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Attractors for the Navier-Stokes equations: localization and approximation

Dedicated to H. Fujita on his 60th birthday

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Introduction

The study of the large time behavior of the solutions of dissipative partial differential equations is a major problem of mathematical physics directly related to the understanding of turbulence. In the simplest regimes the orbits converge to a stationary solution while in turbulent regimes the orbits wander around a global attractor. In the Smale-Ruelle-Takens approach to turbulence, the chaotic behavior is explained by the wandering of the orbits around the attractor which may be a complicated set, possibly a fractal. Existence and properties of the global attractor have been proved for many dissipative equations and since the attractor is the natural object for describing turbulent flows, its determination is a relevant problem. However the attractor $\mathcal{A}$ is expected to be in general a complicated (fractal) set and a rigorous determination is not possible or is of little use at this point. Our object instead here is to derive partial informations on the attractor, namely to localize it in some regions of the phase space and to approximate it.

A general presentation of the attractors for dissipative evolution equations appears in the books by J. Hale [19] and R. Temam [14]. The idea of approximating the global attractor by smooth manifolds has emerged from the study of inertial manifolds and approximate inertial manifolds. We recall (see [5]) that an inertial manifold (IM) is a smooth finite dimensional manifold which attracts all orbits at an exponential rate. When it exists such a manifold $\mathcal{M}$ necessarily contains the global attractor. However the existence results for inertial manifolds (see [5] and the references in Ch. VIII of [14]) rely on a restrictive spectral gap condition not satisfied by all dissipative equations; in particular it is not satisfied by the Navier-Stokes equations, even in space dimension 2.

As an alternative to inertial manifolds when such manifolds do not exist, or are not known to exist, it was proposed in [4] to consider approximate inertial manifolds (AIMs). An AIM is a smooth finite dimensional manifold which attracts all orbits in a thin neighborhood, in particular this neighborhood contains the attractor $\mathcal{A}$. An AIM produces an approximate finite dimensional dynamic and allows for a time uniform approximation of the semigroup under consideration. Another utilization of AIM's, is the determination of new numerical methods called the nonlinear Galerkin methods. The usual Galerkin method consists in projecting the equation to be approximated on the linear space spanned by the functions $w_1, \cdots, w_m$ used in the Galerkin method. Nonlinear Galerkin methods consist in projecting the equation to be approximated on nonlinear smooth finite dimensional manifolds. For the large time approximation of the equation, the approximate inertial manifolds are good candidates for the construction of nonlinear Galerkin methods; and indeed the method based on the AIM by Foias-Manley-Temam [4] has proved to be superior to the usual Galerkin method (see [12], [16]).

While a few AIM's were constructed in [4] and [15], our object here is to present a methodology which produces an infinite sequence of AIM's producing higher and higher orders of accuracy. For the sake of simplicity we restrict ourselves to the two-dimensional Navier-Stokes equations (although the method is much more general (see Marion [20])). The existence of the attractor for these equations was shown in [6] and further properties derived in [1], [2] and [3]. Our approximation and localization procedure consists in determining simple finite dimensional manifolds which attract all the orbits in a thin neighborhood; in particular the global attractor $\mathcal{A}$ lies in such a neighborhood. Actually we determine an infinite number of such manifolds producing thinner and thinner neighborhoods as it will appear below. The utilization of these manifolds for the approximation of turbulent flows is considered elsewhere: see in particular in [11] the study of convergence of new nonlinear Galerkin methods derived from these approximations and in [12] and [16] numerical tests for these methods.

A partial form of the results appearing here was presented in [15]; related results for other equations appear in [9] and [10]; see also the comments in Remark 3.1 hereafter.

In this article we start in Section 1 by recalling a few facts about the Navier-Stokes equations. Also we write the velocity u as the sum p_m+q_m corresponding to the first m modes and the other modes and we write the coupled system of equations for p_m and q_m. Then in Section 2 we describe the principle of the construction of the approximations. In fact we restrict ourselves to one single orbit and show how one can approximate such orbit at higher and higher order of accuracy with simpler orbits lying in finite-dimensional manifolds. Section 3 contains the main results: approximations of an orbit by simpler orbits lying in a finite dimensional manifold; and localization of the attractor $\mathcal{A}$ for the Navier-Stokes equations in a thin neighborhood of such manifolds. The manifolds have an equation of the form

$$Q_m v = \Phi(P_m v)$$

where $\Phi = \Phi_{m,j}$ maps $P_m H$ into $Q_m H$; P_m is the projector in H onto the space $P_m H$ spanned by the first m eigenmodes and $Q_m = I - P_m$. Finally Section 4 contains the detailed proof of a technical result admitted without proof in Section 2. This section contains also our approximation procedure for time derivatives.

Contents

1. Projection of the equations: Behavior of small eddies

We first recall a few facts about the Navier-Stokes equations. In their functional setting the Navier-Stokes equations appear as a differential equation in an infinite dimensional Hilbert space H:

$$\frac{du}{dt} + \nu Au + B(u) = f, \tag{1.1}$$

$$u(0) = u_0. \tag{1.2}$$

Here $u = u(t)$ is a function from $[0, +\infty[$ into H, representing the velocity vector field; $\nu > 0$ is the kinematic viscosity, $f \in H$ represents volume forces. The operator A is an unbounded positive self-adjoint closed operator in H with domain $D(A) \subset H$ and its inverse A^{-1} is compact in H; finally $B(u) = B(u, u)$ where B is a bilinear continuous operator from $D(A) \times D(A)$ into H, which satisfies further continuity properties recalled below.

We denote by $(\cdot, \cdot)$ and $|\cdot|$ the scalar product and the norm in H. We know that we can define the powers A^s of A for all $s \in \boldsymbol{R}$, and A_s maps $D(A^s)$ onto H; $|A^s \cdot|$ is a Hilbert norm on $D(A^s)$. We set $V = D(A^{1/2})$ and denote the norm and the scalar product in V by $\|\cdot\|$, $((\cdot, \cdot))$.

Since A^{-1} is self-adjoint compact in H, there exists an orthonormal basis of H consisting of the eigenvectors w_j of A:

$$\begin{cases} Aw_m = \lambda_m w_m, & m \geq 1, \\ 0 < \lambda_1 \leq \lambda_2 \leq \cdots, & \lambda_m \to \infty \text{ as } m \to \infty. \end{cases} \tag{1.3}$$

Equation (1.1) is the evolution equation for the velocity u for a viscous incompressible fluid in a bounded domain; depending on the choice of A and H, the boundary conditions are the no-slip condition, or a free boundary condition, or the space periodicity.

In space dimension 2, it is well-known that for u_0 given in $D(A^{1/2})$, (1.1), (1.2) possess a unique solution u bounded from $[0, \infty[$ into $D(A^{1/2})$; see [7], [8]. Furthermore u is analytic from $]0, \infty[$ into $D(A)$; the domain of analyticity of u in the complex plan $\boldsymbol{C}$ comprises the region $\Delta(\|u_0\|)$ defined by

$$\Delta(\|u_0\|) = \{\zeta \in \boldsymbol{C},\ \operatorname{Re}\zeta > 0,\ |\operatorname{Im}\zeta| \leq T_0 \text{ if } \operatorname{Re}\zeta \geq T_0, \text{ and } |\operatorname{Im}\zeta| \leq \operatorname{Re}\zeta \text{ if } \operatorname{Re}\zeta \leq T_0\}; \tag{1.4}$$

here $T_0 = T_0(\|u_0\|)$ is a bounded increasing function of ν^{-1}, $|f|$, λ_1^{-1} and $\|u_0\|$; see [13]. If u is a solution of (1.1), (1.2), then we set for $t_* \geq 0$ arbitrary[1)]

$$M_0(t_*) = \operatorname*{Sup}_{s \geq t_*} |u(s)|,\quad M_1(t_*) = \operatorname*{Sup}_{s \geq t_*} \|u(s)\|. \tag{1.5}$$

Finally, let us recall some well-known continuity properties of the operator B that will be repeatedly used: there exist absolute constants c_1, c_2, such that for every $u, v, w \in D(A)$:

$$|B(u, v)| \leq c_1 \begin{cases} |u|^{1/2} \|u\|^{1/2} \|v\|^{1/2} |Av|^{1/2} \\ |u|^{1/2} |Au|^{1/2} \|v\| \end{cases} \tag{1.6}$$

$$|(B(u, v), w)| \leq c_2 |u|^{1/2} \|u\|^{1/2} \|v\|\, |w|^{1/2} \|w\|^{1/2}. \tag{1.7}$$

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1) In the applications the time t_* can be either $t_* = 0$, in which case M_0, M_1 depend on u_0. Or t_* can be a time large enough, after the entrance of the orbit in the absorbing set, in which case M_0, M_1 are independent of u_0; explicit values of M_0, M_1 in term of the other data are given in [4].

Like c_1, c_2 all the quantities c_i, c'_i that will appear subsequently are absolute constants. We recall also that

$$|B(u, v)| \le c_3 \|u\| \|v\| \left\{1 + \log \frac{|Au|^2}{\lambda_1 \|u\|^2}\right\}^{1/2}, \qquad \forall u, v \in D(A), \tag{1.8}$$

$$(B(u, v), v) = 0, \ \forall u, \ v \in D(A). \quad \square \tag{1.9}$$

In the following we consider for $m \in N$ fixed, the space spanned by $w_1, \cdots, w_m$ and we denote by P_m the orthogonal projector in H onto this space, and $Q_m = I - P_m$. We recall that P_m and Q_m are also orthogonal projectors in all the spaces $D(A^s)$ and that they commute with A and its powers. When u is solution of (1.1), (1.2) we write $p_m = P_m u$, $q_m = Q_m u$ and projecting (1.1) on $P_m H$ and $Q_m H$ we find a coupled system of equations satisfied by p_m and q_m:

$$\frac{dp_m}{dt} + \nu A p_m + P_m B(p_m + q_m) = P_m f, \tag{1.10}$$

$$\frac{dq_m}{dt} + \nu A q_m + Q_m B(p_m + q_m) = Q_m f. \tag{1.11}$$

It is clear that p_m which corresponds to the eigenfrequencies $\lambda_1^{-1}, \cdots, \lambda_m^{-1}$ represents large structures in the flow, while q_m, corresponding to the eigenfrequencies $\le \lambda_{m+1}^{-1}$ represents the small structures. At this point the choice of m is arbitrary; a desirable value of m will be determined by the a priori estimate on the attractor established hereafter.

Some a priori estimates on q_m valid for large t, were derived in [4]. They show that the norms of q_m in H and V (and other norms too) are small for large t and large m. More precisely let us consider initial data u_0 in (1.2) satisfying

$$|u_0| \le R_0, \qquad \|u_0\| \le R_1. \tag{1.12}$$

Then we know that there exists a time t_* which depends on R_0, R_1 and the other data ν, $|f|$, λ_1, such that for $t \ge t_*$

$$|u(t)| \le M_0, \qquad \|u(t)\| \le M_1, \tag{1.13}$$

where M_0, M_1 are independent of u_0 but depend on ν, $|f|$, λ_1. Now according to [4], for any orbit of (1.1), after a time t_1 which depends only on ν, $|f|$, λ_1 and on u_0 through R_1, q_m is small in the following sense

(1.14) $|q_m(t)|\leq\kappa_0 L^{1/2}\delta,\quad \|q_m(t)\|\leq\kappa_0 L^{1/2}\delta^{1/2},\quad |Aq_m(t)|\leq\kappa_0 L^{1/2},\quad \forall t\geq t_1.$

Here κ_0 depends only on ν, $|f|$, λ_1 and

(1.15) $$\delta=\frac{\lambda_1}{\lambda_{m+1}},\qquad L=1+\log\frac{\lambda_{m+1}}{\lambda_1}.$$

For later purposes it is noteworthy that the estimates (1.14) are valid for complex times too, for example for $t\in t_1+\Delta(M_1)$ (and κ_0 appropriate). Then by a simple application of Cauchy's formula we find similar estimates for all the time derivatives of q_m in a domain of C slightly smaller than $\Delta(R_1)$. For instance

(1.16) $$\left|\frac{d^j q_m}{dt^j}(t)\right|\leq\kappa_{0,j}L^{1/2}\delta,\quad \left\|\frac{d^j q_m}{dt^j}(t)\right\|\leq\kappa_{0,j}L^{1/2}\delta^{1/2},\quad \left|A\frac{d^j q_m}{dt^j}(t)\right|\leq\kappa_{0,j}L^{1/2},$$

for all $j\geq 1$ and for all t in

(1.17) $$t_1+\frac{1}{2}(T_0(M_1)+\Delta(M_1));$$

$\kappa_{0,j}$ is an appropriate constant depending on ν, $|f|$, λ_1 and j.

2. Principle of the construction of the approximations

In this section we shall consider various approximations of one single orbit, $u=u(t)$ solution of (1.1), (1.2). Then in Section 3 we shall see how this induces localizations and approximations of the attractor.

We consider the orbit $u=u(t)$ solution of (1.1), (1.2) that corresponds to a specific initial value u_0 satisfying (1.12). The integer m in (1.10), (1.11) is fixed and arbitrary. Hence whenever this is possible we shall omit the index m; for example we write p[2] and q instead of p_m and q_m, and $P=P_m$, $Q=Q_m$.

Our first aim is to construct two families of functions $\varphi_{m,j}=\varphi_{m,j}(t)$, $\varphi^i_{m,j}=\varphi^i_{m,j}(t)$, $j\geq 0$, $m\geq 0$, which represent suitable approximations of q_m and $d^i q_m/dt^i$.

For $j=0,1,2,$ we set

(2.1) $$\varphi_{m,0}=0,$$

2) p should not be confused with the pressure which does not appear at all in this article.

$$\nu A\varphi_{m,1}+Q_mB(p_m)=Q_mf, \tag{2.2}$$

$$\nu A\varphi_{m,2}+Q_mB(p_m+\varphi_{m,1})=Q_mf. \tag{2.3}$$

The determination of $\varphi_{m,1}$, $\varphi_{m,2}$ from (2.2), (2.3) is obvious (and reduces to the inversion of A). Then for $j\geq 3$

$$\nu A\varphi_{m,j}+Q_mB(p_m+\varphi_{m,j-1})+\varphi^1_{m,j-2}=Q_mf. \tag{2.4}$$

As in (2.2), (2.3), (2.4) determines explicitly $\varphi_{m,j}$ (with the inversion of A) once $\varphi^1_{m,j-2}$ is known. The construction of $\varphi^1_{m,j-2}$ is involved and will be done in Section 4. At this point we admit the following result shown in Section 4:

$$\left|A\left[\varphi^1_{m,j}(t)-\frac{dq_m}{dt}(t)\right]\right|\leq\kappa^1_j\delta^{j/2}L^{j/2+1/2}, \qquad \forall j\geq 0,\ \forall t\geq t_1, \tag{2.5}$$

where κ^1_j is a constant depending on the data ν, $|f|$, λ_1, R_1; and we prove the following

LEMMA 2.1. *For every $j\geq 0$, there exists a constant κ^0_j depending only on j, ν, $|f|$, λ_1, R_1 such that, for $t\geq t_1$:*

$$|A(\varphi_{m,j}(t)-q_m(t))|\leq\kappa_j\delta^{j/2}L^{j/2+1/2}. \tag{2.6}$$

PROOF. As indicated before we omit the index m; thus $\varphi_j=\varphi_{m,j}$, $q=q_m$.

For $j=0$, (2.6) follows readily from (1.14). Then for $j=1$,

$$\nu A\varphi_1+QB(p)=Qf$$

and we rewrite (1.11) as

$$q'+\nu Aq+QB(p+q)=Qf. \tag{2.7}$$

Thus by difference

$$\begin{aligned}\nu A(\varphi_1-q)&=QB(p+q)-QB(p)+q'\\&=QB(p,q)+QB(q,p+q)+q'.\end{aligned}$$

Thanks to (1.16) written with $j=1$

$$|q'|\leq\kappa\delta L^{1/2}. \tag{2.8}$$

Here and after, κ denotes an unspecified constant κ_j depending on ν, $|f|$, λ_1

and R_1, and c denotes an absolute constant. Using Lemma 2.2 below and (1.13):

(2.9) $$|B(p,q)|+|B(q,p+q)|\leq cL^{1/2}\|p\|\,\|q\|+c|q|^{1/2}|Aq|^{1/2}\|p+q\|.$$

Since $p+q=u$ and since P, Q are projectors in $V=D(A^{1/2})$:

(2.10) $$\begin{cases}\|(p+q)(t)\|=\|u(t)\|\leq M_1, \\ \|p(t)\|\leq\|u(t)\|\leq M_1, \qquad \text{for } t\geq t_1.\end{cases}$$

Hence with (1.14) and since $L\geq 1$:

(2.11) $$\begin{aligned}|B(p,q)|+|B(q,p+q)|&\leq cM_1L^{1/2}(1+L^{1/2})\delta^{1/2}\\ &\leq cM_1L\delta^{1/2}.\end{aligned}$$

Collecting these inequalities we obtain (2.6) for $j=1$:

(2.12) $$|A(\varphi_1-q)|\kappa_1^0\delta^{1/2}L.$$

A (similar) proof is necessary for $j=2$ and then we shall proceed by induction for all $j\geq 2$.

Subtracting (2.7) from (2.3) we obtain

(2.13) $$\begin{aligned}\nu A(\varphi_2-q)&=QB(p+q)-QB(p+\varphi_1)+q'\\ &=QB(p+\varphi_1, q-\varphi_1)+QB(q-\varphi_1, p+q)+q'.\end{aligned}$$

We use (2.8) and, for the terms involving B we proceed as in (2.9)-(2.11):

$$\begin{aligned}|B(p,q-\varphi_1)|&\leq cL^{1/2}\|p\|\,\|q-\varphi_1\|\\ |B(\varphi_1,q-\varphi_1)|&\leq c|\varphi_1|^{1/2}|A\varphi_1|^{1/2}\|q-\varphi_1\|\\ |B(q-\varphi_1,p+q)|&\leq c|q-\varphi_1|^{1/2}|A(q-\varphi_1)|^{1/2}\|p+q\|\\ &\leq cM_1|q-\varphi_1|^{1/2}|A(q-\varphi_1)|^{1/2}.\end{aligned}$$

We recall that

(2.14) $$|\xi|\leq\lambda_{m+1}^{-1/2}\|\xi\|\leq\lambda_{m+1}^{-1}|A\xi|, \qquad \forall\xi\in Q_mD(A),$$

and thanks to (2.12) and the previous inequalities, we infer from (2.13) that

(2.15) $$|A(\varphi_2-q)|\leq\kappa\delta L^{3/2}.$$

We now proceed by induction and assume that (2.6) is valid for the indices $0,\cdots,j-1$ and we want to prove it at order j ($j\geq 3$). We subtract (2.7) from (2.4)

$$\nu A(\varphi_j - q) = QB(p+q) - QB(p+\varphi_{j-1}) + q' - \varphi^1_{j-2}$$
$$= QB(p+\varphi_{j-1}, q-\varphi_{j-1}) + QB(q-\varphi_{j-1}, p+q) + q' - \varphi^1_{j-2}. \tag{2.16}$$

The term $q' - \varphi^1_{j-2}$ is estimated by (2.5) and (2.14):

$$|q' - \varphi^1_{j-2}| \leq \lambda^{-1}_{m+1} \kappa^1_{j-2} \delta^{j/2-1} L^{j/2-1/2}$$
$$\leq \kappa \delta^{j/2} L^{j/2-1/2} \leq \kappa \delta^{j/2} L^{j/2+1/2}.$$

The terms involving B are estimated as before, using Lemma 2.2:

$$|B(p, q-\varphi_{j-1})| \leq c L^{1/2} \|p\| \|q-\varphi_{j-1}\| \leq c L^{1/2} M_1 \|q-\varphi_{j-1}\|$$
$$|B(\varphi_{j-1}, q-\varphi_{j-1})| \leq c |\varphi_{j-1}|^{1/2} |A\varphi_{j-1}|^{1/2} \|q-\varphi_{j-1}\|$$
$$|B(q-\varphi_{j-1}, p+q)| \leq c |q-\varphi_{j-1}|^{1/2} |A(q-\varphi_{j-1})|^{1/2} \|p+q\|$$
$$\leq c M_1 |q-\varphi_{j-1}|^{1/2} |A(q-\varphi_{j-1})|^{1/2}.$$

We use (2.6) at order $j-1$ and (2.14); we observe also that since $\delta L = \delta(1-\log\delta) \leq 1$ for $\delta \leq 1$:

$$|A\varphi_{j-1}| \leq |Aq| + |A(\varphi_{j-1} - q)|$$
$$\leq \kappa L^{1/2} + \kappa \delta^{j/2-1/2} L^{j/2} \leq \kappa L^{1/2}.$$

Collecting all the terms we find, as expected, that $|A(\varphi_j - q)|$ is bounded for $t \geq t_1$ by an expression $\kappa \delta^{j/2} L^{j/2+1/2}$. □

We now prove as announced the

LEMMA 2.2.

$$|B(\xi, \eta)| \leq \begin{cases} c_1 |\xi|^{1/2} |A\xi|^{1/2} \|\eta\|, & \forall \xi \in Q_m D(A), \quad \forall \eta \in V, \\ c_3 L^{1/2} \|\xi\| \|\eta\|, & \forall \xi \in P_m D(A), \quad \forall \eta \in V. \end{cases} \tag{2.17}$$

PROOF. The first inequality (2.17) is a particular case of the second inequality (1.6). For the second inequality (2.17) we use (1.8) and observe that

$$1 + \log \frac{|A\xi|^2}{\lambda_1 \|\xi\|^2} \leq 1 + \log \frac{\lambda_{m+1}}{\lambda_1} = L,$$

since

$$|A\xi| \leq \lambda_m^{1/2} \|\xi\| \leq \lambda_m |\xi|, \qquad \forall \xi \in P_m H. \tag{2.18}$$

REMARK 2.1. We postpone till Section 4 the construction very technical of the $\varphi^1_{m,j}$ satisfying (2.5). We shall find these quantities expressed as analytic functions of p_m, more precisely polynomial type functions[3].

The general idea behind the construction of the $\varphi^1_{m,j}$ is the following. Thanks to (1.1), $u'=du/dt$ is a polynomial type function of u. Similarly by successive differentiations of (1.1) we find that all time derivatives of u are also polynomial functions of u. Then (1.11) gives q'_m in terms of p_m and q_m, and we are able to approximately express q_m as a polynomial type function of p_m, at higher and higher orders of accuracy; first with (2.2), (2.3) and then with more involved expressions.

3. Approximation of the Attractor

As indicated in Remark 2.1 we postpone to Section 4 the construction of the $\varphi^1_{m,j}$. At this point we want to show how the results of Section 2 lead to the construction of approximations of the attractor.

We recall (see C. Foias and R. Temam [6]) that equation (1.1) possesses a global attractor $\mathcal{A}\subset H$ (also called the universal or maximal attractor). This attractor $\mathcal{A}$ is compact, connected and attracts all the orbits, all the bounded sets. It is invariant under the flow $S(t)$, i.e.

$$S(t)\mathcal{A}=\mathcal{A} \qquad \forall t\geq 0, \tag{3.1}$$

where $S(t)$ is the mapping

$$u(0)=u_0\in H \longrightarrow u(t)\in H,$$

$u=u(\cdot)$ being the solution of (1.1), (1.2) (see R. Temam [13]).

The attractor $\mathcal{A}$ which is expected to be in general a complicated (fractal) set, is the mathematical object describing all the permanent regimes, in particular the turbulent ones. It is important for computational and theoretical purposes to be able to approximate it.

We recall that if $u_*\in\mathcal{A}$ then u_* belongs to a complete orbit, i.e. $u_*=u(0)$ where $u=u(t)$ is a solution of (1.1) defined on the whole real axis, $t\in \boldsymbol{R}$. Since $\mathcal{A}$ is compact (in H and V), we can set

$$M_0=\operatorname*{Sup}_{u_*\in\mathcal{A}}|u_*|, \qquad M_1=\operatorname*{Sup}_{u_*\in\mathcal{A}}\|u_*\|.$$

Then for any orbit $u=u(t)$ lying on $\mathcal{A}$, u is an analytic function of $t\in\boldsymbol{C}$,

3) By this we mean here (in infinite dimension) finite sums of multilinear functions of p_m.

for $|\text{Im}\, t| \leq T_0(M_1)$.

As it will appear from the construction of $\varphi_{m,j}^1$ in Section 4, any $\varphi_{m,j}$ is a function of t through $p_m(t)$:

$$\varphi_{m,j}(t) = \Phi_{m,j}(p_m(t))$$

or dropping the indices m as mentioned before

$$\varphi_j(t) = \Phi_j(p(t)), \tag{3.2}$$

where Φ_j maps $P_m D(A)$ into $Q_m D(A)$.

As indicated above any point $u_* \in \mathcal{A}$ is of the form $u_* = u(0)$, where $u(\cdot)$ is solution of (1.1) for all t. Thus with a time shift of $-t_1$, we see that (2.6) applies:

$$|A(\varphi_{m,j}(0) - q_m(0))| \leq \kappa_j^0 \delta^{j/2} L^{j/2+1/2}$$

or

$$|A\Phi_{m,j}(p_m(0)) - Aq_m(0)| \leq \kappa_j^0 \delta^{j/2} L^{j/2+1/2}. \tag{3.3}$$

Alternatively, dropping any reference to time, we have on the attractor

$$|A\Phi_{m,j}(P_m u_*) - AQ_m u_*| \leq \kappa_j^0 \delta^{j/2} L^{j/2+1/2}, \qquad \forall j \geq 0,\ \forall m,\ \forall u_* \in \mathcal{A}. \tag{3.4}$$

With (2.14) this readily implies

$$\begin{cases} |\Phi_{m,j}(P_m u_*) - Q_m u_*| \leq \kappa_j^0 \delta^{j/2+1} L^{j/2+1/2} \\ \|\Phi_{m,j}(P_m u_*) - Q_m u_*\| \leq \kappa_j^0 \delta^{j/2+1/2} L^{j/2+1/2}, \end{cases} \tag{3.5}$$

$\forall j \geq 0$, $\forall m$, $\forall u_* \in \mathcal{A}$.

In conclusion we have proved the following results

THEOREM 3.1. *Let u be any solution of* (1.1), (1.2) *satisfying* (1.12).

Then there exists t_1 depending on $\nu, |f|, \lambda_1$ and R_1, and for every $j \geq 0$ there exists κ_j^0 depending on $\nu, |f|, \lambda_1, R_1$ and j, such that the following holds: for every $m \in N$, there exists a function

$$\Phi_{m,j} : P_m D(A) \longrightarrow Q_m D(A), \tag{3.6}$$

such that

$$\begin{aligned} |\Phi_{m,j}(P_m u(t)) - Q_m u(t)| &\leq \kappa_j^0 \delta^{j/2+1} L^{j/2+1/2} \\ \|\Phi_{m,j}(P_m u(t)) - Q_m u(t)\| &\leq \kappa_j^0 \delta^{j/2+1/2} L^{j/2+1/2}, \\ |A\Phi_{m,j}(P_m u(t)) - Q_m u(t)| &\leq \kappa_j^0 \delta^{j/2} L^{j/2+1/2}, \end{aligned} \tag{3.7}$$

$\forall j \leq 0$, $\forall m \geq 0$, $\forall t \geq t_4$[4]. □

Alternatively we have

THEOREM 3.2. *Let $\mathcal{A}$ denote the global attractor for* (1.1).
For every $m, j \in N$, there exists a function

$$\Phi_{m,j} : P_m D(A) \longrightarrow Q_m D(A)^{5)} \tag{3.8}$$

such that

$$\begin{aligned} |\Phi_{m,j}(P_m u_*) - Q_m u_*| &\leq \kappa_j^0 \delta^{j/2+1} L^{j/2+1/2}, \\ \|\Phi_{m,j}(P_m u_*) - Q_m u_*\| &\leq \kappa_j^0 \delta^{j/2+1/2} L^{j/2+1/2} \\ |A\Phi_{m,j}(P_m u_*) - AQ_m u_*| &\leq \kappa_j^0 \delta^{j/2} L^{j/2+1/2}, \end{aligned} \tag{3.9}$$

$u_* \in \mathcal{A}$, $\forall m, j$, *where κ_j depends on $\nu, |f|, \lambda_1$[6] and j but not on m. Also*

$$\delta = \frac{\lambda_1}{\lambda_{m+1}}, \quad L = 1 + \log \frac{\lambda_{m+1}}{\lambda_1}.$$

REMARK 3.1. (i) The graph of the function $\Phi_{m,j}$ is an *analytic manifold* $\mathcal{M}_j = \mathcal{M}_{m,j}$ (whose explicit equation follows from the explicit expression of φ_α^1). Theorem 3.2 asserts that the attractor $\mathcal{A}$ lies in a neighborhood of $\mathcal{M}_j$ of thickness (in H, V or $D(A)$) given by the right-hand side of (3.9). Also, by Theorem 3.1 any orbit enters this neighborhood in a finite time. Hence Theorem 3.2 is a *localization theorem* for the attractor $\mathcal{A}$.

(ii) Since the constants κ_j^0 are independent of m, we can make the right-hand side of (3.9) (or (3.7)) arbitrarily small by increasing m. Hence by increasing the dimension (m) of the manifold $\mathcal{M}_j = \mathcal{M}_{m,j}$, we can make the neighborhood mentioned above arbitrarily thin.

(iii) For j and j' given, $j < j'$, and for m sufficiently large, $\mathcal{A}$ is closer from $\mathcal{M}_j'$ than from $\mathcal{M}_j$. However since we are not able to compare $\kappa_j^0, \kappa_{j'}^0$, we are not certain, for j, j', m given, $j < j'$, that $\mathcal{A}$ is closer from $\mathcal{M}_j$ than from $\mathcal{M}_{j'}$.

(iv) The manifolds $\mathcal{M}_j = \mathcal{M}_{m,j}$ are *approximate inertial manifolds* for the Navier-Stokes equations. For inertial manifolds see for instance [5] [14] [17] and the references therein. For approximate inertial mani-

4) $t_2 = t_1 + \frac{1}{2} T_0(M_1)$, see (1.17); and of course M_1 depends only on $\nu, |f|, \lambda_1$.

5) The same as in (3.6).

6) Here $R_1 = M_1$ is a function of $\nu, |f|, \lambda_1$; see [14].

folds see [4]; see also in [18] a totally different aspect of the approximation of inertial manifolds.

4. Approximation of time derivatives

Our aim is now to construct the functions φ_j^i used in Section 2, and satisfying (2.5). At the same time we must define two families $\varphi_j^i=\varphi_{m,j}^i$, $\psi_j^i=\psi_{m,j}^i$ where the $\varphi_{m,j}^i$ approximate in some sense d^iq_m/dt^i, while the ψ_j^i approximate d^ip_m/dt^i.

At the step j, i.e. when we want to determine $\varphi_{m,j}$ from (2.4) ($j\geq 3$), we construct the sequence

$$\psi_{j-1-2i}^i,\qquad i=0,\cdots,[(j-1)/2] \tag{4.1}$$

for increasing values of i, and then the sequence

$$\varphi_{j-2i}^i,\qquad i=[(j+1)/2],\cdots,1,0, \tag{4.2}$$

in decreasing order of i. Of course at that stage, the similar sequences with j replaced by $k\leq j-1$ are already defined. If $i=0$,

$$\psi_{j-1}^0=p,\ \forall j \tag{4.3}$$

while

$$\varphi_j^0=\varphi_j \tag{4.4}$$

which will be determined from (2.4) after φ_{j-2}^1 is found ($i=1$ here). For convenience we agree also that

$$\varphi_{(-1)}^k=\varphi_0^k=0,\quad \forall k. \tag{4.5}$$

The sequence (3.1) is determined by the following recursive formula:

$$\psi_{j-3-2i}^{i+1}+\nu A\psi_{j-1-2i}^i+\sum_{k=0}^{i}\binom{i}{k}PB(\psi_{j-1-2k}^k+\varphi_{j-1-2k}^k,\ \psi_{j-1-2i+2k}^{i-k}+\varphi_{j-1-2i+2k}^{i-k})=\begin{cases}Pf, & \text{for } i=0\\ 0, & \text{for } i\geq 1.\end{cases} \tag{4.6}$$

Note that (4.6) gives ψ_{j-3-2i}^{i+1} explicitly and that all the necessary quantities are known when we compute ψ_{j-3-2i}^{i+1} ($i=0,\cdots,[(j-1)/2]$). The expression $A\psi_{j-1-2i}^i$ makes sense since ψ_{j-1-2i}^i belongs to the finite dimensional space P_mH.

Since ψ_s^{i+1} is intended to be some approximation of $d^{i+1}p/dt^{i+1}$ for large t, relation (4.6) ought to be compared to relation (1.10) and the relation obtained by differentiating (1.10) i times; i.e. using Leibnitz formula:

$$(4.7) \qquad \frac{d^{i+1}p}{dt^{i+1}}+\nu A\frac{d^i p}{dt^i}+\sum_{k=0}^{i}\binom{i}{k}PB\Big(\frac{d^k}{dt^k}(p+q),\ \frac{d^{i-k}}{dt^{i-k}}(p+q)\Big) = \begin{cases} Pf, & \text{for } i=0 \\ 0, & \text{for } i\geq 1. \end{cases}$$

The difference between (4.6) and (4.7) is the replacement of $d^k p/dt^k$ and $d^k q/dt^k$ by ψ_s^k and φ_s^k with appropriate values of s.

Similarly, by differentiating (1.10) i times we obtain an analog of (4.7):

$$(4.8) \qquad \frac{d^{i+1}q}{dt^{i+1}}+\nu A\frac{d^i q}{dt^i}+\sum_{k=0}^{i}\binom{i}{k}QB\Big(\frac{d^k(p+q)}{dt^k},\ \frac{d^{i-k}}{dt^{i-k}}(p+q)\Big) = \begin{cases} Qf, & \text{for } i=0, \\ 0, & \text{for } i\geq 1. \end{cases}$$

Once the sequence (4.1) is determined we construct the sequence (4.2) by decreasing order of the index i. We start from (4.5) and then we write

$$(4.9) \qquad \nu A\varphi_{j-2i}^{i}+\varphi_{j-2-2i}^{i+1}+\sum_{i=0}^{k}\binom{i}{k}QB(\psi_{j-1-2k}^{k}+\varphi_{j-1-2k}^{k},\ \psi_{j-1-2i+2k}^{j,k}+\varphi_{j-1-2i+2k}^{j-k}) = \begin{cases} Qf, & \text{for } i=0, \\ 0, & \text{for } i\geq 1. \end{cases}$$

Relations (4.9) allow us to determine $\varphi_{j-2i}^{i}\in QD(A)$ (by inverting A). Note that all the necessary quantities are known when we determine φ_{j-2i}^{i}: they have been computed either at the step $j-1$, or at the step j during the determination of the sequence (4.1); finally φ_{j-2-2i}^{i+1} has been determined at the step j, at the previous value of i since we consider decreasing values of i. Of course (4.9) simply mimics (4.8).

REMARK 4.1. A perusal of the construction above of the φ_j^i, ψ_j^i, shows that these are all polynomial type functions of $p=p_m$ (i.e. finite sums of multilinear continuous functions of p). □

We now proceed with the proof of (2.5). More generally we shall prove by induction the following estimates.

LEMMA 4.1. *For every $j\geq 0$, there exist constants κ_j^1 depending only on $j, \nu, |f|, \lambda_1$ and R_1 such that, for every m and every $t\geq t_2$:*

$$\left|\psi_{j-1-2i}^i(t)-\frac{d^ip}{dt^i}(t)\right|\leq\kappa_j^1\delta^{j/2+1-i}L^{j/2+1/2},\qquad 0\leq i\leq[(j-1)/2] \tag{4.10}$$

$$\left|A\varphi_{j-2i}^i(t)-A\frac{d^iq}{dt^i}(t)\right|\leq\kappa_j^1\delta^{j/2-i}L^{j/2+1/2-i},\qquad 0\leq i\leq[(j+1)/2]. \tag{4.11}$$

PROOF. We first notice that due to (2.14), (2.18), relations (4.10), (4.11) imply similar inequalities with the other norms of H, V, $D(A)$; i.e. changing the name of the constants

$$\left\|\psi_{j-1-2i}^i-\frac{d^ip}{dt^i}\right\|\leq\kappa_j^1\delta^{j/2-i+1/2}L^{j/2+1/2} \tag{4.12}$$

$$\left|A\psi_{j-1-2i}^i-A\frac{d^ip}{dt^i}\right|\leq\kappa_j^1\delta^{j/2-i}L^{j/2+1/2}$$

$$\left|\varphi_{j-2i}^i-\frac{d^iq}{dt^i}\right|\leq\kappa_j^1\delta^{j/2+1-i}L^{j/2+1/2-i}, \tag{4.13}$$

$$\left\|\varphi_{j-2i}^i-\frac{d^iq}{dt^i}\right\|\leq\kappa_j^1\delta^{j/2+1/2-i}L^{j/2+1/2-i}.$$

Also, since

$$\delta^{1/2}L^{i+1/2}\leq c_j \tag{4.14}$$

for $i\leq(j+1)/2$, the relations similar to (4.11), (4.13) hold for the ψ_r^i; in particular[7)]

$$\left\|\psi_{j-1-2i}^i-\frac{d^ip}{dt^i}\right\|\leq\kappa_j^1\delta^{j/2-i}L^{j/2-i}. \tag{4.15}$$

The proof of (4.10), (4.11) (and its consequences (4.12)-(4.15)) will be done by induction. The induction on j and i proceeds in the same order as the determination of the ψ_{j-1-2i}^i, ψ_{j-2i}^i: increasing values of j and, for fixed j, increasing values of i for the ψ_{j-1-2i}^i and then decreasing values of i for the ψ_{j-2i}^i.

For $j=1$, $i=0$ and (4.11), (4.12) are obvious because of (4.3), (4.5). Then assuming that relation (4.10) have been proved up to order $j-1$, we want to prove them at order j. As indicated before we proceed by

7) Compare to (4.13) with j replaced by $j-1$.

induction on i in the order indicated above.

Thanks to (4.3), relation (4.10) is obvious for $i=0$. Assuming that relations (4.10) are proved up to order i, we now prove them at order $i+1$. For that purpose we subtract (4.7) from (4.6):

$$\left|\phi_{j-3-2i}^{i+1}-\frac{d^{i+1}p}{dt^{i+1}}\right|\leq\nu\left|A\phi_{j-1-2i}^{i}-A\frac{d^{i}p}{dt^{i}}\right| \tag{4.16}$$
$$+\sum_{k=0}^{i}\binom{i}{k}\Big|B(\phi_{j-1-2k}^{k}+\varphi_{j-1-2k}^{k},\ \phi_{j-1-2i+2k}^{i-k}+\varphi_{j-1-2i+2k}^{i-k})$$
$$-B\Big(\frac{d^k}{dt^k}(p+q),\ \frac{d^{i-k}}{dt^{i-k}}(p+q)\Big)\Big|.$$

Thanks to the induction assumption and (4.12):

$$\left|A\phi_{j-1-2i}^{i}-A\frac{d^{i}p}{dt^{i}}\right|\leq\kappa\delta^{j/2-i}L^{j/2+1/2}. \tag{4.17}$$

Then using the bilinearity of B, the induction hypotheses and Lemma 2.2 we bound the difference of the B-terms by the sum from $k=0$ to i of quantities of the form:

$$\kappa L^{1/2}\left\|\phi_{j-1-2k}^{k}-\frac{d^{k}p}{dt^{k}}\right\|+\kappa\left\|\varphi_{j-1-2k}^{k}-\frac{d^{k}q}{dt^{k}}\right\| \tag{4.18}$$
$$+\kappa\left|\varphi_{j-1-2k}^{k}-\frac{d^{k}q}{dt^{k}}\right|^{1/2}\left|A\varphi_{j-1-2k}^{k}-A\frac{d^{k}q}{dt^{k}}\right|^{1/2}.$$

As in (4.17), all these terms are bounded by the appropriate powers of δ and L that are necessary for (4.10) (we use also (4.14)).

Then we continue with (4.11) and proceed by decreasing induction on i. For $i=[(j+1)/2]$, $j-2i=0$ or -1, $\varphi_0^i=\varphi_{-1}^i=0$ and (4.11) reduces to

$$\left|A\frac{d^{l}q}{dt^{l}}\right|\leq\kappa L^{1/2}\quad\text{if}\quad j=2l,\ i=l, \tag{4.19}$$
$$\left|A\frac{d^{l+1}q}{dt^{l+1}}\right|\leq\kappa\delta^{-1/2}\quad\text{if}\quad j=2l+1,\ i=l+1.$$

In both cases this follows readily from the third inequality (1.16).

Assuming now that (4.11) has been proved for $i=[(j+1)/2],\cdots,i+1$, we want to prove it at order i. We subtract (4.8) from (4.9) and find

$$\left|A\varphi^i_{j-2i}-A\frac{d^iq}{dt^i}\right|\leq\frac{1}{\nu}\left|\varphi^{i+1}_{j-2-2i}-\frac{d^{i+1}q}{dt^{i+1}}\right|$$
$$+\frac{1}{\nu}\sum_{k=0}^{i}\binom{i}{k}\Big|B(\psi^k_{j-1-2k}+\varphi^k_{j-1-2k},\ \psi^{j-k}_{j-1-2i+2k}+\varphi^{j-k}_{j-1-2i+2k})$$
$$-B\Big(\frac{d^k}{dt^k}(p+q),\ \frac{d^{i-k}}{dt^{i-k}}(p+q)\Big)\Big|.$$

By the induction hypothesis and (4.13)

$$\left|\varphi^{i+1}_{j-2-2i}-\frac{d^{i+1}q}{dt^{i+1}}\right|\leq\kappa\delta^{j/2-i}L^{j/2-1/2-i}$$
$$\leq\kappa\delta^{j/2-i}L^{j/2+1/2-i}. \tag{4.20}$$

Then using the bilinearity of B, the induction hypotheses and Lemma 2.2, we bound the difference of the B-terms by the sum from $k=0$ to i of quantities of the form (4.18). They are all bounded by

$$\kappa\delta^{j/2+1/2-k}L^{j/2+1}+\kappa\delta^{j/2-k}L^{j/2-k}.$$

The worse (largest) terms are obtained for $k=i$:

$$\kappa\delta^{j/2+1/2-i}L^{j/2+1}+\kappa\delta^{j/2-i}L^{j/2-i},$$

and they are both bounded by the desired expression, namely

$$\kappa\delta^{j/2-i}L^{j/2+1/2-i}.$$

The proof is complete. □

REMARK 4.2. We can observe, as we did in Section 3 that ψ^i_{j-1-2i}, ψ^i_{j-2i} depend on t through $p(t)=p_m(t)$:

$$\psi^i_{j-1-2i}(t)=\Psi^i_{j-1-2i}(p(t)),$$
$$\varphi^i_{j-2i}(t)=\Phi^i_{j-2i}(p(t)).$$

Here Ψ^i_{j-1-2i} and Φ^i_{j-2i} are polynomial type functions of p, mapping P_mH into $Q_mD(A)$.

Furthermore we can reinterpret (4.10)-(4.13) as approximation results for the time derivatives of $p=p_m(t)$ and $q=q_m(t)$ and these results complete those in Section 3.

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