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Codimension one foliations of S^3 with only one compact leaf

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§1. Introduction

In this paper, we are concerned with codimension one C^2 foliations of the 3-dimensional sphere S^3 with only one compact leaf.

By Novikov's theorem ([8]), foliations of S^3 contain Reeb components. If a foliation has only one compact leaf, then it is the boundary of a Reeb component. The Reeb component is unique if it is knotted. By Kopell's lemma ([5]), there exists a solid torus containing the Reeb component in its interior such that the restriction of the foliation to its boundary is isomorphic to a linear foliation. Thus we restrict our attention to the foliations $\mathcal{F}$ of knot complements M satisfying the following conditions:

(1) $\mathcal{F}$ is a transversely orientable codimension one C^2 foliation of M transverse to the boundary.

(2) $\mathcal{F}$ has no interior compact leaves.

(3) $\mathcal{F}|\partial M$ is isomorphic to a linear foliation.

Gabai constructed such foliations $\mathcal{F}$ for many kinds of knots ([2], [3], [4]). Since he used sutured manifold hierarchies in the construction, the foliations constructed by him have compact leaves with boundary. When such foliation $\mathcal{F}$ has a compact leaf, the compact leaf is a minimal genus Seifert surface ([14]). Hence the structure of $\mathcal{F}$ is strongly restricted. For example, when M is a fibered knot complement, a minimal genus Seifert surface is unique and the foliation obtained by cutting $\mathcal{F}$ along a compact leaf is a foliated I-bundle.

We give a classification of all foliations of torus knot complement satisfying (1), (2), (3).

When M is a torus knot complement, M is decomposed into two solid tori E_1 and E_2 by cutting along an annulus. A foliation $\mathcal{F}$ of M is said to be standard if $\mathcal{F}$ is transverse to both ∂E_1 and ∂E_2, and $\mathcal{F}|E_1$ and $\mathcal{F}|E_2$ are isomorphic to the product foliations $\{\{*\}\times D^2;\ *\in S^1\}$.

THEOREM. *Let $\mathcal{F}$ be a foliation of a torus knot complement M satis-*

fying (1), (2), (3). *Then* $\mathcal{F}$ *is isomorphic to a standard foliation.*

We prove Theorem in Section 3. In Section 2, a precise definition of standard foliations is given, and we prove the following proposition.

PROPOSITION. *There exists a one-parameter family* $\mathcal{F}_\theta$ $(0<\theta<1)$ *of standard foliations without compact leaves satisfying* (1), (2), (3) *and that*

(a) *If* θ *is rational, then all the leaves of* $\mathcal{F}_\theta|\partial M$ *are compact.*

(b) *If* θ *is irrational, then* $\mathcal{F}_\theta|\partial M$ *has no compact leaves.*

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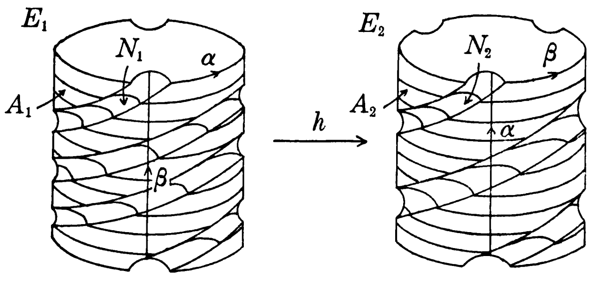
§ 2. The construction of foliations of torus knot complements

Let W_1 be an unknotted solid torus in S^3, and W_2, the closure of the complement of W_1 in S^3. Denote by α and β the meridians of W_1 and W_2, respectively. A simple closed curve in ∂W_1 homotopic to $[\alpha]^m\cdot[\beta]^n\in\pi_1(\partial W_1)$ is called a *torus knot of type* (m, n), denoted by $k(m,n)$, where m and n $(|m|\geqq 2,\ |n|\geqq 2)$ are relatively prime integers. (Furthermore, we assume that $k(m,n)$ is transverse to $\{*\}\times\partial D^2$ of W_1 and W_2 for any $*\in S^1$ when W_1 and W_2 are parametrized by $S^1\times D^2$.)

Let N be a tubular neighborhood of $k(m,n)$. S^3-N is called a *torus knot complement*, denoted by M. Let E_i denote W_i-N for $i=1,2$. We denote $\partial E_i\cap\partial\bar{N}$ by N_i and $\overline{\partial E_i-N_i}$ by A_i for $i=1,2$. Then N_i and A_i are annuli (Figure 1).



$k(m,n)$ intersects the meridian α n times.

$k(m,n)$ intersects the meridian β m times.

Figure 1.

Let F_i be the product foliation of E_i for $i=1, 2$. Then $F_i|A_i$ is the foliation whose leaves are all diffeomorphic to intervals. Therefore there exists a diffeomorphism $h: A_1 \to A_2$ which preserves the leaves of $F_1|A_1$ and $F_2|A_2$. The foliation constructed by attaching F_1 to F_2 by h is called a *standard foliation* of M, denoted by $\mathcal{F}_h$.

The qualitative property of leaves of $\mathcal{F}_h$ can be seen as follows. By reducing each leaf of $F_1|A_1$ to a point, we regard h as a diffeomorphism of S^1 ($\cong \boldsymbol{R}/\boldsymbol{Z}$). For $0 \leqq \theta < 1$, we define the rotation $R(\theta) \in \mathrm{Diff}(S^1)$ by $R(\theta)(x) = x + \theta \bmod \boldsymbol{Z}$. Let f and $g \in \mathrm{Diff}(S^1)$ denote $h^{-1}R(1/m)h$ and $R(1/n)$, respectively. We denote by G the subgroup of $\mathrm{Diff}(S^1)$ generated by f and g. We define the *orbit* $O(p)$ *of* G *passing through* p by $\{\gamma(p); \gamma \in G\}$. Then the qualitative property of orbits of G gives us information about that of leaves of $\mathcal{F}_h$. For example, the existence of a compact leaf of $\mathcal{F}_h$ is equivalent to the existence of a finite orbit of G.

A *linear foliation* is a foliation of T^2 whose leaves wind around T^2 with constant slope. Two foliations are *isomorphic* if there exists a homeomorphism mapping each leaf to a leaf.

$\mathcal{F}_h|\partial M$ is the foliation of T^2 obtained from the product foliation ($S^1 \times [0, 1]$, $\{*\} \times [0, 1]$) by identifying $S^1 \times \{0\}$ and $S^1 \times \{1\}$ by gf. Therefore a standard foliation satisfies the condition (3) if and only if gf is topologically conjugate to a rotation.

Next we construct examples of standard foliations. To simplify the explanation, we assume that $k(m, n)$ is a trefoil knot, that is, $m=3$ and $n=2$.

$\mathrm{PSL}(2, \boldsymbol{R})$ acts on the Poincaré disk D^2 and also on its boundary. Hence $\mathrm{PSL}(2, \boldsymbol{R})$ is considered as a subgroup of $\mathrm{Diff}(S^1)$.

Let h be an element of $\mathrm{PSL}(2, \boldsymbol{R})$ and construct $\mathcal{F}_h$. Denote by O the origin of D^2 and put $A = h^{-1}(O)$. We denote by u the geodesic passing through O and vertical to the segment AO (when $A \neq O$). Denote by l the geodesic passing through A and forming an angle of -60 degrees with the segment AO (when $A \neq O$). Furthermore, let l' denote the geodesic axially symmetric to l with respect to u.

G is classified into the following four cases according to the distance between O and A (Figure 2).

1) $A=O$. G is generated by $R(1/6)$. Since all the orbits of G are finite, all the leaves of $\mathcal{F}_h$ are compact and $\mathcal{F}_h$ satisfies the condition (3). This implies the well-known fact that a torus knot is a fibered knot.

2) *l does not intersect u.* Since gf maps l onto l', gf is a hyperbolic transformation and G has an exceptional minimal set. Thus $\mathcal{F}_h$ has an

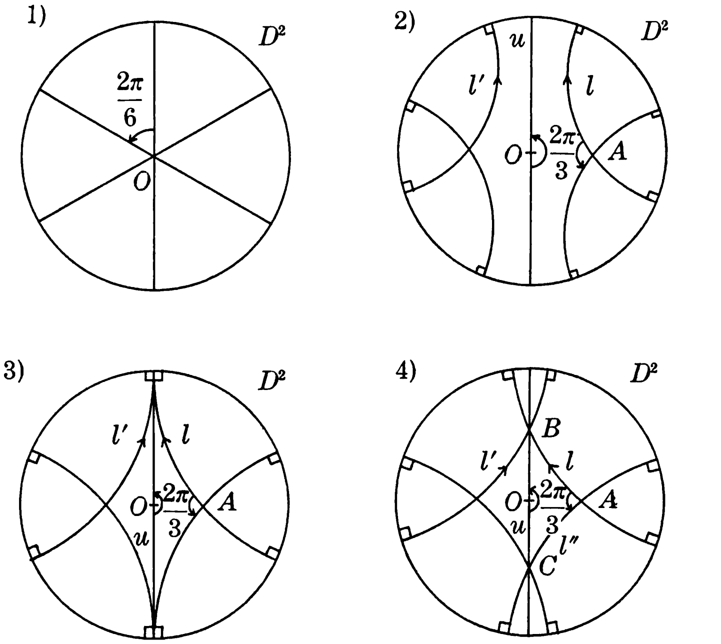


Figure 2.

exceptional minimal set. Since gf has exactly two fixed points, $\mathcal{F}_h$ does not satisfy the condition (3).

3) *l intersects u in ∂D^2.* Since gf maps l onto l', gf is a parabolic transformation and all the orbits of G are dense in S^1. Since gf has a unique fixed point, $\mathcal{F}_h$ is a minimal foliation which does not satisfy the condition (3).

4) *l intersects u in the interior of D^2.* Let B be the intersection point of u and l. Let l'' denote the geodesic passing through A and forming an angle of 60 degrees with the segment AO. Denote by C the intersection point of u and l''. Then gf is an elliptic transformation which fixes B and maps l onto l'.

PROOF OF PROPOSITION. Let h be the element of PSL$(2, \boldsymbol{R})$ such that l intersects u in the interior of D^2 (case 4)) and the angle of $\angle ABO$ is equal to $\pi\theta/6$ for $0<\theta<1$. Denote by $\mathcal{F}_\theta$ the standard foliation $\mathcal{F}_h$. Then $\mathcal{F}_\theta$ satisfies the conditions (1), (2).

First assume that θ is irrational. Then all the orbits of G are dense in S^1 and the rotation number of gf is irrational. Hence $\mathcal{F}_\theta$ is a minimal foliation such that $\mathcal{F}_\theta|\partial M$ has no compact leaves.

Next assume that θ is equal to $6/p$ for some integer p ($p \geqq 7$). Then G is the Fuchsian group of the first kind whose fundamental region is the triangle ABC. Therefore all the orbits of G are dense. Since $(gf)^p$ is an identity map, $\mathcal{F}_\theta$ is a minimal foliation such that all the leaves of $\mathcal{F}_\theta|\partial M$ are compact.

More generally, when θ is rational, $\mathcal{F}_\theta$ is a minimal foliation such that all the leaves of $\mathcal{F}_\theta|\partial M$ are compact. ■

REMARK. B. Raymond constructed codimension 1 real analytic foliations, say $\mathcal{F}$, of torus knot complements M satisfying the following conditions by making use of the Fuchsian group of the second kind ([9]):

1) $\mathcal{F}$ has an exceptional minimal set.
2) $\mathcal{F}$ is transverse to the natural Seifert fibration of M.
3) The holonomy of $\mathcal{F}|\partial M$ is a hyperbolic element of PSL$(2, \boldsymbol{R})$.

The author does not know whether there exists a standard foliation with exceptional minimal sets satisfying the condition (3) of Theorem or not. This is closely related to the differentiability of foliations.

§ 3. The structure of foliations of torus knot complements

In this section we prove the main theorem.

Since M is a torus knot complement, there exists a properly embedded annulus A in M such that the manifold obtained by cutting M along this annulus A is a union of two solid tori E_1 and E_2. For $i=1, 2$, let N_i and A_i denote $\partial M \cap E_i$ and $\overline{\partial E_i - N_i}$, respectively. By the condition (3), the annulus A can be taken so that both of the two connected components of ∂A are transverse to $\mathcal{F}$ or both of them are tangent to $\mathcal{F}$. When ∂A is transverse to $\mathcal{F}$, all the leaves of $\mathcal{F}|N_1$ and $\mathcal{F}|N_2$ are properly embedded arcs, and, when ∂A is tangent to $\mathcal{F}$, all of them are circles.

Since the annulus A is an incompressible surface in M, the following Lemma 1 holds by Roussarie's theorem ([10]).

LEMMA 1. *The annulus A can be deformed by isotopy so that the annulus A satisfies one of the following conditions* a) — c):

a) *A is a leaf of $\mathcal{F}$ and all the leaves of $\mathcal{F}|N_1$ and $\mathcal{F}|N_2$ are circles.*
b) *A is transverse to $\mathcal{F}$, $\mathcal{F}|A$ is a foliation tangent to the boundary, and all the leaves of $\mathcal{F}|N_1$ and $\mathcal{F}|N_2$ are circles.*
c) *A is transverse to $\mathcal{F}$, $\mathcal{F}|A$ is a foliation transverse to the*

boundary, and all the leaves of $\mathcal{F}|N_1$ and $\mathcal{F}|N_2$ are properly embedded arcs.

LEMMA 2. *The cases other than* c) *cannot occur.*

PROOF. *In the case* a). All the leaves of $\mathcal{F}|N_1$ are circles and A_1 is one annular leaf of $\mathcal{F}|E_1$. This contradicts the fact that $\mathcal{F}|E_1$ is transversely orientable. Thus the case a) cannot occur.

In the case b). Let L be a leaf of $\mathcal{F}$ which intersects ∂A. Then $E_1 \cap L$ or $E_2 \cap L$ contains a connected component diffeomorphic to a circle. This connected component is called a *singular loop* (Figure 3).

Let γ be a singular loop of $\mathcal{F}|E_1$. Since $\mathcal{F}|N_1$ is a product foliation, the leaves of $\mathcal{F}|E_1$ close to γ are annuli. If an annular leaf does not have holonomy, then the nearby leaves are annuli. Hence there exists a continuous map $\phi_\gamma : \Sigma \times [0, 1] \to E_1$ satisfying the following conditions ([10]):

1) $\phi_\gamma(\Sigma \times \{0\}) = \gamma$ and $\phi_\gamma|(\Sigma \times (0, 1))$ is an embedding.
2) $\phi_\gamma(\Sigma \times \{t\})$ is an annular leaf of $\mathcal{F}|E_1$ for each $t \in (0, 1)$.
3) $\phi_\gamma(\Sigma \times \{1\})$ is either an annular leaf with holonomy or a singular loop,

where Σ is an annulus. Let L_γ denote $\phi_\gamma(\Sigma \times \{1\})$.

$\mathcal{F}|E_1$ has at most two singular loops. First assume that $\mathcal{F}|E_1$ has one singular loop γ. Then L_γ is an annular leaf with holonomy. Since $\mathcal{F}|N_1$ has no holonomy, $\phi_\gamma(\Sigma \times [0, 1])$ contains N_1. The foliation $\mathcal{F}|E_2$ also has one singular loop γ'. We have $\phi_{\gamma'}(\Sigma \times [0, 1])$ as above. Then $L_{\gamma'}$ is also an annular leaf with holonomy and $\phi_{\gamma'}(\Sigma \times [0, 1])$ contains N_2. Hence $L_\gamma \cup L_{\gamma'}$ is a toral leaf of $\mathcal{F}$ (Figure 3). This contradicts the assumption (2).

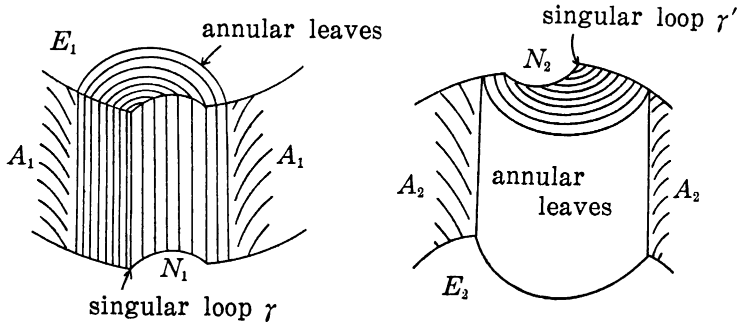


Figure 3.

Next we assume that $\mathcal{F}|E_1$ has two singular loops γ and γ'. Since $\mathcal{F}|N_1$ has no holonomy, L_γ is the singular loop γ'. Hence all the leaves of $\mathcal{F}|A_1$ are circles. Since $\mathcal{F}|E_2$ is a foliation of $S^1\times D^2$ without singular loops and Reeb components, $\mathcal{F}|E_2$ has no null-homotopic closed transversals. Hence a compact leaf of $\mathcal{F}|\partial E_2$ bounds a disk in E_2, and $k(m,n)$ also bounds a disk in S^3. This contradicts the non-triviality of the torus knot.

In the case where $\mathcal{F}|E_1$ has no singular loops, $\mathcal{F}|E_2$ has two singular loops. Therefore this case cannot occur as above.

Thus the case b) cannot occur. ∎

LEMMA 3. *In the case* c), *the annulus A can be taken so that either $\mathcal{F}|E_1$ is a product foliation or all the compact leaves of $\mathcal{F}|E_1$ are parallel to N_1.*

PROOF. First we recall some properties of foliations of $S^1\times D^2$. Let F be a transversely orientable foliation of $S^1\times D^2$ transverse to the boundary and without Reeb components. By results of Thurston ([13]) and Levitt ([6]), F contains a half Reeb component ([12]) if F is not a product foliation.

Furthermore, by Thurston's theorem ([13]), there exist compact 3-submanifolds C_i $(i=1,2,3,\cdots,m)$ with corner satisfying the following conditions:

1) C_i $(i=1,2,3,\cdots,m)$ is obtained by cutting $S^1\times D^2$ along $m-1$ compact leaves.
2) Each $F|C_i$ satisfies the following condition either a) or b).
   a) C_i is diffeomorphic to $\Sigma\times[0,1]$, and $\Sigma\times\{0\}$ and $\Sigma\times\{1\}$ are compact leaves of $F|C_i$. Furthermore, each $\{*\}\times[0,1]$ $(*\in\Sigma)$ is transverse to $F|C_i$.
   b) All the compact leaves of $F|C_i$ are entirely contained in ∂C_i,

where Σ is an annulus. C_i is referred to as a *component* of F and a component of type a) is called an *I-bundle component*.

We return to the proof of Lemma 3. Assume that $\mathcal{F}|E_1$ is not a product foliation and the annulus A is deformed by isotopy with the boundary fixed so that the number of components of $\mathcal{F}|E_1$ is minimal. Let H denote a half Reeb component of $\mathcal{F}|E_1$.

First assume that H is disjoint from N_1. Denote by C_i the next I-bundle component to the half Reeb component H if it exists, where the next component to C_i except for H is not an I-bundle component. Let H' denote $H\cup C_i$. We remove a small neighborhood H'' of H' from E_1 so that the restriction of $\mathcal{F}|E_1$ to $\partial H''$ is a product foliation whose leaves are properly embedded arcs (Figure 4). Next we attach H'' to E_2. In this process, the number of components of $\mathcal{F}|E_1$ decreases. Thus the annulus A can be taken so that the number of components of $\mathcal{F}|E_1$ decreases and this contradicts the assumption. Therefore H intersects N_1. The annular leaf L of H is *parallel* to N_1, that is there exists a submanifold homeomorphic to $S^1\times I\times I$ satisfying $S^1\times I\times\{0\}=L$ and $S^1\times I\times\{1\}=N_1$.

Next we remove a neighborhood of H from E_1 as above. By the above consideration, the remaining foliation has a half Reeb component whose annular leaf is also parallel to N_1, if it is not a product foliation. Thus all the compact leaves of $\mathcal{F}|E_1$ are parallel to N_1 by induction with respect to the number of components of $\mathcal{F}|E_1$. ■

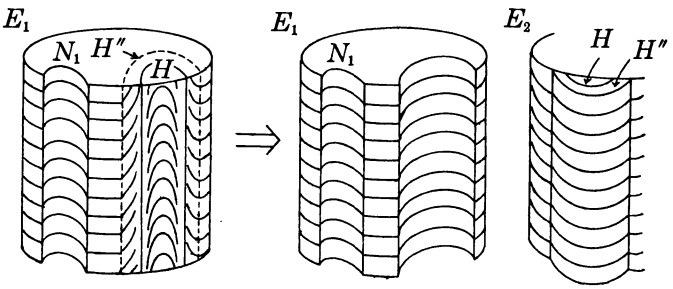


Figure 4.

LEMMA 4. *In the case* c), *if all the compact leaves of* $\mathcal{F}|E_1$ *are parallel to* N_1, *then the number of components of* $\mathcal{F}|E_1$ *decreases by changing the annulus* A *by isotopy.*

PROOF. Since all the compact leaves of $\mathcal{F}|E_1$ are parallel to N_1, there exists a component C_j of $\mathcal{F}|E_1$ which is farthest from N_1 (C_j is not an I-bundle component). Denote by B the annulus $C_j\cap\partial E_2$. Let ∂_+A_2 be one of the connected components of ∂A_2 and let ∂_-A_2 be the other connected component of ∂A_2. Let c denote the compact leaf of $\mathcal{F}|A_2$ nearest to ∂_+A_2. Then c is contained in an annular leaf of $\mathcal{F}|E_2$ and this annular leaf with ∂E_2 bounds a solid torus S. Then there are three posibilities (Figure 5):

1) S contains neither N_2 nor B.
2) S contains B, but not N_2.
3) S contains N_2.

In the case 1). Since c is the nearest compact leaf of $\mathcal{F}|A_2$ to ∂_+A_2, the annular leaf $\overline{\partial S\cap \mathrm{Int}\, E_2}$ is an isolated compact leaf of $\mathcal{F}|E_2$ outside S. Furthermore $\mathcal{F}|(\partial E_2\cap\partial S)$ contains a 2-Reeb component because the orienta-

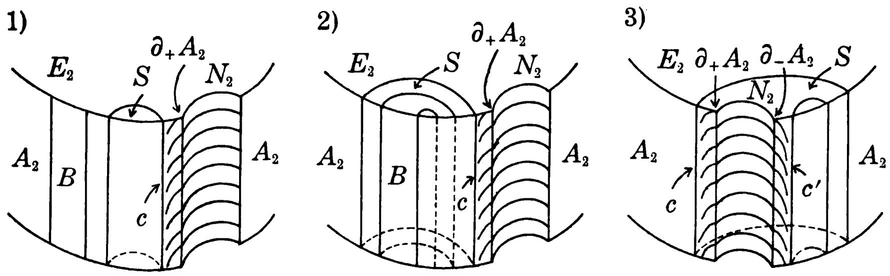


Figure 5.

tions of $\partial(\partial E_2 \cap \partial S)$ induced by the orientation of $\mathcal{F}|\partial E_2$ are opposite. The component of $\mathcal{F}|E_1$ containing this 2-Reeb component is the product of the 2-Reeb component and the interval.

Let S' be a small neighborhood of S such that $\mathcal{F}|(\overline{\partial S' \cap \operatorname{Int} E_2})$ is the product foliation $\{I \times \{*\}\,;\ * \in S^1\}$. We remove S' from E_2, and attach S' to E_1. Let S'' denote the union of components of the resulting foliation of E_1 containing S. The annular leaf of S'' contained in $\partial S''$ is isolated outside S''. Next we remove a small neighborhood of S'' from E_1, and attach this to E_2 as above (Figure 6).

Since at least one component of $\mathcal{F}|E_1$ which is isomorphic to the prod-

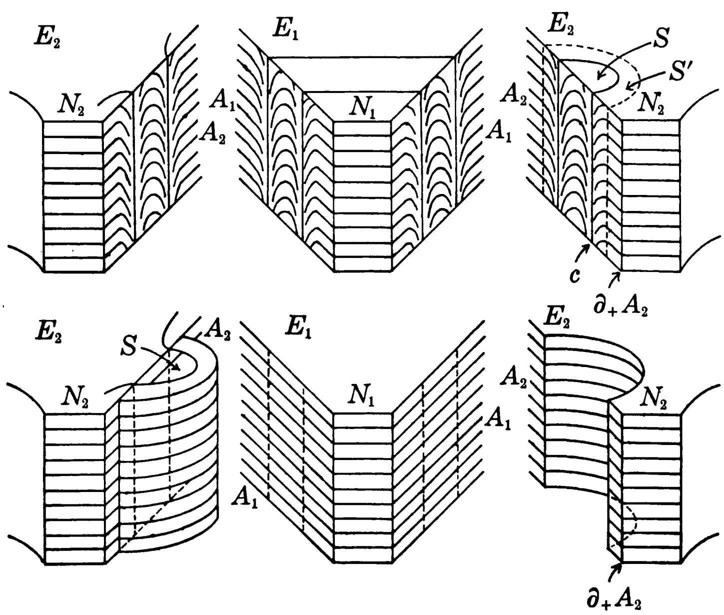


Figure 6.

uct of the 2-Reeb component and the interval vanishes in these processes, the number of components of $\mathcal{F}|E_1$ decreases.

In the case 2). Since $\mathcal{F}$ has no interior compact leaves, the leaves of $\mathcal{F}|E_2$ intersecting ∂B are two annular leaves. At least one of these annular leaves with ∂E_2 bounds a solid torus which contains neither B nor N_2. Thus the number of components of $\mathcal{F}|E_1$ decreases as in the case 1).

In the case 3). Let c' denote the compact leaf of $\mathcal{F}|A_2$ nearest to $\partial_- A_2$. The annular leaf of $\mathcal{F}|E_2$ containing c' does not contain c because $\mathcal{F}$ has no interior compact leaves. Hence the annular leaf containing c' bounds a solid torus with ∂E_2 which does not contain N_2. Thus the number of components of $\mathcal{F}|E_1$ decreases as above. ∎

In the case c), the annulus A can be taken so that $\mathcal{F}|E_1$ is a product foliation by Lemmas 3 and 4. If $\mathcal{F}|E_1$ is a product foliation, then $\mathcal{F}|\partial E_2$ contains no 2-Reeb components. Hence $\mathcal{F}|E_2$ is also a product foliation, and $\mathcal{F}$ is a standard foliation. Thus Theorem is proved.

REMARK. A torus knot complement is a Seifert fibered manifold. By Lemma 2, there exists a fiber transverse to $\mathcal{F}$. If $\mathcal{F}|\partial M$ contains no 2-Reeb components, $\mathcal{F}$ is isomorphic to a foliation transverse to every fiber by Eisenbud, Hirsch and Neumann's theorem ([1]) and Matsumoto's proof ([7]). Hence $\mathcal{F}|E_1$ and $\mathcal{F}|E_2$ are product foliations, and $\mathcal{F}$ is a standard foliation. We can also prove Theorem in this way.

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