J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
36 (1989), 717-727.

On the asymptotic cycles for Axiom A flows

Dedicated to Professor Tosihusa Kimura on his 60th birthday

By Takashi TSUBOI and Seishi TSUKADA

The notion of asymptotic cycles for continuous flows was introduced by Schwartzman ([11]) to represent the asymptotic behavior of the flows. An asymptotic cycle for a flow is defined to be a 1-dimensional homology class of the space and the simplest asymptotic cycles are those given by the closed orbits. The asymptotic cycles for a basic set of an Axiom A flow in the sense of Smale ([14]) were studied by Sigmund ([12]). For such a flow the closed orbits are dense in a basic set and it is known that the set of the asymptotic cycles for a basic set coincides with the closure of the set of the asymptotic cycles given by the closed orbits in the basic set (see Proposition 1.1). Thus it is interesting to know whether there exists a null homologous closed orbit in a basic set when 0 is an asymptotic cycle of the basic set. For a transitive codimension 1 Anosov flow on a closed manifold, Verjovsky ([15]) and Ghys ([6]) showed that it is the case.

The purpose of this paper is to show that for an Axiom A flow on the manifold whose first Betti number is 1, if 0 is an asymptotic cycle of a basic set X, then there exists a null homologous closed orbit in the basic set X (Theorem 2.1). This is generalized to the case where the set of the asymptotic cycles for the basic set is contained in a 1-dimensional linear subspace of the 1-dimensional homology group. By using a result of Fried ([4]), we see that in this case such basic sets have either a cross section or a null homologous closed orbit (Proposition 2.3). (It is easy to see that the existence of null homologous closed orbits implies that 0 is an asymptotic cycle, and that 0 is an asymptotic cycle implies no cross sections.) We also give examples to show that the condition on the first Betti number is necessary (Propositions 3.2 and 3.3). In these examples, 0 is not in the interior of the set of asymptotic cycles, where the interior is considered in the vector subspace spanned by the asymptotic cycles. This leads us to the criterion for the existence of null homologous closed orbits (Theorem 4.1), namely, if 0 is in the interior of the set of asymptotic cycles of a basic set X, then there exists a null homologous closed orbit in X. This observation gives a second proof of the results in § 2.

§ 1. Asymptotic cycles and closed orbits of Axiom A flows.

Let $\varphi_t : M \to M$ be a C^1 Axiom A flow on a compact smooth Riemannian manifold M. According to the spectral decomposition theorem of Smale ([14]), the nonwandering set of an Axiom A flow consists of a finite number of connected components. Each component is either a hyperbolic fixed point, a hyperbolic closed orbit or a component containing more than two closed orbits. We call the last one a basic set.

We review briefly the asymptotic cycles for an Axiom A flow on a basic set X. Let $\varphi_t|X$ denote the restriction of φ_t to X. We define the asymptotic cycles for $\varphi_t|X$ as follows: Let ξ be the vector field associated with the flow φ_t. For a signed measure μ on X invariant under $\varphi_t|X$, there is defined a homology class $A_{\varphi|X}(\mu) \in H_1(M; \boldsymbol{R})$ given by

$$A_{\varphi|X}(\mu)(\omega) = \int_X \langle \xi, \omega \rangle d\mu$$

for any closed 1-form ω on M. Hence, we obtain a homomorphism $A_{\varphi|X}$ from the space of invariant signed measures to $H_1(M; \boldsymbol{R})$. We call the image of a positive normalized invariant measure on X an asymptotic cycle of $\varphi_t|X$. We write the set of asymptotic cycles by $\mathcal{A}_{\varphi|X}$.

A closed orbit O of $\varphi_t|X$ determines the positive normalized invariant measure with support on O (the CO-measure), and this corresponds to the asymptotic cycle $[O]/\mathrm{per}(O) \in H_1(M; \boldsymbol{R})$, where $[O]$ denotes the homology class of O and $\mathrm{per}(O)$, the period of O. For Axiom A flows, CO-measures are dense in the set of normalized invariant measures $\mathcal{M}$ (see [12]). Since $\mathcal{M}$ is compact, so is the set $\mathcal{A}_{\varphi|X}$ of asymptotic cycles, and we have the following proposition.

PROPOSITION 1.1. *Let $\varphi_t : M \to M$ be an Axiom A flow and X, a basic set of φ_t. Then $\mathcal{A}_{\varphi|X}$ coincides with the closure of* $\{[O]/\mathrm{per}(O);$ *O is a closed orbit in $X\}$.*

In the following sections, we study whether $0 \in \mathcal{A}_{\varphi|X}$ implies the existence of null homologous closed orbits in X.

§ 2. Axiom A flows on manifolds M with $\dim H_1(M;\boldsymbol{R})=1$.

In this section we prove the following theorem.

THEOREM 2.1. *Let $\varphi_t: M\to M$ be an Axiom A flow, and X, a basic set of φ_t. Let $\mathcal{A}_{\varphi|X}$ denote the set of asymptotic cycles for $\varphi_t|X$. If* 0 *belongs to $\mathcal{A}_{\varphi|X}$ and* $\dim H_1(M;\boldsymbol{R})=1$, *then there exists a closed orbit in X which is homologous to* 0.

The proof of Theorem 2.1 uses two facts. One is that a long nearly closed orbit in the basic set X can be approximated by a closed orbit, which is stated in Proposition 2.2 below. The other is that for any orbit in X, there is a dense subset of X whose points asymptotically approach to this orbit ([1], [8]). (We give a second proof using the symbolic dynamics in § 4.)

PROPOSITION 2.2 ([1, Theorem 2.4]). *For any positive real number β, there are positive real numbers δ and L for which the following holds: if a real number r satisfies $d(\varphi_r(x), x)\leq\delta$ and $r\geq L$, then there are a point $y\in X$ and a real number r' such that $\varphi_{r'}(y)=y$, $|r'-r|\leq\beta$ and*

$$d(\varphi_t(y), \varphi_t(x))\leq\beta \quad \textit{for} \quad 0\leq t\leq r.$$

PROOF OF THEOREM 2.1. By the Poincaré duality, the rank of $H^1(M;\boldsymbol{Z})$ is 1. Let u be a generator of $H^1(M;\boldsymbol{Z})$. Let $\pi:\hat{M}\to M$ be the infinite cyclic cover corresponding to u. Then we have the lifting $\hat{\varphi}_t:\hat{M}\to\hat{M}$ of φ_t. The manifold $\hat{M}$ has two ends $\{\pm\infty\}$. Put $\hat{X}=\pi^{-1}(X)$. The essential part of the proof is to show that there exists a point $\hat{x}\in\hat{X}$ such that $L^+(\hat{x})\cap\hat{M}\neq\emptyset$, where L^+ denotes the ω-limit set in $\hat{M}\cup\{\pm\infty\}$.

First we assume $L^+(\hat{x})=\{+\infty\}$ for all points $\hat{x}\in\hat{X}$, and deduce a contradiction. (This part can also be deduced from [4, Theorem B].) The class $u\in H^1(M;\boldsymbol{Z})$ is represented by a continuous map $u: M\to S^1$. Let $\hat{u}:\hat{M}\to\boldsymbol{R}$ be a lifting of u. We define a continuous function $\hat{v}$ on $\hat{X}$ by $\hat{v}(\hat{x}, t)=\hat{u}(\hat{\varphi}_t(\hat{x}))-\hat{u}(\hat{x})$. Then if $\pi(\hat{x})$ is contained in a closed orbit O of period T, we have $u([O])=\hat{v}(\hat{x}, T)$. By the assumption that $L^+(\hat{x})=\{+\infty\}$ for all points $\hat{x}\in\hat{X}$, $\hat{v}(\hat{x}, t)\to+\infty$ as $t\to+\infty$.

Let M_0 $(\subset\hat{M})$ be a compact connected fundamental domain of the infinite cyclic cover π. For an appropriate choice of $\hat{u}$, we have $M_0=\hat{u}^{-1}([0, 1])$. For any point $\hat{x}\in M_0$, we can take an open neighborhood $U_{\hat{x}}$ of $\hat{x}$, positive real numbers $t_{\hat{x}}$ and $p_{\hat{x}}$ such that $\hat{v}(w, t_{\hat{x}})>p_{\hat{x}}$ for all $w\in U_{\hat{x}}$. By the compactness of M_0, we can take a finite number of open neighborhoods $U_{x_1}, \cdots, U_{x_k}$ which cover M_0. Define the positive real numbers a and p by

$$a=\max\{t_{x_i};\ 1\leq i\leq k\} \quad \text{and} \quad p=\min\{p_{x_i};\ 1\leq i\leq k\},$$

respectively. Then for any point $\hat{x}\in M_0$, there is a real number $t\in[0, a]$ such that $\hat{v}(\hat{x}, t)>p$. Hence for any point $\hat{x}\in M_0$ and any positive integer n, there is a real number $t\in[0, na]$ such that $\hat{v}(\hat{x}, t)>np$. On the other hand, there exists a positive real number b such that $\hat{v}(\hat{x}, t)>-b$ holds for all $\hat{x}\in M_0$ and $t\in[0, a]$. Let $T_{\hat{x},t}$ be a real number such that $T_{\hat{x},t}\in [0, t]$ and

$$\hat{v}(\hat{x}, T_{\hat{x},t})=\max_{s\in[0,t]}\hat{v}(\hat{x}, s).$$

Then, by the choice of a, we have $0\leq t-T_{\hat{x},t}<a$ and $\hat{v}(\hat{x}, T_{\hat{x},t})>[t/a]p$. Since $\hat{\varphi}_{s+t}(\hat{x})=\hat{\varphi}_s(\hat{\varphi}_t(\hat{x}))$, we have

$$\hat{v}(\hat{x}, t)=\hat{v}(\hat{x}, T_{\hat{x},t})+\hat{v}(\hat{\varphi}T_{\hat{x},t}(\hat{x}), t-T_{\hat{x},t})>[t/a]p-b.$$

If $\pi(\hat{x})$ is contained in a closed orbit of period T, then we have

$$\frac{\hat{v}(\hat{x}, nT)}{nT}\geq\frac{[nT/a]p-b}{nT}\geq\frac{p}{2a}$$

for a sufficiently large positive integer n. By Proposition 1.1, we see that $\mathcal{A}_{\varphi|X}\subset[p/2a, \infty)\subset H_1(M; \boldsymbol{R})$, which contradicts the assumption.

Next we assume that for all points $\hat{x}\in\hat{X}$ we have either $L^+(\hat{x})=\{+\infty\}$ or $L^+(\hat{x})=\{-\infty\}$, and deduce a contradiction. Put

$$B_+=\{\hat{x}\in\hat{X};\ L^+(\hat{x})\ni+\infty\}$$

and

$$B_-=\{\hat{x}\in\hat{X};\ L^+(\hat{x})\ni-\infty\}.$$

For a positive integer n, put

$$B_n=\{\hat{x}\in\hat{X};\ \text{there is } t>0 \text{ such that } \hat{v}(\hat{x}, t)>n\}.$$

If $\hat{x}\in B_+$, then $W^s(\pi^{-1}(\pi(O_{\hat{x}}^-)))\subset B_n$, and $W^s(\pi^{-1}(\pi(O_{\hat{x}}^-)))$ is a dense subset of $\hat{X}$ (see [1], [8]). Here O^- denotes the negative orbit and W^s denotes the stable set. For a subset S of X, $W^s(S)$ is defined by

$$W^s(S)=\{y\in X;\ d(\varphi_t(x), \varphi_t(y))\longrightarrow 0 \quad \text{as} \quad t\rightarrow\infty \text{ for some } x\in S\}.$$

Thus B_n is an open dense subset of $\hat{X}$. Since $M_0\cap B_+=\bigcap_{n\geq1}(M_0\cap B_n)$,

$M_0 \cap B_+$ is a residual subset of M_0. Similarly, $M_0 \cap B_-$ is also a residual subset of M_0. Since M_0 is a Baire space, $(M_0 \cap B_+) \cap (M_0 \cap B_-) \neq \varnothing$. This contradicts the assumption that $B_+ \cap B_- = \varnothing$.

Now by the above argument, there is a point $\hat{x} \in \hat{X}$ which satisfies either $L^+(\hat{x}) \supset \{\pm\infty\}$ or $L^+(\hat{x}) \cap \hat{M} \neq \varnothing$. Since $L^+(\hat{x})$ is connected, $L^+(\hat{x}) \supset \{\pm\infty\}$ implies that $L^+(\hat{x}) \cap \hat{M} \neq \varnothing$. Hence, there is a point $\hat{x} \in \hat{X}$ such that $L^+(\hat{x}) \cap \hat{M} \neq \varnothing$. Then there exists a sequence of real numbers $\{t_n\}_{n=1}^{\infty}$ such that $\hat{\varphi}_{t_n}(\hat{x}) \to \hat{p}$ as $n \to \infty$ for some $\hat{p} \in \hat{M}$. We take positive integers $n > m > 0$ such that $d(\hat{\varphi}_{t_n}(\hat{x}), \hat{\varphi}_{t_m}(\hat{x})) < \delta$ and $t_n - t_m > L$ for the positive real numbers δ and L of Proposition 2.2. Put $y = \pi(\hat{\varphi}_{t_m}(\hat{x}))$ and $T = t_n - t_m$. Then

$$d(y, \varphi_T(y)) = d(\hat{\varphi}_{t_m}(\hat{x}), \hat{\varphi}_{t_n}(\hat{x})) < \delta \quad \text{and} \quad T > L.$$

By Proposition 2.2, there exists a closed orbit O in X which approximates $\varphi_{[0,T]}(y)$ and we have $[O] = 0$.

REMARK. For a basic set X, we define $\dim H_1(X; \boldsymbol{R})$ to be the dimension of the minimal linear subspace of $H_1(M; \boldsymbol{R})$ which contains $\mathcal{A}_{\varphi|X}$. Then, if $\dim H_1(X; \boldsymbol{R}) = 1$, Theorem 2.1 holds.

As an application of Theorem 2.1, we give a criterion for the existence of a cross section for a basic set X. A codimension 1 submanifold K of M is called a cross section for $\varphi_t | X$, if every orbit of $\varphi_t | X$ intersects K transversely. By a result of Fried ([4, Theorem E]), the existence of a cross section is equivalent to the fact that $\mathcal{A}_{\varphi|X}$ is contained in an open half space of $H_1(M; \boldsymbol{R})$. Hence we have the following proposition.

PROPOSITION 2.3. *Let $\varphi_t : M \to M$ be an Axiom A flow, and X, a basic set of φ_t. Suppose that* $\dim H_1(X; \boldsymbol{R}) = 1$. *Then either there exists a null homologous closed orbit in X or $\varphi_t | X$ admits a cross section.*

§ 3. Examples with zero asymptotic cycles.

In this section we construct Axiom A flows φ_t with a basic set X such that $\dim H_1(X; \boldsymbol{R}) = 2$, 0 belongs to $\mathcal{A}_{\varphi|X}$ and no closed orbits in X are homologous to 0. We construct such Axiom A flows with singular points on T^3 and such Axiom A flows without singular points on T^4. For this we use the symbolic dynamics. A similar construction is appeared in Fried ([4, § 3 Note]) in a slightly different context.

Let F be a finite set with the discrete topology, and consider $\Sigma=\prod_{Z} F$ with the product topology. We denote an element $(m_i)_{i\in Z}$ of Σ by $\boldsymbol{m}$, and write $\boldsymbol{m}_i=m_i$. The shift homeomorphism $\sigma:\Sigma\to\Sigma$ is defined by $\sigma(\boldsymbol{m})_i=\boldsymbol{m}_{i+1}$. A finite sequence $(l)=(m_0, \cdots, m_{n-1})$ in F is called a loop and determines a periodic σ orbit $(\cdots, m_{n-1}, m_0, \cdots, m_{n-1}, m_0, \cdots)$ (the j-th component is $m_{j \bmod n}$). Put

$$\Lambda=\Sigma\times[0, 1]/(\boldsymbol{m}, 1)\sim(\sigma(\boldsymbol{m}), 0),$$

and consider the suspension flow $\sigma_t:\Lambda\to\Lambda$ of σ. We call σ_t a hyperbolic symbolic flow.

First we construct an invariant set X embedded in $T^2\times[0, 1]$ such that the flow φ_t on X is topologically conjugate to the hyperbolic symbolic flow σ_t for $F=\{1, 2\}$ and $\dim H_1(X;\boldsymbol{R})=2$. Then the loop (l) determines a closed σ_t orbit and, via the topological conjugacy, a closed orbit $\gamma(l)=\gamma(m_0, \cdots, m_{n-1})$ for φ_t.

Let P_1 and P_2 be the rectangles in $[3, 3]^2$ given by

$$P_1=[-2, -1]\times[-3, 3] \quad \text{and} \quad P_2=[1, 2]\times[-3, 3],$$

respectively. Let Q_1 and Q_2 be the transposed rectangles;

$$Q_1=[-3, 3]\times[1, 2] \quad \text{and} \quad Q_2=[-3, 3]\times[-2, -1].$$

Let $\psi:P_1\cup P_2\to Q_1\cup Q_2$ be the orientation preserving locally affine map preserving both the vertical lines and the horizontal lines which sends P_1 onto Q_1 and P_2 onto Q_2. Put $P_{ij}=\psi^{-1}(Q_i\cap P_j)$ $(i, j=1, 2)$. Let Y be the space obtained from the disjoint union

$$[-3, 3]^2\times[0, 1/2]\cup\left(\bigcup_{i,j=1}^{2} P_{i,j}\right)\times[1/2, 1]$$

by the identification by the inclusion map $(\cup P_{ij})\times\{1/2\}\to[-3, 3]^2\times\{1/2\}$ and by the map $\psi:(\cup P_{ij})\times\{1\}\to[-3, 3]^2\times\{0\}$. Let φ_t denote the suspension semiflow on Y. As in the case of the suspension of the horseshoe diffeomorphism ([14, Proposition 5.3]), φ_t has a unique invariant subset X' where the flow is the hyperbolic symbolic flow for $F=\{1, 2\}$. Y is topologically a 3-dimensional disk with 4 handles. Hence $H_1(Y;\boldsymbol{Z})\cong\boldsymbol{Z}^4$, where the generators a_{ij} of $\boldsymbol{Z}^4$ correspond to $P_{ij}\times[1/2, 1]$ with the direction of the flow. Consider an embedding $\iota: Y\to T^2\times[0, 1]$ such that

$$\iota_* a_{11}=(1, 0), \qquad \iota_* a_{12}=(0, 1),$$

$$\iota_* a_{21} = (0, 1), \qquad \iota_* a_{22} = (-1, 0)$$

in $H_1(T^2 \times [0, 1]; \boldsymbol{Z}) \cong \boldsymbol{Z}^2$. Since Y has the homotopy type of 1-dimensional complex, the semiflow on ιY can be extended to a nonsingular flow on $T^2 \times [0, 1]$ pointing normally outwards along the boundary. Put $X = \iota(X')$. Then we have the following lemma.

LEMMA 3.1. *X is an invariant set for φ_t on $T^2 \times [0, 1]$, $\varphi_t | X$ is the hyperbolic symbolic flow for $F = \{1, 2\}$, 0 belongs to $\mathcal{A}_{\varphi|X}$, and no closed orbits of $\varphi_t | X$ are homologous to 0.*

PROOF. We show the latter two statements.

For a loop $(l) = (m_0, \cdots, m_{n-1})$, the homology class of the closed orbit $\gamma(l)$ is

$$\iota_* a_{m_0 m_1} + \cdots + \iota_* a_{m_{n-2} m_{n-1}} + \iota_* a_{m_{n-1} m_0}.$$

Hence for any loop (l), $[\gamma(l)] \neq 0$.

We show that $0 \in \mathcal{A}_{\varphi|X}$. Put

$$(l_k) = (\overbrace{1, \cdots, 1}^{k}, \overbrace{2, \cdots, 2}^{k}),$$

then $[\gamma(l_k)]/\mathrm{per}(\gamma(l_k)) = (1/2k)(0, 1)$. This tends to 0 as $k \to \infty$. Since $\mathcal{A}_{\varphi|X}$ is closed, $\mathcal{A}_{\varphi|X} \ni 0$.

We extend the flow on X to an Axiom A flow on T^3.

PROPOSITION 3.2. *There exists an Axiom A flow φ_t on T^3 with the basic set X such that 0 belongs to $\mathcal{A}_{\varphi|X}$ and no closed orbits in X are homologous to 0.*

PROOF. Take two copies of $T^2 \times [0, 1]$, one with φ_t and the other with φ_{-t}. We paste these by the identity map of $T^2 \times \{0, 1\}$ and obtain T^3 with the induced flow φ_t. Let X_1 and X_2 denote the invariant sets corresponding to X, and Y_1 and Y_2 be the images of Y containing X_1 and X_2, respectively.

We take a finite family of local cross sections $S_1, \cdots, S_n$ in $T^3 - \mathrm{Int}(Y_1 \cup Y_2)$ satisfying the following conditions: Each S_i is diffeomorphic to a 2-dimensional disk, and, for a small positive real number ε,

$$T^3 - \mathrm{Int}(Y_1 \cup Y_2) \subset \bigcup_{i=1}^{n} \varphi_{[-\varepsilon, \varepsilon]}(S_i) \subset T^3 - (X_1 \cup X_2).$$

We modify the flow φ_t in a small neighborhood of each S_i as follows. Consider the flow f_t on $D^2\times[0,5]$ generated by the vector field η satisfying the following conditions:

(1) $\eta=\partial/\partial z$ along $\partial(D^2\times[0,5])$, where z is the coordinate of $[0,5]$.
(2) f_t has exactly four hyperbolic fixed points at $(0,k)$ $(k=1,2,3,4)$, where the index is $4-k$, and $\{0\}\times[0,5]$ consists of 4 fixed points and 5 regular orbits of f_t.
(3) The 2-dimensional stable manifold of $(0,2)$ intersects $D^2\times\{0\}$ along $\partial D^2_{1/2}\times\{0\}$ and the 2-dimensional unstable manifold of $(0,3)$ intersects $D^2\times\{5\}$ along $\partial D^2_{1/2}\times\{5\}$, where $D^2_{1/2}$ is the disk of radius 1/2 in $D^2=D^2_1$.
(4) The other orbits are either passing $(\operatorname{Int}D^2_{1/2})\times\{0\}$ and having $(0,1)$ as the ω-limit, having $(0,4)$ as the α-limit and passing $(\operatorname{Int}D^2_{1/2})\times\{5\}$, or passing $(x,0)\in(D^2-\operatorname{Int}D^2_{1/2})\times\{0\}$ and $(x,5)\in(D^2-\operatorname{Int}D^2_{1/2})\times\{5\}$.

We replace a small neighborhood of each S_i by the above flow f_t (with the parameter changed appropriately) in such a way that the orbits originally passing through S_i pass the images of $(\operatorname{Int}D^2_{1/2})\times\{0\}$ and $(\operatorname{Int}D^2_{1/2})\times\{5\}$. Note that the flow is unchanged near $X_1\cup X_2$.

Now, the resulted flow φ_t on T^3 have no cycle, and the nonwandering set $\Omega(\varphi_t)$ coincides with $X_1\cup X_2\cup\operatorname{Fix}(\varphi_t)$. So φ_t is an Axiom A flow. By Lemma 3.1, we see that $\dim H_1(X_i;\boldsymbol{R})=2$ for $i=1,2$, $\mathcal{A}_{\varphi|X_i}\ni 0$, and no closed orbits in X_i are homologous to 0.

Now we construct a nonsingular Axiom A flow on $T^4=T^3\times S^1$ with the same basic sets.

PROPOSITION 3.3. *There exists a nonsingular Axiom A flow Φ_t on T^4 such that 0 belongs to $\mathcal{A}_\Phi$ and no closed orbits are homologous to* 0.

PROOF. Let ξ denote the vector field associated with the flow φ_t. We use two vector fields ζ_1 and ζ_2 on S^1 satisfying the following condition: ζ_1 is a nonsingular vector fields, and ζ_2 has exactly two hyperbolic singular points, a source s_0 and a sink s_1.

Let U and V be open sets in T^3 such that $\operatorname{Fix}(\varphi_t)\subset U$, $X_1\cup X_2\subset V$ and $U\cap V=\emptyset$. Let ρ be a smooth function on T^3 such that

$$\rho(x)=\begin{cases}0 & \text{if } x\in T^3-(U\cup V)\\ 1 & \text{if } x \text{ is near } \operatorname{Fix}(\varphi_t)\cup X_1\cup X_2\end{cases}$$

and consider the vector field η on $T^4=T^3\times S^1$ given by

$$\eta(x,s)=\begin{cases}(\xi(x),\rho(x)\zeta_1(s)) & \text{on} \quad U\times S^1\\ (\xi(x),\rho(x)\zeta_2(s)) & \text{on} \quad V\times S^1\\ (\xi(x),0) & \text{elsewhere.}\end{cases}$$

Let Φ_t denote the flow generated by η. A fixed point of φ_t corresponds to a closed orbit of Φ_t, and we have

$$\Omega(\Phi_t)=(\mathrm{Fix}(\varphi_t)\times S^1)\cup((X_1\cup X_2)\times\{s_0,s_1\}).$$

Hence the flow Φ_t is a nonsingular Axiom A flow on T^4. By Proposition 3.2, 0 belongs to $\mathcal{A}_\Phi$, and no closed orbits of Φ_t are homologous to 0.

§ 4. Existence of null homologous closed orbit.

In this section we prove the following theorem.

THEOREM 4.1. *Let $\varphi_t: M\to M$ be an Axiom A flow, and X, a basic set of φ_t. Let V denote the minimal vector subspace which contains $\mathcal{A}_{\varphi|X}$. If 0 belongs to $\mathrm{Int}_V\,\mathcal{A}_{\varphi|X}$, then there is a null homologous closed orbit in X.*

The proof of this theorem follows from the description ([3]) of the basic set by using the symbolic dynamics. For the finite set F and $\Sigma=\prod_{\boldsymbol{Z}} F$ appeared in § 3, let G be a subset of $F\times F$ and consider the subset Σ_G of Σ consisting of the elements $\boldsymbol{m}=(m_i)_{i\in \boldsymbol{Z}}$ such that $(m_{i-1},m_i)\in G$ for all $i\in\boldsymbol{Z}$. The shift homeomorphism σ maps Σ_G to itself and the suspension flow

$$\sigma_t: \Lambda_G \longrightarrow \Lambda_G \quad (\Lambda_G=\Sigma_G\times[0,1]/\sim)$$

of this $\sigma|\Sigma_G$ is called the hyperbolic symbolic flow of finite type. Bowen ([3]) showed that for a basic set X of an Axiom A flow, there exist a hyperbolic symbolic flow (Λ_G,σ_t) of finite type and a semiconjugacy h from Λ_G to X which is finite to one. This hyperbolic symbolic flow and the semiconjugacy can be taken so that there is an embedding ι of the 1-dimensional complex Z with the vertices v_f $(f\in F)$ and the edges $e_{(f,g)}$ $((f,g)\in G)$ in a neighborhood of X and the closed orbit $\gamma(l)$ for an admissible loop $(l)=(m_0,\cdots,m_{n-1})$ $(m_n=m_0,\ (m_{i-1},m_i)\in G$ for $i=1,\cdots,n)$ is homotopic to the curve

$$\iota(e_{(m_0,m_1)}\cdots e_{(m_{n-2},m_{n-1})}e_{(m_{n-1},m_0)})\subset\iota Z\subset M.$$

PROOF OF THEOREM 4.1. First take the hyperbolic symbolic flow of finite type (Λ_G, σ_t) and the semiconjugacy $h: \Lambda_G \to X$. If the dimension of the vector subspace V is n, there are closed orbits $O_0, O_1, \cdots, O_n$ such that the corresponding asymptotic cycles $[O_i]/\mathrm{per}(O_i)$ span a non degenerate n-simplex in $H_1(M; \boldsymbol{R})$ such that the origin belongs to the interior of the n-simplex. Then the origin belongs to the interior of the simplex spaned by $[O_i]$. Let $(L_i)=(m_0^{(i)}, \cdots, m_{n_i-1}^{(i)})$, $m_{n_i}^{(i)}=m_0^{(i)}$ denote the admissible loop corresponding to O_i; $O_i=\gamma(L_i)$. For an element $f \in F$, let $(A_i, m_0^{(i)})=(f, \cdots, m_0^{(i)})$ and $(B_i, f)=(m_0^{(i)}, \cdots, f)$ be admissible paths from f to $m_0^{(i)}$ and from $m_0^{(i)}$ to f, respectively. For a positive integer N, consider the admissible loop

$$(L_i')=(A_i, \overbrace{L_i, \cdots, L_i}^{N}, B_i).$$

Put $O_i'=\gamma(L_i')$. Then the homology class of this orbit is written as follows.

$$[O_i']=N[O_i]+[\gamma(A_i, B_i)].$$

Since the origin belongs to the interior of the simplex spaned by $[O_i]$, it belongs to the interior of the simplex spaned by $[O_i']$ for a large integer N. Since the homology classes $[O_i']$ are integral classes, there exist positive integers k_i such that

$$\sum_{i=0}^{n} k_i[O_i']=0.$$

Let L be the admissible loop given by

$$(L)=(\overbrace{L_0', \cdots, L_0'}^{k_0}, \cdots, \overbrace{L_n', \cdots, L_n'}^{k_n}).$$

Then the closed orbit $O=\gamma(L)$ is homologous to zero.

At the end of this section we show that, by using a result of Fried ([4]), Theorem 4.1 implies Proposition 2.3.

A SECOND PROOF OF PROPOSITION 2.3. Suppose that the above V is 1-dimensional. We take the hyperbolic symbolic flow of finite type (Λ_G, σ_t) and the semiconjugacy h as before. An admissible loop $(l)=(m_0, \cdots, m_{n-1})$ is called minimal if m_i are distinct. By Theorem 4.1, if there are no null homologous closed orbits, the simplex spanned by the homology classes $[\gamma(l_1)]$ and $[\gamma(l_2)]$ for any two minimal loops (l_1) and (l_2) does not contain the origin. Then the set of $[\gamma(l)]$ lies on one of the components of $V-\{0\}$. Now a result of Fried ([4, Theorem H]) says that if there is a rational 1-dimensional cohomology class which is positive on $\gamma(l)$ for all minimal loops, then there is a cross section. By choosing an appropriate cohomology class and applying this result, there is a cross section for $\varphi|X$. The converse is clear; if there is a cross section, there are no null homologous closed orbits in X.

References

[1] Bowen, R., Periodic orbits for hyperbolic flows, Amer. J. Math. **94** (1972), 1-30.

[2] Bowen, R., One-dimensional hyperbolic sets for flows, J. Differential Equations **12** (1972), 173-179.

[3] Bowen, R., Symbolic dynamics for hyperbolic flows, Amer. J. Math. **95** (1973), 429-460.

[4] Fried, D., The geometry of cross sections to flows, Topology **21** (1982), 353-371.

[5] Fuller, F., On the surface of sections and periodic trajectories, Amer. J. Math. **87** (1965), 473-480.

[6] Ghys, E., Codimension one Anosov flows and suspensions, Dynamical systems, Valparaiso 1986, Lecture Notes in Math. vol. 1331, Springer Verlag, Berlin-Heidelberg-New York-London-Paris-Tokyo, 1988, 59-72.

[7] Plante, J., Diffeomorphisms with invariant line bundles, Invent. Math. **13** (1971), 325-334.

[8] Plante, J., Anosov flows, Amer. J. Math. **94** (1972), 729-754.

[9] Plante, J., Anosov flows, transversely affine foliations, and a conjecture of Verjovsky, J. London Math. Soc. (2) **23** (1981), 359-362.

[10] Pugh, C. and M. Shub, The Ω-stability theorem for flows, Invent. Math. **11** (1970), 150-158.

[11] Schwartzman, S., Asymptotic cycles, Ann. of Math. **66** (1957), 270-284.

[12] Sigmund, K., On the space of invariant measures for hyperbolic flows, Amer. J. Math. **94** (1972), 31-37.

[13] Smale, S., Diffeomorphisms with many periodic points, Differential and Combinatorial Topology, Princeton Univ. Press, Princeton, N. J., 1965, 63-80.

[14] Smale, S., Differentiable dynamical systems, Bull. Amer. Math. Soc. **73** (1967), 747-817.

[15] Verjovsky, A., Codimension one Anosov flows, Bol. Soc. Mat. Mexicana **19** (1974), 49-77.

(Received March 23, 1989)

Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113, Japan