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*Large velocities in the relativistic Vlasov-Maxwell equations**)

Dedicated to Professor Hiroshi Fujita on his 60th birthday

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We consider the relativistic Vlasov-Maxwell equations (RVM), which model collisionless plasmas in the kinetic theory. In two earlier papers [1], [2] we proved that no singularity could develop at subrelativistic velocities provided the initial data has compact support. However, if collisions are present, one would expect that the arbitrarily high velocities of a Maxwellian would be generated. Therefore, it is important to see whether the assumption of compact support can be eliminated from RVM. That is the purpose of this paper. In [3] a similar question is studied for the Vlasov-Poisson System.

The RVM system is as follows. Let $f_\alpha(t, x, v)$ be the density of the particles of species α in phase space $(x, v)\in \boldsymbol{R}^3\times\boldsymbol{R}^3$. The momentum is v and the velocity is $\hat{v}_\alpha=v(m_\alpha^2+|v|^2c^{-2})^{-1/2}$. The particles experience only an electromagnetic force $e_\alpha\left(E+\frac{1}{c}\hat{v}_\alpha\wedge B\right)$. Conservation of mass yields the Vlasov equation

$$\frac{\partial f_\alpha}{\partial t}+\hat{v}_\alpha\cdot\nabla_x f_\alpha+e_\alpha\left(E+\frac{1}{c}\hat{v}_\alpha\wedge B\right)\cdot\nabla_v f_\alpha=0.$$

The particles provide charge and current densities

$$\rho(t, x)=4\pi\sum_\alpha\int e_\alpha f_\alpha dv,\qquad j(t, x)=4\pi\sum_\alpha\int \hat{v}_\alpha e_\alpha f_\alpha dv.$$

The electromagnetic field $E(t, x)$, $B(t, x)$ satisfies the standard Maxwell equations

$$\begin{aligned}\partial E/\partial t&=c\nabla\wedge B-j &\qquad \nabla\cdot E&=\rho\\ \partial B/\partial t&=-c\nabla\wedge E &\qquad \nabla\cdot B&=0.\end{aligned}$$

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Initial conditions

$$f_\alpha(0, x, v)=f_{\alpha 0}(x, v), \quad E(0, x)=E_0(x), \quad B(0, x)=B_0(x)$$

are given which satisfy the necessary constraints $\nabla\cdot E_0=\rho_0$, $\nabla\cdot B_0=0$ and the assumption $f_{\alpha 0}(x, v)\geq 0$. The main theorem is as follows.

THEOREM. *Let $f_{\alpha 0}(x, v)$ be C^1 functions such that*

$$0\leq f_{\alpha 0}(x, v)\leq c(1+|v|)^{-7-\delta} \qquad (\textit{for some } \delta>0). \tag{1}$$

Let $E_0(x)$, $B_0(x)$ be C^2 functions which satisfy $\nabla\cdot E_0=\rho_0$ and $\nabla\cdot B_0=0$. Assume the a priori bound

$$\int_{R^3}|v|f_\alpha(t, x, v)dv\leq constant \tag{2}$$

for $x\in R^3$, $0\leq t\leq T$, which is valid for each T and each C^1 solution f_α (as well as for its approximations (13) *uniformly in n). Then there exists a unique C^1 solution for all $t<\infty$ of RVM which satisfies the initial conditions.*

The proof begins with a representation of the field (Lemma 1) from which we obtain a statement that the densities f_α decay to zero as $|v|\to\infty$ (Lemma 2). The main body of the proof is devoted to bounds on the derivatives of the field. This requires the estimate of the kernels in their integral representations (Lemmas 3, 4). An approximation argument completes the proof.

For simplicity we take various constants equal to unity and we take only a single species. Then RVM takes the form

$$\text{RVM}\begin{cases} \partial_t f+\hat{v}\cdot\nabla_x f+(E+\hat{v}\wedge B)\cdot\nabla_v f=0 \\ \partial_t E=\nabla\wedge B-j, \qquad \nabla\cdot E=\rho, \\ \partial_t B=-\nabla\wedge E, \qquad \nabla\cdot B=0, \\ j=\displaystyle\int\hat{v}f dv, \quad \rho=\int f dv, \quad \hat{v}=v(1+|v|^2)^{-1/2}. \end{cases}$$

Since the speed of propagation in x-space is finite $(=c=1)$, we may assume that f_0, E_0, B_0 have compact support in x. However, $f_0(x, v)$ does not have compact support in v but only satisfies (1). We fix any time $T>0$. It suffices to prove the theorem in the time interval $[0, T]$. Any solution f, E, B will have compact support in x.

Our main task is to prove that a presumed solution (f, E, B) is bounded *a priori* in the space $C^1([0, T]\times \boldsymbol{R}^3\times \boldsymbol{R}^3)$.

We have $f(t, x, v)=f_0(X(0, t, x, v),\ V(0, t, x, v))$, where $s\to X(s, t, x, v)$, $s\to V(s, t, x, v)$ are the solutions to $\dot{x}=\hat{v}$, $\dot{v}=E+\hat{v}\wedge B$ with $X=x$, $V=v$ at $s=t$. Hence $\sup_{t,x,v} f(t, x, v)\leq \|f_0\|_\infty<\infty$.

The pointwise bound for the field E, B is proved almost exactly as in [2]. Indeed, from Theorem 3 of [1] we have the representation

$$E=E_0+E_T+E_S,$$
$$B=B_0+B_T+B_S$$

where E_0, B_0 are functionals of the Cauchy data only. One proves that

$$|E_T(t, x)|\leq C\int_{|x-y|\leq t}\int(1+|v|)f(t-|x-y|, y, v)dv\frac{dy}{|y-x|^2}$$
$$\leq c_T$$

exactly as in [2]. Next, E_S^i is for $i=1, 2, 3$ given by

$$E_S^i(t, x)=\int_{|y-x|\leq t}\int\frac{(\omega_i+\hat{v}_i)}{1+\hat{v}\cdot\omega}(Sf)(t-|x-y|, y, v)dv\frac{dy}{|y-x|} \tag{3}$$

where

$$S=\partial_t+\hat{v}\cdot\nabla_x. \tag{4}$$

The Vlasov equation gives $Sf=-\nabla_v\cdot(Kf)$, where $K=E+\hat{v}\wedge B$. If we can justify the integration by parts in v, then we have exactly as in [2] the estimate

$$|E_S^i(t, x)|\leq C\iint_{|y-x|\leq t}(1+|v|)f(t-|y-x|, y, v)dv|K(t-|x-y|, y)|\frac{dy}{|y-x|}$$
$$\leq c_T\left\{1+\int_0^t\sup_x|K(\tau, x)|d\tau\right\}$$

for each $i=1, 2, 3$. Since similar estimates hold for B (cf. [2], pp. 49-50) we conclude that

$$\sup_x|K(t, x)|\leq c_T\left\{1+\int_0^t\sup_x|K(\tau, x)|d\tau\right\}$$

whence

$$\sup_x|E|+\sup_x|B|\leq l_T \tag{5}$$

by an application of the Gronwall inequality.

It remains to justify the v-integration by parts.

LEMMA 1. *$E_S^i(t, x)$ can be written as*

$$E_S^i(t, x) = -\int_{|y-x|\le t}\int \nabla_v\left\{\frac{\omega_i+\hat{v}_i}{1+\hat{v}\cdot\omega}\right\}\cdot K f dv \frac{dy}{|y-x|}.$$

PROOF. We fix x and t, and apply the divergence theorem to the original form (3) for E_S, integrated over $|v|\le R$, R arbitrary. Using (2), we find a sequence $\{R_i\}$ such that $R_i \nearrow \infty$ and $\int_{|v|=R_i} f dS_v \le \text{const. } R_i^{-2}$. Then the "boundary integral" obtained via integration by parts is, for $R=R_i$,

$$B_i \equiv \int_{|v|=R_i} \frac{v}{|v|}\cdot\frac{(\omega_i+\hat{v}_i)}{(1+\hat{v}\cdot\omega)} K f dS_v.$$

Now $\left|\frac{v}{|v|}\cdot K\right| \le |E|$ and we have

$$|\omega_i+\hat{v}_i| \le c(1+\hat{v}\cdot\omega)^{1/2},$$
$$1+\hat{v}\cdot\omega \ge c(1+|v|)^{-2}$$

from [2]. (See also Lemma 3 below.) Thus

$$|B_i| \le c|E| R_i \int_{|v|=R_i} f dS_v \longrightarrow 0 \qquad \text{as } i\to\infty,$$

which completes the proof.

Now that the fields are bounded, we can control the characteristics. We denote

$$M = 7+\delta. \tag{6}$$

Using the bound (5): $\sup_x|E| + \sup_x|B| \le l_T$ we integrate the characteristic equation $\dot{v}=K$ to obtain, for $|v|\ge 2Tl_T$, say,

$$|V(0, t, x, v)| \ge |v| - Tl_T \ge |v|/2.$$

By assumption (1) we therefore have

$$\begin{aligned}0\leq f(t,x,v)&=f_0(X(0,t,x,v),\ V(0,t,x,v))\\&\leq c(1+|V(0,t,x,v)|)^{-M}\\&\leq c_T(1+|v|)^{-M}.\end{aligned}$$

On the set $|v|\leq 2Tl_T$ we have simply

$$\begin{aligned}0\leq f(t,x,v)&=f_0(X(0,t,x,v),\ V(0,t,x,v))\\&=\frac{(1+|v|)^M f_0(X(0,t,x,v),\ V(0,t,x,v))}{(1+|v|)^M}\leq\frac{c_T\|f_0\|_\infty}{(1+|v|)^M}.\end{aligned}$$

Hence

LEMMA 2. *There exists a constant c_T depending on T such that for all x, v and for $0\leq t\leq T$, $f(t,x,v)$ satisfies*

$$0\leq f(t,x,v)\leq c_T(1+|v|)^{-M}$$

where $M=7+\delta$.

Given this pointwise decay for f in v, we can begin to estimate derivatives. The derivatives of f are quite simple, since only the Vlasov characteristics are used. In fact, if we put

$$|f(t)|_1=\sup_{x,v}|\nabla_x f(t,x,v)|+\sup_{x,v}|\nabla_v f(t,x,v)| \tag{7}$$

then from [1] p. 75, we conclude that

$$|f(t)|_1\leq c_T+c_T\int_0^t[1+|E(\tau)|_1+|B(\tau)|_1]|f(\tau)|_1 d\tau. \tag{8}$$

The estimation of the derivatives of the fields is much harder, and will use the now-known decay of Lemma 2. Along the way we require the following.

LEMMA 3.

(a) $1+\hat{v}\cdot\omega\geq\dfrac{1+|v\times\omega|^2}{2(1+|v|^2)}\geq c(1+|v|)^{-2}$

(b) $|\omega+\hat{v}|\leq\sqrt{2}\,(1+\hat{v}\cdot\omega)^{1/2}$

(c) $\left|\dfrac{\partial}{\partial v_k}(\hat{v}\cdot\omega)\right|\leq\dfrac{c(1+\hat{v}\cdot\omega)^{1/2}}{1+|v|}$

(d) $\left|\dfrac{\partial}{\partial v_k}(1+\hat{v}\cdot\omega)^{-\alpha}\right|\leq c(1+|v|)^{2\alpha}\qquad(\alpha>0)$

(e) $\left|\dfrac{\partial \hat{v}_j}{\partial v_k}\right| \leq c(1+|v|)^{-1}$

(f) $\left|\dfrac{\partial}{\partial y_j}(1+\hat{v}\cdot\omega)^{-\alpha}\right| \leq \dfrac{c}{r}(1+|v|)^{2\alpha+1}$ $\qquad \begin{bmatrix}\alpha>0,\ r=|x|, \\ \omega=x/r\end{bmatrix}.$

PROOF. (a) and (b) are done in [2]. (c) is a direct calculation:

$$\begin{aligned}\left|\frac{\partial}{\partial v_k}(\hat{v}\cdot\omega)\right| &= |(1+|v|^2)^{-1/2}\{\omega_k-\hat{v}_k(\hat{v}\cdot\omega)\}| \\ &= |(1+|v|^2)^{-1/2}\{(\omega_k+\hat{v}_k)-\hat{v}_k(1+\hat{v}\cdot\omega)\}| \\ &\leq c(1+|v|^2)^{-1/2}(1+\hat{v}\cdot\omega)^{1/2} \qquad \text{by (b).}\end{aligned}$$

(d) follows from (c) and (a), while (e) is trivial. The computation for (f) is

$$\begin{aligned}\left|\frac{\partial}{\partial y_j}(1+\hat{v}\cdot\omega)^{-\alpha}\right| &= \left|-\alpha(1+\hat{v}\cdot\omega)^{-\alpha-1}\cdot\frac{1}{r}\{\hat{v}_k-\omega_k(\hat{v}\cdot\omega)\}\right| \\ &= \left|\frac{-\alpha}{r}(1+\hat{v}\cdot\omega)^{-\alpha-1}\{(\hat{v}_k+\omega_k)-\omega_k(1+\hat{v}\cdot\omega)\}\right| \\ &\leq \frac{c}{r}(1+|v|)^{2\alpha+1} \qquad \text{by (a) and (b).}\end{aligned}$$

In view of (d), we say that "differentiation does not hurt the estimate (a)".

Next we write the representation for the derivatives of the field from [1], p. 66:

$$\begin{aligned}(9)\qquad \partial_k E^i(t,x) &= (\partial_k E^i)_0(t,x)+\int d(\omega^0,\hat{v})f(t,x,v)dv \\ &+\iint_{|y-x|\leq t} a(\omega,\hat{v})f(t-|y-x|,y,v)dv\frac{dy}{|y-x|^3} \\ &+\iint_{|y-x|\leq t} b(\omega,\hat{v})(Sf)(t-|y-x|,y,v)dv\frac{dy}{|y-x|^2} \\ &+\iint_{|y-x|\leq t} c(\omega,\hat{v})(S^2f)(t-|y-x|,y,v)dv\frac{dy}{|y-x|}\end{aligned}$$

with a similar representation for $B(t,x)$. The first term is a functional of the data only. The kernels a, b, c, d and others will be displayed and estimated in the next lemma.

LEMMA 4 (Estimates of Kernels).

(a) $$a(\omega, \hat{v})=\frac{-3(\omega_i+\hat{v}_i)[\omega_k(1-|\hat{v}|^2)+\hat{v}_k(1+\hat{v}\cdot\omega)]+(1+\hat{v}\cdot\omega)^2\delta_{ik}}{(1+|v|^2)(1+\hat{v}\cdot\omega)^4}$$

from [1] *eq.* (33), *p.* 68. *We use* (a) *and* (b) *of Lemma* 3. *Also* $1-|\hat{v}|^2=(1+|v|^2)^{-1}$. *So,*

$$|a(\omega, \hat{v})|=0\left(\frac{1}{|v|^4(1+\hat{v}\cdot\omega)^{7/2}}+\frac{1}{|v|^2(1+\hat{v}\cdot\omega)^{5/2}}+\frac{1}{|v|^2(1+\hat{v}\cdot\omega)^2}\right)$$
$$\leq c(1+|v|)^3.$$

(b) $$b(\omega, \hat{v})=(1+\hat{v}\cdot\omega)^{-3}(\omega_i+\hat{v}_i)3\omega_k(1+|v|^2)^{-1}+(1+\hat{v}\cdot\omega)^{-2}(\omega_k-2\hat{v}_k)$$
$$+(1+\hat{v}\cdot\omega)^{-1}(\delta_{ik}-\omega_i\omega_k)-(1+\hat{v}\cdot\omega)^{-2}[(\hat{v}_i+\omega_i)-(1+\hat{v}\cdot\omega)\omega_i]$$

by [1] *eq.* (35), *p.* 68. *Estimating each of the* 4 *terms separately, we get*

$$|b(\omega, \hat{v})|=0(|v|^{6-1-2})+0(|v|^4)+0(|v|^2)+0(|v|^{4-1})=0(|v|^4).$$

According to Lemma 3 (d), (e), *we get the same asymptotics when we differentiate with respect to v. Therefore*

(b′) $$|\nabla_v b|\leq c(1+|v|)^4.$$

(c) $c(\omega, \hat{v})=\dfrac{(\omega_i+\hat{v}_i)\omega_k}{(1+\hat{v}\cdot\omega)^2}$ *from* [1] (*top line of p.* 78). *Thus*

$$|c(\omega, \hat{v})|\leq\text{const}(1+\hat{v}\cdot\omega)^{-3/2}\leq\text{const}(1+|v|)^3$$

by Lemma 3 (a) *and* (b). *According to Lemma* 3 (d) *and* (e), *we deduce*

(c′) $|\nabla_v c|=0(|v|)^3$ *and* $(\nabla_v^2 c)=0(|v|^3)$.

(d′) $\hat{c}_{jm}(\omega, \hat{v})=\dfrac{\delta_{jm}-\hat{v}_j\hat{v}_m}{(1+|v|^2)^{1/2}}c(\omega, \hat{v})$ *by* [1], *p.* 78, *line* 5. *Hence*

$$|\hat{c}_{jm}(\omega, \hat{v})|=0(|v|^{-1+3})=0(|v|^2)$$

(e) $$\left|\hat{c}_{jm}(\omega, \hat{v})\frac{\omega_j}{1+\hat{v}\cdot\omega}\right|=0(|v|^{2+2})=0(|v|^4)$$

(e′) $\nabla_v\left\{\hat{c}_{jm}\dfrac{\omega_j}{1+\hat{v}\cdot\omega}\right\}=0(|v|^4)$ *in the same way.*

(f) *Also* $\left|\hat{c}_{jm}\left(\delta_{jk}-\dfrac{\omega_j\hat{v}_l}{1+\hat{v}\cdot\omega}\right)\right|$

$$=\left(c(\omega, \hat{v})\frac{\delta_{jm}-\hat{v}_j\hat{v}_m}{\sqrt{1+|v|^2}}\right)\cdot\left(\delta_{jk}-\frac{\omega_j\hat{v}_l}{1+\hat{v}\cdot\omega}\right)$$

$$=\hat{c}_{jm}\delta_{jk}+\frac{c(\omega, \hat{v})}{\sqrt{1+|v|^2}(1+\hat{v}\cdot\omega)}(\delta_{jm}-\hat{v}_j\hat{v}_m)\omega_j\hat{v}_l$$

$$=0(|v|^2)+\frac{0(|v|^3)}{0(|v|)(1+\hat{v}\cdot\omega)}(\hat{v}_l)(\omega_m-(\hat{v}\cdot\omega)\hat{v}_m)$$

$$=0(|v|^2)+\frac{0(|v|^2)}{(1+\hat{v}\cdot\omega)}((\hat{v}_m+\omega_m)-(1+\hat{v}\cdot\omega)\hat{v}_m)$$

$$=0(|v|^2)+0(|v|^2)\cdot[0(|v|)+0(1)]$$

$$=0(|v|^3).$$

(f′) *Hence* $\left|\frac{\partial}{\partial y_j}\left\{\hat{c}_{jm}\left(\delta_{jk}-\frac{\omega_j\hat{v}_l}{1+\hat{v}\cdot\omega}\right)\right\}\right|\leq\frac{\text{const}}{r}(1+|v|)^4$ *by Lemma* 3 (f).

(*Here we lose one power of* $|v|$ *by differentiation.*)

Finally, the term involving the kernel d *comes from the vertex of the cone via integration by parts in the space variable. Up to a multiplicative constant it is given by*

$$d(\omega^0, \hat{v})=\frac{\omega_k^0(\omega_i^0+\hat{v}_i)}{(1+|v|^2)(1+\hat{v}\cdot\omega^0)^3}$$

with $\omega^0=x/|x|$. *In part* (c) *above it was shown that*

$$c(\omega, \hat{v})\equiv\frac{\omega_k(\omega_i+\hat{v}_i)}{(1+\hat{v}\cdot\omega)^2}=0(|v|^3).$$

Since $(1+|v|^2)(1+\hat{v}\cdot\omega)$ *is bounded below by a positiv constant by Lemma* 3 (a), *we have*

(g) $|d(\omega, \hat{v})|=0(|v|^3)$ *as well.*

Now we estimate each of the terms in the representation (9) for $\partial_k E^i(t, x)$ following the notation of [1], p. 66-67.

The first term $(\partial_k E^i)_0$ is bounded by virtue of hypothesis (1). For the second term involving the kernel d, we have, using Lemmas 2 and 4 (g),

$$\left|\int d(\omega^0, \hat{v})f(t, x, v)dv\right|\leq c_T\int|v|^3\cdot|v|^{-M}dv\leq c_T$$

since $M>7$.

For the fourth term in (9), involving the kernel b, we want to integrate by parts in v. For fixed x, t, the "boundary integral" so obtained is dominated by

$$\frac{(|E|+|B|)}{|y-x|^2}\int_{|v|=R}\left|\frac{v}{|v|}b(\omega,\hat{v})\right|fdS_v.$$

By Lemma 4 (b), $|b(\omega,\hat{v})|=0(|v|^4)$ and $f\leq c_T(1+|v|)^{-M}$ by Lemma 2 again. Hence

$$\int_{|v|=R}|b|fdS_v\leq c_T\int_{|v|=R}|v|^4|v|^{-7-\delta}dS_v$$
$$\leq c_T R^{-1-\delta}\to 0 \qquad \text{as } R\to\infty.$$

Thus for the fourth term in (9) we have

$$\left|\iint_{|y-x|\leq t}b(\omega,\hat{v})(Sf)dv\frac{dy}{|y-x|^2}\right|=\left|\iint_{|y-x|\leq t}\nabla_v b(\omega,\hat{v})\cdot Kfdv\frac{dy}{|y-x|^2}\right|$$
$$\leq c_T\iint_{|y-x|\leq t}|v|^4\cdot|v|^{-7-\delta}dv\leq c_T$$

by Lemma 2, Lemma 4 (b′) and the known bounds (5) on the field.

For the last term $\partial_k E^i_{SS}$ in (9) involving $c(\omega,\hat{v})$ we write $\partial_k E^i_{SS}=$ I+II+III+IV′+IV″ (see [1], eq. (66)).

All y-integrals are taken over $|y-x|\leq t$. First,

$$\text{I}=\iint fK\cdot\nabla_v[(\nabla_v c)(K)]dv\frac{dy}{r}, \qquad \text{so,}$$

$$|\text{I}|\leq\iint|f|(|E|+|B|)^2\{|\nabla_v^2 c|+|\nabla_v c|\}dv\frac{dy}{r}$$
$$\leq c_T\int(1+|v|)^{-M}(1+|v|)^3dv=c_T; \qquad \text{by Lemma 4 }(c').$$

$$\text{II}=\iint\nabla_v c\cdot f\cdot(\partial_t+\hat{v}\cdot\nabla_x)(E+\hat{v}\wedge B)dv\frac{dy}{r}, \qquad \text{so}$$

$$|\text{II}|\leq\iint(|E|_1+|B|_1)\cdot f\cdot|\nabla_v c|dv\frac{dy}{r}$$
$$\leq c_T\left\{\int(1+|v|)^{-M}(1+|v|)^3dv\right\}\left\{\int_0^t(|E|_1+|B|_1)d\tau\right\} \quad \text{by Lemma 4 (c)}$$
$$\leq c_T\int_0^t(|E|_1+|B|_1)d\tau;$$

$$\text{III}=\iint\hat{c}_{jm}\cdot f\cdot\partial_j K^m dv\frac{dy}{r}, \qquad \text{so}$$

$$|\mathrm{III}|\leq c_T\int(1+|v|)^{-M}(1+|v|^2)dv\cdot\int_0^t|K|_1d\tau \quad \text{by Lemma 4 (d)}$$
$$\leq c_T\int_0^t(|E|_1+|B|_1)d\tau;$$
$$\mathrm{IV}'=-\iint\nabla_v\left\{\hat{c}_{jm}K^m\frac{\omega_j}{1+\hat{v}\cdot\omega}\right\}\cdot fKdv\frac{dy}{r}.$$

Here the integration by parts is justified by Lemma 4 (e). Hence

$$|\mathrm{IV}'|\leq c_T\int(1+|v|)^{-M}(1+|v|)^4dv=c_T \quad \text{by Lemma 4 (e}')$$
$$\mathrm{IV}''=-\iint\frac{\partial}{\partial y_l}\left[r^{-1}\hat{c}_{jm}K^m\left(\delta_{jl}-\frac{\omega_j\hat{v}_l}{1+\hat{v}\cdot\omega}\right)\right]fdvdy$$
$$=-\iint\hat{c}_{jm}\left(\delta_{jl}-\frac{\omega_j\hat{v}_l}{1+\hat{v}\cdot\omega}\right)\cdot f\cdot\frac{\partial K^m}{\partial y_l}dv\frac{dy}{r}$$
$$-\iint\frac{\partial}{\partial y_l}\left[r^{-1}\hat{c}_{jm}\left(\delta_{jl}-\frac{\omega_j\hat{v}_l}{1+\hat{v}\cdot\omega}\right)\right]\cdot f\cdot K^mdvdy.$$

By Lemma 4 (f) and 4 (f′),

$$|\mathrm{IV}''|\leq c\iint(1+|v|)^3fdv(|\nabla E|+|\nabla B|)\frac{dy}{r}$$
$$+c\iint(1+|v|)^4fdv(|E|+|B|)\frac{dy}{r^2}$$
$$\leq c_T\left\{\iint(1+|v|)^4(1+|v|)^{-M}dv\right\}\cdot\left\{1+\int_0^t(|E(\tau)|_1+|B(\tau)|_1)d\tau\right\}$$
$$\leq c_T\left\{1+\int_0^t(|E|_1+|B|_1)d\tau\right\}.$$

The last term to estimate in (9) is

$$\partial_kE^i_{TT}=\iint a(\omega,\hat{v})fdv\frac{dy}{r^3}$$

which we break up as in (60)-(63) of [1]:

$$\partial_kE^i_{TT}=\int_0^t\frac{d\tau}{t-\tau}\int_{|\omega|=1}\int a(\omega,\hat{v})f(\tau,x+\omega(t-\tau),v)d\omega dv$$
$$=\int_0^{t-d}+\int_{t-d}^t\equiv\mathrm{I}+\mathrm{II}$$

for some $d>0$ to be chosen below.

For term I we have from Lemma 4 (a) that $a(\omega, \hat{v})=0(|v|^3)$, hence

$$|\mathrm{I}|\leq c_T\int_0^{t-d}\frac{d\tau}{t-\tau}\int|v|^3|v|^{-7-\delta}dv$$
$$\leq c_T\ln\left(\frac{t}{d}\right).$$

We further break up term II as follows:

$$\mathrm{II}=\int_{t-d}^t\int_{|v|>(t-\tau)^{-1/8}}dvd\tau+\int_{t-d}^t\int_{|v|<(t-\tau)^{-1/8}}dvd\tau$$
$$\equiv\mathrm{II}'+\mathrm{II}''.$$

For II′ we have again, by Lemma 4 (a),

$$|\mathrm{II}'|\leq c_T\int_{t-d}^t\frac{d\tau}{t-\tau}\int_{|v|>(t-\tau)^{-1/8}}|v|^3\cdot|v|^{-7}dv$$
$$\leq c_T\int_{t-d}^t\frac{d\tau}{t-\tau}\int_{(t-\tau)^{-1/8}}^\infty p^{-2}dp=c_T\int_{t-d}^t(t-\tau)^{-7/8}d\tau$$
$$=c_T\cdot d^{1/8}.$$

For II″ we recall (cf. [1], p. 67) that $\int_{|\omega|=1}ad\omega=0$. Hence

$$|\mathrm{II}''|\leq c_T\sup_{\tau\leq t}\|\nabla_x f(\tau)\|_\infty\cdot\int_{t-d}^t d\tau\int_{|v|<(t-\tau)^{-1/8}}|v|^3dv$$
$$\leq c_T\sup_{\tau\leq t}\|\nabla_x f(\tau)\|_\infty\cdot\int_{t-d}^t(t-\tau)^{-3/4}d\tau.$$

Therefore,

$$|\mathrm{II}|\leq c_T\left[d^{1/8}+d^{1/4}\sup_{\tau\leq t}\|\nabla_x f(\tau)\|_\infty\right]$$

and hence also

$$|\partial_k E^i_{TT}|\leq c_T\left[\ln\frac{t}{d}+d^{1/8}+d^{1/4}\sup_{\tau\leq t}\|\nabla_x f(\tau)\|_\infty\right].$$

Choosing $d=\left(\sup_{\tau\leq t}\|\nabla_x f(\tau)\|_\infty\right)^{-4}$ we get

$$|\partial_k E^i_{TT}|\leq c_T\left[1+\ln\left(\sup_{\tau\leq t}\|\nabla_x f(\tau)\|_\infty\right)\right]$$

provided $\sup_{\tau\le t}\|\nabla_x f(\tau)\|_\infty$ is bounded away from 2, say, which we can assume.

Putting these estimates together, we have shown that

$$(10)\qquad \begin{aligned} &|E(t)|_1+|B(t)|_1\\ &\le c_T\left[1+\ln^*\left(\sup_{\tau\le t}\|\nabla_x f(\tau)\|_\infty\right)\right]\\ &\quad +c_T\int_0^t(|E(\tau)|_1+|B(\tau)|_1)d\tau \end{aligned}$$

where $\ln^* x=\begin{cases} x & \text{if } x\le 1\\ 1+\ln x & \text{if } x>1.\end{cases}$

The last term in (10) can now be dropped by an application of the Gronwall inequality, so that we have

$$(11)\qquad |E(t)|_1+|B(t)|_1\le c_T\left[1+\ln^*\left(\sup_{\tau\le t}\|\nabla_x f(\tau)\|_\infty\right)\right].$$

Inserting this estimate into (8), we derive a bound for $|f(t)|_1$, and hence for $|E(t)|_1+|B(t)|_1$ too. This completes the C^1-estimates.

Now the proof of the theorem is concluded using the iteration scheme of [1], [2]. Namely we set

$$(12)\qquad f^{(0)}(t,x,v)=f_0(x,v);\ E^{(0)}(t,x)=E_0(x),\ B^{(0)}(t,x)=B_0(x)$$

while for $n=1,2,\cdots$, $f^{(n)}$ is the solution of

$$(13)\qquad \begin{aligned} &\partial_t f^{(n)}+\hat{v}\cdot\nabla_x f^{(n)}+\{E^{(n-1)}+\hat{v}\wedge B^{(n-1)}\}\cdot\nabla_v f^{(n)}=0\\ &f^{(n)}(0,x,v)=f_0(x,v); \end{aligned}$$

$$(14)\qquad \begin{aligned} &\partial_t E^{(n)}=\nabla\wedge B^{(n)}-j^{(n)};\ \nabla\cdot E^{(n)}=\rho^{(n)}\\ &\partial_t B^{(n)}=-\nabla\wedge E^{(n)};\ \nabla\cdot B^{(n)}=0\\ &E^{(n)}(0,x)=E_0(x),\ B^{(n)}(0,x)=B_0(x) \end{aligned}$$

where $\rho^{(n)}(t,x)=\int f^{(n)}dv;\ j^{(n)}(t,x)=\int\hat{v}f^{(n)}dv.$

The argument is the same as that of Section 5 of [1], and this concludes the proof of the theorem.

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