rho(t,x,dy)$ denotes the distribution of $Y(t,x,\theta\,;q)$ on $\boldsymbol{R}^n$.

3. Estimates of Malliavin covariance.

Following the idea in the last part of the previous section, we restrict our interest to the regularity estimation of the solution Y to (2.12) for the moment. First of all, the same method as is used in Kusuoka-Stroock [2] and Taniguchi [6] leads us to the following two lemmas.

(3.1) LEMMA. *$Y(t,x\,;q)\in \boldsymbol{D}(\boldsymbol{R}^n)$ and*

(3.2) $$DY(t,x\,;q)(h)=\sum_{i=1}^d \int_0^t V_i^{(1)}[Y(s,x\,;q),\pi(s)]DY(s,x\,;q)(h)\circ d\theta_i(s)$$

$$+\int_0^t V_0^{(1)}[Y(s,x;q),\pi(s)]DY(s,x;q)(h)ds$$

$$+\sum_{i=1}^d \int_0^t V_i[Y(s,x;q),\pi(s)]h_i'(s)ds\,,$$

where $V_i^{(1)}$ *is the derivative (an* $n\times n$ *matrix) of* $V_i[x,p]$ *with respect to* x.

(3.3) LEMMA. $Y(t,x;q)$ *is differentiable in* x *and letting* $J(t,x;q)$ *be its Jacobian matrix, we have*

$$J(t,x;q)=I+\sum_{i=1}^d \int_0^t V_i^{(1)}[Y(s,x;q),\pi(s)]J(s,x;q)\circ d\theta_i(s) \tag{3.4}$$

$$+\int_0^t V_0^{(1)}[Y(s,x;q),\pi(s)]J(s,x;q)ds\,,$$

and moreover,

$$J^{-1}(t,x;q)=I-\sum_{i=1}^d \int_0^t J^{-1}(s,x;q)V_i^{(1)}[Y(s,x;q),\pi(s)]\circ d\theta_i(s) \tag{3.5}$$

$$-\int_0^t J^{-1}(s,x;q)V_0^{(1)}[Y(s,x;q),\pi(s)]ds\,.$$

By Lemma (3.1), Malliavin covariance matrix $A(t,x;q)$ of $Y(t,x;q)$ can be defined. In order to apply the Malliavin calculus, we have to obtain the L^p-estimates of $\{\det A(t,x;q)\}^{-1}$. To do so, define an $n\times n$ matrix $\tilde{A}(t,x;q)$ by

$$\tilde{A}(t,x;q)=(J^{-1}(t,x;q)DY(t,x;q),J^{-1}(t,x;q)DY(t,x;q))_H\,. \tag{3.6}$$

Then, the equality

$$\tilde{A}(t,x;q)=\sum_{i=1}^n \int_0^t \{J^{-1}(s,x;q)V_i[Y(s,x;q),\pi(s)]\}^{\otimes 2}ds \tag{3.7}$$

holds. ($v^{\otimes 2}=v\otimes v$ for $v\in \boldsymbol{R}^n$.) Using this $\tilde{A}$, we can express $A(t,x;q)$ by

$$A(t,x;q)=J(t,x;q)\tilde{A}(t,x;q)^tJ(t,x;q)\,. \tag{3.8}$$

The boundedness of $V_i^{(1)}, i=0,1,\cdots,d$ in (3.5) implies that

$$E\left[\left\{\sup_{0<t\leq T}|J^{-1}(t,x;q)|\right\}^p\right]<\infty \qquad \text{for given } T>0 \text{ and any } p>1.$$

(For an $n\times n$ matrix $M=(M_{ij})$, $|M|=\left(\sum_{i,j=1}^n M_{ij}^2\right)^{1/2}$.) And from this and the equality (3.8), it follows that

$$E[\{\det A(t,x;q)\}^{-p}]\leqq C_p(E[\{\det \tilde{A}(t,x;q)\}^{-2p}])^{1/2} \tag{3.9}$$

for some $C_p>0$. This makes us realize that it is essential to obtain the L^p-estimates of $\{\det \tilde{A}(t,x;q)\}^{-1}$.

Now, let us proceed to the estimation of the matrix $\tilde{A}(t,x;q)$ following the above reasoning. For this purpose, stochastic Taylor expansion of the integrand in (3.7) is quite powerful (cf. [2]), but before that, we should prepare a lemma to handle the integrands with measures as parameters.

(3.10) LEMMA. *Let $\Phi\in C^\infty(\mathcal{M}(\boldsymbol{R}^n))$ and $\mathcal{F}$ be the operator defined after* (2.6). *Then, for the probability distribution $\pi(t)$ in* (2.8),

$$\frac{d}{dt}\Phi[\pi(t)]=\int_{R^n}L[\pi(t)](\mathcal{F}\Phi)[y,\pi(t)]\pi(t,dy)\,. \tag{3.11}$$

PROOF. In the same way as in the proof of Proposition (2.10), let $u>t\geqq 0$. Then,

$$\begin{aligned}\Phi[\pi(u)]-\Phi[\pi(t)]&=\int_0^1\frac{\partial}{\partial s}\Phi[(1-s)\pi(t)+s\pi(u)]ds\\&=\int_0^1\frac{\partial}{\partial s}\Phi[\pi(t)+s(\pi(u)-\pi(t))]ds\\&=\int_0^1\mathcal{D}\Phi[\pi(t)+s(\pi(u)-\pi(t))](\pi(u)-\pi(t))ds\\&=\int_0^1\langle\mathcal{F}\Phi[\cdot,\pi(t)+s(\pi(u)-\pi(t))]-\mathcal{F}\Phi[\cdot,\pi(t)],\pi(u)-\pi(t)\rangle ds\\&\quad+\langle\mathcal{F}\Phi[x,\pi(t)],\pi(u)-\pi(t)\rangle.\end{aligned}$$

As for the first term of the last line, the integrand

$$\begin{aligned}&\langle\mathcal{F}\Phi[\cdot,\pi(t)+s(\pi(u)-\pi(t))]-\mathcal{F}\Phi[\cdot,\pi(t)],\pi(u)-\pi(t)\rangle\\&=\int_t^u\langle(L[\pi(\xi)]\mathcal{F}\Phi)[\cdot,\pi(t)+s(\pi(u)-\pi(t))]-(L[\pi(\xi)]\mathcal{F}\Phi)[\cdot,\pi(t)],\pi(\xi)\rangle d\xi\end{aligned}$$

and for each $x\in\boldsymbol{R}^n$ and $\xi\in[t,u]$,

$$\begin{aligned}&|(L[\pi(\xi)]\mathcal{F}\Phi)[x,\pi(t)+s(\pi(u)-\pi(t))]-(L[\pi(\xi)]\mathcal{F}\Phi)[x,\pi(t)]|\\&=\left|\int_0^1\frac{\partial}{\partial v}(L[\pi(\xi)]\mathcal{F}\Phi)[x,\pi(t)+v\cdot s(\pi(u)-\pi(t))]dv\right|\\&=\left|\int_0^1 s\langle\mathcal{F}(L[\pi(\xi)]\mathcal{F}\Phi)[x,\cdot,\pi(t)+v\cdot s(\pi(u)-\pi(t))],\pi(u)-\pi(t)\rangle dv\right|\end{aligned}$$

$$=\left|\int_0^1 s\int_t^u \langle (L_y[\pi(\xi)]\mathscr{F}(L_x[\pi(\xi)]\mathscr{F}\Phi))[x,\cdot,\pi(t)+v\cdot s(\pi(u)-\pi(t))],\pi(\eta)\rangle d\eta dv\right|$$

$$\leqq sC(u-t)$$

with a constant $C>0$ independent of $x\in \boldsymbol{R}^n, u, t$ and $\xi\in[t,u]$ by (i) and (ii) of Definition (2.4). Therefore, as u approaches t,

$$\frac{1}{u-t}\int_0^1 \langle \mathscr{F}\Phi[\cdot,\pi(t)+s(\pi(u)-\pi(t))]-\mathscr{F}\Phi[\cdot,\pi(t)],\pi(u)-\pi(t)\rangle ds$$

tends to 0 and so, recalling the equality (2.3), we have

$$\frac{d}{dt}\Phi[\pi(t)]=\lim_{u\to t}\left\langle \mathscr{F}\Phi[x,\pi(t)],\frac{\pi(u)-\pi(t)}{u-t}\right\rangle$$

$$=\langle L[\pi(t)]\mathscr{F}\Phi[\cdot,\pi(t)],\pi(t)\rangle . \qquad \text{Q. E. D.}$$

Define an operator Q on $C^\infty(\mathscr{M}(\boldsymbol{R}^n))$ by

$$(Q\Phi)[p]=\int_{R^n} L[p]\mathscr{F}\Phi[x,p]p(dx), \quad x\in\boldsymbol{R}^n, p\in\mathscr{M}(\boldsymbol{R}^n) \text{ and } t\geqq 0. \tag{3.12}$$

Then, (3.11) is rewritten as

$$\frac{d}{dt}\Phi[\pi(t)]=(Q\Phi)[\pi(t)]. \tag{3.13}$$

In view of the definition of C^∞, the above Q can operate on every $V[\cdot,\cdot]\in C^\infty(\boldsymbol{R}^n\times\mathscr{M}(\boldsymbol{R}^n);\boldsymbol{R}^n)$ as many times as desired.

Let $\mathscr{A}=\{\emptyset\}\cup\bigcup_{l=1}^{\infty}\{0,1,\cdots,d\}^l$ be the space of multi-indices and for $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_l)\in\mathscr{A}$, define $|\alpha|$, $\|\alpha\|$, α' and α_* by l, $l+\#\{i;\alpha_i=0\}$, $(\alpha_1,\cdots,\alpha_{l-1})$ and α_l respectively. ($|\alpha|=0$ if $\alpha=\emptyset$ and $\alpha'=\emptyset$ if $|\alpha|=1$.) Then, $V_{(\alpha)}[\cdot,\cdot]$ can be defined for $V[\cdot,\cdot]\in C^\infty(\boldsymbol{R}^n\times\mathscr{M}(\boldsymbol{R}^n);\boldsymbol{R}^n)$ and $\alpha\in\mathscr{A}$ inductively as below:

$$V_{(\alpha)}[x,p]=\begin{cases} V[x,p] & \text{if } \alpha=\emptyset, \\ [V_{\alpha_*}, V_{(\alpha')}][x,p] & \text{if } \alpha\neq\emptyset,\ \alpha_*\neq 0, \\ [V_0, V_{(\alpha')}][x,p]+(QV_{(\alpha')})[x,p] & \text{if } \alpha\neq\emptyset,\ \alpha_*=0, \end{cases} \tag{3.14}$$

where $[V,W]=VW-WV$ with V and W regarded as vector fields and

$$(QV)[x,p]=\int_{R^n} L_y[p]\mathscr{F}V[x,y,p]p(dy). \tag{3.15}$$

The subscript of $L_y[p]$ indicates that $L[p]$ acts on $\mathcal{F}V$ as a function of the second parameter y.

We are now ready to execute stochastic Taylor expansion for the integrand $J^{-1}(t, x; q)V_i[Y(t, x; q), \pi(t)]$ of (3.7).

(3.16) PROPOSITION. *For* $V[\cdot, \cdot]\in C^{\infty}(\boldsymbol{R}^n\times\mathcal{M}(\boldsymbol{R}^n); \boldsymbol{R}^n)$ *and* $L\in\boldsymbol{N}$,

$$J^{-1}(t, x; q)V[Y(t, x; q), \pi(t)]=\sum_{\|\alpha\|\leqq L-1}\theta^{(\alpha)}(t)V_{(\alpha)}[x, q]+R_L(t, x, q, V), \tag{3.17}$$

where $\theta^{(\alpha)}(t)=1$ *if* $\alpha=\emptyset$ *and* $=\int_0^t\theta^{(\alpha')}(s)\circ d\theta_{\alpha_*}(s)$ *if* $\alpha\neq\emptyset$ $(\theta_0(t)=t)$. R_L *is expressed as a sum of multiple Stratonovich integrals of order* $\geqq L$. *(Integral with respect to dt is thought of as of order 2 in terms of Stratonovich integral.) And for given* $0<\varepsilon\leqq 1$, *there exist positive constants* C *and* γ *depending on* L *and* ε *such that*

$$\sup_{0<t\leqq 1}\mu\left(\int_0^{t/K}|R_L(s, x, q, V)|^2ds/t^L\geqq 1/K^{L+1-\varepsilon}\right)\leqq C\exp(-K^{\gamma}/(1+M)^2) \tag{3.18}$$

for any $K\geqq 1$, *where*

$$M=\sup\{\|V_k[\cdot, p]\|_{C_b^2};\ 0\leqq k\leqq d,\ p\in\mathcal{P}(\boldsymbol{R}^n)\}$$
$$\vee\sup\{\|V_{(\alpha)}[\cdot, p]\|_{C_b^2};\ |\alpha|\leqq L+1,\ p\in\mathcal{P}(\boldsymbol{R}^n)\}.$$

PROOF. Applying (2.12), (3.5) and (3.13) adequately, we have

$$\begin{aligned}
&d(J^{-1}(t, x; q)V[Y(t, x; q), \pi(t)])\\
&=\circ dJ^{-1}(t, x; q)V[Y(t, x; q), \pi(t)]\\
&\quad +J^{-1}(t, x; q)V^{(1)}[Y(t, x; q), \pi(t)]\circ dY(t, x; q)\\
&\quad +J^{-1}(t, x; q)(QV)[Y(t, x; q), \pi(t)]dt\\
&=-\sum_{i=1}^{d}J^{-1}(t, x; q)V_i^{(1)}[Y(t, x; q), \pi(t)]V[Y(t, x; q), \pi(t)]\circ d\theta_i(t)\\
&\quad -J^{-1}(t, x; q)V_0^{(1)}[Y(t, x; q), \pi(t)]V[Y(t, x; q), \pi(t)]dt\\
&\quad +J^{-1}(t, x; q)\sum_{i=1}^{d}V^{(1)}[Y(t, x; q), \pi(t)]V_i[Y(t, x; q), \pi(t)]\circ d\theta_i(t)\\
&\quad +J^{-1}(t, x; q)V^{(1)}[Y(t, x; q), \pi(t)]V_0[Y(t, x; q), \pi(t)]dt\\
&\quad +J^{-1}(t, x; q)(QV)[Y(t, x; q), \pi(t)]dt
\end{aligned}$$

$$= \sum_{i=1}^{d} J^{-1}(t, x; q)[V_i, V][Y(t, x; q), \pi(t)]\circ d\theta_i(t)$$
$$+J^{-1}(t, x; q)\{[V_0, V][Y(t, x; q), \pi(t)]+(QV)[Y(t, x; q), \pi(t)]\}dt.$$

Therefore,

$$J^{-1}(t, x; q)V[Y(t, x; q), \pi(t)] \tag{3.19}$$
$$= V[x, q]+\sum_{i=1}^{d}\int_0^t J^{-1}(s, x; q)[V_i, V][Y(s, x; q), \pi(s)]\circ d\theta_i(s)$$
$$+\int_0^t J^{-1}(s, x; q)\{[V_0, V][Y(s, x; q), \pi(s)]+(QV)[Y(s, x; q), \pi(s)]\}ds.$$

Since both $[V_i, V]$ and $[V_0, V]+QV$ belong to $C^\infty(\boldsymbol{R}^n\times\mathcal{M}(\boldsymbol{R}^n); \boldsymbol{R}^n)$ again, (3.19) can be applied to each integrand repeatedly until (3.17) is obtained. The estimate (3.18) is the same as Theorem (2.12) of [2] except slight difference in the definition of M and we 'quote' only the outline of the proof. Of course our definition of $V_{(\alpha)}$'s is also different from that in [2] in that our $V_{(\alpha)}$'s contain the operator Q, but such difference does not matter in this proof, since all the ingredients are bounded in our case. First, we divide R_L into two terms:

$$R_L(t, x, q, V)=\sum_{|\alpha|=L} S^{(\alpha)}(t, q, V)+\sum_{|\alpha|<L, \|\alpha\|\geqq L}\theta^{(\alpha)}(t)V_{(\alpha)}[x, q],$$

where $\theta^{(\alpha)}$ and $S^{(\alpha)}$ are multiple Stratonovich integrals with respect to $d\theta_{\alpha_1}, d\theta_{\alpha_2}, \cdots$ and $d\theta_{\alpha_l}$ with $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_l)$ of the constant 1 and $J^{-1}(t, x; q)V_{(\alpha)}[Y(t, x; q), \pi(t)]$, respectively. Since it holds that

$$\int_0^{t/K}|f(s)|^2dt\leqq\frac{2}{2L-\varepsilon}\Big(\sup_{0<s\leqq1}|f(s)|^2/s^{L-\varepsilon/2}\Big)\cdot(t/K)^{L+1-\varepsilon/2}, \qquad 0\leqq t\leqq 1$$

for any $f\in C([0, 1])$, all we have to check is that both

$$\mu\Big(\frac{2}{2L-\varepsilon}\sup_{0<s\leqq1}|S^{(\alpha)}(s, q, V)|^2/s^{L-\varepsilon/2}\geqq K^{\varepsilon/2}\Big)$$

and

$$\mu\Big(\frac{2}{2L-\varepsilon}\sup_{0<s\leqq1}|\theta^{(\alpha)}(s)|^2/s^{L-\varepsilon/2}\geqq K^{\varepsilon/2}\Big)$$

are dominated by the expression like the right hand side of (3.18). And this is done by using Theorem (A.5) of [2] which states that for any given $\varepsilon>0$ and multiple Stratonovich integral $S^{(\alpha)}(t, Z)$ of $Z(t)=Z_0+\sum_k\int_0^t Y_k d\theta_k$, $\alpha\in\mathcal{A}$ with $\|\alpha\|=L\geqq1$, there exist positive constants C and

λ such that

$$\mu\Big(\sup_{0<t\le 1}|S^{(\alpha)}(t,Z)|/t^{L/2-\varepsilon}\geqq K^{2L},\ \sup_{0<t\le 1}|Z(t)|\leqq K,\ \sum_{k=1}^{d}\int_0^1 |Y_k(t)|^2 dt\leqq K^2\Big)$$
$$\leqq C\exp(-\lambda K),\qquad K>0.$$

Q. E. D.

For each $L\in \boldsymbol{N}$ and $\eta\in S^{n-1}$, define

$$\mathcal{V}_L(x,q,\eta)=\sum_{k=1}^{d}\sum_{\|\alpha\|<L}((V_k)_{(\alpha)}[x,q],\eta)^2_{R^n},$$

$$\mathcal{V}_L(x,q)=\inf_{\eta\in S^{n-1}}(\mathcal{V}_L(x,q,\eta)\wedge 1)$$

and

$$\lambda(t,x;q)=\inf_{\eta\in S^{n-1}}(\eta,\tilde{A}(t,x;q)\eta)_{R^n}.$$

Then, there exist positive constants C_L and ν_L such that for any $t\in(0,1]$, $x\in \boldsymbol{R}^n$ and $K\geqq 1$,

$$\mu(\lambda(t/K,x;q)\leqq t^L/K^{L+1})\leqq C_L\exp(-(\mathcal{V}_L(x,q)^{L+2}K^{L+1})^{\nu_L}/(1+M)^2),\tag{3.20}$$

where $M=\sup\{\|(V_k)_{(\alpha)}[\cdot,p]\|_{C_b^2};\ 0\leqq k\leqq d,\ |\alpha|\leqq L+1,\ p\in\mathcal{P}(\boldsymbol{R}^n)\}$. This inequality is the same as (2.18) of [2] and can be proved by the techniques used there. Indeed, in view of the previous proposition,

$$(\eta,\tilde{A}(t/K,x;q)\eta)^{1/2}_{R^n}\geqq\Big(\sum_{k=1}^{d}\int_0^{t/K}\Big\{\sum_{\|\alpha\|\le L-1}((V_k)_{(\alpha)}[x,q],\eta)_{R^n}\theta^{(\alpha)}(s)\Big\}^2 ds\Big)^{1/2}$$
$$-\Big(\sum_{k=1}^{d}\int_0^{t/K}|R_L(s,x,q,V_k)|^2 ds\Big)^{1/2}.$$

Therefore,

$$2\lambda(t/K,x;q)\geqq\inf_{\eta\in S^{n-1}}\sum_{k=1}^{d}\int_0^{t/K}\Big\{\sum_{\|\alpha\|\le L-1}((V_k)_{(\alpha)}[x,q],\eta)_{R^n}\theta^{(\alpha)}(s)\Big\}^2 ds$$
$$-2\sum_{k=1}^{d}\int_0^{t/K}|R_L(s,x,q,V_k)|^2 ds$$
$$\geqq\mathcal{V}_L(x,q)\inf\Big\{\int_0^{t/K}\Big(\sum_{\|\alpha\|\le L-1}b_\alpha\theta^{(\alpha)}(s)\Big)^2 ds;\ \sum_{\|\alpha\|\le L-1}b_\alpha^2=1\Big\}$$
$$-2\sum_{k=1}^{d}\int_0^{t/K}|R_L(s,x,q,V_k)|^2 ds.$$

Hence, for any $0<\varepsilon<1$,

$$\mu(\lambda(t/K, x, q) \leqq t^L/K^{L+1-\varepsilon})$$

$$\leqq \mu\Big(\frac{1}{2}\mathcal{V}_L(x, q) \inf\Big\{\int_0^{t/K}\Big(\sum_{\|\alpha\| \leq L-1} b_\alpha \theta^{(\alpha)}(s)\Big)^2 ds\,;\ \sum_{\|\alpha\| \leq L-1} b_\alpha^2 = 1\Big\} \leqq 2t^L/K^{L+1-\varepsilon}\Big)$$

$$+\mu\Big(\sum_{k=1}^{d} \int_0^{t/K} |R_L(s, x, q, V_k)|^2 ds \geqq t^L/K^{L+1-\varepsilon}\Big).$$

By (3.18), the second term is dominated by $d \cdot C\exp(-(K/d)^r/(1+M)^2)$ and the first term is dominated by $C_L\exp(-\{\mathcal{V}_L(x, q)K^{1-\varepsilon}/4\}^{\nu_L})$ in view of Theorem (A.6) of [2] stating that given $L \geqq 1$, there exist C_L and $\nu_L > 0$ such that for any $0 < t < 1$ and $K > 0$,

$$\mu\Big(\inf\Big\{\int_0^t\Big(\sum_{\|\alpha\| \leq L-1} b_\alpha \theta^{(\alpha)}(s)\Big)^2 ds\,;\ \sum_{\|\alpha\| \leq L-1} b_\alpha^2 = 1\Big\} \leqq t^L/K\Big) \leqq C_L\exp(-K^{\nu_L}).$$

Now we can state a proposition on the regularity of $Y(t, x\,; q)$.

(3.21) PROPOSITION. *Define* $U_L(q) = \{x \in \boldsymbol{R}^n\,;\ \mathcal{V}_L(x, q) > 0\}$ *for each* $L \in \boldsymbol{N}$ *and* $U(q) = \bigcup_{L=1}^{\infty} U_L(q)$. *Then, the distribution* $\rho(t, x, dy)$ *of* $Y(t, x\,; q)$ *has its smooth density function for* $x \in U(q)$. *In fact, there exists* $\rho \in C^\infty([0, \infty) \times U(q) \times \boldsymbol{R}^n)$ *such that*

$$\rho(t, x, dy) = \rho(t, x, y)dy\,. \tag{3.22}$$

Moreover, for any $T \geqq 1$ *and* $k \in \boldsymbol{N} \cup \{0\}$, *there exist* G_k, σ_k *and* $\mu_k > 0$ *such that for any* $m \in \boldsymbol{N} \cup \{0\}$ *and* $\alpha, \beta \in \mathcal{A}$ *satisfying* $m + |\alpha| + |\beta| \leqq k$,

$$|\partial_t^m \partial_x^\alpha \partial_y^\beta \rho(t, x, y)| \leqq G_k \exp(-\sigma_k \|y - x\|^2)/(\mathcal{V}(x, q)^{L+2} t^L)^{\mu_k}\,, \tag{3.23}$$

$(t, x, y) \in (0, T] \times U_L(q) \times \boldsymbol{R}^n$.

PROOF. Let $x \in U_L(q)$ and $\mathcal{V}_L(x, q) = c > 0$. Then, since $\tilde{A}(t, x\,; q)$ is non-decreasing in t, it follows from (3.20) that

$$\begin{aligned}\mu(\lambda(t, x\,; q)/t^L \leqq 1/K) &\leqq \mu(\lambda(t/K^{1/L+1}, x\,; q)/t^L \leqq 1/K)\\ &\leqq C_L\exp(-(c^{L+2}K)^{\nu_L}/(1+M)^2)\end{aligned} \tag{3.24}$$

for $0 < t \leqq 1$ and $K \geqq 1$. The last expression of (3.24) can be written as $C_L\exp(-r_L K^{\nu_L})$ where $r_L = c^{\nu_L(L+2)}/(1+M)^2 > 0$ is a constant independent of K. Under this notation, standard arguments yield the estimation:

$$E[\{\lambda(t, x\,; q)\}^{-p}\,;\ t^L/\lambda(t, x\,; q) \geqq K] \leqq (K^p C_L \exp(-\beta_L K^{\nu_L}) + \tilde{C}_{L.p}\beta_L^{-p/\nu_L})/t^{pL}$$

with some $\tilde{C}_{L.p} > 0$. And so, there exists $B_{L.p} > 0$ such that

$$\|1/\lambda(t, x ; q)\|_{L^p(\mu)} \leqq B_{L,p} c^{-(L+2)} t^{-L}.$$

Noting that $\det \tilde{A}(t, x ; q) \geqq \lambda(t, x ; q)^n$, we have

$$E[\{\det \tilde{A}(t, x ; q)\}^{-p}] \leqq E[\lambda(t, x ; q)^{-np}] \\ \leqq B_{L,np} c^{-n(L+2)} t^{-nL}$$

for all $p \in \boldsymbol{N}$. From the above estimation combined with (3.9), for any $p \in \boldsymbol{N}$, there exists a positive constant $A_{L,p}$ such that

(3.25) $$\|1/\det A(t, x ; q)\|_{L^p(\mu)}^p \leqq A_{L,p} c^{-n(L+2)} t^{-nL}.$$

Hence, Proposition (1.7) guarantees the existence of smooth density function ρ. The smoothness of ρ in (t, x) can also be seen easily.

The estimate (3.23) is shown by using the so-called 'Malliavin's integration-by-parts formula' and the estimate (3.25). (For details, see Corollary (3.25) of [2].) Q. E. D.

We are now ready to present our main theorem.

(3.26) THEOREM. *Let* $(V_k)_{(\alpha)}, k \in \{1, \cdots, d\}, \alpha \in \mathcal{A}$ *be the vectors defined in* (3.14) *and assume that*

(3.27) $$\operatorname{span}\{(V_k)_{(\alpha)}[x, q] ; 1 \leqq k \leqq d, \alpha \in \mathcal{A}\} = \boldsymbol{R}^n \quad \text{for any } x \in \operatorname{supp} q.$$

(i) *If, moreover, there exists an* $L \geqq 1$ *such that*

(3.28) $$\operatorname{span}\{(V_k)_{(\alpha)}[x, q] ; 1 \leqq k \leqq d, \alpha \in \mathcal{A}, \|\alpha\| \leqq L-1\} = \boldsymbol{R}^n \quad \text{for any } x \in \operatorname{supp} q$$

and $\varepsilon_L \equiv \inf_{x \in \operatorname{supp} q} \left\{ \inf_{\eta \in S^{n-1}} \left(\sum_{k=1}^{d} \sum_{\|\alpha\| \leqq L-1} ((V_k)_{(\alpha)}[x, q], \eta)^2 \right) \wedge 1 \right\} > 0$, *then there exists* $\pi \in C^\infty([0, \infty) \times \boldsymbol{R}^n)$ *which satisfies*

(3.29) $$\pi(t, dy) = \pi(t, y) dy,$$

where $\pi(t, dy)$ *is the distribution of the unique solution* $X(t, \theta)$ *to the generalized McKean equation* (2.8).

(ii) *In particular, if* $\operatorname{supp} q$ *is compact in* $\boldsymbol{R}^n$, *then only the first condition* (3.27) *is required for the above density function* $\pi(t, y)$ *to exist.*

PROOF. First note that (3.27) implies that $\operatorname{supp} q \subset U(q)$ from the definition of $U(q)$. And so the smooth density function $\rho(t, x, y)$ of Y exists for any $(t, x, y) \in (0, \infty) \times \operatorname{supp} q \times \boldsymbol{R}^n$ by Proposition (3.21). Using the equality (2.14) and Fubini's theorem, we can see the existence of the density function $\pi(t, y)$ defined by

(3.30) $$\pi(t, y)=\int_{R^n}\rho(t, x, y)q(dx).$$

All we have to show is the smoothness of the function $\pi(t, y)$ defined above. Since ρ is smooth, it suffices to see that the derivatives of $\rho(t, x, y)$ are integrable with respect to $q(dx)$. In the case (i), the condition (3.28) is equivalent to saying that $\operatorname{supp} q\subset U_L(q)$ and by the estimation (3.23) and the definition of ε_L, the estimate

(3.31) $$|\partial_t^m\partial_x^\alpha\partial_y^\beta\rho(t, x, y)|\leqq G_k(\varepsilon_L^{L+2}t^L)^{-\mu_k}, \quad (t, x, y)\in(0, T]\times\operatorname{supp} q\times \boldsymbol{R}^n$$

holds. In the case (ii), the compactness of $\operatorname{supp} q$ guarantees the existence of L satisfying the conditions in (i). Q. E. D.

(3.32) REMARK. The sufficient conditions in Theorem (3.26) do not look like those on Lie algebra because $(V_k)_{(\alpha)}$'s contain also the operator Q. But with the help of partial hypoellipticity which is mentioned before, the result in the theorem are regarded in a sense as the projection of the conditions on Lie algebra (cf. [2], [5]): For a time-inhomogeneous stochastic differential equation

$$dZ_t=\sum_{k=1}^{d} W_k(Z_t, t)\circ d\theta_k(t)+W_0(Z_t, t)dt,$$

define

$$L(z, t)=\operatorname{span}\{[\tilde{W}_{k_j}, [\cdots, [\tilde{W}_{k_1}, \tilde{W}_{k_0}]\cdots]](z, t);$$
$$1\leqq k_0\leqq d, 0\leqq k_i\leqq d, 1\leqq i\leqq j, j=0, 1, \cdots\},$$

where $\tilde{W}_k=W_k$ if $k=1, 2, \cdots, d$ and $=W_0+\partial/\partial t$ if $k=0$. If the equality

$$(\pi_*)_{(z,0)}[L(z, 0)]=\boldsymbol{R}^n$$

holds for any $z\in\boldsymbol{R}^n$ with the natural projection $\pi:[0, \infty)\times\boldsymbol{R}^n\to\boldsymbol{R}^n$, then the distribution of the solution Z_t has a smooth density function. Applying this sufficient condition to $W_k(x, t)=V_k[x, \pi(t)]$, the same result as Theorem (3.26) can be obtained. From this point of view, what is shown in this section is how the ordinary Malliavin calculus on $\boldsymbol{R}^n$ is carried out directly to the time-inhomogeneous equations like McKean equations not by handling them on the manifold $[0, \infty)\times\boldsymbol{R}^n$ as is done in the argument of partial hypoellipticity.

4. Application to the original McKean equations.

In this section, the results of the previous section are applied to the original type of McKean equations of Section 2, that is, the equation (2.1) with

$$(4.1) \qquad V_i[x, p]=\int_{R^{n\times m}} V_i(x, y_1, \cdots, y_m)p^{\otimes m}(dy_1, \cdots, dy_m), \qquad i=1, \cdots, d,$$

where $x\in \boldsymbol{R}^n$, $p\in \mathcal{P}(\boldsymbol{R}^n)$ and $m\in \boldsymbol{N}$.

Since the above $V_i[\cdot, \cdot]$'s belong to the class $C^\infty(\boldsymbol{R}^n\times \mathcal{M}(\boldsymbol{R}^n); \boldsymbol{R}^n)$, Theorem (3.26) holds also in this case, and the operator Q can be written in more explicit forms.

Let us now introduce the classes of functions in $C^\infty(\boldsymbol{R}^n\times \mathcal{M}(\boldsymbol{R}^n); \boldsymbol{R}^n)$ which have the expression like (4.1) with some $m\in \boldsymbol{N}$. That is,

$$(4.2) \qquad \mathcal{I}^m(\boldsymbol{R}^n)$$
$$=\Big\{ V\in C^\infty(\boldsymbol{R}^n\times \mathcal{M}(\boldsymbol{R}^n); \boldsymbol{R}^n);\ \text{with some } V\in C_b^\infty(\boldsymbol{R}^{n\times(m+1)}; \boldsymbol{R}^n),$$
$$V[x, p]=\int_{R^{n\times m}} V(x, y_1, \cdots, y_m)p^{\otimes m}(dy_1, \cdots, dy_m), x\in \boldsymbol{R}^n, p\in \mathcal{P}(\boldsymbol{R}^n)\Big\}.$$

For a vector $V[\cdot, \cdot]\in \mathcal{I}^m(\boldsymbol{R}^n)$, its Fréchet derivative is

$$(4.3) \qquad \mathcal{D}V[x, p](p')$$
$$=\sum_{k=1}^m \int_{R^{n\times m}} V(x, y_1, \cdots, y_m)p^{\otimes(k-1)}\otimes p'\otimes p^{\otimes(m-k)}(dy_1, \cdots, dy_m)$$

and so,

$$(4.4) \qquad \mathcal{F}V[x, z, p]=\sum_{k=1}^m \int_{R^{n\times(m-1)}} V(x, y_1, \cdots, y_{k-1}, z, y_{k+1}, \cdots, y_m)$$
$$\times p^{\otimes(m-1)}(dy_1, \cdots, dy_{k-1}, dy_{k+1}, \cdots, dy_m).$$

In this way, $\mathcal{F}^j V[x, z_1, \cdots, z_j, p], j\leqq m$ can also be defined as the sum of all the vectors obtained by integrating $V(x, y_1, \cdots, y_m)$ in $(m-j)$ parameters out of $\{y_1, \cdots, y_m\}$ with respect to p and replacing the remaining parameters by $z_k, k=1, \cdots, j$ in arbitrary orders.

In view of (3.12) and (3.13), we have

$$(4.5) \quad \frac{d}{dt}V[x, \pi(t)]=(QV)[x, \pi(t)]$$

$$=\int_{R^n} L_z[\pi(t)]\mathcal{F}V[x,z,\pi(t)]\pi(t)(dz)$$

$$=\sum_{k=1}^{m}\int_{R^{n\times m}}(L_k[\pi(t)]V)(x,y_1,\cdots,y_m)\pi(t)^{\otimes m}(dy_1,\cdots,dy_m),$$

where $L_k[\pi(t)]$ acts on $V_i(x, y_1, \cdots, y_m)$ with respect to the k-th parameter y_k. Since QV_i again belongs to $\mathcal{J}^{m'}(\boldsymbol{R}^n)$ for some $m'\in\boldsymbol{N}$, we can let Q operate as many times as desired. However, the number of integrations (denoted by m) increases at each stage, for $L_k[\pi(t)]$ itself contains $\pi(t)$.

In order to see how the actual calculation is carried out, let us consider the case that all the $V_k[\cdot,\cdot], k=0,1,\cdots,d$ in the McKean equation (2.1) belong to the class $\mathcal{J}^1(\boldsymbol{R}^n)$. In this case,

$$V_k[x,p]=\int_{R^n}V_k(x,y)p(dy),\quad k=0,1,\cdots,d \qquad \text{for } p\in\mathcal{M}(\boldsymbol{R}^n)\,, \tag{4.6}$$

and from (4.3) and (4.4),

$$\mathcal{D}V_k[x,p](p')=\int_{R^n}V_k(x,y)p'(dy)\,, \tag{4.7}$$

$$\mathcal{F}V_k[x,z,p]=V_k(x,z) \qquad \text{independent of } p\in\mathcal{M}(\boldsymbol{R}^n)\,. \tag{4.8}$$

Now, let us proceed to the calculation of $V_{(\alpha)}[x,p]$, $\alpha\in\mathcal{A}$ for $V\in\mathcal{J}^1(\boldsymbol{R}^n)$. But to do so, it is necessary for us to get the explicit form of the operator $L[p]$ when $p\in\mathcal{P}(\boldsymbol{R}^n)$.

$$\begin{aligned}L[p]&=\frac{1}{2}\sum_{k=1}^{d}V_k[y,p]^2+V_0[y,p]\\&=\frac{1}{2}\sum_{i,j=1}^{n}\Big(\sum_{k=1}^{d}V_k^i[y,p]V_k^j[y,p]\Big)\frac{\partial^2}{\partial y_i\partial y_j}\\&\quad+\sum_{j=1}^{n}\Big(\frac{1}{2}\sum_{k=1}^{d}\sum_{i=1}^{n}V_k^i[y,p]\frac{\partial V_k^j}{\partial y_i}[y,p]+V_0^j[y,p]\Big)\frac{\partial}{\partial y_j}\\&=\int_{R^{2n}}\Big\{\frac{1}{2}\sum_{i,j=1}^{n}\Big(\sum_{k=1}^{d}V_k^i(y,u)V_k^j(y,v)\Big)\frac{\partial^2}{\partial y_i\partial y_j}\\&\quad+\sum_{j=1}^{n}\Big(\frac{1}{2}\sum_{k=1}^{d}\sum_{i=1}^{n}V_k^i(y,u)\frac{\partial V_k^j}{\partial y_i}(y,v)+V_0^j(y,u)\Big)\frac{\partial}{\partial y_j}\Big\}p^{\otimes 2}(du,dv)\,.\end{aligned} \tag{4.9}$$

For simplicity, let the last expression be denoted by

$$\int_{R^{2n}}\{A(y,u,v)\partial^2+B(y,u,v)\partial\}p^{\otimes 2}(du,dv)\,, \tag{4.10}$$

and using this notation, define a linear map $\mathcal{R}: C^\infty(\boldsymbol{R}^{n\times(m+1)}; \boldsymbol{R}^n)\to C^\infty(\boldsymbol{R}^{n\times(m+3)}; \boldsymbol{R}^n)$ by

$$(4.11)\qquad (\mathcal{R}V)(x, y_1, \cdots, y_m, u, v) = \sum_{k=1}^{m} \{A(y_k, u, v)\partial_k^2 V(x, y_1, \cdots, y_m) + B(y_k, u, v)\partial_k V(x, y_1, \cdots, y_m)\}.$$

Then, for each $V\in\mathcal{J}^m(\boldsymbol{R}^n)$,

$$(4.12)\qquad QV[x, p] = \int_{\boldsymbol{R}^{n\times(m+2)}} \mathcal{R}V(x, y_1, \cdots, y_m, u, v)p^{\otimes(m+2)}(dy_1, \cdots, dy_m, du, dv),$$

that is, $QV\in\mathcal{J}^{m+2}(\boldsymbol{R}^n)$.

Suppose that for $\alpha\in\mathcal{A}$ and $V\in\mathcal{J}^1(\boldsymbol{R}^n)$, $V_{(\alpha')}$ belongs to $\mathcal{J}^m(\boldsymbol{R}^n)$ and

$$(4.13)\qquad V_{(\alpha')}[x, p] = \int_{\boldsymbol{R}^{n\times m}} W(x, y_1, \cdots, y_m)p^{\otimes m}(dy_1, \cdots, dy_m)$$

with some $W\in C_b^\infty(\boldsymbol{R}^{n\times(m+1)}; \boldsymbol{R}^n)$. (For $\alpha=(\alpha_1, \cdots, \alpha_l)\in\mathcal{A}$, $\alpha'=(\alpha_1, \cdots, \alpha_{l-1})$ and $\alpha_*=\alpha_l$.) If $\alpha_*\neq 0$, then

$$\begin{aligned} V_{(\alpha)} &= [V_{\alpha_*}, V_{(\alpha')}][x, p] \\ &= V_{(\alpha')}^{(1)}[x, p]V_{\alpha_*}[x, p] - V_{\alpha_*}^{(1)}[x, p]V_{(\alpha')}[x, p] \\ &= \int_{\boldsymbol{R}^{n\times(m+1)}} \{W^{(1)}(x, y_1, \cdots, y_m)V_{\alpha_*}(x, z) - V_{\alpha_*}^{(1)}(x, z)W(x, y_1, \cdots, y_m)\} \\ &\qquad \times p^{\otimes(m+1)}(dy_1, \cdots, dy_m, dz), \end{aligned}$$

where $W^{(1)}$ and $V_{\alpha_*}^{(1)}$ are derivatives in the first parameter and so, $V_{(\alpha)}\in\mathcal{J}^{m+1}(\boldsymbol{R}^n)$. If $\alpha_*=0$, then

$$V_{(\alpha)} = [V_0, V_{(\alpha')}][x, p] + (QV_{(\alpha')})[x, p]$$

and since $[V_0, V_{(\alpha')}]\in\mathcal{J}^{m+1}(\boldsymbol{R}^n)$ and $QV_{(\alpha')}\in\mathcal{J}^{m+2}(\boldsymbol{R}^n)$, $V_{(\alpha)}$ belongs to $\mathcal{J}^{m+2}(\boldsymbol{R}^n)$. Thus, we have the following proposition as a conclusion.

(4.14) PROPOSITION. *If $V_k[\cdot, \cdot], k=0, 1, \cdots, d$ belong to $\mathcal{J}^1(\boldsymbol{R}^n)$, then, for each $V\in\mathcal{J}^1(\boldsymbol{R}^n)$, the vector $V_{(\alpha)}$ defined in* (3.14) *is an element of $\mathcal{J}^{\|\alpha\|+1}(\boldsymbol{R}^n)$. Suppose further that $V_{(\alpha')}[x, p]$ has the expression* (4.13). *Then $V_{(\alpha)}$ is calculated explicitly:*

$$(4.15)\quad V_{\langle\alpha\rangle}[x,p] = \begin{cases} \int_{R^{n\times(m+1)}}\{W^{(1)}(x,y_1,\cdots,y_m)V_{\alpha_*}(x,z)-V^{(1)}_{\alpha_*}(x,z)W(x,y_1,\cdots,y_m)\} \\ \quad\times p^{\otimes m+1}(dy_1,\cdots,dy_m,dz) \qquad if\ \alpha_*\neq 0, \\ \int_{R^{n\times(m+2)}}\{W^{(1)}(x,y_1,\cdots,y_m)V_0(x,u)-V^{(1)}_0(x,u)W(x,y_1,\cdots,y_m) \\ \quad +\mathcal{R}W(x,y_1,\cdots,y_m,u,v)\}p^{\otimes m+2}(dy_1,\cdots,dy_m,du,dv) \quad if\ \alpha_*\neq 0, \end{cases}$$

where $\mathcal{R}$ *is the operator defined in* (4.11).

Next, assume in addition that the initial distribution $q=\delta_\xi$ with some fixed point $\xi\in\boldsymbol{R}^n$. In this case, the McKean equation (2.1) takes the following form.

$$(4.16)\quad X(t,\theta)=\xi+\sum_{i=1}^{d}\int_0^t V_i[X(s,\theta),\pi(s)]\circ d\theta_i(s)+\int_0^t V_0[X(s,\theta),\pi(s)]ds,$$

where $\pi(0,dy)=q(dy)=\delta_\xi(dy)$. With the help of Proposition (4.14), very simple expression of $V_{\langle\alpha\rangle}[x,q]$ can be obtained.

(4.17) LEMMA. *Let* $V\in\mathcal{J}^1(\boldsymbol{R}^n)$ *and* $q=\delta_\xi$ *with* $\xi\in\boldsymbol{R}^n$. *If* $V_{\langle\alpha'\rangle}[x,p]$ *is expressed as in* (4.13) *for any* $p\in\mathcal{P}(\boldsymbol{R}^n)$, *then*

$$(4.18)\quad V_{\langle\alpha\rangle}[x,q] = \begin{cases} W^{(1)}(x,\xi,\cdots,\xi)V_{\alpha_*}(x,\xi)-V^{(1)}_{\alpha_*}(x,\xi)W(x,\xi,\cdots,\xi) \qquad if\ \alpha_*\neq 0, \\ W^{(1)}(x,\xi,\cdots,\xi)V_0(x,\xi)-V^{(1)}_0(x,\xi)W(x,\xi,\cdots,\xi)+\mathcal{R}W(x,\xi,\cdots,\xi) \\ \qquad\qquad if\ \alpha_*=0. \end{cases}$$

Noting that $\operatorname{supp} q=\{\xi\}$ is compact, we can see that the following corollary of Theorem (3.26) (ii) holds.

(4.19) COROLLARY. *For* $V\in\mathcal{J}^1(\boldsymbol{R}^n)$, *define* $V_{\langle\alpha\rangle}[x,q]$ *inductively for every* $\alpha\in\mathcal{A}$ *as in Lemma* (4.17). *If* $\operatorname{span}\{(V_k)_{\langle\alpha\rangle}[\xi,q];k=1,\cdots,d$ *and* $\alpha\in\mathcal{A}\}=\boldsymbol{R}^n$, *then the solution* $X(t,\theta)$ *to the McKean equation* (4.16) *has a smooth density function.*

(4.20) REMARK. The above argument is easily extended to the case $V_k\in\mathcal{J}^m(\boldsymbol{R}^n)$, $k=0,1,\cdots,d$ with $m\geqq 2$. In that case, $V_{\langle\alpha\rangle}$ belongs to the class $\mathcal{J}^{m(\|\alpha\|+1)}(\boldsymbol{R}^n)$ for $V\in\mathcal{J}^m(\boldsymbol{R}^n)$.

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