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# *On the integration of involutive systems of non-linear partial differential equations*

Dedicated to Professor Hiroshi Fujita on his 60th birthday

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## Introduction

Extending the classical Darboux's method, we have given a method of integration in [7, 8], which enables us, if it may be successfully applied, to solve an involutive system of partial differential equations by integrating *ordinary* differential equations. It may be applied only to those systems of which Cartan characters of order $\geqq 2$ vanish. The principal aim of this paper is to extend it so that the method may be applied to some class of involutive systems of which Cartan characters of higher orders do not necessarily vanish. To do so, we must investigate the *analytic* structure, besides the algebraic one, of an involutive system in a quite different way from that in [7, 8].

*Throughout this paper, all notions such as functions, manifolds, vector bundles are assumed to be those of infinitely differentiable* (*smooth*) *ones*, unless expressly indicated otherwise.

Let us describe our method roughly. Let $\mathcal{R}_1$ be an involutive system of first order with $n$ independent variables. Denote its Cartan characters by $s_0, s_1, \cdots, s_n$. We assume that $s_1=\cdots=s_l>0$, $s_{l+1}=\cdots=s_n=0$ with $1\leqq l<n$. Let $M_P$ be the characteristic module of $\mathcal{R}_1$ at $P$. It is a homogeneous submodule of a finitely generated module $L_P$ over a polynomial ring with $n$ variables. Let $M_P=\bigcap_{j=1}^{\nu(P)} Q_{j,P}$ be an irredundant primary decomposition in $L_P$, $Q_{j,P}$ be $\mathfrak{P}_{j,P}$-primary. Then each $\mathfrak{P}_{j,P}$ is a homogeneous prime ideal of projective dimension $l-1$ (cf. Theorem 4.1). We assume moreover that each $\mathfrak{P}_{j,P}$ is generated by linear forms and the exponent of each $Q_{j,P}$ is equal to 1. Under the regularity conditions such as $\nu=\nu(P)=$const., the Monge characteristic system $\Delta^k(\mathfrak{P}_j)$ of order $k$ is well-defined corresponding to each family $\mathfrak{P}_j=\{\mathfrak{P}_{j,P}\}$ $(1\leqq j\leqq\nu)$. Each $\Delta^k(\mathfrak{P}_j)$ contains the Pfaffian subsystem $\Theta_k$ generated by contact forms of

orders $\leqq k$. We say that $\varDelta^k(\mathfrak{P}_j)$ is *principally integrable* if there exist its integrals $\{g_\gamma\}$ such that the Pfaffian system $\varDelta^k(\mathfrak{P}_j)$ is generated by the 1-forms $dg_\gamma$ and $\Theta_k$. Our method of integration is stated as follows (Theorem 6.2): *If, for some* $k\geqq 1$, $\nu-1$ *Monge characteristic systems* $\varDelta^k(\mathfrak{P}_1), \cdots, \varDelta^k(\mathfrak{P}_{\nu-1})$ *are principally integrable, then any local solution of* $\mathcal{R}_1$ *can be obtained by solving ordinary differential equations.* The most crucial procedure in obtaining solutions is to construct an involutive system admitting a given one as a solution and having Cauchy characteristics. This is carried out by applying Theorem 5.2, of which proof contains the most substantial analytic discussion in the paper.

The paper composed of six sections. §§1–2 are concerned with an involutive symbol and its characteristic module. §3 is devoted to recalling the notion of involutive differential systems, and §4 to introducing new notions related to the Monge characteristic systems and important in this paper. In §5, we give a method of constructing a new involutive systems. In §6, we establish our method of integration.

## §1. An involutive symbol and its characteristic module

Let $V$ and $E$ be real vector spaces of dimensions $m$ and $n$, respectively. As usual, we denote by $E^*$ the dual space to $E$, and by $S^kE^*$ the $k$-th symmetric product of $E^*$. By a *symbol of order* $k$, we shall mean a vector subspace of $V\otimes S^kE^*$. The *l-th prolongation* $p^lG_k$ of a symbol $G_k$ of order $k$ is defined by $p^lG_k=(G_k\otimes S^lE^*)\cap V\otimes S^{k+l}E^*$.

We first recall the notion of involutive symbols (cf. e.g. Kuranishi [14], §6). Let $G_1$ be a symbol of order 1; $G_1\subset V\otimes E^*=\mathrm{Hom}(E, V)$. Given a basis $\{e_1, \cdots, e_n\}$ of $E$, we denote by $E_i$ the subspace of $E$ spanned by $e_1, \cdots, e_i$; $E_0=\{0\}\subset E_1\subset\cdots\subset E_n=E$. Let $G_{1,i}$ be the subspace of $G_1$ consisting of those elements which annihilate any $e\in E_i$. The inequality $\dim pG_1\leqq\sum_{i=0}^{n}\dim G_{1,i}$ always holds. A symbol $G_1$ is said to be *involutive* if there exists a basis $\{e_1, \cdots, e_n\}$ such that $\dim pG_1=\sum_{i=0}^{n}\dim G_{1,i}$. In this case such a basis $\{e_1, \cdots, e_n\}$ is called a *regular basis for* $G_1$. When $G_1$ is involutive, the set of all regular bases for $G_1$ is an open dense subset of the manifold consisting of all bases of $E$. Furthermore the integers $g_i=\dim G_{1,i}$ $(0\leqq i\leqq n)$ are determined independently of the choice of a regular basis $\{e_1, \cdots, e_n\}$. The *Cartan characters* $s_i$ of an involutive symbol $G_1$ are defined by $s_i=g_{i-1}-g_i$ $(1\leqq i\leqq n)$. They are ordered by the inequalities $s_1 \geqq s_2 \geqq \cdots \geqq s_n \geqq 0$.

Let $D = \mathrm{Ann}(G_1)$ be the annihilator of $G_1$ in $V^* \otimes E$. Given a basis $\{e_1, \cdots, e_n\}$, we set $D_i = D \cap (V^* \otimes E_i)$.

LEMMA 1.1. *In order that a symbol $G_1$ of order 1 be involutive, it is necessary and sufficient that there exist a basis $\{e_1, \cdots, e_n\}$ of $E$ and a basis $\{u_1, \cdots, u_m\}$ of $V^*$ such that we can find a basis*

$$\{\phi_\alpha^1 \ (1 \leqq \alpha \leqq \kappa_1), \cdots, \phi_\alpha^n \ (1 \leqq \alpha \leqq \kappa_n)\}$$

*of $D$ satisfying the following four conditions:*

(i) $0 \leqq \kappa_1 \leqq \kappa_2 \leqq \cdots \leqq \kappa_n \leqq m$.

(ii) *For each $i = 1, \cdots, n$, $D_i$ is spanned by $D_{i-1}$ and $\phi_\alpha^i$ $(1 \leqq \alpha \leqq \kappa_i)$.*

(iii) *$\phi_\alpha^i$'s take the form: when $\kappa_{j-1} < \alpha \leqq \kappa_j$ (here $\kappa_0 = 0$),*

$$\phi_\alpha^i = u_\alpha \otimes e_i - \psi_\alpha^i \quad \textit{where } \psi_\alpha^i \in V^* \otimes E_{j-1}.$$

(iv) *The set $\{\phi_\alpha^i \circledcirc e_j;\ 1 \leqq j \leqq i \leqq n,\ 1 \leqq \alpha \leqq \kappa_i\}$ forms a basis of $D \circledcirc E$ (=the annihilator of $pG_1$ in $V^* \otimes S^2E$),* $\circledcirc$ being the symmetric product.

*Under the circumstances, the Cartan character $s_i$ is equal to $m - \kappa_i$ $(1 \leqq i \leqq n)$.*

Let us consider a symbol $G_k$ of order $k \geqq 1$. It defines canonically a symbol $\hat{G}_1 = G_k \subset (V \otimes S^{k-1}E^*) \otimes E^*$ of order 1. $G_k$ is called an *involutive* symbol if $\hat{G}_1$ is involutive in the above sense. The *Cartan characters* of $G_k$ are defined to be those of $\hat{G}_1$.

PROPOSITION 1.2 ([14] § 9). *If $G_k$ is involutive, then the prolongation $pG_k$ is also involutive.*

We shall now state some fundamental facts concerning the characteristic module of an involutive symbol. Let $R$ be the graded ring $R = \sum_{k=0}^{\infty} R_k$ where $R_k = S^kE$ (the symmetric algebra over $E$), and $L$ be the graded $R$-module $L = \sum_{k=0}^{\infty} L_k$ where $L_k = V^* \otimes S^kE$. We denote by $\mathfrak{X}$ the ideal $\sum_{k \geqq 1} R_k$ in $R$. The *characteristic module* of an involutive symbol $G_k$ is defined to be that smallest (homogeneous) submodule $M$ of $L$ which contains the annihilator of $G_k$ in $L_k$ and which possesses the property "$\mathfrak{X} z \subset M$ $(z \in L)$ implies $z \in M$" (Kakié [8]). The characteristic module of any prolongation $p^lG_k$ coincides with that of $G_k$.

In the sequel, we shall use the following notations and elementary facts in commutative algebra. Let $K=\sum_{j=0}^{\infty} K_j$ be a finitely generated graded $R$-module. The Hilbert characteristic polynomial of $K$ will be denoted by $P(K, x)$ (cf. Sérre [18], Zariski-Samuel [19]). We shall denote by $\mu(K)$ the leading coefficient of $P(K, x)$ divided by $r!$, where $r=\deg P(K, x)$.

The set of all associated prime ideals of $K$ will be denoted by $\mathrm{Ass}(K)$. For a submodule $H$ of $K$, we denote by $\mathfrak{r}_K(H)$ the ideal in $R$ consisting of those elements $f\in R$ which satisfy $f^q K\subset H$ for some positive integer $q=q(f)$.

LEMMA 1.3. *Let $H$ be a homogeneous submodule of $K$, and $H=\bigcap_{j=1}^{\nu} Q_j$ be an irredundant primary decomposition in $K$ with $Q_j$ being $\mathfrak{P}_j$-primary. Then the following are valid*:

(I) *The set $\{\mathfrak{P}_1, \cdots, \mathfrak{P}_\nu\}$ coincides with* $\mathrm{Ass}(K/H)$.

(II) $\max\{\text{proj dim}\,\mathfrak{P}_j;\ 1\leqq j\leqq \nu\}$ *is equal to* $r=\deg P(K/H, x)$, *and*

$$\mu(K/H)=\sum_{\text{proj dim}\,\mathfrak{P}_j=r} \mu(K/Q_j).$$

(III) *The transformation $h_f$ in $K/H$ defined by $h_f(z)=f\cdot z$ is injective if and only if $f\notin \bigcup_{j=1}^{\nu}\mathfrak{P}_j$.*

(IV) $\mathfrak{r}_K(H)=\bigcap_{j=1}^{\nu}\mathfrak{P}_j$.

PROOF. See the following references. (I): Bourbaki [1], §2; (II): Zariski-Samuel [19], Chap. VII, §12 and Kakié [8], §1; (III) and (IV): Bourbaki [1], §1.

Given an involutive symbol $G_k$, we denote by $M$ its characteristic module. Observe that $L$ is a finitely generated $R$-module, and hence a Noetherian module. Let $M=\bigcap_{j=1}^{\nu} Q_j$ be an irredundant primary decomposition in $L$, and $Q_j$ be $\mathfrak{P}_j$-primary. The Cartan characters of $G_k$ will be denoted by $s_1, \cdots, s_n$.

THEOREM 1.4. *If $G_k$ is an involutive symbol of order $k$, then the following are valid*:

(I) *Any ideal $\mathfrak{P}_j$ does not coincides with $\mathfrak{X}$.*

(II) *$M_{k+l}=M_k \circledcirc S^l E$ for any $l\geqq 0$, where $M_q=M\cap L_q$.*

(III) *If $s_1\geqq\cdots\geqq s_l>0$, $s_{l+1}=\cdots=s_n=0$, then*

(a) *the maximum of projective dimensions of the prime ideals* $\mathfrak{P}_j$ *is equal to* $l-1$, *and*

(b) $\sum_{\text{proj dim}\,\mathfrak{P}_j=l-1}\mu(L/Q_j)=s_l$.

PROOF. See Kakié [8], §3.

THEOREM 1.5. *If* $G_1$ *is an involutive symbol of order* 1, *then the following are valid*:

(I) *For any* $e\in E$ *outside a finite number of proper subspaces,* $M_0=\{z\in L_0;\, z\otimes e\in M_1\}$. *Moreover* $\dim L_0/M_0=s_1$.

(II) *If* $s_1=\cdots=s_l>0$, $s_{l+1}=\cdots=s_n=0$ *with* $1\leqq l<n$, *then all the ideals* $\mathfrak{P}_j$ *have the equal projective dimension* $l-1$.

PROOF. By Theorem 1.4 (II), $M_1=\mathrm{Ann}(G_1)$. On account of Theorem 1.4 (I), we can apply Lemma 1.3 (III) to deduce the first assertion of (I).

What we have just proved enables us to take a basis $\{e_1,\cdots,e_n\}$ of $E$ regular for $G_1$ such that $M_0=\{z\in L_0;\, z\otimes e_1\in M_1\}$. By Lemma 1.1, we get $\dim M_0=m-s_1$.

Let us prove (II). We can choose a basis $\{e_1,\cdots,e_n\}$ of $E$ and a basis $\{u_1,\cdots,u_m\}$ of $V^*$ satisfying the conditions of Lemma 1.1. Let $E_l$ be the vector subspace spanned by $e_1,\cdots,e_l$, $V'$ be the subspace spanned by $u_{\kappa_1+1},\cdots,u_m$. Let $R'$ be the symmetric algebra $\sum_{j=0}^{\infty}S^jE_l$. $R'$ is a subring of $R$. Using Theorem 1.4 (II) and bearing in mind (I) just proved, we can prove without any difficulty that

$$L=(V'\otimes R')\oplus M\quad\text{(direct sum as } R'\text{-modules)}.$$

Now let $N$ denote the intersection of those $Q_j$ with $\text{proj dim}\,\mathfrak{P}_j=l-1$. By Theorem 1.4 (III) and Lemma 1.3 (I), each ideal $\mathfrak{P}_j$ with $\text{proj dim}\,\mathfrak{P}_j=l-1$ is a minimal element in $\mathrm{Ass}(L/M)$. Therefore the uniqueness theorem of an irredundant primary decomposition indicates that the components $Q_j$ with $\mathfrak{P}_j$ being of projective dimension $l-1$ are uniquely determined, and are homogeneous submodules of $L$. Thus $N$ is a uniquely defined homogeneous submodule containing $M$. Since $\mathrm{Ass}(L/N)$ is the set $\{\mathfrak{P}_j;\, \text{proj dim}\,\mathfrak{P}_j=l-1\}$ by Lemma 1.3 (I), the required result follows if $M=N$ holds. We prove it by showing that $M\neq N$ leads us to a contradiction. Applying Lemma 1.3 (II), we find that $\mu(L/M)=\mu(L/N)$ and that the polynomials $P(L/M,x)$ and $P(L/N,x)$ have the same degree $l-1$ (with their leading coefficient being equal). Take a homogeneous element $z$ in $N$ not belonging to $M$. On account of the above direct sum decomposition, $L/M$ may be regarded as a free $R'$-module. Hence the homomorphism $h: R' \to N/M$ sending $a'$ to $a' \cdot z$ is injective. Thus the $R$-module $N/M$ contains an $R'$-module $h(R')$ of which Hilbert characteristic polynomial is evidently of degree $l-1$. This implies that $P(N/M, x)$ must be a polynomial of degree $l-1$ with positive leading coefficient. Since $P(L/M, x) = P(L/N, x) + P(N/M, x)$, it follows that $\mu(L/M) \neq \mu(L/N)$. This contradicts with $\mu(L/M) = \mu(L/N)$. Q.E.D.

## § 2. Typical involutive subsymbols of an involutive symbol

Let $G_1$ be an involutive symbol of order 1. Denote its characteristic module by $M$, and its Cartan characters by $s_1, \cdots, s_n$. The quotient module $L/M$ will be denoted by $M^\sharp$, and the vector space $L_k/M_k$ by $M_k^\sharp$. Let $M = \bigcap_{j=1}^{\nu} Q_j$ be an irredundant primary decomposition in $L$, and $Q_j$ be $\mathfrak{P}_j$-primary. For brevity we write $c(\mathfrak{P}_j) = \mathfrak{P}_j \cap R_1$ (subspace of $E$). In this section the $(k-1)$-th prolongation of $G_1$ will be denoted by $G_k$.

For a vector $e \in E$, we denote by $\sigma_{k,e}$ the linear mapping from $V \otimes (S^{k+1}E^*)$ to $V \otimes (S^k E^*)$ defined by

$$\sigma_{k,e}(v \otimes \xi_1 \circledcirc \cdots \circledcirc \xi_{k+1}) = \langle e, \xi_{k+1} \rangle v \otimes \xi_1 \circledcirc \cdots \circledcirc \xi_k \qquad (v \in V, \xi_i \in E^*).$$

Taking its restriction, we have a mapping $\sigma_{k,e}: G_{k+1} \to G_k$.

For each $k = 1, 2, \cdots$ and each $\mathfrak{P}_j$, let us introduce a symbol of order $k$

$$C_k(\mathfrak{P}_j) = \{\text{the subspace of } G_k \text{ spanned by } \sigma_{k,e}(G_{k+1}) \ (e \in c(\mathfrak{P}_j))\}.$$

We also introduce homogeneous submodules of $L$ and of $M^\sharp$

$$N^{(j)} = \{z \in L;\ c(\mathfrak{P}_j) z = \{0\}\}, \quad N^{(j)\sharp} = N^{(j)}/M \qquad (1 \leqq j \leqq \nu).$$

Let $U$ be a subspace of a vector space $W$. The annihilator of $U$ in $W^*$ will be denoted by $\mathrm{Ann}_{W^*}(U)$. In the following lemma, we use the canonical identification of the dual space to $G_k$ with $M_k^\sharp$ (cf. Theorem 1.4 (II)).

LEMMA 2.1. *Let $G_1$ be an involutive subspace. The following are true*:

(I) $\mathrm{Ann}_{L_k}(C_k(\mathfrak{P}_j)) = N^{(j)} \cap L_k$, $\mathrm{Ann}_{M_k^\sharp}(C_k(\mathfrak{P}_j)) = N^{(j)\sharp} \cap M_k^\sharp$ $(k \geqq 1)$.

(II) *If $c(\mathfrak{P}_\beta)$ is not contained in $\bigcup_{j \neq \beta} c(\mathfrak{P}_j)$, and if $\mathfrak{P}_\beta L \subset Q_\beta$, then*

$N^{(\beta)}=\bigcap_{j\neq\beta} Q_j$. *In particular* $\mathrm{Ann}_{L_k}(C_k(\mathfrak{P}_\beta))=\left(\bigcap_{j\neq\beta} Q_j\right)\cap L_k$ $(k\geqq 1)$.

PROOF. For the proof of (I), see Kakié [8], § 6 and [11], Lemma 4.3. Observing that there is an element $e\in c(\mathfrak{P}_\beta)$ not belonging to $\bigcup_{j\neq\beta}\mathfrak{P}_j$, and applying Lemma 1.3 (III), we can readily verify (II). Q.E.D.

PROPOSITION 2.2. *Let $G_1$ be an involutive symbol such that $s_1=\cdots=s_l>0$, $s_{l+1}=\cdots=s_n=0$ with $1\leqq l<n$. If $Q_\beta$ is a primary component such that $\mathfrak{P}_\beta L\subset Q_\beta$ and that the ideal $\mathfrak{P}_\beta$ is generated by linear forms (that is, by $c(\mathfrak{P}_\beta)$), then*

(I) *the symbol $C_1(\mathfrak{P}_\beta)\subset V\otimes E^*$ is involutive, and its Cartan characters $s_i^{(\beta)}$ are given by $s_i^{(\beta)}=s_i-\mu(L/Q_\beta)$ $(1\leqq i\leqq l)$, $s_i^{(\beta)}=0$ $(l<i\leqq n)$;*

(II) *the characteristic module $M^{(\beta)}$ of $C_1(\mathfrak{P}_\beta)$ admits an irredundant primary decomposition $M^{(\beta)}=\bigcap_{j\neq\beta} Q_j$. Moreover $M^{(\beta)}=N^{(\beta)}$;*

(III) *the vector space $N_0^{(\beta)\sharp}=N^{(\beta)\sharp}\cap M_0^\sharp$ is of dimension $\mu(L/Q_\beta)$.*

PROOF. By Theorem 1.5 (II), $\mathrm{proj\,dim}\,\mathfrak{P}_\beta=n-l$, and hence $\dim c(\mathfrak{P}_\beta)=n-l$. Let $V^{(\beta)*}=\{z\in L_0;\ c(\mathfrak{P}_\beta)z\subset M\}$. Define a subspace $G_1'$ of $G_1$ by

$$G_1'=\{z\in G_1;\ \langle z, u\otimes e\rangle=0 \ \text{ for any } v\in V^{(\beta)*}, e\in E\},$$

$\langle\ ,\ \rangle$ being the duality pairing. We show that it is an involutive symbol. Let $\{e_1,\cdots,e_n\}$ be a regular basis for $G_1$ such that $c(\mathfrak{P}_\beta)$ and the space spanned by $e_1,\cdots,e_l$ have the intersection $\{0\}$. We can choose a basis $\{u_1,\cdots,u_m\}$ of $V^*$ in such a way that $\{u_\alpha;\ 1\leqq\alpha\leqq\kappa_1\}$ gives a basis of $M_0$, $\{u_\alpha;\ 1\leqq\alpha\leqq\kappa_1+d\}$ is a basis of $V^{(\beta)*}$, and $D=\mathrm{Ann}(G_1)$ admits a basis $\{\phi_\alpha^i;\ 1\leqq\alpha\leqq\kappa_i,\ i=1,\cdots,n\}$ satisfying the conditions of Lemma 1.1. It is easily seen that a basis of $D'=\mathrm{Ann}(G_1')$ is obtained by adjoining the $\{\phi_\alpha^i\}$ the elements $\{u_\alpha\otimes e_i;\ \kappa_1<\alpha\leqq\kappa_1+d,\ 1\leqq i\leqq l\}$. We put ${}'\phi_\alpha^i=\phi_\alpha^i$ $(1\leqq\alpha\leqq\kappa_i,\ i=1,\cdots,n)$, ${}'\phi_\alpha^i=u_\alpha\otimes e_i$ $(\kappa_1<\alpha\leqq\kappa_1+d,\ i=1,\cdots,l)$. The assumption $c(\mathfrak{P}_\beta)L_0\subset M_1$ implies that $u_\alpha\otimes e_i$ $(\kappa_1<\alpha\leqq\kappa_1+d,\ l<i\leqq n)$ can be expressed as linear combinations of the $\{{}'\phi_\alpha^i\}$, and hence that the basis $\{{}'\phi_\alpha^i\}$ of $D'$ satisfies the conditions (i)-(iii) of Lemma 1.1. Therefore $G_1'$ is involutive, and its Cartan characters $s_i'$ are found to be $s_i'=s_i-d$ $(1\leqq i\leqq l)$, $s_i'=0$ $(l<i\leqq n)$, where $d=\dim N_0^{(\beta)\sharp}$.

Let $M'$ be the characteristic module of $G_1'$. We shall show that $M'=\bigcap_{j\neq\beta} Q_j$. Let $M'=\bigcap_{j=1}^{\nu'} Q_j'$ be an irredundant primary decomposition in $L$, and $Q_j'$ be $\mathfrak{P}_j'$-primary. By Theorem 1.5 (II), every prime ideal $\mathfrak{P}_j'$ is of projective dimension $l-1$. Since $M'\supset M$, any component $Q_k'$ contains

$\bigcap_{j=1}^{\nu} Q_j$. Hence applying Lemma 1.3 (IV), we have $\mathfrak{P}'_k \supset \bigcap_{j=1}^{\nu} \mathfrak{P}_j$. This implies that $\mathfrak{P}'_k$ contains some $\mathfrak{P}_j$. Since both prime ideals have the same dimension, they must coincide. Thus $\nu' \leqq \nu$, and we may assume that $\mathfrak{P}'_k = \mathfrak{P}_k$ $(k=1, \cdots, \nu')$. We assert that $Q'_k \supset Q_k$ $(k=1, \cdots, \nu')$. If $\nu=1$, this is obvious. If $\nu>1$, for each $k=1, \cdots, \nu'$, we can take an element $f \in \bigcap_{j \neq k} \mathfrak{P}_j$ not belonging to $\mathfrak{P}_k$. There is a positive integer $q$ such that $f^q L \subset Q_j$ $(j \neq k)$. Let $z \in Q_k$. Then $h_{f^q}(z) \in M \subset Q'_k$. Applying Lemma 1.3 (III), we have $z \in Q'_k$. Thus the required assertion is shown.

On the other hand, since $c(\mathfrak{P}_\beta)$ is not contained in $\bigcup_{j \neq \beta} c(\mathfrak{P}_j)$, we can choose an element $e$ of $c(\mathfrak{P}_\beta)$ not belonging to any $\mathfrak{P}_j$ with $j \neq \beta$. Bearing in mind the facts deduced from Theorem 1.4 (II) and Theorem 1.5 (I), we can readily see that $h_e(M') \subset M$. Accordingly we can apply Lemma 1.3 (III) to conclude that $M' \subset \bigcap_{j \neq \beta} Q_j$. Hence by the same reasoning as above, we know that each component $Q_j$ $(j \neq \beta)$ contains some $Q'_k$. Combining this with the fact already obtained, we see that $\nu' = \nu - 1$, and the set $\{Q'_j\}$ coincides with $\{Q_j; j \neq \beta\}$.

Theorem 1.4 (II) and Lemma 2.1 indicate that the annihilators in $L_1$ of $G'_1$ and $C_1(\mathfrak{P}_\beta)$ are $\left(\bigcap_{j \neq \beta} Q_j\right) \cap L_1$. This means that $G'_1 = C_1(\mathfrak{P}_\beta)$. Consequently, since $M^{(\beta)} = N^{(\beta)}$ by Lemma 2.1 (II), the proof will be complete if we show $d = \mu(L/Q_\beta)$. In virtue of Theorem 1.4 (III) and Theorem 1.5 (II), $s_l = \sum_j \mu(L/Q_j)$, $s'_l = \sum_{j \neq \beta} \mu(L/Q_j)$. As we have seen, $s'_l = s_l - d$. From these it follows that $d = \mu(L/Q_\beta)$. Q.E.D.

LEMMA 2.3. *Let $G_1$ be an involutive symbol such that $s_1 = \cdots = s_l > 0$, $s_{l+1} = \cdots = s_n = 0$ $(1 \leqq l < n)$.*

(I) *If, for each $\beta = 1, \cdots, r$, $c(\mathfrak{P}_\beta)$ is not contained in $\bigcup_{j \neq \beta} c(\mathfrak{P}_j)$, and $\mathfrak{P}_\beta L \subset Q_\beta$, then $N^{(1)\sharp} + \cdots + N^{(r)\sharp}$ is a direct sum in $M^\sharp$.*

(II) *If each $\mathfrak{P}_j$ is generated by linear forms, and if $\mathfrak{P}_j L \subset Q_j$ $(1 \leqq j \leqq \nu)$, then $M^\sharp = N^{(1)\sharp} \oplus \cdots \oplus N^{(\nu)\sharp}$ (direct sum).*

PROOF. Let $e_\beta$ be a vector of $c(\mathfrak{P}_\beta)$ not belonging to $\bigcup_{j \neq \beta} c(\mathfrak{P}_j)$ $(1 \leqq \beta \leqq r)$. Assume that $z^{(1)} + \cdots + z^{(r)} = 0$ with $z^{(\beta)} \in N^{(\beta)\sharp}$. Then $e_2 \cdots e_r z^{(1)} = 0$. Since $e_2 \cdots e_r \notin \mathfrak{P}_1$, we can apply Lemma 1.3 (III) to see that $z^{(1)} \in Q_1^\sharp$. It follows from this and Lemma 2.1 (II) that $z^{(1)} \in \bigcap_{j=1}^{\nu} Q_j^\sharp = \{0\}$, and hence that $z^{(1)} = 0$. Similarly we can show that $z^{(\beta)} = 0$ for any $\beta = 1, \cdots, r$. This proves (I). To show (II), we first observe that the assumption of (II) implies that every $\mathfrak{P}_j$ satisfies the condition of (I). Therefore the proof will be complete if we show that $M_0^\sharp = N_0^{(1)\sharp} \oplus \cdots \oplus N_0^{(\nu)\sharp}$. By Proposition 2.2 (III), $\dim N_0^{(j)\sharp} = \mu(L/Q_j)$. Theorem 1.5 and Theorem 1.4 (III) indicate that their sum is equal to $s_l = s_1 = \dim M_0^\sharp$. This implies our required result. Q.E.D.

For later applications, we need to generalize Proposition 2.2.

PROPOSITION 2.4. *Assume that $G_1$ is involutive, and that $s_1 = \cdots = s_l > 0$, $s_{l+1} = \cdots = s_n = 0$ with $1 \leqq l < n$. Let $Q_{j_1}, \cdots, Q_{j_r}$ be primary components such that $\mathfrak{P}_{j_q} L \subset Q_{j_q}$ and that each $\mathfrak{P}_{j_q}$ is generated by linear forms $(q=1, \cdots, r)$. Let $G_k'$ be the symbol $C_k(\mathfrak{P}_{j_1}) \cap \cdots \cap C_k(\mathfrak{P}_{j_r}) \subset V \otimes S^k E^*$. Then, for each $k=1, 2, 3, \cdots$, $G_k'$ is an involutive symbol of order $k$, and the characteristic module $M'$ of $G_k'$ admits an irredundant primary decomposition $M' = \bigcap_{j \neq j_1, \cdots, j_r} Q_j$ in $L$.*

PROOF. We may assume that $j_1 = 1, \cdots, j_r = r$. We first consider the case when $r=1$. Proposition 2.1 (II) indicates that if $k=1$, the statement is true. Assume that $k>1$. Let $M^{(1)}$ be the characteristic module of $G_k'$. By Lemma 2.1, $\mathrm{Ann}_{L_k}(G_k')$ coincides with $\left[\bigcap_{j \neq 1} Q_j\right] \cap L_k$, and the latter space is $M_k^{(1)}$. In virtue of Theorem 1.4 (II), $M_k^{(1)} = M_1^{(1)} \circledcirc S^{k-1} E$. Since the latter space is the annihilator in $L_k$ of the $(k-1)$-th prolongation $p^{k-1} G_1'$, we conclude that $G_k' = p^{k-1} G_1'$. Hence $G_k'$ is involutive by Proposition 1.2, and the characteristic module of $G_k'$ is that of $G_1'$.

We next consider the case when $r>1$. Let $V^{(\beta)*}$ be the space defined in the proof of Proposition 2.2. Lemma 2.3 (I) indicates that the sum of $N_0^{(\beta)\sharp} = V^{(\beta)*}/M_0$ $(1 \leqq \beta \leqq r)$ is a direct sum. Introduce the symbol

$$G_1'' = \left\{ z \in G_1;\ \langle z, u \otimes e \rangle = 0 \quad \text{for any } u \in \bigcup_{\beta=1}^{r} V^{(\beta)*}, e \in E \right\}.$$

In the same manner as in the proof of Proposition 2.2, we can verify that $G_1''$ is an involutive symbol with its Cartan characters $s_i'' = s_i - \sum_{\beta=1}^{r} d_\beta$ $(1 \leqq i \leqq l)$, $s_i'' = 0$ $(l < i \leqq n)$, where $d_\beta = \mu(L/Q_\beta)$, and that its characteristic module $M''$ admits an irredundant primary decomposition $M'' = \bigcap_{j=r+1}^{\nu} Q_j$.

We show that $G_1' = G_1''$. As we have seen in the proof of Proposition 2.2, $\mathrm{Ann}_{L_1}(C_1(\mathfrak{P}_\beta)) = \mathrm{Ann}_{L_1}(G_1) + V^{(\beta)*} \otimes E$. It follows that the annihilator in $L_1$ of the two spaces $G_1'$ and $G_1''$ are the same. Consequently they must coincide. This proves the assertion when $k=1$. Assume $k>1$. It is sufficient to prove that $G'_k$ is the $(k-1)$-th prolongation of $G'_1$. By Lemma 2.1, $\mathrm{Ann}_{L_q}(G'_q)=\sum_{\beta=1}^{r} N_q^{(\beta)}$ $(q\geqq 1)$. We have already seen that $N_k^{(\beta)}=N_1^{(\beta)}\circledcirc S^{k-1}E$. From these it follows that $\mathrm{Ann}_{L_k}(G'_k)=\mathrm{Ann}_{L_1}(G'_1)\circledcirc S^{k-1}E$. This implies that $G'_k=p^{k-1}G'_1$. Q.E.D.

## § 3. An involutive system of non-linear partial differential equations

This section is devoted to recalling some fundamental facts concerning involutive differential systems. For details about the notions without indicating any reference, refer to Kuranishi [14], Pommaret [16].

Let $\mathcal{E}$ be a fibered manifold over a manifold $X$ with projection $\pi$, and $J_k=J_k(\mathcal{E})$ the bundle of $k$-jets of its sections. The natural projection from $J_k$ onto $J_l$ will be denoted by $\pi_l^k$ $(k\geqq l\geqq 0)$, and $\pi\circ\pi_0^k$ by $\pi_{-1}^k$. Let $(x, y)=(x_1, \cdots, x_n, y_1, \cdots, y_m)$ be a fibered chart of $\mathcal{E}$, where $n=\dim X$, $n+m=\dim\mathcal{E}$. We denote by $I_k$ an ordered $k$-tuple of integers $1, 2, \cdots, n$, and by $S_k$ the set of $k$-tuples $I_k=(i_1, \cdots, i_k)$ with $i_1\leqq\cdots\leqq i_k$. For each $I_q=(i_1, \cdots, i_q)$, we define a function $p_\alpha^{I_q}=p_\alpha^{i_1\cdots i_q}$ on $J_k$ by $p_\alpha^{I_q}(j_x^k(f))=\partial^q f_\alpha(x)/\partial x_{i_1}\cdots\partial x_{i_q}$, $j_x^k(f)$ being the $k$-jet at $x$ of a section $f$ of $\mathcal{E}$, $f_\alpha$ being $y_\alpha$-coordinate of $f$ $(q\leqq k)$. The functions $p_\alpha^{i_1\cdots i_q}$ are symmetric with respect to any transposition of $i_1, \cdots, i_q$. With a fibered chart $(x, y)$ associates a local coordinate system $(x, y, p)$ of $J_k$ given by

$$(x_i, y_\alpha, p_\alpha^{I_q};\ 1\leqq i\leqq n,\ 1\leqq\alpha\leqq m,\ 1\leqq q\leqq k,\ I_q\in S_q).$$

For convenience we write $y_\alpha=p_\alpha^{I_0}$, and set $S_0=\{I_0\}$.

A *non-linear system of partial differential equations of order $k$ on $\mathcal{E}$* is, by definition, a fibered submanifold of the fibered manifold $J_k(\mathcal{E})$ over $X$ with projection $\pi_{-1}^k$. A system $\mathcal{R}_k$ of order $k$ may be described locally by the equations

$$\mathcal{R}_k:\quad F_\beta(x, y, p)=0 \qquad (\beta=1, \cdots, r) \tag{3.1}$$

with the Jacobian matrix $\partial(F_1, \cdots, F_r)/\partial(y, p)$ being of rank $r=\{\mathrm{codim}\,\mathcal{R}_k$ in $J_k\}$ everywhere on $\mathcal{R}_k$. A solution of $\mathcal{R}_k$ corresponds to a section of $\mathcal{E}$ described locally by $y_\alpha=y_\alpha(x)$ $(1\leqq\alpha\leqq m)$ such that $y_\alpha(x)$ together with their derivatives $p_\alpha^{i_1\cdots i_q}=\partial^q y_\alpha(x)/\partial x_{i_1}\cdots\partial x_{i_q}$ satisfy the equations of (3.1).

Given a system $\mathcal{R}_k$, let $\mathcal{R}_{k+l}$ denote its $l$-th (total) prolongation. It is a system of order $k+l$, and can be defined locally by

$$\mathcal{R}_{k+l}: \quad F_\beta=0,\ \partial_\sharp^{i_1}\cdots\partial_\sharp^{i_q}F=0\ \ (\beta=1,\cdots,r,\ q=1,\cdots,l,\ 1\leqq i_1,\cdots,i_q\leqq n),$$

where $\partial_\sharp^i$ denotes the total differentiation with respect to $x_i$.

Let $V(\mathcal{E})$ be the kernel of the mapping $\pi_*: T\mathcal{E}\to TX$, where $T\mathcal{E}$ denotes the tangent bundle to $\mathcal{E}$. For brevity we write $T=TX$, $T^*=T^*X$ (the cotangent bundle), $V=V(\mathcal{E})$. We shall make the convention that the bundles such as $T$, $V$ denote also their pull-backs by the corresponding projections. Let

$$\varepsilon_k: V\otimes S^kT^*\longrightarrow TJ_k;\quad \varepsilon_k^*: T^*J_k\longrightarrow V^*\otimes S^kT\quad (k=0,1,2,\cdots)$$

be the bundle homomorphism over $J_k$ defined by

$$\varepsilon_k((\partial/\partial y_\alpha)\otimes dx_{i_1}\circledcirc\cdots\circledcirc dx_{i_k})=\partial/\partial p_\alpha^{i_1\cdots i_k};\quad \varepsilon_k^*(dx_i)=0,$$
$$\varepsilon_k^*(dp_\alpha^{i_1\cdots i_q})=\begin{cases} dy_\alpha\otimes(\partial/\partial x_{i_1})\circledcirc\cdots\circledcirc(\partial/\partial x_{i_k}) & \text{if } q=k,\\ 0 & \text{if } 0\leqq q<k.\end{cases}$$

The homomorphism $\varepsilon_k^*$ is the dual mapping of $\varepsilon_k$.

Let $G_k$ be the *symbol* of $\mathcal{R}_k$. It is a family of vector spaces over $\mathcal{R}_k$, and may be described by

$$G_k=\{\chi\in V\otimes S^kT^*;\ \langle\chi,\varepsilon_k^*(dF_\beta)\rangle=0\quad(\beta=1,\cdots,r)\},$$

where $\langle\ ,\ \rangle$ is the duality pairing. Let $G_{k+1}$ be the prolongation of $G_k$; it is a family of vector spaces over $\mathcal{R}_k$ locally described by

$$G_{k+1}=\left\{\chi\in V\otimes S^{k+1}T^*;\ \langle\chi,\varepsilon_k^*(dF_\beta)\circledcirc\partial/\partial x_i\rangle=0\ \begin{pmatrix}\beta=1,\cdots,r\\ i=1,\cdots,n\end{pmatrix}\right\}.$$

The symbol $G_k$ is said to be *involutive* if each fiber $G_{k,P}$ over $P\in\mathcal{R}_k$ is an involutive symbol of order $k$ in the sense of §1. The following criterion of involutiveness is fundamental (cf. Kuranishi [14], Goldschmidt [5], Pommaret [16]).

THEOREM 3.A. *A non-linear system $\mathcal{R}_k$ of order $k$ is involutive if and only if the following three conditions are satisfied:*

(i) *$G_{k+1}$ is a vector bundle over $\mathcal{R}_k$,*

(ii) *the symbol $G_k$ is involutive,*

(iii) *the mapping $\pi_k^{k+1}: \mathcal{R}_{k+1}\to\mathcal{R}_k$ is surjective.*

*Note.* If $\mathcal{R}_k$ is involutive, its symbol $G_k$ is a vector bundle.

Assume that $\mathcal{R}_k$ is involutive. The Cartan characters $s_1(P),\cdots,s_n(P)$ of the symbol $G_{k,P}$ are constant on $P\in\mathcal{R}_k$. The Cartan characters

$s_1, \cdots, s_n$ of $\mathcal{R}_k$ are defined by $s_1 = s_1(P), \cdots, s_n = s_n(P)$.

Involutiveness keeps to hold under the prolongation (cf. Matsuda [15]).

THEOREM 3.B. *If $\mathcal{R}_k$ is involutive, so are its prolongations $\mathcal{R}_{k+l}$.*

By the *contact forms* on $J_k$, we shall mean the 1-forms defined by

$$\theta_\alpha^{I_q} = dp_\alpha^{I_q} - \sum_{i=1}^n p_\alpha^{I_q i} dx_i \qquad (1 \leqq \alpha \leqq m, 0 \leqq q < k, I_q \in S_q). \tag{3.2}$$

Given a system $\mathcal{R}_k$, let $\Sigma_k = \Sigma(\mathcal{R}_k)$ be the differential ideal on $J_k$ generated by all (smooth) functions vanishing everywhere on $\mathcal{R}_k$ and the contact forms (3.2). If $\mathcal{R}_k$ is described by (3.1), $\Sigma_k$ is the exterior differential system generated by the 0-forms $F_\beta$, the 1-forms $dF_\beta, \theta_\alpha^{I_q}$, and the 2-forms $d\theta_\alpha^{I_q}$ as an algebraic ideal in the ring of differential forms on $J_k$. A solution of $\mathcal{R}_k$ corresponds to an $n$-dimensional integral manifold $\mathcal{M}$ of $\Sigma_k$ such that $\dim(\pi_{-1}^k)_* T_P(\mathcal{M}) = n$ for any $P \in \mathcal{M}$.

Let $\Omega$ be the module of 1-forms on $X$. Its pull-backs will be denoted by the same symbol. $\Omega$ admits a local basis $dx_1, \cdots, dx_n$.

Let $I^n(\Sigma_k, \Omega)$ denote the set of those $n$-dimensional integral elements $E_n$ on which $dx_1, \cdots, dx_n$ are linearly independent. The differential system $\Sigma_k$ is said to be *involutive at $P \in \mathcal{R}_k$ with independence condition $\Omega$* if there is an element $E_n \in I^n(\Sigma_k, \Omega)$ of origin $P$, and if any such $E_n$ admits a regular chain $E_0 \subset E_1 \subset \cdots \subset E_n$ ending with it (cf. Kähler [6], Cartan [3], Kuranishi [13]).

THEOREM 3.C. *A system $\mathcal{R}_k$ is involutive if and only if $\Sigma(\mathcal{R}_k)$ is involutive at each $P \in \mathcal{R}_k$ with independence condition $\Omega$.*

*Note.* Let $\rho: I^n(\Sigma_k, \Omega) \to \mathcal{R}_k$ be the map that assigns $E_n$ its origin. Then $\rho$ is surjective if and only if so is $\pi_k^{k+1}: \mathcal{R}_{k+1} \to \mathcal{R}_k$. More precisely there is a canonical bijection between $\{E_n \in I^n(\Sigma_k, \Omega); E_n \text{ is of origin } P\}$ and $(\pi_k^{k+1}|_{\mathcal{R}_{k+1}})^{-1}(P)$.

For a manifold $Y$, we shall denote by $\wedge^q Y$ the module of $q$-forms on $Y$. By a *Pfaffian system $\Theta$ of rank $r$ on $Y$*, we shall mean a submodule of the $\wedge^0 Y$-module $\wedge^1 Y$ such that for each $y \in Y$, there exist $r$ everywhere linearly independent 1-forms which generate $\Theta$ on a neighborhood of $y$. Such a set of $r$ 1-forms will be called a (*local*) *basis* of *$\Theta$ around $y$*.

Let $\Theta_k = \Theta(\mathcal{R}_k)$ be the submodule of $\wedge^1 \mathcal{R}_k$ generated by the restrictions to $\mathcal{R}_k$ of the contact forms $\theta$'s in (3.2). It can be verified that if

$\mathcal{R}_k$ is involutive, $\Theta_k$ is a Pfaffian system on $\mathcal{R}_k$. The integral elements of $\Theta_k$ correspond in one-to-one manner to those of $\Sigma_k$; Thus we may write $I^n(\Theta_k, \Omega)=I^n(\Sigma_k, \Omega)$.

The involutiveness is what is called a local notion. In later discussions, we shall need to consider locally defined systems. We say that a system $\mathcal{R}_k$ is a *system of order k defined on an open set* $\mathcal{U}$ of $J_k$ if $\mathcal{R}_k$ is a fibered submanifold of $\pi^k_{-1}: \mathcal{U}\to\pi^k_{-1}(\mathcal{U})$. We also say that $\mathcal{R}_k$ is *involutive at* $P\in\mathcal{R}_k$ if it is a system defined on a neighborhood of $P$ in $J_k$ and if it is involutive, that is, it satisfies the conditions (i)-(iii) of Theorem 3.A over the neighborhood.

## § 4. The Monge characteristic systems

Let $\mathcal{R}_k$ be an involutive system of order $k$ on $\mathcal{E}$, and $G_k$ be its symbol. We shall keep the same notations as in § 3. We shall also use the notations: $T_x$=the tangent space to $X$ at $x$, $V_y$=the fiber of $V(\mathcal{E})$ over $y$, $T_PJ_k$=the tangent space to $J_k$ at $P$, etc.

Given a point $P\in\mathcal{R}_k$, we write $x=\pi^k_{-1}(P)$, $y=\pi^k_0(P)$. Let $R_P$ be the ring $\sum_{q=0}^{\infty} R_{q,P}$ where $R_{q,P}=S^qT_x$, and $L_P$ be the graded $R_P$-module $\sum_{q=0}^{\infty} L_{q,P}$ where $L_{q,P}=V_y^*\otimes S^qT_x$.

The *characteristic module* $M_P$ *of* $R_k$ *at* $P\in\mathcal{R}_k$ is, by definition, the characteristic module of the symbol $G_{k,P}\subset V_y\otimes S^kT_x^*$ (cf. § 1). The quotient module $L_P/M_P$ will be denoted by $M_P^\sharp$.

Let $M_P=\bigcap_{j=1}^{\nu(P)} Q_{j,P}$ be an irredundant primary decomposition in $L_P$, $Q_{j,P}$ be $\mathfrak{P}_{j,P}$-primary. We put $c(\mathfrak{P}_{j,P})=\mathfrak{P}_{j,P}\cap R_{1,P}\subset T_x$. The vectors in $c(\mathfrak{P}_{j,P})$ have a certain special properties (See Kakié [10].).

Using each $c(\mathfrak{P}_{j,P})$, we define the space $C(\mathfrak{P}_{j,P})$ to be the subspace of $G_{k,P}$ spanned by $\sigma_{k,e}(G_{k+1,P})$ $(e\in c(\mathfrak{P}_{j,P}))$, where $\sigma_{k,e}$ is the linear mapping from $V_y\otimes S^{k+1}T_x^*$ to $V_y\otimes S^kT_x^*$ defined similarly as in § 2. (Note that $G_{k+1,P}=pG_{k,P}$.)

We shall call a vector $v\in T_PJ_k$ a *Monge characteristic vector of* $\mathcal{R}_k$ *at* $P$ if $v\in E_n$ for some $E_n\in I^n(\Sigma_k, \Omega)$ and if $(\pi^k_{-1})_*(v)\in\bigcup_j c(\mathfrak{P}_{j,P})$ (cf. [10]).

Let $B(\mathfrak{P}_{j,P})$ be the subspace of $T_PJ_k$ spanned by the Monge characteristic vectors $v$ of $\mathcal{R}_k$ at $P$ with $(\pi^k_{-1})_*(v)\in c(\mathfrak{P}_{j,P})$. It is indeed a subspace of $T_P\mathcal{R}_k$. Let $D(\mathfrak{P}_{j,P})$ be the annihilator of $B(\mathfrak{P}_{j,P})$ in $T_P^*\mathcal{R}_k$.

Let $\tilde{\varepsilon}_k: G_{k,P}\to T_P\mathcal{R}_k$ be the homomorphism obtained by restricting $\varepsilon_k$ (see § 3). Identifying the dual space to $G_{k,P}$ with $M_{k,P}^\sharp=L_{k,P}/M_{k,P}$, where

$M_{k,P}=M_P \cap L_{k,P}$, we have its dual map $\bar{\varepsilon}_k^* : T_P^*\mathcal{R}_k \to M_{k,P}^\sharp$. We also need to introduce the modules $N_P^{(j)\sharp}=\{z \in M_P^\sharp;\ c(\mathfrak{P}_{j,P})z=\{0\}\}$ (cf. § 2). The vector space $N_P^{(j)\sharp} \cap M_{k,P}^\sharp$ will be denoted by $N_{k,P}^{(j)\sharp}$ (cf. § 2).

LEMMA 4.1. *Under the above circumstances, the following are true*:

(I) $\bar{\varepsilon}_k^*(D(\mathfrak{P}_{j,P}))=\mathrm{Ann}_{M_{k,P}^\sharp} C(\mathfrak{P}_{j,P})=N_{k,P}^{(j)\sharp}$.

(II) *Let* $\{\omega_i; 1 \leqq i \leqq n-l_j\}$ *be a basis of the annihilator of* $c(\mathfrak{P}_{j,P})$ *in* $T_x^*X$. *Then* $D(\mathfrak{P}_{j,P}) \cap \mathrm{Ker}\,\bar{\varepsilon}_k^*$ *is generated by* $\omega_i$ $(1 \leqq i \leqq n-l_j)$ *and the values at* $P$ *of the* 1-*forms in* $\Theta(\mathcal{R}_k)$, *the* $\omega$'*s being regarded as vectors in* $T_P^*\mathcal{R}_k$.

PROOF. See Kakié [10], Lemma 3.1.

To define Monge characteristic systems, we shall assume the following regularity conditions: (a) the number $\nu=\nu(P)$ is constant on $\mathcal{R}_k$, (b) for each family $\mathfrak{P}_j=\{\mathfrak{P}_{j,P}; P \in \mathcal{R}_k\}$, the family $\boldsymbol{c}(\mathfrak{P}_j)=\{c(\mathfrak{P}_{j,P});\ P \in \mathcal{R}_k\}$ forms canonically a vector bundle, (c) for each $\mathfrak{P}_j$, the family $\boldsymbol{C}(\mathfrak{P}_j)=\{C(\mathfrak{P}_{j,P}); P \in \mathcal{R}_k\}$ forms a vector subbundle of $G_k$; equivalently, $\dim C(\mathfrak{P}_{j,P})$ remains constant on $\mathcal{R}_k$.

Then the family $\boldsymbol{B}(\mathfrak{P}_j)=\{B(\mathfrak{P}_{j,P}); P \in \mathcal{R}_k\}$ is a vector subbundle of $T\mathcal{R}_k$, and the family $\boldsymbol{D}(\mathfrak{P}_j)=\{D(\mathfrak{P}_{j,P}); P \in \mathcal{R}_k\}$ is that of $T^*\mathcal{R}_k$.

The *Monge characteristic system* $\Delta^k(\mathfrak{P}_j)$ (*of order* $k$) *of* $\mathcal{R}_k$ *corresponding to* $\mathfrak{P}_j$ is defined to be the Pfaffian system on $\mathcal{R}_k$ generated by all sections of $\boldsymbol{D}(\mathfrak{P}_j)$.

Let $\boldsymbol{c}(\mathfrak{P}_j)^\perp$ denote the annihilator bundle of $\boldsymbol{c}(\mathfrak{P}_j)$ in the induced bundle $(\pi_{-1}^k|_{\mathcal{R}_k})^*T^*X$. By Lemma 4.1, the sections of $\boldsymbol{c}(\mathfrak{P}_j)^\perp$ and the 1-forms in $\Theta(\mathcal{R}_k)$ belong to $\Delta^k(\mathfrak{P}_j)$. Moreover $\mathrm{rank}\,\Delta^k(\mathfrak{P}_j)=\mathrm{rank}\,\Theta_k+\mathrm{rank}\,\boldsymbol{c}(\mathfrak{P}_j)+\dim N_{k,P}^{(j)\sharp}$.

Assume that the prolongations $\mathcal{R}_{k+l}$ satisfy the above regularity conditions. (Indeed they satisfy (a), (b) if so does $\mathcal{R}_k$.) Then we can define Monge characteristic systems of higher orders: The *Monge characteristic system* $\Delta^q(\mathfrak{P}_j)$ *of order* $q$ $(q \geqq k)$ *corresponding to* $\mathfrak{P}_j$ is defined to be the Monge characteristic system of $\mathcal{R}_q$ corresponding to $\mathfrak{P}_j$.

DEFINITION. A function $f$ defined (locally) on $\mathcal{R}_q$ is called an *integral* of $\Delta^q(\mathfrak{P}_j)$ if the 1-form $df$ belongs to $\Delta^q(\mathfrak{P}_j)$.

Let $Z^q(\mathfrak{P}_j)$ be the module of those vector fields $\boldsymbol{v}$ on $\mathcal{R}_q$ which satisfy "$\langle \boldsymbol{v}, \tau \rangle=0$ on $\mathcal{R}_q$ for any $\tau \in \Delta^q(\mathfrak{P}_j)$". A function $f$ is an integral of $\Delta^q(\mathfrak{P}_j)$ if and only if $\boldsymbol{v}(f)=0$ for any $\boldsymbol{v} \in Z^q(\mathfrak{P}_j)$. Hence we can apply the classical theory to obtain its integrals (cf. Forsyth [4]). For a module $Z$ of vector fields, denote by $p_0Z$ the module of vector fields generated by $v$, $[v, w]$ where $v, w \in Z$. Here the bracket is the usual one defined by $[v, w](f) = v(w(f)) - w(v(f))$. We set $\hat{Z}^q(\mathfrak{P}_j) = \bigcup_{r=0}^{\infty} p_0^r Z^q(\mathfrak{P}_j)$.

DEFINITION. We say that $\Delta^q(\mathfrak{P}_j)$ is *p-regular around* $P \in \mathcal{R}_q$ if the module $\hat{Z}^q(\mathfrak{P}_j)$ is generated locally around $P$ by some number of vector fields which are everywhere linearly independent.

If $\Delta^q(\mathfrak{P}_j)$ is $p$-regular, $\hat{Z}^q(\mathfrak{P}_j)$ defines an involutive distribution in the terminology of Chevalley [17], and hence we can find a finite number of its integrals $\{f_\gamma : \gamma = 1, \cdots, r\}$ (defined around $P$) called a *fundamental system of the integrals* such that any integral admits an expression $f = \Phi(f_1, \cdots, f_r)$. We emphasize that the integrals can be obtained by integrating ordinary differential equations (cf. [4]).

Let us introduce one more notion, which is especially important in our method of integration.

DEFINITION. We say that $\Delta^q(\mathfrak{P}_j)$ is *principally integrable around* $P \in \mathcal{R}_q$ if it admits a finite number of integrals $\{f_\gamma\}$ such that it is generated by $\Theta(\mathcal{R}_q)$ and the 1-forms $df_\gamma$ around $P$. In this case we can find a set $\{u_\delta\}$ of integrals such that $\{\varepsilon_q^*(du_\delta)\}$ gives a basis of the module $\varepsilon_q^*(\Delta^q(\mathfrak{P}_j))$ around $P$ (cf. Lemma 4.1). We call such a set $\{u_\delta\}$ a *complete set of principal integrals.*

We shall give some fundamental facts concerning the integrals.

PROPOSITION 4.2. *If $f$ is an integral of $\Delta^q(\mathfrak{P}_j)$, then its pull-back $(\pi_q^{q+1})^*f$ under the map $\pi_q^{q+1} : \mathcal{R}_{q+1} \to \mathcal{R}_q$ is an integral of $\Delta^{q+1}(\mathfrak{P}_j)$.*

PROOF. We can readily prove that for any Monge characteristic vector $v$ of $\mathcal{R}_{q+1}$ with $(\pi_{-1}^{q+1})_* v \in c(\mathfrak{P}_j)$, $\langle v, d(\pi_q^{q+1})^* f \rangle = 0$ (cf. Kakié [7]).

Q.E.D.

PROPOSITION 4.3. *Let $P_0 \in \mathcal{R}_q$. If $\Delta^q(\mathfrak{P}_j)$ is principally integrable around $P_0$, and if $N_{q+1,P'}^{(j)\#} = R_{1,P'} \cdot N_{q,P'}^{(j)\#}$ $(P' = \pi_k^q(P_0))$, then $\Delta^{q+1}(\mathfrak{P}_j)$ is principally integrable around any point $\tilde{P}_0 \in \mathcal{R}_{q+1}$ over $P_0$.*

PROOF. Let $E_n^{(0)}$ be the element of $I^n(\Theta_q, \Omega)$ with origin $P_0$ which corresponds to the point $\tilde{P}_0$ (see Note to Theorem 3.C). The principal integrability implies that there exist $l = n - \operatorname{rank} c(\mathfrak{P}_j)$ integrals $w_1, \cdots, w_l$ of $\Delta^q(\mathfrak{P}_j)$ such that $dw_1, \cdots, dw_l$ are linearly independent on $E_n^{(0)}$.

SUBLEMMA. *Under the above circumstances, there exists $l$ total differentiations* $\lambda_i=\sum_{i'=1}^{n} c^i_{i'}\partial^{i'}_{\#}$ $(1\leqq i\leqq l)$ *with the $c$'s being functions on* $\mathcal{R}_{q+1}$ *such that the following hold*: (i) *Put* $e_i=\sum_{i'=1}^{n} c^i_{i'}\partial/\partial x_{i'}$. *Then* $e_i$ $(1\leqq i\leqq l)$ *are linearly independent, and* $e_i(\tilde{P})\notin c(\mathfrak{P}_{j,P})$ *for any* $\tilde{P}\in\mathcal{R}_{q+1}$ *near* $\tilde{P}_0$, $P=\pi^{q+1}_k(\tilde{P})$; (ii) *If $f$ is an integral of* $\Delta^q(\mathfrak{P}_j)$ *around* $P_0$, *then* $\lambda_i(f)$ $(1\leqq i\leqq l)$ *are integrals of* $\Delta^{q+1}(\mathfrak{P}_j)$ *around* $\tilde{P}_0$. *Here the total differentiation* $\partial^i_{\#}f$ *is defined to be the restriction to* $\mathcal{R}_{q+1}$ *of the function* $\partial^i_{\#}\tilde{f}$ *of* $(q+1)$*-jets where* $\tilde{f}$ *is an extention of $f$ to a neighborhood of* $\mathcal{R}_q$ *in* $J_q$.

PROOF OF SUBLEMMA. For brevity we write $\Theta_q=\Theta(\mathcal{R}_q)$, $\rho=\pi^{q+1}_q|_{\mathcal{R}_{q+1}}$. Let $\omega_i$ be the 1-form $\sum_{i'=1}^{n}(\partial^{i'}_{\#}w_i)dx_{i'}$ on $\mathcal{R}_{q+1}$ $(1\leqq i\leqq l)$. The $\omega_i$'s are 1-forms in $\Delta^{q+1}(\mathfrak{P}_j)$ which are linearly independent around $\tilde{P}_0$, and $d(\rho^*_q w_i)\equiv\omega_i$ $(\mathrm{mod}\,\Theta_{q+1})$. We can choose 1-forms $\omega_i=\sum_{i'=1}^{n} b^{i'}_i dx_{i'}$ $(l<i\leqq n)$ on $\mathcal{R}_{q+1}$ in such a way that $\omega_i$ $(1\leqq i\leqq n)$ are linearly independent around $\tilde{P}_0$. Let $C=(c^i_{i'})$ be the inverse matrix of the coefficient matrix of the $\omega_i$'s. Let us put $\lambda_i=\sum_{i'=1}^{n} c^i_{i'}\partial^{i'}_{\#}$ $(1\leqq i\leqq n)$. It is readily seen that (i) holds true. To verify (ii), we observe that for a function $g$ on $\mathcal{R}_q$,

$$d(\rho^*_q g)\equiv\sum_{i=1}^{n}(\partial^i_{\#}g)dx_i \qquad (\mathrm{mod}\,\Theta_{q+1}) \qquad \text{on } \mathcal{R}_{q+1}.$$

Accordingly, from the way the $\lambda_i$, $\omega_i$ are defined, we find that

$$d(\rho^*_q f)\equiv\sum_{i=1}^{n}\lambda_i(f)\omega_i \qquad (\mathrm{mod}\,\Theta_{q+1}) \qquad \text{on } \mathcal{R}_{q+1}.$$

Now in virtue of Theorems 3.B and 3.C, for any point $P\in\mathcal{R}_{q+1}$ near $\tilde{P}_0$, there is an element $E_n\in I^n(\Theta_{q+1},\Omega)$ of origin $P$. By Proposition 4.2, $\rho^*_q f$ is an integral of $\Delta^{q+1}(\mathfrak{P}_j)$. Hence $\langle v, d(\rho^*_q f)\rangle=0$ for any Monge characteristic vector $v\in E_n$ with $(\pi^{q+1}_{-1})_*(v)\in c(\mathfrak{P}_{j,P})$, or equivalently with $\langle v,\omega_i\rangle=0$ $(1\leqq i\leqq l)$. Therefore, from the above formula, we find that $\lambda_i(f)$ $(l<i\leqq n)$ vanish at any $P\in\mathcal{R}_{q+1}$ near $\tilde{P}_0$. Hence we get

$$d(\rho^*_q f)\equiv\sum_{i=1}^{l}\lambda_i(f)d(\rho^*_q w_i) \qquad (\mathrm{mod}\,\Theta_{q+1}).$$

Taking exterior differentiation, we obtain

$$\sum_{i=1}^{l}d(\lambda_i(f))\wedge d(\rho^*_q w)\equiv 0 \qquad (\mathrm{mod}\,\Theta_{q+1}, d\Theta_{q+1}).$$

Let $E_n \in I^n(\Theta_{q+1}, \Omega)$. Since $dw_i$ restricted to $E_n$ are linearly independent, the above formula indicates that the 1-forms $d(\lambda_i(f))$ restricted to $E_n$ are expressed as linear combinations of $dw_i$'s. This implies that for any Monge characteristic vector $v \in E_n$ with $(\pi^k_{-1})_*(v) \in c(\mathfrak{P}_{j,P})$, $\langle v, d(\lambda_i(f))\rangle = 0$. From the definition it follows that $\lambda_i(f)$ are integrals of $\Delta^{q+1}(\mathfrak{P}_j)$. Thus we have proved Sublemma.

Now we can choose a complete set $\{u_\delta\}$ of principal integrals of $\Delta^q(\mathfrak{P}_j)$ such that $du_\delta$'s restricted to $E_n^{(0)}$ vanish, equivalently, $(\partial^i_\# u_\delta)(\tilde{P}_0) = 0$ $(1 \leqq i \leqq n)$. The functions $g^i_\delta = \lambda_i(u_\delta)$ are integrals of $\Delta^{q+1}(\mathfrak{P}_j)$ by Sublemma, and

$$\tilde{\varepsilon}^*_{q+1}(dg^i_\delta)(\tilde{P}_0) = \sum_{i'=1}^{n} c^{i'}_i \tilde{\varepsilon}^*_q(du_\delta)(P_0) \circledcirc \partial/\partial x_{i'}.$$

By the assumption on $N^{(j)\sharp}_{q+1}$, this implies that $\{\tilde{\varepsilon}^*_{q+1}(dg^i_\delta)\}$ gives a basis of $\tilde{\varepsilon}^*_{q+1}(\Delta^{q+1}(\mathfrak{P}_j))$ (cf. Lemma 4.1). It follows that $\Delta^{q+1}(\mathfrak{P}_j)$ is generated by the 1-forms $dg^i_\delta$, $d(\rho^*_q(w_i))$, and $\Theta_{q+1}$ around $\tilde{P}_0$, which means that it is principally integrable. Q.E.D.

We finally note that all the discussions in this section hold true for a locally defined system $\mathcal{R}_k$, or a locally defined Monge characteristic system.

## § 5. A method of constructing new involutive systems

Let $\mathcal{R}_1$ be an involutive system *of order* 1 on $\mathcal{E}$. We shall denote by $s_1, \cdots, s_n$ its Cartan characters, and by $M_P$ its characteristic module at $P \in \mathcal{R}_1$. Let $M_k$ denote the family $\{M_{k,P} = M_P \cap L_{k,P}; P \in \mathcal{R}_1\}$ of vector spaces $(k = 0, 1, 2, \cdots)$. Each $M_k$ is indeed a vector bundle over $\mathcal{R}_1$ (cf. Note to Theorem 3.A, Theorem 1.3 (II), and Theorem 1.4 (I)). Let $M_P = \bigcap_{j=1}^{\nu(P)} Q_{j,P}$ be an irredundant primary decomposition in $L_P$, and $Q_{j,P}$ be $\mathfrak{P}_{j,P}$-primary. We shall assume that $\mathcal{R}_1$ satisfies the following conditions:

(H-1) $\mathcal{R}_1$ is involutive, and $s_1 = \cdots = s_l > 0$, $s_{l+1} = \cdots = s_n = 0$ with $1 \leqq l < n$.

(H-2) Let $P_0$ be a point on $\mathcal{R}_1$. We can find a neighborhood $\mathcal{U}$ of $P_0$ in $\mathcal{R}_1$ such that the following three conditions are satisfied: (i) The number $\nu = \nu(P)$ remains constant on $\mathcal{U}$. (ii) Each family $c(\mathfrak{P}_j) = \{c(\mathfrak{P}_{j,P}); P \in \mathcal{U}\}$ is a vector bundle *of rank* $n-l$ over $\mathcal{U}$ $(1 \leqq j \leqq \nu)$. (iii) The exponent of each $Q_{j,P}$ is equal to 1; that is, $\mathfrak{P}_{j,P} L_{j,P} \subset Q_{j,P}$ $(1 \leqq j \leqq \nu, P \in \mathcal{U})$.

(H-1) implies that $\operatorname{proj\,dim} \mathfrak{P}_{j,P} = l-1$ by Theorem 1.4, and hence that the components $Q_{j,P}$ are uniquely determined. As in § 4, we put

$$N_P^{(j)}=\{z\in L_P;\ c(\mathfrak{P}_{j,P})z\subset M_P\},\quad N_P^{(j)\sharp}=N_P^{(j)}/M_P.$$

LEMMA 5.1. *If* (H-1) *and* (H-2) *hold, then, for any integer* $k\geqq 0$, (I) $M_{k,P}=\bigoplus_{j=1}^{\nu} N_{k,P}^{(j)\sharp}$, (II) $\dim N_{k,P}^{(j)\sharp}=\text{const.}$, (III) $N_{k+1,P}^{(j)\sharp}=R_{1,P}\cdot N_{k,P}^{(j)\sharp}$.

PROOF. By Lemma 2.3, (I) holds true. (I) implies that the sum of $\dim N_{k,P}^{(j)\sharp}$ is equal to $\dim M_{k,P}^{\sharp}=\text{const.}$ Since the dimensions are upper-semicontinuous, they must be constant. The assertion (III) follows at once from (I) and the fact $R_{1,P}\cdot M_{k,P}^{\sharp}=M_{k+1,P}^{\sharp}$. Q.E.D.

Lemmas 4.1 and 5.1 indicate that the Monge characteristic systems of $\mathcal{R}_1$ including of higher orders are well-defined locally over $\mathcal{U}$ (cf. § 4).

Fixing a system $\mathcal{R}_1$, we shall denote its $(k-1)$-th prolongation by $\mathcal{R}_k$. The Monge characteristic system of $\mathcal{R}_1$ of order $k$ corresponding to $\mathfrak{P}_j=\{\mathfrak{P}_{j,P}\}$ will be denoted by $\Delta^k(\mathfrak{P}_j)$.

For a set $\Phi$ of functions (locally) defined on $\mathcal{R}_k$, we denote by $\mathcal{R}_k[\Phi]$ the set of those points $P\in\mathcal{R}_k$ at which every function of $\Phi$ vanishes.

By a *non-characteristic integral manifolds* $\mathcal{M}$ *of* $\Sigma(\mathcal{R}_k)$, we shall mean a (locally closed) integral manifold $\mathcal{M}$ such that, for each $\tilde{P}\in\mathcal{M}$, $(\pi_{-1}^k)_*T_{\tilde{P}}\mathcal{M}$ is of dimension $\dim\mathcal{M}$, and does not meet any space $c(\mathfrak{P}_{j,P})$ except the zero vector, where $P=\pi_1^k(\tilde{P})$.

We shall now state our main theorem, which gives a method of constructing new involutive systems of which solutions are those of a given one.

THEOREM 5.2. *Let* $\mathcal{R}_1$ *be a system satisfying the conditions* (H-1) *and* (H-2). *Assume that, for some integer* $k\geqq 1$, *the* $r$ *Monge characteristic systems* $\Delta^k(\mathfrak{P}_j)$ $(1\leqq j\leqq r)$ *are p-regular and principally integrable around a point* $\tilde{P}_0\in\mathcal{R}_k$ *over* $P_0$. *Let* $\Phi^{(j)}=\{f_\gamma^{(j)};1\leqq\gamma\leqq n_j\}$ *be a set of integrals of* $\Delta^k(\mathfrak{P}_j)$ $(j=1,\cdots,r)$ *such that the following conditions* (i)-(iii) *are satisfied:*

( i ) *The 1-forms* $df_\gamma^{(j)}$ $(1\leqq\gamma\leqq n_j, j=1,\cdots,r)$ *are linearly independent modulo* $\Omega$ *around* $\tilde{P}_0$.

(ii) $\Phi^{(j)}$ *contains a complete set of principal integrals of* $\Delta^k(\mathfrak{P}_j)$ $(j=1,\cdots,r)$.

(iii) *For each* $j=1,\cdots,r$, *the manifold* $\mathcal{R}_k[\Phi^{(j)}]$ *admits a submanifold* $\mathcal{F}^{(j)}$ *passing through* $\tilde{P}_0$ *and satisfying* (a) $\mathcal{F}^{(j)}$ *is swept out by a family*

$\{\mathcal{M}_t^l; t\in T\}$ *of non-characteristic $l$-dimensional integral manifolds of* $\Sigma(\mathcal{R}_k)$, (b) *for a set* $\{g_\delta^{(j)}; 1\leqq\delta\leqq m_j\}$ *of integrals of* $\Delta^k(\mathfrak{P}_j)$ *which together with* $\Phi^{(j)}$ *forms a fundamental system of the integrals, the* 1*-forms* $dg_\delta^{(j)}$ *restricted to* $\mathcal{F}^{(j)}$ *form a basis of the module of* 1*-forms on* $\mathcal{F}^{(j)}$ $(j=1, \cdots, r)$.

*Then* $\mathcal{S}_k=\mathcal{R}_k[\Phi^{(1)}, \cdots, \Phi^{(r)}]$ *is an involutive system of partial differential equations of order $k$ defined on a neighborhood of* $\tilde{P}_0$, *and its characteristic module* $\tilde{M}_{\tilde{P}}$ *at* $\tilde{P}\in\mathcal{S}_k$ *admits an irredundant primary decomposition* $\tilde{M}_{\tilde{P}}=\bigcap_{j=r+1}^{\nu} Q_{j,P}$ *where* $P=\pi_1^k(\tilde{P})$.

The proof of this theorem contains the most crucial discussions in this paper. In the proof, besides the facts stated already, we shall also use the following proposition due to Kuranishi [14] (cf. Matsuda [15]).

PROPOSITION 5.3. *If* $\mathcal{R}_1$ *is an involutive system of order* 1 *on* $\mathcal{E}$, *then there exists a fibered chart* $(x, y)$ *of* $\mathcal{E}$ *such that the coordinates* $(x, y, p)$ *of* $J_1$ *associated with it enables us to describe* $\mathcal{R}_1$ *locally around a point* $P_0\in\mathcal{R}_1$ *by the equations*

$$\phi_\alpha^0=0\ \ (\alpha=1, \cdots, \kappa_0),\ \phi_\alpha^1=0\ \ (\alpha=1, \cdots, \kappa_1),\ \cdots,\ \phi_\alpha^n=0\ \ (\alpha=1, \cdots, \kappa_n)$$

*in which the following hold*: (i) $0\leqq\kappa_0\leqq\kappa_1\leqq\cdots\leqq\kappa_n\leqq m$, *and* $\kappa_i=m-s_i$ $(1\leqq i\leqq n)$; (ii) *each* $\phi_\alpha^i$ *takes the form* $\phi_\alpha^i=p_\alpha^i-\psi_\alpha^i\ ('p^i, \cdots, 'p^0, x)$, *where* $'p^i=(p_{\kappa_i+1}^i, \cdots, p_m^i)$, $p_\alpha^0=y_\alpha$; (iii) *each prolongation* $\mathcal{R}_k$ *of* $\mathcal{R}_1$ *can be described by*

$$\begin{cases}\phi_\alpha^i=0 & (\alpha=1, \cdots, \kappa_i,\ i=0, 1, \cdots, n),\\ \partial_\#^{i_1}\partial_\#^{i_2}\cdots\partial_\#^{i_q}\phi_\alpha^i=0 & (\alpha=1, \cdots, \kappa_i,\ 1\leqq q<k, 1\leqq i_1\leqq\cdots\leqq i_q\leqq i\leqq n).\end{cases}$$

A fibered chart $(x, y)$ given by Proposition 5.3 is said to be *regular with respect to* $\mathcal{R}_1$.

Under the circumstances of Proposition 5.3, one can define the *Cartan character* $s_0$ *of order* 0 *of* $\mathcal{R}_1$ by $s_0=m-\kappa_0$.

*Hereafter we shall always assume that the hypotheses* (H-1) *and* (H-2) *hold.* By $(x, y, p)$ we shall indicate a coordinate system of $J_k$ associated with a fibered chart which is regular with respect to $\mathcal{R}_1$.

For brevity we denote $\Theta_k=\Theta(\mathcal{R}_k)$. It is a Pfaffian system on $\mathcal{R}_k$ generated by the restrictions of the contact forms $\theta_\alpha^{I_q}$ $(1\leqq\alpha\leqq m, 0\leqq q< k, I_q\in S_q)$ (cf. § 3). Let $S_q^\#$ denote the set $\{(i_1, \cdots, i_q)\in S_q; 1\leqq i_1\leqq\cdots\leqq i_q\leqq l\}$. Using the equations of $\mathcal{R}_k$ given by Proposition 5.3, we can show without any difficulty that the 1-forms

$$\theta_\alpha^{I_0}\ (m-s_0<\alpha\leqq m-s_1),\quad \theta_\alpha^{I_q}\ (m-s_1<\alpha\leqq m,\ 0\leqq q<k,\ I_q\in S_q^\sharp)$$

restricted to $\mathcal{R}_k$ form a local basis of $\Theta_k$. Our first step to prove Theorem 5.2 is to find a more convenient basis of $\Theta_k$.

Let $\Omega(\mathfrak{P}_j)$ denote the module of sections of $c(\mathfrak{P}_j)^\perp$ over $\mathcal{U}$. We shall regard an element of $\Omega(\mathfrak{P}_j)$ as a 1-form on $\mathcal{R}_k$.

LEMMA 5.4. *Assume that, for each* $j=1,\cdots,\nu$, *we are given a basis* $\{\omega_1^{(j)},\cdots,\omega_l^{(j)}\}$ *of* $\Omega(\mathfrak{P}_j)$ *on a neighborhood of a point* $\tilde{P}_0\in\mathcal{R}_k$ *over* $P_0$. *Then we can find a basis of* $\Theta_k$ *around* $\tilde{P}_0$

$$\theta_{\alpha,I_0}^{(0)}\ (1\leqq\alpha\leqq\mu_0),\ \theta_{\alpha,I_q}^{(j)}\ (1\leqq\alpha\leqq\mu_j,\ 0\leqq q<k,\ I_q\in S_q^\sharp;\ j=1,\cdots,\nu)$$

*where* $\mu_0=s_0-s_1$, $\mu_j=\mu(L_{P_0}/Q_{j,P_0})$ *such that*

(0) *for each* $\tilde{P}\in\mathcal{R}_k$ *near* $\tilde{P}_0$, $\theta_{\alpha,I_0}^{(0)}(\tilde{P})\in T_y^*\mathcal{E}\subset T_{\tilde{P}}^*\mathcal{R}_k$ $(y=\pi_0^k(\tilde{P}))$, *and* $\varepsilon_0^*(\theta_{\alpha,I_0}^{(0)}(\tilde{P}))\in M_{0,P}$ ($\varepsilon_0^*$ *being the map defined in* § 3),

(i) $d\theta_{\alpha,I_0}^{(0)}\equiv 0\quad (\mathrm{mod}\,\theta_{\beta,I_0}^{(j)}\ (1\leqq\beta\leqq\mu_j,\ j=0,1,\cdots,\nu))$.

(ii) *When* $0\leqq q\leqq k-2$,

$$d\theta_{\alpha,I_q}^{(j)}\equiv\sum_{i=1}^{l}\theta_{\alpha,I_qi}^{(j)}\wedge\omega_i^{(j)}\ (\mathrm{mod}$$

$(※)_{q+1}$ $\theta_{\beta,I_0}^{(0)}\ (1\leqq\beta\leqq\mu_0),\ \theta_{\beta,I_r}^{(j')}\ (1\leqq\beta\leqq\mu_{j'},\ 0\leqq r\leqq q,\ I_r\in S_r^\sharp,\ j'=1,\cdots,\nu))$,

*where the* $\theta_{\alpha,I_r}^{(j)}$ *are assumed to be defined for any* $I_r=(i_1,\cdots,i_r)$ *in such a way that they are symmetric with respect to any transposition of* $i_1,\cdots,i_r$,

(iii) *for each* $A\in T_{k-1}^{(j)}=\{(\alpha,I_{k-1});\ 1\leqq\alpha\leqq\mu_j,\ I_{k-1}\in S_{k-1}^\sharp\}$,

$$d\theta_A^{(j)}\equiv\sum_{i=1}^{l}\tau_{A;i}^{(j)}\wedge\omega_i^{(j)}\quad(\mathrm{mod}\,\Theta_k),$$

*where, for each* $j=1,\cdots,\nu$, *the* 1-*form* $\tau_{A;i}^{(j)}$ *are linearly independent modulo* $\Theta_k+\Omega$.

*Under the circumstances, the following are valid:*

(I) *For any* 1-*form* $\theta\in\Theta_k$,

$$d\theta\equiv 0\quad(\mathrm{mod}\,d\theta_A^{(j)}\ (A\in T_{k-1}^{(j)},\ 1\leqq j\leqq\nu),\ \Theta_k).$$

(II) *For each* $j=1,\cdots,\nu$, *the Monge characteristic system* $\Delta^k(\mathfrak{P}_j)$ *is generated by* $\Theta_k$, $\omega_i^{(j)}$, $\tau_{A;i}^{(j)}$ $(1\leqq i\leqq l,\ A\in T_{k-1}^{(j)})$.

(III) *The* 1-*forms* $\tau_{A;i}^{(j)}$ $(1\leqq i\leqq l,\ A\in T_{k-1}^{(j)},\ 1\leqq j\leqq\nu)$ *are linearly inde-*

*pendent modulo* $\Theta_k+\Omega$.

PROOF. We divide the proof into several steps.

(1°) Let $\mathcal{R}_0=\pi_0^1(\mathcal{R}_1)$. It is a submanifold of $\mathcal{E}$ (cf. Proposition 5.3). Let $V(\mathcal{R}_0)$ be the kernel of the bundle epimorphism $\pi_*: T\mathcal{R}_0\to TX$; It is a vector bundle over $\mathcal{R}_0$. Let $V^*$ denote the pull-back of $V^*(\mathcal{E})$ by the projection $\pi_0^1: \mathcal{R}_1\to\mathcal{E}$. Define $V^{(-1)*}$ to be the annihilator bundle in $V^*$ of the pull-back of $V(\mathcal{R}_0)$ by the map $\pi_0^1: \mathcal{R}_1\to\mathcal{R}_0$. Applying Proposition 5.3, we can readily see that $\operatorname{rank} V^{(-1)*}=m-s_0$, and $V^{(-1)*}$ is a subbundle of $M_0$. Let $N_0^{(j)}$ be the family $\{N_{0,P}; P\in\mathcal{U}\}$ $(1\leqq j\leqq\nu)$. By Lemma 5.1 they are vector bundles, which admit $M_0$ as a subbundle. Let us choose vector subbundles $V^{(j)*}$ of $V^*$ over $\mathcal{U}$ such that $M_0=V^{(-1)*}\oplus V^{(0)*}$, $N_0^{(j)}=M_0\oplus V^{(j)*}$ $(1\leqq j\leqq\nu)$. Then, in virtue of Lemma 5.1 and Proposition 2.2, $V^*=\bigoplus_{j=-1}^{\nu} V^{(j)*}$, $\operatorname{rank} V^{(j)*}=\mu_j$.

(2°) We denote the module of sections of a vector bundle $W$ by $\Gamma(W)$, and we call a function on $J_k$ obtained by pulling back a function on $J_q$ a function of $q$-jets. From Proposition 5.3 and Theorem 1.3 (II), we can readily deduce the following:

(a) There exists a set $\{\chi_a; a=1, \cdots, m-s_0\}$ of functions of 0-jets defined around $\tilde{P}_0$ such that each $\chi_a$ vanishes everywhere on $\mathcal{R}_k$, and that the elements $\left\{\sum_{\alpha=1}^{m}\partial\chi_a/\partial y_\alpha dy_\alpha\right\}$ form a basis of $\Gamma(V^{(-1)*})$ around $P_0$.

(b) For each $q=1, 2, \cdots, k$, there exists a set $\{\phi_a; a=1, \cdots, \rho_q\}$ of functions of $q$-jets defined around $\tilde{P}_0$ such that $\phi_a$'s vanish everywhere on $\mathcal{R}_k$, and that the elements $\{\varepsilon_q^*(d\phi_a)\}$ gives a basis of $\Gamma(M_q)$ around $P_0$. (Here it is tacitly included in the condition that the coefficients of $dp_\alpha^{I_q}$ of the 1-forms $d\phi_a$ are functions of 1-jets.)

(3°) For each $j=1, \cdots, \nu$, we adjoin the $\omega_i^{(j)}$ $(1\leqq i\leqq l)$ $n-l$ elements $\omega_i^{(j)}$ $(l<i\leqq n)$ of $\Omega$ so that they form a local basis of $\Omega$. Let $\{\boldsymbol{v}_i^{(j)}; 1\leqq i\leqq n\}$ be the basis of $\Gamma((\pi_{-1}^1|_{\mathcal{R}_1})^*TX)$ dual to $\{\omega_i^{(j)}\}$. In terms of the local coordinates, they have the expressions

$$\boldsymbol{v}_i^{(j)}=\sum_{i'=1}^{n} a_{ii'}^{(j)}(P)\partial/\partial x_{i'} \qquad (i=1, \cdots, n),$$

the coefficients $a$'s being functions defined locally on $\mathcal{R}_1$.

Let $\{\boldsymbol{u}_\beta^{(j)}; 1\leqq\beta\leqq\mu_j\}$ be a basis of $\Gamma(V^{(j)*})$ around $P_0$ $(j=-1, 0, 1, \cdots, \nu)$, where $\mu_{-1}=m-s_0$. In terms of the local basis $\{dy_\alpha; 1\leqq\alpha\leqq m\}$ of $\Gamma(V^*)$, they are described by

$$u_\beta^{(j)}=\sum_{\alpha=1}^{m} b_{\beta\alpha}^{(j)}(P)dy_\alpha \qquad (1\leqq\beta\leqq\mu_j;\ -1\leqq j\leqq\nu).$$

We write $\tilde{S}_q=\{(i_1, \cdots, i_q);\ 1\leqq i_1, \cdots, i_q\leqq n\}$ if $q\geqq 1$, $\tilde{S}_0=\{I_0\}=S_0$. Let us define the 1-forms on $\mathcal{R}_k$

$$\theta_{\beta,I_q}^{(j)}=\sum_{1\leqq\alpha\leqq m}\ \sum_{I'_q\in\tilde{S}_q} a_{I'_q,I_q}^{(j)} b_{\beta\alpha}^{(j)}\theta_\alpha^{I'_q}$$
$$(\beta=1, \cdots, \mu_j, 0\leqq q<k, I_q\in\tilde{S}_q;\ j=-1, 0, 1, \cdots, \nu)$$

where $a_{I'_q,I_q}^{(j)}=a_{i'_1 i_1}^{(j)}\cdots a_{i'_q i_q}^{(j)}$ when $I_q=(i_1, \cdots, i_q)$, $I'_q=(i'_1, \cdots, i'_q)$, $q\geqq 1$, $a_{I'_0,I_0}^{(j)}=1$, and the contact forms $\theta_\alpha^{I_q}$ are assumed to be defined for any $I_q\in\tilde{S}_q$ in such a manner that they are symmetric with respect to any permutation of $I_q$. Clearly $\theta_{\beta,I_q}^{(j)}=\theta_{\beta,I'_q}^{(j)}$ if $I'_q$ is a permutation of $I_q$, and the 1-forms $\theta$'s just defined generate the Pfaffian system $\Theta_k$ around $\tilde{P}_0$.

Let $\Theta_q$ denote the submodule of $\Theta_k$ generated by the contact forms $\{\theta_\alpha^{I_r};\ 1\leqq\alpha\leqq m, 0\leqq r<q, I_r\in S_r\}$ restricted to $\mathcal{R}_k$. Then

$$d\theta_{\beta,I_q}^{(j)}\equiv\sum_{i=1}^{n}\theta_{I_q i}^{(j)}\wedge\omega_i^{(j)} \qquad (\mathrm{mod}\ \Theta_{q+1}).$$

Furthermore the following are valid:

$$(5.1)\quad \begin{cases}\theta_{\alpha,I_0}^{(-1)}\equiv 0;\ \theta_{\alpha,I_q}^{(-1)},\ \theta_{\alpha,I_q}^{(0)}\equiv 0 \quad (\mathrm{mod}\ \Theta_{q+1}) \text{ if } q\geqq 1;\\ \theta_{\alpha,I_q}^{(j)}\equiv 0 \quad (\mathrm{mod}\ \Theta_{q+1}) \text{ if } 1\leqq j\leqq\nu, q\geqq 1, \text{ and any permutation of } I_q\notin S_q^\sharp.\end{cases}$$

In fact, using (a) in (2°), we readily see that $\theta_{\alpha,I_0}^{(-1)}\equiv 0\ (\mathrm{mod}\ \Omega)$. Since $\varepsilon_q^*(\theta_{\alpha,I_q}^{(j)})\in\Gamma(M_q)$, we can apply (b) in (1°) to prove that $\theta_{\alpha,I_q}^{(j)}\equiv 0\ (\mathrm{mod}\ \Theta_{q+1}+\Omega)$. Observing that, for any point $\tilde{P}\in\mathcal{R}_k$, there is $E_n\in I^n(\Theta_{q+1}, \Omega)$ of origin $P$ (Theorems 3.B and 3.C), we can show that (5.1) holds true.

Using (5.1), we can verify by induction on $q=1, 2, \cdots, k$ that $\Theta_q$ is generated by $(\divideontimes)_q$. The number of 1-forms in $(\divideontimes)_k$ is equal to $\mathrm{rank}\,\Theta_k$. Hence $(\divideontimes)_k$ is a basis of $\Theta_k$ around $\tilde{P}_0$. It is now clear that (i), (ii) hold true. To prove (iii), we set

$$\zeta_{\beta,I_k}^{(j)}=\sum_{1\leqq\alpha\leqq m}\ \sum_{I'_k\in\tilde{S}_k} a_{I'_k,I_k}^{(j)} b_{\beta,\alpha}^{(j)} dp_\alpha^{I'_k}$$

($dp$'s being their restrictions to $\mathcal{R}_k$). Then

$$d\theta_{\beta,I_{k-1}}^{(j)}\equiv\sum_{i=1}^{n}\zeta_{\beta,I_{k-1}i}^{(j)}\wedge\omega_i^{(j)} \qquad (\mathrm{mod}\ \Theta_k).$$

In exactly the same manner as we prove (5.1), we can show that

$$\zeta^{(j)}_{\beta,I_{k-1}i}\equiv 0 \quad (\mathrm{mod}\, \Theta_k+\Omega) \quad \text{if } l<i\leqq n.$$

Accordingly we have

$$d\theta^{(j)}_{\beta,I_{k-1}}\equiv \sum_{i=1}^{l}\zeta^{(j)}_{\beta,I_{k-1}i}\wedge\omega^{(j)}_i+\sum_{i,i'=1}^{n}C^{ii'}_{\beta,I_{k-1}}\omega^{(j)}_i\wedge\omega^{(j)}_{i'} \quad (\mathrm{mod}\, \Theta_k).$$

The existence of $E_n\in I^n(\Theta_k, \Omega)$ of origin $P\in\mathcal{R}_k$ implies that there is an integral element $E_{n-l}$ of $\Theta_k$ with origin $P$ on which $\omega^{(j)}_i$ $(l<i\leqq n)$ are linearly independent. Hence $C^{ii'}_{\beta,I_{k-1}}$ must vanish at $P$ if $i$ and $i'>l$. Denoting the 1-form $\zeta^{(j)}_{\beta,I_{k-1}i}$ added a suitable linear combination of $\omega^{(j)}_i$ by $\tau^{(j)}_{\beta,I_{k-1};i}$, we see that (ii) holds.

Let us prove the last part. (I) is an immediate consequence of (i)-(iii). From (iii) we know that the 2-forms $\sum_{i=1}^{n}\tau^{(j)}_{A;i}\wedge\omega^{(j)}_i$ vanish if restricted to any element $E_n\in I^n(\Theta_k, \Omega)$. Since $\langle v, \omega^{(j)}_i\rangle=0$ $(1\leqq i\leqq l)$ for any Monge characteristic vector $v$ with $(\pi^k_{-1})_*(v)\in c(\mathfrak{P}_j)$, it follows that at any such vector $v$ in $E_n$ annihilate the 1-forms $\tau^{(j)}_{A;i}$. From the very definition, we find that they belong to $\Delta^k(\mathfrak{P}_j)$. The 1-forms $\tau^{(j)}_{A;i}$, $\omega^{(j)}_i$ $(1\leqq i\leqq l, A\in T^{(j)}_{k-1})$ in $\Delta^k(\mathfrak{P}_j)$ are linearly independent $\mathrm{mod}\,\Theta_k$, and the number of them is equal to $\mathrm{rank}\,\Delta^k(\mathfrak{P}_j)-\mathrm{rank}\,\Theta_k$ (cf. Lemmas 4.1 and 5.1), which imply (II). We finally show (III). For each $j=1, \cdots, \nu$, the sections $\tilde{\varepsilon}^*_k(\tau^{(j)}_{A;i})$ are linearly independent around $\tilde{P}_0$. Lemmas 4.1 and 5.1 indicate that $\sum_{j=1}^{\nu}\tilde{\varepsilon}^*_k(\Delta^k(\mathfrak{P}_j))$ is a direct sum. Therefore (III) holds. Q.E.D.

LEMMA 5.5. *Under the circumstances of Lemma* 5.4, *if* $\Delta^k(\mathfrak{P}_\beta)$ *is principally integrable around* $\tilde{P}_0$, *then we can find* 2-*forms* $\eta^{(\beta)}_A$ $(A\in T^{(\beta)}_{k-1})$ *defined around* $\tilde{P}_0$ *on* $\mathcal{R}_k$ *such that*

( i ) $\eta^{(\beta)}_A\equiv d\theta^{(\beta)}_A \quad (\mathrm{mod}\,\Theta_k)$,

( ii ) $d\eta^{(\beta)}_A\equiv \sum_{B\in T^{(\beta)}_{k-1}}\zeta^B_A\wedge\eta^{(\beta)}_B \quad (\mathrm{mod}\,\Theta_k)$,

(iii) *For any vector field* $v$ *in* $\hat{Z}^k(\mathfrak{P}_\beta)$, *the interior products* $v\,\lrcorner\,\eta^{(\beta)}_A$ *vanish.*

PROOF. Let $\mu^{(\beta)}$ denote $\mathrm{rank}\,\Delta^k(\mathfrak{P}_\beta)-\mathrm{rank}\,\Theta_k$. By the principal integrability, there exist integrals $\{g_\gamma; 1\leqq\gamma\leqq\mu^{(\beta)}\}$ such that $\Delta^k(\mathfrak{P}_\beta)$ is generated by the 1-forms $\{dg_\gamma\}$ and $\Theta_k$ around $\tilde{P}_0$. We can choose 1-forms $\tau'_{A;i}$, $\omega'_i$ such that they can be expressed as linear combinations of $\{dg_\gamma\}$ with function coefficients, and that $\tau'_{A;i}\equiv\tau^{(\beta)}_{A;i}$, $\omega'_i\equiv\omega^{(\beta)}_i$ $(\mathrm{mod}\,\Theta_k)$. Define $\eta^{(\beta)}_A=\sum_{i=1}^{l}\tau'_{A;i}\wedge\omega'_i$. Then the 2-forms $\eta^{(\beta)}_A$ clearly satisfy (i) and (iii). It remains to show that (ii) holds. We can choose 1-forms $\omega_i$ $(l<i\leqq n)$ in $\Omega$ so that the 1-forms $\{dg_\gamma\}$, $\{\tau_A^{(j)}; A\in T_{k-1}^{(j)}, 1\leqq i\leqq l, j\neq\beta\}$, $\{\theta_{\alpha,I_q}^{(j)}\}$, $\{\omega_i\}$ form a local basis of the module of 1-forms on $\mathcal{R}_k$. Bearing in mind Lemma 5.4, we have

$$d\eta_A^{(\beta)}\equiv\sum_{B\in T_{k-1}^{(\beta)}}\zeta_A^B\wedge\eta_B^{(\beta)}+\sum_{j\neq\beta}\sum_{B\in T_{k-1}^{(j)}}\zeta_{A,j}^B\wedge\eta_B^{(j)}\qquad(\mathrm{mod}\,\Theta_k),$$

where $\eta_B^{(j)}=\sum_{i=1}^{l}\tau_{B;i}^{(j)}\wedge\omega_i^{(j)}$ $(j\neq\beta)$. Here we may assume that the $\zeta_{A,j}^B$'s contain no term involving the $\theta$'s. Fixing $\beta$ and $A$, we denote by $\tilde\omega$ the second term on the right side. The above formula implies that $\tilde\omega\equiv 0$ $(\mathrm{mod}\, dg_\gamma\wedge dg_\delta\ (1\leqq\gamma,\delta\leqq\mu^{(\beta)}), \Theta_k)$. Let us decompose $\zeta_{A,j}^B={}'\zeta_{A,j}^B+{}''\zeta_{A,j}^B$ with ${}'\zeta_{A,j}^B\equiv 0$ (mod $dg_\gamma$'s), ${}''\zeta_{A,j}^B\equiv 0$ $(\mathrm{mod}\,\tau_A^{(j)}\ (j\neq\beta), \omega_i)$. Denote by $\tilde\omega'$, $\tilde\omega''$ the 3-forms obtained from $\tilde\omega$ by replacing the $\zeta$'s by the corresponding $'\zeta$'s and $''\zeta$'s, respectively. Since the terms containing $\tau_{A;i}^{(j)}\wedge\tau_{A';i'}^{(j')}$ does not appear in $\tilde\omega'$, $\tilde\omega''$ must be zero. In other words, we may assume that the $\zeta_{A,j}^B$'s are linear combinations only of the $dg_\gamma$'s. If some $\zeta_{A,j}^B$ is not zero, after expressing in terms of the basis of 3-forms, $\tilde\omega$ contains a non-zero term const. $\zeta_{A;i}^{(j)}\wedge dg_\gamma\wedge\omega_{i'}$. This contradicts with the property mentioned above. Thus $\zeta_{A,j}^B$'s must be zero, and hence (ii) holds true.

Q.E.D.

LEMMA 5.6. *Assume that $\Delta^k(\mathfrak{P}_\beta)$ is p-regular and principally integrable around $\tilde P_0$. Let $\eta_A^{(\beta)}$ $(A\in T_{k-1}^{(\beta)})$ be 2-forms given by Lemma 5.5. Given a non-characteristic l-dimensional integral manifold $\mathcal{M}^l$ of $\Theta_k$ passing through $\tilde P_0$, we set $\mathcal{N}=\bigcup_{\tilde P\in\mathcal{M}^l}C_{\tilde P}$, where $C_{\tilde P}$ denotes the integral manifold of the involutive distribution $\hat Z^k(\mathfrak{P}_\beta)$ passing through $\tilde P$. Then the manifold $\mathcal{N}$ possesses the following property: For any $\tilde P\in\mathcal{N}$ sufficiently near $\tilde P_0$, there exists an n-dimensional contact element $E_n\subset T_{\tilde P}\mathcal{N}$ with $\dim(\pi_{k-1}^k)_*E_n=n$ such that the 1-forms in $\Theta_k$ and the 2-forms $\eta_A^{(\beta)}$ vanish on $E_n$.*

PROOF. For brevity we write: $\Delta=\Delta^k(\mathfrak{P}_\beta)$, $\eta_A=\eta_A^{(\beta)}$, $Z=Z^k(\mathfrak{P}_\beta)$, $\hat Z=\hat Z^k(\mathfrak{P}_\beta)$. Let $\{u_\delta; 1\leqq\delta\leqq\mu\}$ be a complete set of principal integrals of $\Delta$, and $\{w_i; 1\leqq i\leqq l\}$ be integrals of $\Delta$ such that $\Delta$ is generated by the 1-forms $du_\delta$'s, $dw_i$'s, $\Theta_k$. We can choose integrals $\{g_\gamma; 1\leqq\gamma\leqq\mu'\}$ so that the integrals $u_\delta$'s, $w_i$'s, $g_\gamma$'s form a fundamental set of integrals of $\Delta$. Then the manifold $\mathcal{N}$ can be described by

$$\begin{cases} \tilde{u}_\delta = u_\delta - \phi_\delta(w_1, \cdots, w_l) = 0 & (\delta = 1, \cdots, \mu), \\ \tilde{g}_\gamma = g_\gamma - \psi_\gamma(w_1, \cdots, w_l) = 0 & (\gamma = 1, \cdots, \mu'). \end{cases}$$

We may assume that $\tilde{u}_\delta = u_\delta$, $\tilde{g}_\gamma = g_\gamma$. From the integrals $g_\gamma$, we have the 1-forms $\theta_\gamma \in \Theta_k$ defined by

$$dg_\gamma \equiv \theta_\gamma \quad (\mathrm{mod}\, du_\delta\text{'s},\ dw_i\text{'s}) \quad (1 \leqq \gamma \leqq \mu').$$

Clearly the 1-forms $\theta_\gamma$ are linearly independent. Furthermore the following hold true.

(5.2) $\quad \langle \hat{v}, \theta_\gamma \rangle = 0$ for any $\hat{v} \in \hat{Z}$; $v \lrcorner d\theta_\gamma \equiv 0 \quad (\mathrm{mod}\, \Theta_k)$ for any $v \in Z$.

In fact, the first assertion is obvious. To prove the second, we use a basis $\{\theta^{(j)}_{\alpha, I_q}\}$ of $\Theta_k$ given by Lemma 5.4. For some functions $C^\alpha_j$, we have $\theta_\gamma \equiv \sum_{j,\alpha} C^\alpha_j \theta^{(j)}_{\alpha, I_{k-1}}$ $(\mathrm{mod}(\divideontimes)_{k-1})$. Since $d\theta_\gamma \in \Delta$, taking exterior differentiation, we get $\sum_{j,\alpha} C^\alpha_j d\theta^{(j)}_{\alpha, I_{k-1}} \equiv 0$ $(\mathrm{mod}\, \Delta)$. Taking into account of the properties of the basis, we can show that the coefficients $C^\alpha_j$ with $j \neq \beta$ must vanish at each $\tilde{P} \in \mathcal{R}_k$. Thus $\theta_\gamma$ is a linear combination of the $\theta^{(\beta)}_{\alpha, I_{k-1}}$'s and $(\divideontimes)_{k-1}$. This implies our required assertion.

Let us choose $\theta'_\varepsilon \in \Theta_k$ $(1 \leqq \varepsilon \leqq \nu')$ so that $\theta'_\varepsilon$'s, $\theta_\gamma$'s form a basis of $\Theta_k$ around $\tilde{P}_0$. Then the 1-forms $du_\delta$'s, $dw_i$'s, $dg_\gamma$'s, $\theta'_\varepsilon$'s are linearly independent. Since the restrictions of $dw_i$ to $\mathcal{M}^l$ are linearly independent, we can construct vector fields $e_i$ $(1 \leqq i \leqq l)$ tangent to $\mathcal{N}$ such that $\langle e_i, \theta'_\varepsilon \rangle = 0$ on $\mathcal{N}$ $(1 \leqq \varepsilon \leqq \nu')$, and that the $e_i(\tilde{P})$'s span the tangent space $T_{\tilde{P}}\mathcal{M}^l$ for each $\tilde{P} \in \mathcal{M}^l$. Observe that any vector field tangent to $\mathcal{N}$ can be expressed as a linear combination of the $e_i$'s and a field of $\hat{Z}$, and that $\langle e_i, \theta \rangle = 0$ on $\mathcal{M}^l$ for any $\theta \in \Theta_k$.

Let $v$ be a vector field in $Z$. Applying the well-known formulae (Kobayashi-Nomizu [12], p. 36, Proposition 3.11), we have

$$\begin{aligned} v(\langle e_i, \theta_\gamma \rangle) &= e_i(\langle v, \theta_\gamma \rangle) + \langle [v, e_i], \theta_\gamma \rangle + \langle v \wedge e_i, d\theta_\gamma \rangle, \\ v(\langle e_i \wedge e_j, \eta_A \rangle) &= e_i(\langle v \wedge e_j, \eta_A \rangle) + e_j(\langle e_i \wedge v, \eta_A \rangle) + \langle [v, e_i] \wedge e_j, \eta_A \rangle \\ &\quad + \langle [e_i, e_j] \wedge v, \eta_A \rangle + \langle [e_j, v] \wedge e_i, \eta_A \rangle + \langle v \wedge e_i \wedge e_j, d\eta_A \rangle. \end{aligned}$$

For simplicity we write $F^\gamma_i = \langle e_i, \theta_\gamma \rangle$, $F^A_{ij} = \langle e_i \wedge e_j, \eta_A \rangle$, and put $\Phi = \{F^\gamma_i;\ 1 \leqq i \leqq l, 1 \leqq \gamma \leqq \mu'\}$, $\Psi = \{F^A_{ij};\ 1 \leqq i, j \leqq l, A \in T^{(\beta)}_{k-1}\}$. On account of (5.2), (iii) of Lemma 5.5, and the fact $[e_i, v] \equiv 0$ $(\mathrm{mod}\, e_1, \cdots, e_l, \hat{Z})$, we obtain from the formulae

$$(5.3) \qquad \begin{cases} v(F^\gamma_i) \equiv 0 & (\mathrm{mod}\, \Phi) \text{ on } \mathcal{N} \quad (1 \leqq i \leqq l, 1 \leqq \gamma \leqq \mu'), \\ v(F^A_{ij}) \equiv 0 & (\mathrm{mod}\, \Phi, \Psi) \text{ on } \mathcal{N} \quad (1 \leqq i, j \leqq l, A \in T^{(\beta)}_{k-1}). \end{cases}$$

Since $\hat{Z}=\bigcup_{r=0}^{q} p_0^r Z$ for some $q$ (cf. § 4), the equations in (5.3) hold true for any vector field $\hat{v}$ of $\hat{Z}$. Consequently the functions $F_i^{\gamma}$, $F_{ij}^{A}$ remains constant on each integral manifold $\mathcal{C}_P$ of the distribution $\hat{Z}$. Combining this with the fact that the functions vanish on $\mathcal{M}^l$, we conclude that they must vanish everywhere on $\mathcal{N}$.

Let $\tilde{P}\in\mathcal{N}$. There exist Monge characteristic vectors $e_i\in T_{\tilde{P}}\mathcal{N}$ $(l<i\leqq n)$ with $(\pi_{-1}^k)_*(e_i)$'s being linearly independent. Let $E_n$ be a contact element spanned by $e_i(\tilde{P})$ $(1\leqq i\leqq l)$ and $e_i$ $(l<i\leqq n)$. Bearing in mind what we have proved above, we can easily verify that $E_n$ satisfies the required conditions. Q.E.D.

PROPOSITION 5.7. *Assume that the circumstances of Theorem* 5.2 *hold. Then, for any point* $\tilde{P}\in\mathcal{S}_k=\mathcal{R}_k[\Phi^{(1)},\cdots,\Phi^{(r)}]$, *there exists that n-dimensional contact element* $E_n\subset T_{\tilde{P}}\mathcal{S}_k$ *with* $\dim(\pi_{-1}^k)_*E_n=n$ *which is an integral element of* $\Theta_k$, *equivalently, of* $\Sigma(\mathcal{R}_k)$.

PROOF. We shall use a basis $\{\theta_{\alpha,I_q}^{(j)}\}$ of $\Theta_k$ and 1-forms $\tau_{A;i}^{(j)}$ given by Lemma 5.4. Let $\Pi^{(j)}=\{u_\delta^{(j)}:1\leqq\delta\leqq\mu^{(j)}\}$ be a complete set of principal integrals contained in $\Phi^{(j)}$, and $\Psi^{(j)}=\{g_\gamma^{(j)};1\leqq\gamma\leqq m^{(j)}\}$ be the set $\Phi^{(j)}-\Pi^{(j)}$. Let $\{w_i^{(j)};1\leqq i\leqq l\}$ be integrals such that $\Delta^k(\mathfrak{P}_j)$ is generated by the 1-forms $d\Pi^{(j)}$, $dw_i^{(j)}$'s, $\Theta_k$. We define 1-forms $\theta_\gamma^{(j)}\in\Theta_k$ by

$$dg_\gamma^{(j)}\equiv\theta_\gamma^{(j)}\qquad(\mathrm{mod}\,d\Pi^{(j)},\,dw_i^{(j)}\ (1\leqq i\leqq l)).$$

Let $\eta_A^{(j)}$ be 2-forms given by Lemma 5.5 $(1\leqq j\leqq r)$. We denote $\mathcal{R}_k[\Phi^{(j)}]$ by $\mathcal{S}_k^{(j)}$, and the inclusion $\mathcal{S}_k^{(j)}\hookrightarrow\mathcal{R}_k$ by $\iota_j$. Let us prove that $\iota_j^*(\theta_\gamma^{(j)})=0$ $(\gamma=1,\cdots,m^{(j)})$, $\iota_j^*(\eta_A^{(j)})\equiv0$ $(\mathrm{mod}\,\iota_j^*\Theta_k)$ $(A\in T_{k-1}^{(j)})$ $(j=1,\cdots,r)$. Denote by $\mathcal{N}_t$ the submanifold of $\mathcal{S}_k^{(j)}$ defined by $\mathcal{N}_t=\bigcup_{\tilde{P}\in\mathcal{M}_t^l}\mathcal{C}_{\tilde{P}}$. The hypothesis (iii) of Theorem 5.2 implies that $\mathcal{S}_k^{(j)}$ is swept out by $\mathcal{N}_t$ $(t\in T)$. By Lemma 5.6, for each $\tilde{P}\in\mathcal{N}_t$, there is a contact element $E_n\subset T_{\tilde{P}}\mathcal{N}_t$ with $\dim(\pi_{-1}^k)_*E_n=n$ on which the 1-forms in $\Theta_k$ and the 2-forms $\eta_A^{(j)}$ vanish. On the other hand, from the definition and Lemma 5.5, it follow that

$$\theta_\gamma^{(j)}\equiv\sum_{i=1}^{l}C_\gamma^i dw_i^{(j)}\ (\mathrm{mod}\,d\Phi^{(j)}),\ \eta_A^{(j)}\equiv\sum_{1\leqq i<i'\leqq l}C_A^{ii'}dw_i^{(j)}\wedge dw_{i'}^{(j)}\ (\mathrm{mod}\,\Theta_k,\,d\Pi^{(j)}),$$

the $C$'s being functions defined locally on $\mathcal{R}_k$. The existence of such an element $E_n$ on which $dw_i^{(j)}$ $(1\leqq i\leqq l)$ are linearly independent implies that the coefficients $C$'s vanish at each point $P\in\mathcal{S}_k^{(j)}$. Hence our required assertions follow.

The hypothesis (i) implies that the 1-forms $\theta_\gamma^{(j)}$ $(1\leqq\gamma\leqq m^{(j)}, 1\leqq j\leqq r)$ are linearly independent. Therefore we may choose 1-forms $\theta_\varepsilon$ $(1\leqq\varepsilon\leqq m^*)$ in $\Theta_k$ such that the 1-forms $\theta_\gamma^{(j)}$'s, $\theta_\varepsilon$'s form a basis of $\Theta_k$ around $\tilde{P}_0$. Let $\iota: \mathcal{S}_k \hookrightarrow \mathcal{R}_k$ be the inclusion. As is readily seen, the 1-forms

$$\iota^*(dx_i),\ \iota^*(\theta_\varepsilon),\ \iota^*(\tau_{A;i}^{(j)}) \qquad (1\leqq i\leqq l, 1\leqq\varepsilon\leqq m, A\in T_{k-1}^{(j)}, r<j\leqq\nu)$$

form a basis of the module of 1-forms on $\mathcal{S}_k$. Hence, for each $\tilde{P}\in\mathcal{S}_k$, we can construct $n$ vectors $e_1, \cdots, e_n\in T_{\tilde{P}}\mathcal{S}_k$ such that

$$\det(\langle e_i, dx_{i'}\rangle)_{i,i'=1,\dots,n}\neq 0,\ \langle e_i, \theta_\varepsilon(\tilde{P})\rangle=0 \quad (1\leqq\varepsilon\leqq m^*),$$
$$\langle e_i, \tau_{A;i'}^{(j)}(P)\rangle=0 \quad (1\leqq i'\leqq l, A\in T_{k-1}^{(j)}, r<j\leqq\nu).$$

Let $E_n$ be the contact element spanned by $e_1, \cdots, e_n$. Clearly any 1-form of $\Theta_k$ vanish on $E_n$, and so do the 2-forms $d\theta_A^{(j)}$ $(r<j\leqq\nu)$. Furthermore what we have just proved above indicates that the 2-forms $\eta_A^{(j)}$ vanish on $E_n$. From these facts, it also follows that $d\theta$ vanish on $E_n$ for any $\theta\in\Theta_k$. Consequently $E_n$ is an integral element of $\Theta_k$. Q.E.D.

We are now in a position to complete the proof of Theorem 5.2. It is clear that $\mathcal{S}_k$ is a (locally closed) fibered submanifold of $\mathcal{R}_k$. Let $H_k$ denote the symbol of $\mathcal{S}_k$. Denoting the $(k-1)$-th prolongation of $G_1$ by $G_k$ we have

$$H_k=\{\chi\in G_k|_{\mathcal{S}_k};\ \langle\chi, \varepsilon_k^*(df_\gamma^{(j)})\rangle=0 \quad (1\leqq\gamma\leqq n_j, 1\leqq j\leqq r)\}.$$

In virtue of Lemmas 4.1, 5.1, and the assumption (i), $H_{k,\tilde{P}}$ is found to be $\bigcap_{r<j\leqq\nu} C_k(\mathfrak{P}_{j,P})$, where $P=\pi_1^k(\tilde{P})$, and we use the same notation as in §2. Hence, by Proposition 2.4, $H_{k,P}$ is an involutive symbol. Moreover, by Lemmas 4.1 and 5.1, $\dim H_{k,P}=$const., that is, $H_k$ is a vector bundle. It follows that the prolongation $pH_k$ is a vector bundle.

Finally Proposition 5.6 implies that the projection $\pi_k^{k+1}: p\mathcal{S}_k\to\mathcal{S}_k$ is surjective (see Note after Theorem 3.C). Thus we have shown that the system $\mathcal{S}_k$ of order $k$ satisfies the conditions of Theorem 3.1, and hence $\mathcal{S}_k$ is an involutive system. The remaining assertion is an immediate consequence of Proposition 2.4. Q.E.D.

## §6. The method of integration

Let $\mathcal{R}_1$ be an involutive system of order 1 on $\mathcal{E}$. We shall keep the same notations as in §5. We are concerned with its *local* and *smooth* solutions. The solutions of $\mathcal{R}_1$ correspond to $n$-dimensional integral manifold of the differential system $\Sigma(\mathcal{R}_1)$ or of its prolongation $\Sigma(\mathcal{R}_k)$. By an integral manifold $\mathcal{M}$ of $\Sigma(\mathcal{R}_k)$, we shall always mean one such that $(\pi^k_{-1})_* T_P\mathcal{M}$ is of dimension $\dim \mathcal{M}$ for each $P \in \mathcal{M}$.

The Cartan-Kähler theorem asserts that an *analytic* involutive system admits *analytic* solutions (cf. Cartan [3], Kähler [6], Kuranishi [14]). However the existence of smooth solutions of a smooth system $\mathcal{R}_1$ has not yet been shown in general. Here we consider only a system $\mathcal{R}_1$ satisfying the conditions (H-1) and (H-2) stated in § 5. Then the existence of its smooth solutions may be shown by solving the following two problems: (1°) To find a non-characteristic $l$-dimensional integral manifold $\mathcal{M}^l$ of $\Sigma(\mathcal{R}_1)$; (2°) To find an $n$-dimensional integral manifold of $\Sigma(\mathcal{R}_1)$ passing through such a given $\mathcal{M}^l$ (the Cauchy problem). The problem (1°) is easy to solve; it can be solved by using a part of the Cartan-Kähler procedure. The Cauchy problem (2°) is much more crucial; we can solve it by applying the result in our previous work [11]. In this way we are led to the following existence theorem.

THEOREM 6.1. *Assume that* (H-1) *and* (H-2) *hold.* (I) *For any given non-characteristic l-dimensional integral manifold, there exists a unique n-dimensional integral manifold of* $\Sigma(\mathcal{R}_1)$ *passing through it.*

(II) $\mathcal{R}_1$ *admits smooth solutions; the solution can be parametrized by* $s_0$ *arbitrary constants and* $s_l$ *arbitrary functions of l variables* (*A* $C^\infty$ *Cartan-Kähler theorem*).

Let us now consider our main problem: (the integration problem) To reduce the solution of $\mathcal{R}_1$ to integrating *ordinary* differential equations. Our method of integration is obtained by seeking sufficient conditions under which the problems (1°), (2°) may be solved by integrating ordinary differential equations and algebraic operations.

THEOREM 6.2. *Let* $\mathcal{R}_1$ *be a system satisfying* (H-1) *and* (H-2). *Assume that, for some integer* $k \geqq 1$, $\nu - 1$ *Monge characteristic systems, say,* $\Delta^k(\mathfrak{P}_1), \cdots, \Delta^k(\mathfrak{P}_{\nu-1})$ *are p-regular and principally integrable around a point* $\tilde{P}_0 \in \mathcal{R}_k$ *over* $P_0$. *Then any solution of* $\mathcal{R}_1$ *of which k-jet at* $x_0 = \pi^k_{-1}(\tilde{P}_0)$ *is sufficiently close to* $\tilde{P}_0$ *can be obtained by solving ordinary differential equations.*

REMARK. If $\Delta^k(\mathfrak{P}_j)$ is principally integrable, so is $\Delta^{k+q}(\mathfrak{P}_j)$ for any $q \geqq 1$ (Proposition 4.4 and Lemma 5.1). Accordingly, making use of Monge characteristic systems *of equal order* gives rise to no essential assumption in the theorem except for the regularity condition.

If $\nu=1$ in the theorem, the assertion is substantially a consequence of the following fact.

PROPOSITION 6.3. *Let $\mathcal{S}_k$ be an involutive system of order $k$ on $\mathcal{E}$. Denote by $l$ the greatest integer $i$ such that its Cartan character of order $i$ is non-zero. Assume that the following four conditions are satisfied:* (i) $1\leqq l<n$, (ii) *the characteristic module $N_P$ of $\mathcal{S}_k$ at $P\in\mathcal{S}_k$ is itself $\mathfrak{P}_P$-primary,* (iii) $\mathfrak{P}_P L_P\subset N_P$, (iv) *the family $\{c(\mathfrak{P}_P); P\in\mathcal{S}_k\}$ is a vector bundle of rank $n-l$.*

*Then through any point $P\in\mathcal{S}_k$, there passes a unique $(n-l)$-dimensional Cauchy-Cartan characteristic manifold of the differential system $\Sigma(\mathcal{S}_k)$. Moreover it is obtained by solving ordinary differential equations.*

PROOF. Let $\Gamma$ be the Cauchy-Cartan characteristic system of $\Sigma(\mathcal{S}_k)$ (cf. Cartan [3]). Applying the theorem in Kakié [9], we can readily show that $\Gamma$ is a Pfaffian system of corank $n-l$. (Indeed $\Gamma$ coincides with the Monge characteristic system $\Delta^k(\mathfrak{P})$ of $\mathcal{S}_k$.) As is well-known, $\Gamma$ is completely integrable, and hence its integral manifolds may be obtained by integrating ordinary differential equations (cf. Cartan [2-3]).
Q.E.D.

PROOF OF THEOREM 6.3. Let $\sigma$ be a solution of $\mathcal{R}_1$. What we must prove is that $\sigma$ is obtained by solving ODE's (=ordinary differential equations). Let $\mathcal{M}^n$ be the integral manifold of $\Sigma(\mathcal{R}_1)$ corresponding to $\sigma$, and $\tilde{\mathcal{M}}^n$ be that of $\Sigma(R_k)$; $\tilde{\mathcal{M}}^n$ is obtained from $\mathcal{M}^n$ by prolongation. We may assume that $P_0\in\mathcal{M}^n$, $\tilde{P}_0\in\tilde{\mathcal{M}}^n$. It suffices to prove that $\tilde{\mathcal{M}}^n$ can be obtained by solving ODE's.

Let $\{\theta^{(j)}_{\alpha,I_q}\}$ be a basis of $\Theta_k$ around $\tilde{P}_0$ given by Lemma 5.4. For each $j=1,\cdots,\nu-1$, by integrating ODE's, we can obtain a fundamental system $\{g^{(j)}_\gamma\ (1\leqq\gamma\leqq n_j),\ h^{(j)}_\delta\ (1\leqq\delta\leqq q_j),\ w^{(j)}_i\ (1\leqq i\leqq l)\}$ of integrals of $\Delta^k(\mathfrak{P}_j)$ defined around $\tilde{P}_0$ such that (i) the 1-forms $dg^{(j)}_\gamma$ are linearly independent modulo $\theta^{(0)}_{\alpha,I_0}$ $(1\leqq\alpha\leqq\mu_0)$ and $\Omega$, (ii) $dh^{(j)}_\delta(\tilde{P}_0)\in\{\text{span of } \theta^{(0)}_{\alpha,I_0}(P_0)\ (1\leqq\alpha\leqq\mu_0)\}$, (iii) $dw^{(j)}_i(\tilde{P}_0)\in\{\text{span of } \theta^{(0)}_{\alpha,I_0}(P_0)\text{'s and } \omega(P_0)\ (\omega\in\Omega(\mathfrak{P}_j)\}$.

Let us take a non-characteristic $l$-dimensional integral manifold $\mathcal{M}^l_0$ lying on the $\mathcal{M}^n$ and passing through $P_0$.

SUBLEMMA 1. *The manifold $\mathcal{M}^l_0$ can be obtained by solving* ODE's. *Moreover, for each $j=1,\cdots,\nu-1$, by solving* ODE's *and algebraic opera-*

*tions, we can construct a family* $\{\mathcal{M}_t^l\}$ *of non-characteristic l-dimensional integral manifolds depending smoothly on* $q_j$*-parameters* $t \in T$, *where* $T$ *is an open neighborhood of the origin of the* $q_j$*-dimensional Euclidean space, such that* $\mathcal{M}_t^l = \mathcal{M}_0^l$ *if* $t$ *is the origin, and that* $\mathcal{N}^{(j)} = \bigcup_{t \in T} \mathcal{M}_t^l$ *is a* $(q_j + l)$*-dimensional manifold with* $dh_\delta^{(j)}(P_0)$*'s and* $dw_i^{(j)}(P_0)$*'s (considered as vectors in* $T_{P_0}^* \mathcal{R}_1$*) being linearly independent on* $T_{P_0}^* \mathcal{N}^{(j)}$. (The proof will be given later.)

Denote by $\tilde{\mathcal{M}}_t^l$ the integral manifold of $\Sigma(\mathcal{R}_k)$ obtained from $\mathcal{M}_t^l$ by prolongation. (Note that each $\mathcal{M}_t^l$ is non-characteristic.) We put $\mathcal{F}^{(j)} = \bigcup_{t \in T} \tilde{\mathcal{M}}_t^l$. Then $\mathcal{F}^{(j)}$ is a $(q_j + l)$-dimensional manifold on which the $dh_\delta^{(j)}$'s, $dw_i^{(j)}$'s are linearly independent around $\tilde{P}_0$. Hence we can construct functions $f_\gamma^{(j)} = g_\gamma^{(j)} - \chi_\gamma^{(j)}$ ($h^{(j)}$'s, $w^{(j)}$'s) such that $f_\gamma^{(j)}$'s vanish everywhere on $\mathcal{F}^{(j)}$ $(1 \leqq j < \nu)$.

SUBLEMMA 2. *The 1-forms* $df_\gamma^{(j)}$ $(1 \leqq \gamma \leqq n_j, 1 \leqq j < \nu)$ *are linearly independent modulo* $\Omega$ *around* $\tilde{P}_0$.

Let us put $\Phi^{(j)} = \{f_\gamma^{(j)}; 1 \leqq \gamma \leqq n_j\}$. Then we can apply Theorem 5.2 to deduce that the system $\mathcal{S}_k = \mathcal{R}_k[\Phi^{(1)}, \cdots, \Phi^{(\nu-1)}]$ is an involutive system of order $k$, and that $\mathcal{S}_k$ satisfies the assumptions of Proposition 6.3. Let $C_{\tilde{P}}$ denote the $(n-l)$-dimensional Cauchy-Cartan characteristic manifold of $\Sigma(\mathcal{S}_k)$ passing through a point $\tilde{P}$. Then $\hat{\mathcal{M}}^n = \bigcup_{\tilde{P} \in \tilde{\mathcal{M}}^l} C_{\tilde{P}}$ is an $n$-dimensional integral manifold of $\Sigma(\mathcal{S}_k)$, and hence of $\Sigma(\mathcal{R}_k)$. Applying Proposition 6.1 (I) to the integral manifolds $\pi_1^k(\hat{\mathcal{M}}^n)$, $\pi_1^k(\tilde{\mathcal{M}}^n)$ passing through $\mathcal{M}_0^l$, we know that $\hat{\mathcal{M}}^n = \tilde{\mathcal{M}}^n$. Consequently $\tilde{\mathcal{M}}^n$ can be obtained by integrating ODE's. Thus the proof will be complete if we show Sublemmas.

PROOF OF SUBLEMMA 1. We can choose a fibered chart $(x, y)$ regular with respect to $\mathcal{R}_1$ in such a way that $x_i = 0$ on $\mathcal{M}_0^l$ $(l < i \leqq n)$ and $x_i(P_0) = 0$ $(1 \leqq i \leqq n)$. We indicate the variables $x_1, \cdots, x_l$ by $x'$. Let $\phi_\alpha^i = p_\alpha^i - \psi_\alpha^i = 0$ $(1 \leqq \alpha \leqq \kappa_i; i = 0, 1, \cdots, n)$ be a regular local equation of $\mathcal{R}_1$ around $P_0$ given by Proposition 5.3. Here $\kappa_1 = \cdots = \kappa_l = m - s_l$, $\kappa_{l+1} = \cdots = \kappa_n = m$. In accordance with the general theory (cf. Kuranishi [14], Matsuda [15]), introduce initial data

$$(6.1) \qquad \text{(I)} \ y_\alpha|_{x'=0} = c_\alpha \ (\kappa_0 < \alpha \leqq \kappa_l), \qquad \text{(II)} \ y_\alpha = g_\alpha^0(x') \ (\kappa_l < \alpha \leqq m)$$

the $c$'s being constants, and consider the differential equations with unknown functions $y_\alpha = y_\alpha(x')$ $(\kappa_0 < \alpha \leqq \kappa_l)$

$$\phi^i_\alpha = 0 \quad (\kappa_0 < \alpha \leqq \kappa_l,\ i = 1, \cdots, l) \tag{6.2}$$

in which (6.1)-(II) and $p^j_\alpha = \partial g^0_\alpha(x')/\partial x_j$ $(\kappa_l < \alpha \leqq m, 1 \leqq j \leqq l)$ are assumed to be substituted. We assume that the $k$-jet at $\pi^1_{-1}(P_0)$ determined by (6.1), (6.2) is sufficiently near $\tilde{P}_0$. Careful reading of the proof of the Cartan-Kähler theorem in [14] or [15] indicates that, for any given data (6.1), we can obtain a unique solution of (6.2) satisfying (6.1)-(I) by solving ODE's.

Now consider an $l$-dimensional submanifold $\mathcal{M}^l$ of $J_1$ sufficiently close to $\mathcal{M}^l_0$ such that $x_i = 0$ on $\mathcal{M}^l$ $(l < i \leqq n)$. It is described by

$$x_i = 0 \ (l < i \leqq n), \quad y_\alpha = g^0_\alpha(x'), \quad p^i_\alpha = g^i_\alpha(x') \ (1 \leqq \alpha \leqq m, 1 \leqq i \leqq n). \tag{6.3}$$

$\mathcal{M}^l$ is an integral manifold of $\Sigma(\mathcal{R}_1)$ if and only if

$$\begin{cases} g^i_\alpha(x') = \partial g^0_\alpha(x')/\partial x_i & (1 \leqq i \leqq l, 1 \leqq \alpha \leqq m); \\ g^i_\alpha(x') = [\phi^i_\alpha](x') & (0 \leqq i \leqq n, 1 \leqq \alpha \leqq \kappa_i), \end{cases} \tag{6.4}$$

where $[\phi^i_\alpha]$ denotes the function obtained from $\phi^i_\alpha$ by substituting (6.3).

For each data (6.1), let $y_\alpha = g^0_\alpha(x')$ $(\kappa_0 < \alpha \leqq \kappa_l)$ be the (unique) solution of (6.2) satisfying (6.1)-(I), and $g^0_\alpha(x')$ $(1 \leqq \alpha \leqq \kappa_0)$ be the function obtained from $\phi^0_\alpha$ by substituting $x_i = 0$ $(l < i \leqq n)$, (6.1)-(II), $y_\alpha = g^0_\alpha(x)$ $(\kappa_0 < \alpha \leqq \kappa_l)$. The $g^0_\alpha(x')$'s define an (integral) manifold $\mathcal{M}^l$ by (6.4). Let $A$ be the mapping which assigns each data (6.1) the integral manifold $\mathcal{M}^l$. Then $A$ gives a one-to-one correspondence between the data (6.1) and the integral manifolds $\mathcal{M}^l$ of $\Sigma(\mathcal{R}_1)$. From this fact it follows in particular that $\mathcal{M}^l_0$ may be obtained by solving ODE's.

To construct a required family $\{\mathcal{M}^l_t\}$, we assume that the $\mathcal{M}^l_0$ corresponds to the data (6.1) under $A$. We may also assume that $c_\alpha = 0$. Taking into account of the property (0) in Lemma 5.4, we see that the matrix $\left(\langle \partial/\partial y_\alpha, dh^{(j)}_\delta \rangle; \begin{matrix} \alpha = \kappa_0 + 1, \cdots, \kappa_1 \\ \gamma = 1, \cdots, q_j \end{matrix}\right)$ has rank $q_j$ at $P_0$. We may assume that the first $q$ column vectors are linearly independent. Let $\mathcal{M}^l_t$ be the integral manifold which corresponds under the mapping $A$ to the data (6.1) in which $c_{\kappa_0+\gamma} = t_\gamma$ $(1 \leqq \gamma \leqq q_j)$. It is easy to see that the family $\{\mathcal{M}^l_t\}$ is a required one.

PROOF OF SUBLEMMA 2. For each $\beta = 1, \cdots, \nu - 1$, we may assume that $\Pi^{(\beta)} = \{f^{(\beta)}_\gamma; 1 \leqq \gamma \leqq m_\beta\}$ is a complete set of principal integrals of

$\Delta^k(\mathfrak{P}_\beta)$. We first observe that $\tilde{\varepsilon}_k^*(df_\gamma^{(\beta)})$ $(1\leqq\beta<\nu, 1\leqq\gamma\leqq m_\beta)$ are linearly independent at $\tilde{P}_0$ (cf. Lemmas 4.1, 5.1). Let us next show that, for each fixed $\beta$, the 1-forms $df_\delta^{(\beta)}$ $(m_\beta<\delta\leqq n_\beta)$ can be expressed as linear combinations of $\theta_{\alpha,I_q}^{(\beta)}$'s, $\theta_{\alpha,I_0}^{(0)}$'s, $d\Pi^{(\beta)}$, $dw_i^{(\beta)}$'s. In fact, each $df_\delta^{(\beta)}$ can be expressed as a linear combination of the local basis $\theta_{\alpha,I}^{(j)}$'s, $d\Pi^{(\beta)}$, $dw_i^{(\beta)}$'s of $\Delta^k(\mathfrak{P}_\beta)$. Let $C_{\delta,j}^{\alpha,I_q}$ be the coefficient of $\theta_{\alpha,I_q}^{(j)}$. Denote by $\lambda_\delta^{(\beta)}$ the 2-form $\sum_j\sum_{\alpha,I_q}C_{\delta,j}^{\alpha,I_q}d\theta_{\alpha,I_q}^{(j)}$. Taking exterior differentiation, we get $\lambda_\delta^{(\beta)}\equiv 0$ (mod $\Delta^k(\mathfrak{P}_\beta)$). Bearing in mind the properties of the $\theta$'s stated in Lemma 5.4, we can deduce from these formulae that the $C_{\delta,j}^{\alpha,I_{k-1}}$ $(j\neq 0,\beta)$ vanish around $\tilde{P}_0$. Next, taking into account of this fact, we find that

$$\lambda_\delta^{(\beta)}\equiv 0 \quad (\text{mod}\, \theta_{\alpha,I_{k-1}}^{(\beta)}\text{'s},\ (\text{※})_{k-1},\ d\Pi^{(\beta)},\ dw_i^{(\beta)}\text{'s}).$$

In the same manner as above, we see that the $C_{\delta,j}^{\alpha,I_{k-2}}$ $(j\neq 0,\beta)$ must vanish. Proceeding step by step, we have the required result. Furthermore the coefficients of $dw_i^{(\beta)}$ in the expression of $df_\delta^{(\beta)}$ vanish at $\tilde{P}_0$, since there is an integral element $E_n$ of $\Theta_k$ on which the $df_\gamma^{(\beta)}$'s vanish, and on which the $dw_i^{(\beta)}$'s are linearly independent.

Combining these facts, we conclude that the $df_\gamma^{(j)}$'s are linearly independent modulo $\Omega$ around $\tilde{P}_0$. Q.E.D.

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