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# *Formal series solutions of singular systems of linear differential equations and singular matrix pencils*

Dedicated to Tosihusa Kimura on the occasion of his 60th birthday

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**Abstract.** A system of $n$ linear differential equations $xy'(x)=x^{-s}B(x)y(x)$ is considered, in which $s$ is a positive integer and $B(x)$ is a formal power series. In a preceding paper [6] the author has developed a method for evaluating a formal fundamental matrix solution to the system. This method leads to the algebraic treatment of certain singular matrix pencils. In the present paper the author provides some transformation algorithms for singular matrix pencils which are used in all parts of the method for constructing the formal fundamental matrix solution.

## Introduction.

Let an $n$ by $n$ system of formal linear differential equations

$$xy'(x)=x^{-s}B(x)y(x)$$

be given, in which $s$ is a positive integer and $B(x)$ a formal power series in $x$. The problem is how to calculate a formal fundamental matrix solution of the form $Y(x)=H(x)x^{J}e^{Q(x)}$. Here the exponent $Q(x)$ is a diagonal matrix containing polynomials in negative fractional powers of $x$, $J$ is a constant Jordan matrix commuting with $Q(x)$, and $H(x)$ is a formal power series in a positive fractional power of $x$. In a preceding paper [6] the author has developed the theoretical foundations for a new method by which a formal fundamental solution is computed by columns In the first step of this method the exponential part is determined. After this, the problem can be reduced to the computation of single formal logarithmic solutions as treated in the author's paper [5]. Both parts of this method are based on the algebraic treatment of certain matrices $A_{m}^{(r)}(\lambda)$ containing the leading coefficients of $B(x)$ and a linear complex parameter.

The fundamental properties of matrices which are polynomials in $\lambda$ can be read from their Smith's canonical form as it is done in the paper [5]. In fact, Smith's canonical form can algorithmically be evaluated—

see [1], for instance—. From a practical point of view, however, the evaluation of Smith's canonical form is not recommendable because the canonical form as well as the corresponding transformation matrices contain polynomials of higher degree: this will lead to severe storage requirements. As the parameter $\lambda$ occurs only linearly, the $A_m^{(r)}(\lambda)$ are so-called singular matrix pencils which can be transformed into Kronecker's canonical form by means of an algorithm developed by van Dooren [4]. We will be contented with some weaker properties than Kronecker's canonical form.

In Section 1 we report on some present results concerning the exponential part in the formal fundamental solution: it turns out that we need the defect numbers and generalized characteristic polynomials of the $A_m^{(r)}(\lambda)$. In Section 2 we provide the algorithm announced by which the defect numbers and characteristic polynomials can be determined. As compared with van Dooren [4], we introduce two modifications. First we use rational transformation matrices instead of unitary ones, because we intend to realize the algorithm by a computer algebra system. Secondly, whenever we have constructed a transformation of an $A_m^{(r)}(\lambda)$ we want to use this result for a transformation of the subsequent $A_{m+1}^{(r)}(\lambda)$: this is desirable with regard to the large size of the matrices. On the other hand, the latter modifications give rise to weaker structures of the transformed matrices than obtained by van Dooren's method. We will call these structures "property (C)" and "property (R)" suggesting the words "columns" and "rows", respectively.

Section 3 is devoted to the computation of the formal logarithmic part in the formal fundamental solution. We refer to a constructional method presented in the author's paper [5]. This method is based on an appropriate treatment of certain systems of linear homogeneous algebraic equations the coefficient matrices of which are derived from the $A_m^{(0)}(\lambda)$. Here we describe the method in a formally simplified way by considering polynomial identities instead of using augmented coefficient matrices. The main subject of this section, however, is the practical treatment of the linear algebraic systems. It turns out that the transformations from Section 2 are very useful for the evaluation both of the formal monodromy $J$ and of the power series coefficients of $H(x)$.

## 1. A surview on former results.

The system considered has the form

$$xy'(x)=x^{-s}\Big(\sum_{\nu=0}^{\infty}x^{\nu}B_{\nu}\Big)y(x), \tag{1.1}$$

where $s\in N_{+}$, the $B_{\nu}$ are constant $n$ by $n$-matrices. The formal fundamental matrix to be determined has the form

$$Y(x)=H(x)x^{J}e^{Q(x)},$$

where with a certain integer $p\in N_{+}$

(1.2)
$$Q(x)=\operatorname{diag}(q_1(x),\cdots,q_n(x)),\qquad q_k(x)=\sum_{\nu=1}^{sp}\alpha_{k,\nu/p}x^{-\nu/p},$$
$J$ a constant $n$ by $n$-Jordan matrix commuting with $Q$, $H(x)$ a formal power series in $x^{1/p}$.

The matrices $A_m^{(r)}(\lambda)$ $(r\in\{0,1,\cdots,s\},\ m\in N)$ are defined in the following way—where $I$ is the $n$ by $n$ identity matrix, $\lambda$ is a complex parameter—:

$$A_m^{(r)}(\lambda)=\begin{pmatrix}
B_0 & & & & & & & \\
B_1 & B_0 & & & & & 0 & \\
\vdots & \ddots & \ddots & & & & & \\
B_{s-r}-\lambda I & \cdots & B_1 & B_0 & & & & \\
\vdots & \ddots & & \ddots & \ddots & & & \\
B_s & \cdots & B_{s-r}-\lambda I & \cdots & B_1 & B_0 & & \\
\vdots & B_s-I & & \ddots & & \ddots & \ddots & \\
\vdots & & \ddots & & \ddots & & \ddots & \ddots \\
B_m & \cdots & B_s-(m-s)I & \cdots & B_{s-r}-\lambda I & \cdots & B_1 & B_0
\end{pmatrix}.$$

For $r\in\{0,1,\cdots,s\}$ we define

$$d_m^{(r)}:=\min\{\dim\mathfrak{N}(A_m^{(r)}(\lambda)):\lambda\in C\}\qquad(m\in N),\quad d_{-1}^{(r)}:=0, \tag{1.3}$$

where the symbol $\mathfrak{N}$ denotes the null-space, and further

(1.4) $\chi_m^{(r)}(\lambda):=$greatest common divisor of all subdeterminants of the order $(m+1)n-d_m^{(r)}$ in $A_m^{(r)}(\lambda)$ $(m\in N)$.
$\chi_{-1}^{(r)}(\lambda):=1.$

For each $r$ the sequence of the $d_m^{(r)}$ $(m\in N\cup\{-1\})$ has the following behaviour—see [2] and [5]:

(1.5) LEMMA. *There exists a unique* $N_r \in N \cup \{-1\}$ *such that*

$$\forall m < N_r \quad d_m^{(r)} < d_{m+1}^{(r)}; \quad \forall m \geqq N_r \quad d_m^{(r)} = d_{N_r}^{(r)} =: d^{(r)}.$$

*There are the a priori estimates*

$$N_r \leqq (s-r)n-1, \qquad d^{(r)} \leqq (s-r)n.$$

In the case $r>0$, the polynomials $\chi_m^{(r)}(\lambda)$ for $m=N_r$ and $N_{r+1}$ serve for the evaluation of the coefficients $\alpha_{k,r}$ in the $q_k$ from (1.2). The basic result is comprehended in the following theorem proved by Schäfke and Volkmer [3]:

(1.6) THEOREM. *Let* $r \in \{1, \cdots, s\}$ *be given. Then for all* $m \geqq N_r$

$$\chi^{(r)}(\lambda) := \chi_{m+1}^{(r)}(\lambda) / \chi_m^{(r)}(\lambda)$$

*is a polynomial independent of m. Further*

$$\deg \chi^{(r)} = \#\{k \in \{1, \cdots, n\} : \partial q_k \leqq r\}.$$

*The zeros of* $\chi^{(r)}$ *are*

$$-\frac{1}{r}\alpha_{k,r} \text{ for those } q_k \text{ with } \partial q_k \leqq r.$$

Here $\partial q_k$ denotes the maximal rational index $\sigma$ with $\sigma_{k,\sigma} \neq 0$, or $\partial q_k := 0$ if $q_k = 0$. The theorem remains valid for any rational $r = \rho/p$ with $\rho, p \in N_+$, $r<s$, and for systems (1.1) containing a power series in $x^{1/p}$. For the definition of $\chi^{(r)}$ in those cases, we change the variable $x$ into $t$ with $t^p = x$ and consider the polynomial $\tilde{\chi}^{(\rho)}$ corresponding to the transformed differential system: then we define $\chi^{(r)}(\lambda) := \tilde{\chi}^{(\rho)}(p\lambda)$. It turns out that this definition is independent of the $p$ chosen.

Theorem (1.6) is employed for a stepwise evaluation of all $q_k$'s. Suppose there is $0 < r \in Q$ such that all coefficients $\alpha_{k,\sigma}$, $r < \sigma \leqq s$, of a $q_k$ are known. Then $\alpha_{k,r}$ is determined by Theorem (1.6) applied to the system

$$xw'(x) = \left(x^{-s} \sum_{\nu=0}^{\infty} x^\nu B_\nu + \sum_{r<\sigma\leqq s} \sigma \alpha_{k,\sigma} x^{-\sigma} I\right) w(x), \tag{1.7}$$

see [6], Section 1. It is important for practice that the denominators $p$ in the fractional powers of $x$ are as small as possible. Otherwise, the matrices $A_m^{(r)}(\lambda)$ occurring become too large. In [6], Section 3 the author describes an algorithm which has the purpose that each $q_k$ is represented with a minimal $p$ dependent on the $q_k$ considered.

The matrices $A_m^{(0)}(\lambda)$ are responsible for the formal logarithmic part in (1.2), namely $J$ and $H(x)$. A preliminary result concerning the polynomial $\chi^{(0)}$ has been proved in [6]:

(1.8) THEOREM. *For all* $m \geqq N_0$

$$\chi^{(0)}(\lambda) := \chi_{m+1}^{(0)}(\lambda)/\chi_m^{(0)}(\lambda+1)$$

*is a polynomial independent of* $m$. *Further,*

$$\deg \chi^{(0)} = \#\{k \in \{1, 2, \cdots, n\}\colon\ q_k = 0\}.$$

*The zeros of* $\chi^{(0)}$ *modulo* $\boldsymbol{Z}$ *are the diagonal elements of* $J$ *in the columns corresponding to* $q_k = 0$ *in* $Q$.

The definition of $\chi^{(0)}$ is extended to systems (1.1) with a formal power series in $x^{1/p}$ ($p \in N_+$) as it has been done for the $\chi^{(r)}$, $r > 0$. Let $q$ be a polynomial in a negative fractional power of $x$. Then Theorem (1.8), applied to the system

$$xw'(x) = \left(x^{-s} \sum_{\nu=0}^{\infty} x^\nu B_\nu - x\frac{d}{dx}q(x)I\right)w(x), \tag{1.9}$$

can serve for a test whether $q$ occurs as a $q_k$ in $Q$ or not, and is used in the algorithm for the minimal choices of $p$.

## 2. Singular matrix pencils.

Let $A$ and $C$ be some rectangular $n$ by $m$ matrices with complex coefficients, $n$ and $m$ being arbitrary positive integers at the moment. Then the family $(A - \lambda C)_{\lambda \in C}$ is called a singular matrix pencil. For each $\lambda$ fixed, $A - \lambda C$ corresponds to a linear map from $\boldsymbol{C}^m$ into $\boldsymbol{C}^n$ and the transposed matrix $(A - \lambda C)^t$ to a linear map from $\boldsymbol{C}^n$ into $\boldsymbol{C}^m$. We define

$$d = d(A, C) := \min\{\dim \mathcal{N}(A - \lambda C)\colon\ \lambda \in \boldsymbol{C}\},$$
$$\tilde{d} = \tilde{d}(A, C) := \min\{\dim \mathcal{N}((A - \lambda C)^t)\colon\ \lambda \in \boldsymbol{C}\},$$

where $\mathcal{N}$ denotes the null-space (kernel) of a linear map. Obviously, $\tilde{d} = d + (n - m)$. Further we introduce

$\chi(\lambda)=\chi(A, C;\ \lambda):=$greatest common divisor of all $(m-d)\times(m-d)$-subdeterminants of $A-\lambda C$.

The basic property of $\chi$ is that for any $\lambda_0 \in \boldsymbol{C}$

$$\dim \mathcal{N}(A-\lambda_0 C)>d \Longleftrightarrow \chi(\lambda_0)=0.$$

With a view to block decompositions it also makes sense to admit $n=0$ or $m=0$. In the case $n=0$, for instance, we set $d=m$, $\tilde{d}=0$, $\chi(\lambda)=1$. In the following lemma, which quotes a well-known result, $n$ and $m$ are positive:

(2.1) LEMMA. *Let $P$, $Q$ be constant invertible matrices of size $n$ by $n$ and $m$ by $m$, respectively, and*

$$\tilde{A}-\lambda\tilde{C}:=P(A-\lambda C)Q \qquad (\lambda \in \boldsymbol{C}).$$

*Then*

$$d(\tilde{A}, \tilde{C})=d(A, C), \qquad \chi(\tilde{A}, \tilde{C};\ \lambda)=\chi(A, C;\ \lambda).$$

(2.2) DEFINITION. An $n$ by $m$ matrix pencil $A-\lambda C$ is said to have *property* (C) iff there exists a block decomposition

$$A-\lambda C=\left(\begin{array}{c|c} A_I-\lambda C_I & 0 \\ \hline \ast & A_{II}-\lambda C_{II} \end{array}\right)\begin{array}{l}\}n_I \\ \}n_{II}\end{array}$$
$$\qquad\underbrace{\qquad\quad}_{m_I}\ \underbrace{\qquad\quad}_{m_{II}}$$

and

$$\operatorname{rank} C_I=m_I, \qquad \forall\lambda \in \boldsymbol{C} \quad \operatorname{rank}(A_{II}-\lambda C_{II})=n_{II}.$$

(2.3) THEOREM. *Suppose $A-\lambda C$ has property* (C). *Then*

$$d(A, C)=m_{II}-n_{II}, \qquad \chi(A, C;\ \lambda)=\chi(A_I, C_I;\ \lambda).$$

PROOF. Since $C_I$ has full column rank it is easy to see that $d(A_I, C_I)=0$. Further, $d(A_{II}, C_{II})=m_{II}-n_{II}$ and $\chi(A_{II}, C_{II};\ \lambda)=1$. Therefore the first assertion of the theorem is obvious. Next we have to consider all determinants of square submatrices of $A-\lambda C$ of size $(m-d)^2=(m_I+n_{II})^2$. First we restrict ourselves to those such matrices which have both an $m_I\times m_I$-subblock inside of $A_I-\lambda C_I$ and an $n_{II}\times n_{II}$-subblock inside of $A_{II}-\lambda C_{II}$. The corresponding determinant is the product of determinants defining $\chi(A_I, C_I;\ \lambda)$ and $\chi(A_{II}, C_{II};\ \lambda)$, respectively. Therefore the greatest common divisor of these special subdeterminants is $\chi(A_I, C_I; \lambda)$ because $\chi(A_{II}, C_{II}; \lambda)=1$. If any $(m_I+n_{II})^2$-submatrix of $A-\lambda C$ does not have a full $m_I \times m_I$-block in $A_I-\lambda C_I$ and a full $n_{II}\times n_{II}$-block in $A_{II}-\lambda C_{II}$ then its determinant vanishes owing to the block structure of $A-\lambda C$.

(2.4) THEOREM. *Let $A-\lambda C$ be an arbitrary $n$ by $m$ pencil with $n, m>0$. Then there exists a rational algorithm to construct constant invertible matrices $P$, $Q$ of size $n$ by $n$ and $m$ by $m$, respectively, such that the pencil*

$$\tilde{A}-\lambda\tilde{C} := P(A-\lambda C)Q$$

*has property* (C).

PROOF. The rational algorithm which we will present is a modification of [4], Algorithm 4.1. —Given an arbitrary constant $n$ by $m$ matrix $A$, the well-known Gaussian elimination method applied to the rows or the columns of $A$, yields a constant invertible $n$ by $n$ matrix $P$ or, respectively, an invertible $m$ by $m$ matrix $Q$ such that

$$PA=\begin{pmatrix} 0 \\ A_R \end{pmatrix}\}r, \qquad AQ=\Big(\underbrace{A_C}_{r} \;\Big|\; 0\Big),$$

where $r$ denotes the rank of $A$. Moreover, $A_R$ is a generalized upper-triangular, $A_C$ a generalized lower-triangular matrix. The latter means that $A_C=: (\tilde{a}_{i,j})$

$$\text{(2.5)}\qquad \begin{pmatrix} 0 & 0 & & \\ x & & 0 & \\ | & 0 & & \\ | & x & & \\ | & | & \cdots & x \\ | & | & & | \end{pmatrix}, \; x \text{ means some nonzero number,}$$

that is there exist $1\leqq n_1< \cdots <n_r\leqq n$ such that $\tilde{a}_{n_j,j}\neq 0$ $(j=1, \cdots, r)$ and $\tilde{a}_{i,j}=0$ for all $i<n_j$, $j\in\{1, \cdots, r\}$. —Such transformations, which are called compressions to full row (column) rank, are used in the following algorithm.

(2.6) *Algorithm.* Let an $n$ by $m$ pencil $A-\lambda C$ be given.

At first, construct $Q_1$ such that

$$CQ_1=(C_2|0),\quad C_2 \text{ has full column rank.}$$

If $\operatorname{rank} C_2=m$ —the zero columns in $CQ_1$ do not occur— then set $P=I$, $Q=Q_1$ and stop. Otherwise, let

$$AQ_1=:(A_2|A_1)$$

be subdivided in the same manner as $CQ_1$ above. Then we comprime $A_1$ to full row rank by using an invertible $n$ by $n$ matrix $P_1$. We define

$$P_1A_1=:\begin{pmatrix} 0 \\ A_{1,1} \end{pmatrix},\qquad P_1AQ_1=:\begin{pmatrix} \tilde{A}_{2,2} & 0 \\ \tilde{A}_{2,1} & A_{1,1} \end{pmatrix}.$$

Let $P_1CQ_1$ be subdivided in the same manner,

$$P_1CQ_1=:\begin{pmatrix} \tilde{C}_{2,2} & 0 \\ \tilde{C}_{2,1} & 0 \end{pmatrix}.$$

If $\tilde{C}_{2,2}$ has full column rank then $P_1(A-\lambda C)Q_1$ has property (C). Otherwise, the first step of our algorithm is applied to the subpencil $\tilde{A}_{2,2}-\lambda\tilde{C}_{2,2}$, and so on.

The final form of $\tilde{A}-\lambda\tilde{C}$ obtained by this algorithm is noted in van Dooren's paper [4], Formula (4.2). Obviously, property (C) is satisfied.

As an application to singular systems of differential equations (1.1), we use the preceding results to evaluate the $d_m^{(r)}$ which are the quantities $d$ corresponding to the matrix pencils $A_m^{(r)}(\lambda)$. For $r$ fixed, we evaluate the $d_m^{(r)}$ $(m=-1, 0, 1, 2, \cdots)$ successively until $d_{m+1}^{(r)}=d_m^{(r)}$ occurs for the first time: then $m=N_r$ is achieved. For this purpose it is convenient to modify the algorithm (2.6) in such way that the transformation of $A_{m+1}^{(r)}(\lambda)$ into property (C) requires only a few steps whenever the analogous transformation for $A_m^{(r)}(\lambda)$ is present. The latter situation is the starting point for the next algorithm.

(2.7) *Algorithm.* Suppose the matrix pencil $A-\lambda C$ has the following structure:

$$A-\lambda C=\left(\begin{array}{c|c} G-\lambda H & \begin{array}{c} 0 \\ \hline A'-\lambda C' \end{array} \end{array}\right)\begin{array}{l}\}\tilde{n} \\ \}n' \end{array},$$
$$\qquad\qquad\tilde{m}\qquad\quad m'$$

further let $P', Q'$ be invertible such that

$$P'(A'-\lambda C')Q' = \left(\begin{array}{c|c} A'_I - \lambda C'_I & 0 \\ \hline * & A'_{II} - \lambda C'_{II} \end{array}\right) \begin{array}{l} \}n'_I \\ \}n'_{II} \end{array}$$
$$\qquad\qquad\qquad\qquad m'_I \qquad\qquad m'_{II}$$

has property (C). Then the full matrix $A-\lambda C$ is transformed as follows. First, set

$$P_0 := \begin{pmatrix} I_{\hat{n}} & 0 \\ 0 & P' \end{pmatrix}, \qquad Q_0 := \begin{pmatrix} I_{\hat{m}} & 0 \\ 0 & Q' \end{pmatrix}$$

and compute

$$P_0(A-\lambda C)Q_0 =: \left(\begin{array}{c|c|c} & \multicolumn{2}{c}{0} \\ \cline{2-3} G'_I - \lambda H'_I & A'_I - \lambda C'_I & 0 \\ \hline * & * & A'_{II} - \lambda C'_{II} \end{array}\right).$$

We assume that $C'_I$ has got a generalized triangular form (2.5) by a preceding algorithm. Then we apply a sequence of Gaussian elimination steps to the columns of

$$\left(\begin{array}{c|c} H'_I & \begin{array}{c} 0 \\ \hline C'_I \end{array} \end{array}\right),$$

starting with the "$x$"-elements in $C'_I$ as pivots. As a result, we obtain an invertible $(m+m'_I)^2$-matrix $Q_I^{(1)}$ such that

$$\left(\begin{array}{c|c} H'_I & \begin{array}{c} 0 \\ \hline C'_I \end{array} \end{array}\right) Q_I^{(1)} = \left(\begin{array}{c|c} C_I^{(2)} & 0 \end{array}\right),$$

$C_I^{(2)}$ has full column rank. After this modified first step we apply the rest of algorithm (2.6) to construct $P_I, Q_I$ such that

$$P_I \left(\begin{array}{c|c} G'_I - \lambda H'_I & \begin{array}{c} 0 \\ \hline A'_I - \lambda C'_I \end{array} \end{array}\right) Q_I =: \left(\begin{array}{c|c} \begin{array}{c} A_I - \lambda C_I \\ \hline * \end{array} & \begin{array}{c} 0 \\ \hline A_{III} - \lambda C_{III} \end{array} \end{array}\right)$$

has property (C). Finally, we set

$$P := P_0 \left( \begin{array}{c|c} P_I & 0 \\ \hline 0 & I_{n'_{II}} \end{array} \right), \qquad Q := \left( \begin{array}{c|c} Q_I & 0 \\ \hline 0 & I_{m'_{II}} \end{array} \right) Q_0.$$

Then $P(A-\lambda C)Q$ has property (C) because

$$A_{II} - \lambda C_{II} := \left( \begin{array}{c|c} A_{III} - \lambda C_{III} & 0 \\ \hline \ast & A'_{II} - \lambda C'_{II} \end{array} \right)$$

has full row rank for all $\lambda$.

Next we introduce a property dual to (C):

(2.8) DEFINITION. An $n$ by $m$ matrix pencil $A-\lambda C$ is said to have *property* (R) iff there exists a block decomposition

$$A - \lambda C = \left( \begin{array}{c|c} \tilde{A}_I - \lambda \tilde{C}_I & 0 \\ \hline \ast & \tilde{A}_{II} - \lambda \tilde{C}_{II} \end{array} \right) \begin{array}{l} \}\tilde{n}_I \\ \}\tilde{n}_{II} \end{array}$$
$$\qquad\qquad \tilde{m}_I \qquad\qquad \tilde{m}_{II}$$

and

$$\forall \lambda \in \boldsymbol{C} \quad \operatorname{rank}(\tilde{A}_I - \lambda \tilde{C}_I) = \tilde{m}_I, \quad \operatorname{rank} \tilde{C}_{II} = \tilde{n}_{II}.$$

(2.9) THEOREM. *Suppose $A-\lambda C$ has property* (R). *Then*

$$\tilde{d}(A, C) = \tilde{n}_I - \tilde{m}_I, \qquad \chi(A, C; \lambda) = \chi(\tilde{A}_{II}, \tilde{C}_{II}; \lambda).$$

The proof is analogous with that of Theorem (2.3).

(2.10) THEOREM. *Let $A-\lambda C$ be an arbitrary $n$ by $m$ pencil with $n, m>0$. Then there exists a rational algorithm to construct constant invertible matrices $P$, $Q$ of size $n$ by $n$ and $m$ by $m$, respectively, such that*

$$\tilde{A} - \lambda \tilde{C} := P(A - \lambda C)Q$$

*has property* (R).

The corresponding algorithm proceeds in a similar way as (2.6): but in the present case the column compressions in $C$ must be changed into row compressions, the row compressions in $A$ are changed into column compressions. For more details see van Dooren [4], Algorithm 4.5.

In the following we discuss the result obtained by a combination of both algorithms.

(2.11) THEOREM. *Suppose a matrix pencil* $A-\lambda C$ *has property* (C) *and* $n_I, m_I>0$. *Further let* $P_I, Q_I$ *be invertible matrices of size* $n_I^2, m_I^2$, *respectively, such that*

$$P_I(A_I-\lambda C_I)Q_I=\left(\begin{array}{c|c} \tilde{A}_I-\lambda\tilde{C}_I & 0 \\ \hline /\!/\!/\!/\!/ & \tilde{A}_{II}-\lambda\tilde{C}_{II} \end{array}\right)\begin{array}{l}\}\tilde{n}_I \\ \}\tilde{n}_{II}\end{array}$$
$$\qquad\qquad\tilde{m}_I \qquad\qquad \tilde{m}_{II}$$

*has property* (R).—*Then* $\tilde{n}_{II}=\tilde{m}_{II}$. *Further* $\chi(A, C;\ \lambda)=1$ *iff* $\tilde{n}_{II}=\tilde{m}_{II}=0$. *Otherwise,* $\tilde{C}_{II}$ *is invertible and*

$$\chi(A, C;\ \lambda)=(\det(-\tilde{C}_{II}))^{-1}\det(\tilde{A}_{II}-\lambda\tilde{C}_{II}).$$

PROOF. A combination of the theorems (2.3) and (2.9) yields

$$\chi(A, C;\ \lambda)=\chi(A_I, C_I;\ \lambda)=\chi(\tilde{A}_{II}, \tilde{C}_{II};\ \lambda).$$

Further, because $C_I$ has full column rank, also $P_IC_IQ_I$ has full column rank: by using the block structure of $P_IC_IQ_I$ we conclude that $\tilde{C}_{II}$ has full column rank. On the other hand, $\tilde{C}_{II}$ also has full row rank because of property (R), and therefore $\tilde{n}_{II}=\tilde{m}_{II}$ holds. If $\tilde{n}_{II}$ is positive then $\tilde{C}_{II}$ is an invertible matrix: this implies $d(\tilde{A}_{II}, \tilde{C}_{II})=0$, $\chi(\tilde{A}_{II}, \tilde{C}_{II};\ \lambda)=(\det(-\tilde{C}_{II}))^{-1}\det(\tilde{A}_{II}-\lambda\tilde{C}_{II})$.

As already mentioned, we use the algorithms (2.6) and (2.7) for the successive evaluation of the $d_m^{(r)}$ $(m=0, 1, 2, \cdots)$, $r\in\{0, 1, \cdots, s-1\}$ being fixed. With regard to Algorithm (2.7) we modify the sequence of the $A_m^{(r)}(\lambda)$ $(m=0, 1, 2, \cdots)$ as follows. We set

$$\hat{A}_0^{(r)}(\lambda):=B_0$$

$$\hat{A}_{m+1}^{(r)}(\lambda):=\left(\begin{array}{c|c} B_0 & 0\cdots\cdots 0 \\ \hline \vdots & \\ B_{s-r}-\lambda I & \\ \vdots & \\ B_s+(m+1)I & \hat{A}_m^{(r)}(\lambda) \\ \vdots & \\ B_{m+1} & \end{array}\right) \qquad (m=0, 1, 2, \cdots).$$

Then each $\hat{A}^{(r)}_{m+1}(\lambda)$ has a subdivision as required in Algorithm (2.7) where $\hat{A}^{(r)}_m(\lambda)$ plays the role of $A'-\lambda C'$. Let $C^{(0)}_m$ be defined by the relations

$$A^{(0)}_m(\lambda)=: A^{(0)}_m(0)-\lambda C^{(0)}_m \qquad (m\in N).$$

Then it can be shown by induction that

$$\hat{A}^{(r)}_m(\lambda)=\begin{cases} A^{(r)}_m(\lambda)+mC^{(0)}_m & \text{in general,}\\ A^{(0)}_m(\lambda-m) & \text{in the case } r=0.\end{cases} \tag{2.12}$$

Lutz and Schäfke ([2], Lemma 3.1) have proved that in the case $r\geqq 1$ (and obviously for $r=0$) each $d^{(r)}_m$ remains unchanged when $A^{(r)}_m(\lambda)$ is replaced by $A^{(r)}_m(\lambda)-\alpha C^{(0)}_m$ with an arbitrary $\alpha\in \boldsymbol{C}$. The same is true for the polynomials $\chi^{(r)}_m$ in the case $r\geqq 1$ —see Schäfke and Volkmer [3], Corollary 1 (p. 92).

Summing up, we compute $d^{(r)}$, $N_r$ and $\chi^{(r)}$ by the following method.
*Step* 1: For $m=0, 1, 2, \cdots$ we transform the $\hat{A}^{(r)}_m(\lambda)$ into property (C) in order to obtain $d^{(r)}_m$. As soon as $d^{(r)}_{m+1}=d^{(r)}_m$ $(m\geqq -1)$ for the first time, we set $N_r:=m$ and go to Step 2.
*Step* 2: Provided $N_r\geqq 0$, we transform the submatrices $A_I-\lambda C_I$ in the results of Step 1 for $m=N_r$ and $N_r+1$ into property (R). Then we obtain $\chi^{(r)}_{N_r}(\lambda)$ and $\chi^{(r)}_{N_r+1}(\lambda)$ or $\chi^{(0)}_{N_0}(\lambda-N_0)$ and $\chi^{(0)}_{N_0+1}(\lambda-N_0-1)$, respectively.

Sometimes it is desirable to determine

$$\deg \chi^{(r)}=\deg \chi^{(r)}_{N_r+1}-\deg \chi^{(r)}_{N_r}$$

without performing Step 2. This is the case during the construction of a $q_k$ according to [6], Section 3, when it is checked whether a $\chi^{(r)}(\lambda)\equiv 1$ or not. For brevity, the proofs of the following results will not be very detailed. We need Kronecker's canonical form (K.c.f.) of a singular pencil, for which we use the notations given in [4], Formula (2.5).

(2.13) LEMMA. *Let the matrix pencil $A-\lambda C$ have property* (C), *and let the Kronecker's canonical form to $A-\lambda C$ be given by* [4], (2.5), *where we write $\tilde{d}$ and $d$ instead of $q$ and $p$. Then the*

*K.c.f. to $A_I-\lambda C_I$ is composed of the $L^P_{\eta_i}$ $(i=1, \cdots, \tilde{d})$ and $\lambda I-J$,*
*K.c.f. to $A_{II}-\lambda C_{II}$ is composed of the $L_{\varepsilon_i}$ $(i=1, \cdots, d)$ and $\lambda N-I$.*

In order to prove this lemma, we have to show that the hatched block in $A-\lambda C$ as in (2.2) can be cancelled by a constant transformation. For this we apply van Dooren's Algorithm 4.1 in [4] to the block $A_I-\lambda C_I$ in order to obtain the special form [4], (4.2) in which the block $\lambda B_{l+1,l+1}-A_{l+1,l+1}$ does not occur. Starting with the latter form, the block in the left under corner of $A-\lambda C$ is cancelled by a similar method as in [4], Lemma 3.2.

In fact, Lemma (2.13) yields an alternative proof for Theorem (2.3) which we had proved without using the K.c.f.. —Our main result concerning deg $\chi^{(r)}$ reads:

(2.14) THEOREM. *Let $\hat{A}_m^{(r)}(\lambda)$ for $m=N_r$ and $m=N_r+1$ be transformed into property* (C). *Let the corresponding quantities $n_I$ be denoted $n_I(\hat{A}_m^{(r)})$. Then*

$$\deg \chi^{(r)}=n_I(A_{N_r+1}^{(r)})-n_I(A_{N_r}^{(r)}).$$

PROOF. Owing to (1.6) and (1.8), $\deg \chi^{(r)}$ is the difference between the sizes of $\lambda I-J$ occurring in the K. c. forms to $\hat{A}_{N_r+1}^{(r)}(\lambda)$ and $\hat{A}_{N_r}^{(r)}(\lambda)$. We conclude from Lemma (2.13) that

$$n_I=\text{size of } (\lambda I-J)+\sum_{i=1}^{\tilde{d}}(\eta_i+1)$$

with $\tilde{d}=d^{(r)}$ for both matrices. It remains to show that the numbers $\eta_i$ coincide. By the theory of K.c.f. to a pencil $A-\lambda C$ —see [1]— the $\eta_i$ are characterized by the existence of $x_i(\lambda)\in\mathcal{N}((A-\lambda C)^t)$ $(i=1,\cdots,\tilde{d})$ which are polynomials in $\lambda$ of degree $\eta_i$ and are linearly independent for each $\lambda$ such that the sum of the $\eta_i$ is minimal. In the case $r=0$ the matrix $\hat{A}_{N+1}^{(0)}(\lambda)$ contains $\hat{A}_N^{(0)}(\lambda-1)$ in the left upper corner $(N:=N_0$, for brevity). Therefore, starting from polynomials $x_i(\lambda)$ of degree $\eta_i$ $(i=1,\cdots,d^{(0)})$ which are in $\mathcal{N}(\hat{A}_N^{(0)}(\lambda)^t)$ and are linearly independent for each $\lambda$, we get

$$\hat{x}_i(\lambda):=\begin{pmatrix} x_i(\lambda-1)\\ 0\end{pmatrix}\in\mathcal{N}(\hat{A}_{N+1}^{(0)}(\lambda)^t) \text{ with the analogous properties,}$$

and vice versa because $d_{N+1}^{(0)}=d_N^{(0)}$. —In the case $r>0$ the construction of the corresponding $\hat{x}_i(\lambda)$ is somewhat more tedious. It is possible to proceed in a similar way as in the proof of [2], Lemma 3.1.

## 3. Computation of formal logarithmic solutions.

Suppose $q(x)$ is a polynomial in $x^{-1/p}$ and occurs $\mu$-times as a $q_k(x)$ in the exponential part of a formal fundamental solution (1.2). According to [6], Section 2, we can construct the corresponding $\mu$ columns in $H(x)x^J$ in the following way. We substitute $t^p=x$ in the system

$$xw'(x)=(x^{-s}B(x)-xq'(x)I)w(x)$$

in order to obtain a system of the form (1.1). Then the reduced problem is the computation of a formal logarithmic $n$ by $\mu$-matrix solution

$$\tilde{W}(x)=\tilde{H}(x)x^{\tilde{J}} \tag{3.1}$$

to the system (1.1), where $\tilde{H}(x)$ is a formal power series in $x$, $\tilde{J}$ a constant $\mu$ by $\mu$-Jordan matrix and the columns of $\tilde{W}(x)$ are $C$-linearly independent. In the following constructions the expressions $A_m^{(r)}(\lambda)$, $d^{(r)}$, $N_r$, $\chi_m^{(r)}$, and $\chi^{(r)}$ are needed with $r=0$ merely: we omit the superscripts (0), for brevity. The following fundamental lemma is implicitly contained in [5], Section 1.1:

(3.2) LEMMA. *Let* $k\in N_+$, $\lambda_0\in C$, $h_{\kappa+1,\nu}\in C^n$ $(\nu\in N,\ \kappa=0,1,\cdots,k-1)$ *be given. We set*

$$h_\nu(\lambda):=\sum_{\kappa=0}^{k-1}(\lambda-\lambda_0)^\kappa h_{\kappa+1,\nu}\quad(\nu\in N),$$

$$h^\kappa(x):=\sum_{\nu=0}^{\infty}x^\nu h_{\kappa,\nu}\quad(\kappa=1,2,\cdots,k),\qquad J_k:=\begin{pmatrix}\lambda_0 & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda_0\end{pmatrix}(k,k)$$

*and, finally, defined by columns,*

$$W(x):=(h^1(x),h^2(x),\cdots,h^k(x))x^{J_k}.$$

*Then $W$ is a formal $n$ by $k$-matrix solution to* (1.1) *iff*

$$\forall m\in N\qquad A_m(\lambda)(h_\nu(\lambda))_{\nu=0}^m\in(\lambda-\lambda_0)^k C^{(m+1)n}. \tag{3.3}$$

PROOF. Let

$$Z(x,\lambda):=\sum_{\kappa=0}^{k-1}(\lambda-\lambda_0)^\kappa h^{\kappa+1}(x).$$

Then by definition of the $A_m(\lambda)$ the relation (3.3) is equivalent with the following one

$$\left(x\frac{d}{dx}-x^{-s}B(x)\right)x^\lambda Z(x,\lambda)=x^\lambda\left(x\frac{d}{dx}-x^{-s}B(x)+\lambda I\right)Z(x,\lambda)\in(\lambda-\lambda_0)^k x^{\lambda-s}\boldsymbol{O}^n,$$

where $\boldsymbol{O}$ denotes the space of formal power series in $x$. Generally the term on the left hand side has the form

$$F(x, \lambda) := x^{\lambda-s} \sum_{\kappa=0}^{k} (\lambda-\lambda_0)^{\kappa} f_{\kappa}(x), \qquad f_{\kappa} \in \boldsymbol{O}^n.$$

It is easy to show that

$$F(x, \lambda) \in (\lambda-\lambda_0)^k x^{\lambda-s} \boldsymbol{O}^n \quad \text{iff} \quad \frac{\partial^{\kappa}}{\partial \lambda^{\kappa}} F(x, \lambda) \Big|_{\lambda=\lambda_0} = 0 \quad \text{for } \kappa=0, 1, \cdots, k-1.$$

Further we notice that the differentiation with respect to $\lambda$ and to $x$ can be interchanged, and obtain that (3.3) is equivalent with

$$\begin{aligned} 0 &= \left(x\frac{d}{dx} - x^{-s}B(x)\right)\left(\frac{\partial^{\kappa}}{\partial \lambda^{\kappa}} x^{\lambda} Z(x, \lambda)\right)\Big|_{\lambda=\lambda_0} \\ &= \left(x\frac{d}{dx} - x^{-s}B(x)\right)\kappa!\ x^{\lambda_0} \sum_{j=0}^{\kappa} \frac{1}{(\kappa-j)!} (\log x)^{\kappa-j} h^{j+1}(x) \qquad (\kappa=0, 1, \cdots, k-1). \end{aligned}$$

Obviously, the expression

$$x^{\lambda_0} \sum_{j=0}^{\kappa} \frac{1}{(\kappa-j)!} (\log x)^{\kappa-j} h^{j+1}(x)$$

is the column no. $(\kappa+1)$ in $W(x)$.

An alternative proof avoiding the differentiation of formal logarithmic series with respect to $\lambda$ can be deduced from [5], Section 1.1. —The solvability of the infinite linear system (3.3) has been discussed in [5], Hilfssatz 1.30. The polynomial $\chi$ mentioned there is $\chi_{N+1}$ $(N := N_0)$ in our present notation. We want to replace this $\chi_{N+1}$ by our present $\chi = \chi^{(0)}$ as defined by (1.8).

(3.4) LEMMA. *Suppose* $\lambda_0 \in C$ *and* $\sigma \in N$ *are given such that* $\chi(\lambda_0+m) \neq 0$ *for all* $m \in N$ *with* $m \geqq \sigma+1$. *We set*

$$v(\lambda_0) := \sum_{j=0}^{\sigma} (\textit{multiplicity of } \lambda_0+j \textit{ as a zero of } \chi).$$

*Then for all* $m \in N$, $m \geqq \sigma+1$:

(i) $\chi_N(\lambda_0+m) \neq 0$, $\chi_{N+1}(\lambda_0+m) \neq 0$,

(ii) $v(\lambda_0) =$ *multiplicity of* $\lambda_0$ *as a zero of* $\chi_{N+m}$.

PROOF. Since $\chi_N(\lambda)$ divides $\chi_{N+1}(\lambda)$, we have only to prove that $\chi_{N+1}(\lambda_0+m) \neq 0$ for $m \in N$, $m \geqq \sigma+1$. Suppose $\chi_{N+1}(\lambda_0+m) = 0$: then because $\chi_{N+1}(\lambda) = \chi(\lambda)\chi_N(\lambda+1)$ and $\chi(\lambda_0+m) \neq 0$, also $\lambda_0+m+1$ is a zero of $\chi_N$ and

$\chi_{N+1}$. By the same argument we could prove that all $\lambda_0+\nu$, $\nu\in N$, $\nu\geqq m$ are zeros of $\chi_N$: this is impossible. Part (ii) is proved by the relations

$$\chi_{N+m}(\lambda)=\left(\prod_{j=0}^{m-1}\chi(\lambda+j)\right)\chi_N(\lambda+m), \qquad \chi_N(\lambda_0+m)\neq 0,$$

see (1.8).

(3.5) DEFINITION. For arbitrary $\lambda_0\in C$, $m\in N$, $k\in N_+$ we set

$$\begin{aligned}\mathcal{N}_k(A_m,\lambda_0) &:=\{X_m=X_m(\lambda)\colon C\to C^{n(m+1)} \text{ polynomial in } \lambda \text{ of degree} \\ &\qquad \leqq k-1 \text{ with } A_m(\lambda)X_m(\lambda)\in(\lambda-\lambda_0)^k C^{n(m+1)}\},\\ \mathcal{M}_{m,k}(\lambda_0) &:=\{(h_\nu)_{\nu=1}^m=(h_\nu(\lambda))_{\nu=0}^m\colon C\to C^{n(m+1)} \text{ for which further} \\ &\qquad h_\nu\ (\nu=m+1,\cdots,m+N+1) \text{ exist such that} \\ &\qquad (h_\nu)_{\nu=0}^{m+N+1}\in\mathcal{N}_k(A_{m+N+1},\lambda_0)\}.\end{aligned}$$

Here $N:=N_0$ as in Lemma (1.5). Finally, let

$$\begin{aligned}\mathcal{M}_{\infty,k}(\lambda_0) &:=\{(h_\nu(\lambda))_{\nu=0}^\infty \text{ of degree } \leqq k-1 \text{ satisfying (3.3)}\}\\ &=\{(h_\nu(\lambda))_{\nu=0}^\infty\colon \forall m\in N\ (h_\nu(\lambda))_{\nu=0}^m\in\mathcal{M}_{m,k}(\lambda_0)\}.\end{aligned}$$

It is easy to show that each $\mathcal{N}_k(A_m,\lambda_0)$ $(m\in N)$ is isomorphic with $\mathcal{N}(A_m^{[k]}(\lambda_0))$ as defined in [5], Definition 1.17, by means of the natural isomorphism

$$(3.6)\qquad X_m(\lambda)=\sum_{\kappa=0}^{k-1}(\lambda-\lambda_0)^\kappa X_{m,\kappa+1}\in\mathcal{N}_k(A_m,\lambda_0)\longmapsto(X_{m,\kappa})_{\kappa=1}^k\in\mathcal{N}(A_m^{[k]}(\lambda_0)).$$

In an analogous way, each $\mathcal{M}_{m,k}(\lambda_0)$ is isomorphic with $\mathcal{M}_m^{[k]}(\lambda_0)$ according to [5], Definition 1.27. Therefore the following lemma is an immediate consequence of [5], Hilfssatz 1.30.

(3.7) LEMMA. *Suppose $\lambda_0\in C$ and $\sigma\in N$ are such that $\chi(\lambda_0+m)\neq 0$ for all $m\in N$, $m\geqq\sigma+1$. Then for each $k\in N_+$, $m\in N$, $m\geqq\sigma+1$*

$$\mathcal{M}_{\infty,k}(\lambda_0)\simeq\mathcal{M}_{m,k}(\lambda_0)\ \text{by}\ (h_\nu(\lambda))_{\nu=0}^\infty\longmapsto(h_\nu(\lambda))_{\nu=0}^m.$$

The next step in our program is the construction of practicable representations of the spaces $\mathcal{M}_{\sigma,k}(\lambda_0)$ $(k\in N_+)$. After this, we shall give an algorithm for extending any $(h_\nu(\lambda))_{\nu=0}^\sigma\in\mathcal{M}_{\sigma,k}(\lambda_0)$ to a sequence $(h_\nu(\lambda))_{\nu=0}^\infty\in\mathcal{M}_{\infty,k}(\lambda_0)$. First we prove a more general lemma.

(3.8) LEMMA. *Suppose $A-\lambda C$ is an arbitrary $l$ by $r$ matrix pencil, further $Y\colon C\to C^l$ is a polynomial of degree $\leqq k$ and $\lambda_0\in C$ is fixed. For*

*$X: \boldsymbol{C}\to\boldsymbol{C}^r$ let* ($*$) *denote the following statement*:

$$(*)\quad \begin{cases} X \text{ is a polynomial of degree } \leqq k-1, \\ (A-\lambda C)X(\lambda)-Y(\lambda)\in(\lambda-\lambda_0)^k\boldsymbol{C}^l. \end{cases}$$

(i) *If $A-\lambda_0 C$ is surjective then there exists an $X$ satisfying* ($*$).

(ii) *If $A-\lambda_0 C$ is injective then there is at most one $X$ satisfying* ($*$).

For the proof we note that ($*$) is equivalent with

$$(A-\lambda_0 C)\sum_{\kappa=0}^{k-1}(\lambda-\lambda_0)^\kappa X_\kappa-(\lambda-\lambda_0)\sum_{\kappa=0}^{k-1}(\lambda-\lambda_0)^\kappa CX_\kappa-\sum_{\kappa=0}^{k}(\lambda-\lambda_0)^\kappa Y_\kappa\in(\lambda-\lambda_0)^k\boldsymbol{C}^l.$$

This provides a recursion formula for the $X_\kappa$ ($\kappa=0,1,\cdots,k-1$) which is always solvable in the case (i) or admits at most one solution in the case (ii).

During the algorithms required for the evaluation of $d$ and $N$ we had to transform the pencil $A_N(\lambda)$ into property (C) as described in Section 2. Therefore we are given invertible matrices $P_N$ and $Q_N$ such that

$$P_N A_N(\lambda) Q_N =: \left(\begin{array}{c|c} A_I-\lambda C_I & 0 \\ \hline & A_{II}-\lambda C_{II} \end{array}\right), \tag{3.9}$$

where $C_I$ has full column rank and $A_{II}-\lambda C_{II}$ has full row rank for any $\lambda$.

(3.10) THEOREM. *Suppose $\lambda_0\in\boldsymbol{C}$ and $\sigma\in\boldsymbol{N}$ are given such that $\chi(\lambda_0+m)\neq 0$ for all $m\in\boldsymbol{N}$, $m\geqq\sigma+1$. We extend $P_N, Q_N$ from* (3.9) *to*

$$P_{N+1+\sigma}:=\begin{pmatrix} I_{(\sigma+1)n} & 0 \\ 0 & P_N \end{pmatrix},\qquad Q_{N+1+\sigma}:=\begin{pmatrix} I_{(\sigma+1)n} & 0 \\ 0 & Q_N \end{pmatrix}$$

*and set $\tilde{A}_{N+1+\sigma}(\lambda):=P_{N+1+\sigma}A_{N+1+\sigma}(\lambda)Q_{N+1+\sigma}$. Then $\tilde{A}_{N+1+\sigma}(\lambda)$ has the following block structure*:

$$\tilde{A}_{N+1+\sigma}(\lambda)=:\left(\begin{array}{c|c|c} & 0 & 0 \\ \cline{2-3} D_\sigma(\lambda) & A_I-(\lambda+\sigma+1)C_I & 0 \\ \hline & & A_{II}-(\lambda+\sigma+1)C_{II} \end{array}\right). \tag{3.11}$$

*We consider the following submatrix of $\tilde{A}_{N+1+\sigma}(\lambda)$*:

$$(3.11)' \qquad \hat{A}_{N+1+\sigma}(\lambda) := \left( \begin{array}{c|c} D_\sigma(\lambda) & \begin{array}{c} 0 \\ \hline A_I-(\lambda+\sigma+1)C_I \end{array} \end{array} \right).$$

*We define* $\mathcal{N}_k(\hat{A}_{N+1+\sigma}, \lambda_0)$ *in an analogous way as we had done for* $A_m$ *in* (3.5). *Then for each* $k$

$$\mathcal{N}_k(\hat{A}_{N+1+\sigma}, \lambda_0) \simeq \mathcal{M}_{\sigma,k}(\lambda_0)$$

*by the isomorphism*

$$X(\lambda) = \begin{pmatrix} X_\sigma(\lambda) \\ X_I(\lambda) \end{pmatrix} \longmapsto X_\sigma(\lambda),$$

*where* $X_\sigma(\lambda)$ *consists in the leading* $(\sigma+1)n$ *components of* $X(\lambda)$.

PROOF. The block decomposition in (3.11) is an easy consequence of (3.9) and the block structure of $A_{N+1+\sigma}(\lambda)$, namely

$$A_{N+1+\sigma}(\lambda) = \left( \begin{array}{c|c} \underbrace{\quad /\!/\!/ \quad}_{(\sigma+1)n} & \begin{array}{c} 0 \\ \hline A_N(\lambda+\sigma+1) \end{array} \end{array} \right).$$

Next we prove the characterization of $\mathcal{M}_{\sigma,k}(\lambda_0)$. Obviously

$$Y_{N+1+\sigma} \in \mathcal{N}_k(A_{N+1+\sigma}, \lambda_0) \Longleftrightarrow X_{N+1+\sigma} := Q_{N+1+\sigma}^{-1} Y_{N+1+\sigma} \in \mathcal{N}_k(\tilde{A}_{N+1+\sigma}, \lambda_0).$$

Now let

$$(3.13) \qquad Y_{N+1+\sigma} =: \begin{pmatrix} Y_\sigma \\ Y_N \end{pmatrix}, \qquad X_{N+1+\sigma} =: \begin{pmatrix} X_\sigma \\ X_I \\ X_{II} \end{pmatrix},$$

corresponding to the subdivisions (3.12) and (3.11), respectively, into column blocks. The special structure of $Q_{N+1+\sigma}$ yields $X_\sigma = Y_\sigma$. This leads to the criterion:

> $X_\sigma \in \mathcal{M}_{\sigma,k}(\lambda_0)$ *iff there exist* $X_I, X_{II}$ *such that* $X_{N+1+\sigma}$ *as in* (3.13) *satisfies* $X_{N+1+\sigma} \in \mathcal{N}_k(\tilde{A}_{N+1+\sigma}, \lambda_0)$.

The matrix $A_{II}-(\lambda_0+\sigma+1)C_{II}$ has full row rank because of property (C) in (3.9). Therefore, Lemma (3.8), (i) provides the weaker criterion

*$X_\sigma \in \mathcal{M}_{\sigma,k}(\lambda_0)$ iff there exists $X_I$ such that*

$$X := \begin{pmatrix} X_\sigma \\ X_I \end{pmatrix} \in \mathcal{N}_k(\hat{A}_{N+1+\sigma}, \lambda_0).$$

Finally, $\chi_N(\lambda_0+\sigma+1) \neq 0$ because of Lemma (3.4), (i), where $\chi_N$ is the characterizing polynomial corresponding to the pencil $A_N(\lambda)$ as well as to $A_I - \lambda C_I$. Consequently $A_I - (\lambda_0+\sigma+1)C_I$ is injective, and Lemma (3.8), (ii) implies that the linear map $X \longmapsto X_\sigma$ leading from $\mathcal{N}_k(\hat{A}_{N+1+\sigma}, \lambda_0)$ onto $\mathcal{M}_{\sigma,k}(\lambda_0)$ is injective.

Our next aim is the extension of a given $X_\sigma(\lambda) = (h_\nu(\lambda))_{\nu=0}^{\sigma} \in \mathcal{M}_{\sigma,k}(\lambda_0)$ to a full sequence $(h_\nu(\lambda))_{\nu=0}^{\infty} \in \mathcal{M}_{\infty,k}(\lambda_0)$. We describe the constructional step leading from $(h_\nu(\lambda))_{\nu=0}^{m}$ to $(h_\nu(\lambda))_{\nu=0}^{m+1}$ $(m \geqq \sigma)$. For this we subdivide $A_{N+m+2}(\lambda)$ into

$$A_{N+m+2}(\lambda) = \left(\begin{array}{c|c} A_m(\lambda) & 0 \\ \hline & A_{N+1}(\lambda+m+1) \end{array}\right).$$

Then we apply Theorem (3.10) to $\lambda_0+m+1$ in the place of $\lambda_0$ and to $\sigma=0$. We extend the matrices $P_{N+1}$, $Q_{N+1}$ as defined in the theorem, to

$$P_{N+2+m} = \begin{pmatrix} I_{(m+1)n} & 0 \\ 0 & P_{N+1} \end{pmatrix}, \qquad Q_{N+2+m} = \begin{pmatrix} I_{(m+1)n} & 0 \\ 0 & Q_{N+1} \end{pmatrix}$$

and obtain

(3.14)
$$P_{N+2+m} A_{N+2+m}(\lambda) Q_{N+2+m} =: \tilde{A}_{N+2+m}(\lambda)$$
$$= \left(\begin{array}{c|c|c} A_m(\lambda) & 0 & 0 \\ \hline E_m(\lambda) & \hat{A}_{N+1}(\lambda+m+1) & 0 \\ \hline & & A_{II} - (\lambda+m+1)C_{II} \end{array}\right)$$

with $\hat{A}_{N+1}$ as in the theorem with respect to $\sigma=0$.

(3.15) THEOREM. *Let $\lambda_0 \in \boldsymbol{C}$ and $\sigma \in \boldsymbol{N}$ with $\chi(\lambda_0+m) \neq 0$ for all $m \in \boldsymbol{N}$, $m \geqq \sigma+1$ be given. Further, let $k \in \boldsymbol{N}_+$, $m \geqq \sigma$ and $(h_\nu(\lambda))_{\nu=0}^{m} \in \mathcal{M}_{m,k}(\lambda_0)$ be fixed. Then there exists a unique polynomial $X : \boldsymbol{C} \to \boldsymbol{C}^r$ of degree $\leqq k-1$ such that*

(*)
$$\hat{A}_{N+1}(\lambda+m+1)X(\lambda) + E_m(\lambda)(h_\nu(\lambda))_{\nu=0}^{m} \in (\lambda-\lambda_0)^k \boldsymbol{C}^l,$$

*where r, l denote the numbers of columns and rows in $\hat{A}_{N+1}$, respectively. Let $X(\lambda)$ be subdivided into*

$$X(\lambda)=\begin{pmatrix} X_0(\lambda) \\ X_I(\lambda) \end{pmatrix}, \qquad X_0(\lambda)\in C^n.$$

*Then $h_{m+1}(\lambda):=X_0(\lambda)$ is the unique complement by which $(h_\nu(\lambda))_{\nu=0}^m$ is extended to $(h_\nu(\lambda))_{\nu=0}^{m+1}\in\mathcal{M}_{m+1,k}(\lambda_0)$.*

PROOF. $m\geqq\sigma$ implies $\chi_{N+1}(\lambda_0+m+1)\neq 0$ and further $\mathcal{M}_{0,k}(\lambda_0+m+1)=\{0\}$ for any $k\in N$ —see Lemma (3.4) and [5], Hilfssatz 1.29.— We conclude from Theorem (3.10) that $\hat{A}_{N+1}(\lambda_0+m+1)$ is injective: therefore the system (*) admits at most one solution $X$ of degree $\leqq k-1$ as it was proved in Lemma (3.8). On the other hand, $\mathcal{M}_{m+1,k}(\lambda_0)$ is isomorphic with $\mathcal{M}_{m,k}(\lambda_0)$ by an analogous isomorphism as mentioned in Lemma (3.7). This means that the given $(h_\nu(\lambda))_{\nu=0}^m$ can be supplied by further $h_{m+1}(\lambda),\cdots,h_{N+m+2}(\lambda)$ such that

$$X_{N+m+2}(\lambda):=(h_\nu(\lambda))_{\nu=0}^{N+m+2}\in\mathcal{N}_k(A_{N+m+2},\lambda_0).$$

Then $Y_{N+m+2}(\lambda):=Q_{N+m+2}^{-1}X_{N+m+2}(\lambda)\in\mathcal{N}_k(\tilde{A}_{N+m+2},\lambda_0)$. Obviously $Y_{N+m+2}(\lambda)$ can be subdivided into

$$Y_{N+m+2}(\lambda)=\begin{pmatrix} (h_\nu(\lambda))_{\nu=0}^m \\ X(\lambda) \\ \\ X_{II}(\lambda) \end{pmatrix}=\begin{pmatrix} (h_\nu(\lambda))_{\nu=0}^m \\ X_0(\lambda) \\ X_I(\lambda) \\ X_{II}(\lambda) \end{pmatrix},$$

and $X_0(\lambda)=h_{m+1}(\lambda)$ owing to the structure of $Q_{N+m+2}$.

So far the construction of sequences $(h_\nu(\lambda))_{\nu=0}^\infty$ of $\lambda$-degree $\leqq k-1$ satisfying (3.3) for $\lambda_0\in C$, $k\in N_+$ fixed. According to Lemma (3.2) each sequence of this kind corresponds to a formal logarithmic solution with only one Jordan block $J_k$. The next problem is to construct

$$W_\tau(x)=(h^{1,\tau}(x),\cdots,h^{k_\tau,\tau}(x))x^{J_{k_\tau}} \qquad (\tau=1,\cdots,t)$$

each corresponding to $\lambda_0$ such that the formal power series $h^{1,1}(x),\cdots,h^{1,t}(x)$ are linearly independent and $\sum_{\tau=1}^t k_\tau$ is maximal —see [5], Satz 1.7—. Owing to Lemma (3.7), this problem is reduced to the evaluation of $t\in N$, $k_\tau\in N_+$ $(\tau=1,\cdots,t)$ and

$$X_\sigma^\tau(\lambda)=(h_\nu^\tau(\lambda))_{\nu=0}^\sigma=\left(\sum_{\kappa=0}^{k_\tau-1}(\lambda-\lambda_0)^\kappa h_{\kappa+1,\nu}\right)_{\nu=0}^\sigma\in\mathcal{M}_{\sigma,k_\tau}(\lambda_0)$$

$(\tau=1, \cdots, t)$ such that

(3.16) $$(h_{1,\nu}^1)_{\nu=0}^\sigma, \cdots, (h_{1,\nu}^t)_{\nu=0}^\sigma \text{ linearly independent}$$

and $\sum_{\tau=1}^t k_\tau$ is maximal. We set

$$\mathcal{M}_{\sigma,0}(\lambda_0):=\{0\}, \qquad \varepsilon_k:=\dim\mathcal{M}_{\sigma,k}(\lambda_0) \qquad (k\in N).$$

Obviously two canonical inclusions are valid:

(3.17.1) $$(\lambda-\lambda_0)\mathcal{M}_{\sigma,k}(\lambda_0)\subseteq\mathcal{M}_{\sigma,k+1}(\lambda_0) \qquad (k\in N),$$

(3.17.2) $$\mathcal{M}_{\sigma,k+1}(\lambda_0)/(\lambda-\lambda_0)\mathcal{M}_{\sigma,k}(\lambda_0)\hookrightarrow\mathcal{M}_{\sigma,k}(\lambda_0)/(\lambda-\lambda_0)\mathcal{M}_{\sigma,k-1}(\lambda_0) \quad (k\in N_+),$$

where the inclusion (3.17.2) is generated by omitting the $(\lambda-\lambda_0)^k$-term in any element of $\mathcal{M}_{\sigma,k+1}(\lambda_0)$. As a consequence we get

(3.18) $$\varepsilon_k\leqq\varepsilon_{k+1} \quad (k\in N), \qquad \varepsilon_{k+1}-\varepsilon_k\leqq\varepsilon_k-\varepsilon_{k-1} \quad (k\in N_+).$$

It has been proved in [5], Satz 1.19 and Hilfssatz 1.29, 3) that there exists a $K\in N$ such that $\varepsilon_k=v(\lambda_0)$ for all $k\geqq K$, where $v(\lambda_0)$ is defined as in Lemma (3.4). If we choose $K$ minimal then

(3.19) $$\varepsilon_K=v(\lambda_0), \qquad \varepsilon_{k-1}<\varepsilon_k \quad (k\leqq K).$$

For practical application, we use that the $\varepsilon_k$ are the dimensions of the $\mathcal{N}_k(\hat{A}_{N+1+\sigma},\lambda_0)$ $(k=1, 2, 3, \cdots)$ as defined in Theorem (3.10), and that $v(\lambda_0)$ is well-known. Provided $v(\lambda_0)>0$, we evaluate $\varepsilon_k$ for $k=1, 2, 3, \cdots$ until $\varepsilon_k=v(\lambda_0)$ is achieved for the first time. We set

$$t_j:=\varepsilon_{K+1-j}-\varepsilon_{K-j} \quad (j=0, 1, \cdots, K), \ \ t:=t_K,$$

such that $t_0=0$, $t_K=\varepsilon_1$, and further

$$k_\tau:=K-j \qquad (t_j+1\leqq\tau\leqq t_{j+1};\ j=0, 1, \cdots, K-1).$$

The first $X_\sigma^\tau\in\mathcal{M}_{\sigma,k_\tau}(\lambda_0)$ $(\tau=1, \cdots, t_1)$ are determined as a base of $\mathcal{M}_{\sigma,K}(\lambda_0)$ with respect to $(\lambda-\lambda_0)\mathcal{M}_{\sigma,K-1}(\lambda_0)$. If $K>1$, these $X_\sigma^\tau$ without their $(\lambda-\lambda_0)^{K-1}$-terms are contained in $\mathcal{M}_{\sigma,K-1}(\lambda_0)$ and linearly independent with respect to $(\lambda-\lambda_0)\mathcal{M}_{\sigma,K-2}(\lambda_0)$ —see (3.17.2)—: we evaluate additional $X_\sigma^\tau\in\mathcal{M}_{\sigma,K-1}(\lambda_0)$ $(\tau=t_1+1, \cdots, t_2)$ to get a full base of $\mathcal{M}_{\sigma,K-1}(\lambda_0)$ with respect to $(\lambda-\lambda_0)\mathcal{M}_{\sigma,K-2}(\lambda_0)$, and so on. By this construction, the properties (3.16) are obvious. Of course, this method is quite analogous with the construction of principal vectors to a constant square matrix: in the special case that all $k_\tau$ ($\tau=1, \cdots, t$) are equal ($t=1$, for instance), it is possible to begin with the construction of a base for $\mathcal{M}_{\sigma,1}(\lambda_0)$.

The foregoing constructions have to be applied to appropriate $\lambda_1, \cdots, \lambda_m$ such that the corresponding formal logarithmic solutions altogether provide a formal logarithmic $n$ by $\mu$ matrix solution (3.1) with full column rank. $\mu$ has to coincide with $\deg \chi$, as mentioned in Theorem (1.8). We choose $\lambda_1, \cdots, \lambda_m$ the zeros of $\chi$ with the properties

$$\begin{cases} \lambda_i - \lambda_j \notin \boldsymbol{Z} \quad (i \neq j), \\ \textit{each zero of } \chi \textit{ has the form } \lambda_i + \kappa \quad (\kappa \in N,\ i \in \{1, 2, \cdots, m\}). \end{cases} \tag{3.20}$$

Then by definition of $v(\lambda_i)$ in Lemma (3.4) it is clear that

$$\sum_{i=1}^{m} v(\lambda_i) = \deg \chi.$$

On the other hand, our construction provides $v(\lambda_i)$ linearly independent formal solutions for each $\lambda_i$. Because the $\lambda_i$ are distinct modulo $\boldsymbol{Z}$, all these solutions together are linearly independent. We note that in [5], Satz 1.32 the choice of $\lambda_i$ is the same as in (3.20) but with $\chi_{N+1}(\lambda)$ in the place of $\chi(\lambda) = \chi_{N+1}(\lambda)/\chi_N(\lambda+1)$. The $\lambda_i$ in [5] can differ from the present ones by some nonpositive integers, but the corresponding formal solutions are the same.

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