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A quaternionic L-value congruence

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I. Introduction

In this paper we will prove a sharpened form of an L-value congruence which appears in [3] by using the results of P. Deligne and K. Ribet in [4].

Let p, r and q be prime numbers such that $p\equiv r\equiv -q\equiv 3 \bmod 4$ and for which one has Legendre symbols $\left(\frac{p}{q}\right)=\left(\frac{r}{q}\right)=-1$. By [3, Prop. 4.1.3], there is a unique complex Galois extension N of $\boldsymbol{Q}$ for which $G=\mathrm{Gal}(N/\boldsymbol{Q})$ is isomorphic to the quaternion group H_8 of order eight, and such that N is ramified at exactly p, r, q and infinity. Let V be the two dimensional irreducible representation of G, and let $L(s, V)$ be the Artin L-function of V. Define $\mathcal{S}$ to be the set of prime numbers l for which $l\equiv 1 \bmod 4$ and $\left(\frac{l}{r}\right)=\left(\frac{l}{p}\right)=-\left(\frac{l}{q}\right)=1$. If $d>1$ is an odd square-free integer, we will denote by $t_d(*)$ the primitive quadratic Dirichlet character of conductor d.

THEOREM 1.1. *For $l\in\mathcal{S}$, $L(0, V)$ and $L(0, V\otimes t_l)$ are rational integers exactly divisible by* 8 *and* 16, *respectively. One has*

$$L(0, V\otimes t_l)\equiv 2L(0, V) \quad \mathit{mod}\ 64\ \boldsymbol{Z}.$$

The motivation for Theorem 1.1 is the conjecture that $\Omega_m(L/K)=W_{L/K}$ if L/K is a finite normal extension of number fields, where $\Omega_m(L/K)$ is the multiplicative invariant of L/K, and $W_{L/K}$ is the Cassou-Noguès Fröhlich class of L/K. (See [3], [1], [2].) More precisely, Theorem 1.1 is related to the compatibility of the $\Omega_m(L/K)=W_{L/K}$ conjecture under twisting, in the following way.

For $l\in\mathcal{S}$ there is a unique extension $N[l]/\boldsymbol{Q}$ other than $N/\boldsymbol{Q}$ for which $\mathrm{Gal}(N[l]/\boldsymbol{Q})$ is isomorphic to H_8 and for which $N[l] \subseteq N(\sqrt{l})$. In

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[3] it was shown that if $\Omega_m(N[l]/Q)=W_{N[l]/Q}$ for a sufficiently large finite set $\mathcal{S}_0$ of l in $\mathcal{S}$, then $\Omega_m(N[l]/Q)=W_{N[l]/Q}$ for all l in $\mathcal{S}$. The analytic ingredient in the proof was the existence of a finite Galois extension H of $\boldsymbol{Q}$, depending only on N, such that if $l\in\mathcal{S}$ is unramified in H, then the Frobenius conjugacy class of l in $\mathrm{Gal}(H/Q)$ determines $L(0, V\otimes t_l)$ mod $64\,\boldsymbol{Z}$. Theorem 1.1 shows one can take $H=\boldsymbol{Q}$.

The finite subset $\mathcal{S}_0$ discussed above is effectively computable from p, r and q, but is quite large in general. To show that $\mathcal{S}_0$ can be taken to be any non-empty subset of $\mathcal{S}$, one would need to combine Theorem 1.1 with a sharpened form of the algebraic result in [3, Corollary 4.3.3]. The proof of [3, Theorem 4.3.4] suggests how to go about sharpening [3, Corollary 4.3.3], but we will not undertake this here.

Of course, it would be desirable to show that one can take the set $\mathcal{S}_0$ above to be empty, or more generally to show $\Omega_m(L/Q)=W_{L/Q}$ for all H_8-extensions $L/\boldsymbol{Q}$. To prove Theorem 1.1, we will need only the congruences proved by Deligne and Ribet in [4] via Hilbert modular Eisenstein series. It seems likely that one will also have to use Hilbert modular theta series and cusp forms in order to study the $\Omega_m(L/Q)=W_{L/Q}$ conjecture for arbitrary H_8-extensions $L/\boldsymbol{Q}$ by the methods in this paper and in [3].

Theorem 1.1 will be proved by the reduction-of-level method discussed in [3]. Because this method can be applied to prove other L-value congruences, we now outline it by means of the present case. We will discuss only two-adic congruences, though one could similarly consider p-adic congruences for odd primes p.

The L-function $L(s, V\otimes t_l)$ equals $L(s, \chi_2\otimes\psi_l)$ for an odd ray class character $\chi_2\otimes\psi_l$ of $K=\boldsymbol{Q}(\sqrt{pr})$. We analyze $L(0, V\otimes t_l)=L(0, \chi_2\otimes\psi_l)$ by finding a linear combination $\epsilon=\sum a_\chi\chi$ of ray class characters of K with the following properties. First, the values of ϵ lie in a finite extension L of $\boldsymbol{Q}_2$ and are two-adically close to 0. Second, $L(0,\epsilon)=\sum a_\chi L(0,\chi)$ equals $L(0,\chi_2\otimes\psi_l)$ plus $\sum b_\xi L(0,\xi)$ for some linear combination $\sum b_\xi\xi$ of ray class characters ξ for which the conductor of the primitive character ξ' associated to ξ properly divides that of $\chi_2\otimes\psi_l$. One shows that $L(0,\epsilon)$ is two-adically close to zero using either Shintani's formulas [6], as in [3], or by using the extra two-adic divisibilities of Deligne and Ribet, as in the present paper. The principal advantage of using Deligne and Ribet's results is in avoiding the elementary 'error terms' which arose in [3, Prop. 6.3.10] from Shintani's formulas. One then has

$$(1.1) \qquad \begin{aligned} L(0, \chi_2 \otimes \psi_l) &= L(0, \epsilon) - \sum b_\xi L(0, \xi) \\ &\equiv -\sum b_\xi L(0, \xi) \text{ modulo a high power of 2.} \end{aligned}$$

If ξ' is a primitive character inducing an imprimitive character ξ, one has $L(0, \xi) = E(\xi, \xi')L(0, \xi')$, where $E(\xi, \xi')$ is a finite product of Euler factors of $L(s, \xi)$ evaluated at $s=0$. Substituting this into (1.1), one is reduced to evaluating each $L(0, \xi')$ modulo a high power of 2, this power depending on $b_\xi E(\xi, \xi')$. One may try to do this by evaluating $L(0, \xi')$ explicitly modulo the required power of 2, or by applying the reduction of level method to ξ', where now ξ' has strictly smaller conductor than our original character $\chi_2 \otimes \psi_l$.

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II. A special case of Deligne and Ribet's results

Let K be a totally real number field. Let F be a non-zero integral ideal of K. Let G_F be the strict ray class group of K conductor F. Define G to be the strict ray class group of K of conductor $F^\infty 2^\infty$. For ϵ a complex valued function on G_F, define

$$L(s, \epsilon) = \sum \epsilon(x) \ \text{Norm}(x)^{-s}$$

for $\text{Re}(s) > 1$, where the sum is over integral prime to F ideals x of K. Then $L(s, \epsilon)$ has a meromorphic continuation to all $s \in \boldsymbol{C}$, and $L(s, \epsilon)$ is holomorphic except possibly at $s=1$, where it can have at most a simple pole. In particular, $L(0, \epsilon)$ is well-defined. By a Theorem of Siegel [7], if ϵ takes values in a $\boldsymbol{Q}$ vector space W, then $L(0, \epsilon)$ lies in W.

Let $\mathcal{N} : G \longrightarrow \boldsymbol{Z}_2^*$ be the continuous character whose value on the image in G of a prime to $2F$ ideal is its norm. For $c \in G$, define ϵ_c to be the function on G for which $\epsilon_c(g) = \epsilon(cg)$, where the product cg is computed in G_F. If ϵ takes values in a finite extension L of $\boldsymbol{Q}_2$, define

$$\Delta_c(0, \epsilon) = L(0, \epsilon) - \mathcal{N}(c) L(0, \epsilon_c)$$

so that $\Delta_c(0, \epsilon) \in L$. A function ϵ on G_F is even (resp. odd) if for each real place v of K and each $g \in G_F$, we have $\epsilon(\sigma_v g) = \epsilon(g)$ (resp. $\epsilon(\sigma_v g) = -\epsilon(g)$), where σ_v is a Frobenius element for v in G_F.

The following Theorem results from [4, Thm. 8.4 and Prop. 8.8] and from the linearity of $\Delta_c(0,\epsilon)$ in ϵ.

THEOREM 2.1 (Deligne and Ribet). *Suppose F is not the trivial ideal. Let ϵ be an odd function from G_F to the ring of integers $\mathcal{O}_L$ of a finite extension L of $\boldsymbol{Q}_2$. Then $\Delta_c(0,\epsilon)$ lies in $2^r\mathcal{O}_L$ for all c in G, where $r=[K:\boldsymbol{Q}]$.*

COROLLARY 2.2. *Let $K(\mu_{2^\infty})$ be the extension of K generated by all roots of unity of two-power order. Let K_F be the strict ray class field of K of conductor F. Suppose that K_F and $K(\mu_{2^\infty})$ are disjoint over K. (This will be the case if F is prime to 2 and K is unramified above 2.) Let d be an element of G_F. Then there is an element c of G with image d in G_F for which $\mathcal{N}(c)=1$ in $\boldsymbol{Z}_2^*$. For this c and ϵ as in Theorem 2.1, we have $\Delta_c(0,\epsilon)=L(0,\epsilon-\epsilon_c)\in 2^r\mathcal{O}_L$. If ϵ is an odd character of G_F then $L(0,\epsilon-\epsilon_c)=(1-\epsilon(d))L(0,\epsilon)$, where $\epsilon(d)=\epsilon(c)$.*

PROOF. Let K' be the maximal abelian extension of K which is unramified at all finite primes of K which are relatively prime to $2F$. Then $\mathrm{Gal}(K'/K)=G_{2^\infty F\infty}=G$. The existence of c follows from the fact that the norm $\mathcal{N}:G\longrightarrow \boldsymbol{Z}_2^*=\mathrm{Gal}(\boldsymbol{Q}(\mu_{2^\infty})/\boldsymbol{Q})$ corresponds to restriction of automorphisms from K' to $\boldsymbol{Q}(\mu_{2^\infty})$. If ϵ is a character of G_F, then $L(s,\epsilon_c)=\sum \epsilon_c(x)\,\mathrm{Norm}(x)^{-s}=\sum \epsilon(c)\epsilon(x)\,\mathrm{Norm}(x)^{-s}=\epsilon(d)L(s,\epsilon)$, so $L(0,\epsilon_c)=\epsilon(d)L(0,\epsilon)$.

III. Characters of some ray class groups of $\boldsymbol{Q}(\sqrt{pr})$

Let $K=\boldsymbol{Q}(\sqrt{pr})$. Suppose that f and f' are integral ideals of K, and that χ is a character of the strict ray class group G_f. If f' is divisible by the conductor of the primitive character χ' which induces χ, then we will denote by ${}_{f'}\chi$ the character of $G_{f'}$ induced by χ'.

Let P_p and P_r be the prime ideals of K over p and r, respectively Since $\left(\frac{q}{pr}\right)=\left(\frac{q}{p}\right)\left(\frac{q}{r}\right)=\left(\frac{p}{q}\right)\left(\frac{r}{q}\right)=-1\cdot -1=1$, the prime q splits in K as $P_q\bar{P}_q$. Let P_∞ and $\bar{P}_\infty$ be the two infinite places of K. Fix $l\in\mathcal{S}$. Since $\left(\frac{l}{pr}\right)=1$, l splits in K as $P_l\bar{P}_l$.

PROPOSITION 3.1. *The group G_l/G_l^2 is isomorphic to $(\boldsymbol{Z}/2)^3$. The even characters of G_l/G_l^2 are induced by the elements of $S_l=\{\psi_1,\psi_l',\psi_l'',\psi_l\}$, where ψ_l' (resp. ψ_l'') is a primitive quadratic character of conductor P_l (resp. $\bar{P}_l$), ψ_1 is the trivial character, and $\psi_l=\psi_l'\psi_l''=t_l\circ\mathrm{Norm}_{K/Q}$ is prim-*

itive and quadratic of conductor $l\mathcal{O}_K=P_l\bar{P}_l$. *The odd characters of* G_l/G_l^2 *have the form* ${}_l\chi_1\ \phi$, *where* ϕ *is in* S_l *and* χ_1 *is the primitive odd quadratic character of conductor* 1 *corresponding by class field theory to the quadratic extension* $K(\sqrt{-p})$ *of* K.

PROOF. By genus theory, the class number of K is odd. Let J be the multiplicative group of elements of K^* which are prime to $P_l\bar{P}_l$. Define $h: J\longrightarrow(\boldsymbol{Z}/2)^4$ by letting $h(\alpha)$ for α in J have coordinates the quadratic residue symbols of α at P_l, $\bar{P}_l$, P_∞ and $\bar{P}_\infty$. The ideal content map induces an isomorphism

$$\frac{(\boldsymbol{Z}/2)^4}{h(\mathcal{O}_K^*)}=G_l/G_l^2. \tag{3.1}$$

where $\mathcal{O}_K^*$ is the unit group of K. Under this isomorphism, the even characters of G_l/G_l^2 are those induced by characters of $(\boldsymbol{Z}/2)^4$ which are trivial on the factors corresponding to P_∞ and $\bar{P}_\infty$ as well as on $h(\mathcal{O}_K^*)$.

By [3, Lemma 4.2.1], there is a totally positive fundamental unit ϵ_{pr} for K. The extension $K(\sqrt{-p})=K(\sqrt{-r})$ of K is quadratic and unramified at all finite primes of K since $-p\equiv-r\equiv1 \bmod 4$. Hence the quadratic character χ_1 associated to $K(\sqrt{-p})/K$ is odd of conductor 1. By Kummer theory, $K(\sqrt{-p})=K(\sqrt{-\epsilon_{pr}})$. Now $\left(\frac{l}{p}\right)=1$ by assumption, so the primes P_l and $\bar{P}_l$ over l in K split in $K(\sqrt{-p})=K(\sqrt{-\epsilon_{pr}})$. Hence $-\epsilon_{pr}$ is a square at both P_l and $\bar{P}_l$. Since $l\equiv1 \bmod 4$, it follows that all the units of K are squares at P_l and $\bar{P}_l$. Proposition 3.1 now follows from the isomorphism (3.1).

PROPOSITION 3.2. *Let* $f=P_pP_rP_q\bar{P}_q=P_pP_rq$. *The group* $\tilde{G}_f=G_f/G_f^4$ *is isomorphic to* $(\boldsymbol{Z}/4)\oplus(\boldsymbol{Z}/2)^3$. *The representation* V *in Theorem* 1.1 *has the form* $\mathrm{Ind}_{K/\boldsymbol{Q}}\chi_2$ *for a faithful, odd, primitive quartic character* χ_2 *of* G_f. *The character* χ_2^2 *is induced by a primitive even quadratic character* μ_q *of conductor* $q\mathcal{O}_K$, *where* μ_q *corresponds by class field theory to the extension* $K(\sqrt{q})/K$. *There is a unique primitive odd quadratic character* χ_3 *(resp.* χ_3'*) of conductor* $P_pP_rP_q$ *(resp.* $P_pP_r\bar{P}_q$*). Let* χ_1 *be as in Proposition* 3.1. *Then there is a unique primitive odd quartic character* χ_q *of conductor* $q\mathcal{O}_K$ *such that* ${}_f\chi_q=\chi_1\chi_2\chi_3$. *There is a unique odd primitive quartic Dirichlet character* λ_q *of conductor* q *if* $q\equiv5 \bmod 8$ *and of conductor* pq *if* $q\equiv1 \bmod 8$ *such that* χ_q *is the primitive character associated to* $\lambda_q\circ\mathrm{Norm}_{K/\boldsymbol{Q}}$. *The odd characters of* G_f *which factor through* $\tilde{G}_f$ *are* *induced by the elements of the following set of primitive characters:*

$$S=\{\chi_1, \mu_q\chi_1, \chi_2, \chi_2^{-1}=\mu_q\chi_2, \chi_3, \chi_3', \chi_q, \chi_q^{-1}=\mu_q\chi_q\}.$$

The set of even characters of G_f factoring through $\tilde{G}_f$ is ${}_f\chi_1\cdot S$.

PROOF. Let I be the multiplicative group of elements of K which are relatively prime to f. Let $g: I\longrightarrow(\mathcal{O}_K/f)^*\oplus\{\pm 1\}\oplus\{\pm 1\}=B$ be the homomorphism such that the first coordinate of $g(\alpha)$ is the residue class of α mod f, and the last two coordinates of $g(\alpha)$ are the signs of α at P_∞ and $\bar{P}_\infty$. Let $g': I\longrightarrow B/B^4=B'$ be the composition of g with the quotient map $B\longrightarrow B/B^4$. Since K has odd class number, g' induces an isomorphism

$$B'/g'(\mathcal{O}_K^*)=G_f/G_f^4. \tag{3.2}$$

For P a prime ideal of K and j a positive integer, define $F(P,j)=(\mathcal{O}_K/P)^*/((\mathcal{O}_K/P)^*)^j$. For $P=P_\infty$ or $\bar{P}_\infty$, let $F(P,2)=F(P,4)=\{\pm 1\}$. Since $p\equiv r\equiv -1 \bmod 4$ and the primes over p, r and q in K have residue field degree 1, we have

$$B'=F(P_p,2)\times F(P_r,2)\times F(P_q,4)\times F(\bar{P}_q,4)\times F(P_\infty,2)\times F(\bar{P}_\infty,2). \tag{3.3}$$

As in the proof of Proposition 3.1, we have $K(\sqrt{-p})=K(\sqrt{-\epsilon_{pr}})$ when ϵ_{pr} is a totally positive fundamental unit for K. By quadratic reciprocity, $\left(\frac{q}{p}\right)=\left(\frac{p}{q}\right)$, and $\left(\frac{p}{q}\right)=-1$ by assumption. Hence $-\epsilon_{pr}$ is a non-square at each of the primes P_q and $\bar{P}_q$ over q in K. Since $q\equiv 1 \bmod 4$, it follows that ϵ_{pr} is a non-square at P_q and $\bar{P}_q$. Hence the image ϵ of ϵ_{pr} in $F(P_q,4)$ is a generator of $F(P_q,4)$. Let $\bar{\epsilon}$ be the image of ϵ in $F(\bar{P}_q,4)$ under the nontrivial automorphism of $\mathrm{Gal}(K/Q)$. The image of -1 generates $F(P_p,2)$ and $F(P_r,2)$ since P_p and P_r have residue field degree 1 and $p\equiv r\equiv 3 \bmod 4$. Define $\partial=1$ if ϵ_{pr} is a square at P_p and let $\partial=-1$ otherwise. Let $\gamma=2$ if $q\equiv 5 \bmod 8$ and let $\gamma=4$ if $q\equiv 1 \bmod 8$. We now claim that in terms of the factorization of B' in (3.3), the images of -1 and ϵ_{pr} under g' are given by the following table.

TABLE 3.3. *The images of -1 and ϵ_{pr} under g'.*

	$F(P_p,2)$	$F(P_r,2)$	$F(P_q,4)$	$F(\bar{P}_q,4)$	$F(P_\infty,2)$	$F(\bar{P}_\infty,2)$
-1	-1	-1	ϵ^γ	$\bar{\epsilon}^\gamma$	-1	-1
ϵ_{pr}	∂	$-\partial$	ϵ	$\bar{\epsilon}^{-1}$	1	1

The entries for -1 in this table are clear. Since ϵ_{pr} is totally positive, the last two components of $g'(\epsilon_{pr})$ equal 1. Because $\epsilon_{pr}^{\sigma}=\epsilon_{pr}^{-1}$ if σ is the nontrivial element of $\mathrm{Gal}(K/Q)$, the third and fourth components of $g'(\epsilon_{pr})$ are as indicated. The first component of $g'(\epsilon_{pr})$ is ∂ by definition. Suppose that the second component of $g'(\epsilon_{pr})$ were equal to ∂ rather than $-\partial$. Then there would be a quadratic extension K' of K which ramifies only over P_p and P_r. This K' would be Galois over Q; but since $p\equiv r\equiv 3 \bmod 4$, there are no quartic Galois extensions of Q ramified only over p and r. This establishes Table 3.3.

From Table 3.3 one sees that $g'(\mathcal{O}_K^*)$ is isomorphic to $(Z/4)\oplus(Z/2)$. Hence (3.2) shows G_f/G_f^4 is isomorphic to $(Z/4)+(Z/2)^3$. We will now give the local components of the characters of B' which by (3.2) induce the characters of G_f appearing in Proposition 3.2.

Let Φ be a character of order 4 of $F(P_q, 4)$. Define $\bar{\Phi}$ to be the character $\Phi\circ\sigma$ of $F(\bar{P}_q, 4)$. For $P\in\{P_p, P_r, P_\infty, \bar{P}_\infty\}$ and $j=2$ or 4, let $(-)=(-)_P$ be the (unique, quadratic) non-trivial character of $F(P,j)$, and let $1=1_P$ be the trivial character of $F(P,j)$.

TABLE 3.4. *Local components of some characters of B'.*

	$F(P_p, 2)$	$F(P_r, 2)$	$F(P_q, 4)$	$F(\bar{P}_q, 4)$	$F(P_\infty, 2)$	$F(\bar{P}_\infty, 2)$
${}_f\chi_1$	1	1	1	1	$(-)$	$(-)$
${}_f\chi_2$	$(-)$	$(-)$	Φ	$\bar{\Phi}^{-1}$	$(-)$	$(-)$
${}_f\chi_3$	$(-)$	$(-)$	$(-)$	1	$(-)$	$(-)$
${}_f\chi_q$	1	1	Φ^{-1}	$\bar{\Phi}^{-1}$	$(-)$	$(-)$

One readily verifies from Table 3.3 that the characters of Table 3.4 are trivial on $g'(-1)$ and $g'(\epsilon_{pr})$, and hence that they give characters of G_f. The character ${}_f\chi_2=\chi_2$ is primitive, and satisfies $\chi_2\circ\sigma=\chi_2^{-1}$. Furthermore, the restriction of χ_2 to rational ideals relatively prime to prq is the Dirichlet character mod prq induced by the Legendre symbol $\left(\frac{pr}{*}\right)$. Since χ_2 is odd, it follows from [5, exercise 7] that the fixed field N' of χ_2 is a quaternion extension of Q which ramifies at exactly p, r, q and ∞. By [3, Prop. 4.1.3], we must have $N'=N$, and so $V=\mathrm{Ind}_{K/Q}\chi_2$ is the unique two-dimensional irreducible representation of $\mathrm{Gal}(N/Q)$. The remaining assertions in Proposition 3.2 are readily verified from Table 3.4.

The following Corollary will be used in the proof of Theorem 1.1.

COROLLARY 3.5. *Let i be a fixed fourth root of unity. There is an element d of G_{fl} such that ${}_{fl}\psi(d)=1$ for $\psi\in\{\psi_1, \psi_l', \psi_l'', \psi_l\}$, ${}_{fl}\chi_2(d)={}_{fl}\chi_q(d)=i$ and ${}_{fl}\chi_1(d)={}_{fl}\chi_3(d)=-1$. For this d one has ${}_{fl}\mu_q(d)=-1$.*

PROOF. From Table 3.4 it is clear that we can construct a $d'\in B'$ which has nontrivial components only at the factors $F(\bar{P}_q, 4)$ and $F(P_\infty, 2)$ of B' such that ${}_f\chi_2(d')={}_f\chi_q(d')=i$ and ${}_f\chi_1(d')={}_f\chi_3(d')=-1$. Since ${}_f\mu_q=({}_f\chi_2)^2$, we have ${}_f\mu_q(d')=-1$. Let d'' be the image of d' in G_f/G_f^4. Let G_l^+ be the wide ideal class group of K of conductor $l\mathcal{O}_K=P_l\bar{P}_l$. By Proposition 3.1, the character group of $G_{l,2}^+=G_l^+/(G_l^+)^2$ is $\{{}_l\psi_1, {}_l\psi_l', {}_l\psi_l'', {}_l\psi_l\}$. Let $K_{l,2}^+$ (resp. $K_{f,4}$) be the abelian extension of K corresponding to $G_{l,2}^+$ (resp. G_f/G_f^4) by class field theory. Since $K_{f,4}/K$ is unramified over l, and every nontrivial subextension of $K_{l,2}^+/K$ is ramified over l, the fields $K_{f,4}$ and $K_{l,2}^+$ are disjoint over K. Hence since $\mathrm{Gal}(K_l^+/K)=G_{l,2}^+$ and $\mathrm{Gal}(K_{f,4}/K)=G_f/G_f^4$ are both quotients of G_{fl}, we can find a lift d of d'' to G_{fl} such that $\psi(d)=1$ for all characters ψ of $G_{l,2}^+$. This d satisfies all our requirements.

IV. Proof of Theorem 1.1

Throughout this section we will assume the notations of Propositions 3.1 and 3.2, and we will let V and t_l be as in Theorem 1.1.

LEMMA 4.1. *The numbers $L(0, V)=L(0, \chi_2)$ and $L(0, V\otimes t_l)=L(0, \chi_2\psi_l)$ are rational integers which are exactly divisible by 8 and by 16, respectively.*

PROOF. Since $V=\mathrm{Ind}_{K/Q}\chi_2$ and $V\otimes t_l=\mathrm{Ind}_{K/Q}\chi_2\psi_l$ we have $L(0, V)=L(0, \chi_2)$ and $L(0, V\otimes t_l)=L(0, \chi_2\psi_l)$. In [3, Prop. 4.3.7] it is shown that $L(0, V\otimes t_l)$ is a rational integer exactly divisible by 16. When one replaces l by 1, the same arguments show $L(0, V)$ is a rational integer exactly divisible by 8.

DEFINITION 4.2. Let f and $d\in G_{fl}$ be as in Corollary 3.5. Let $F=fl$ in Theorem 2.1, and let $c\in G=G_{2^\infty F^\infty}$ be defined as in Corollary 2.2 for the above $d\in G_F$. Let S and S_l be as in Propositions 3.1 and 3.2. Let h be the odd function from G_F to $Q_2(i)$ defined by

$$h=\frac{1}{2}\sum_{\chi\in S}\sum_{\psi\in S_l}{}_F(\chi\psi).$$

LEMMA 4.3. *The values of h lie in* $16\,\boldsymbol{Z}_2$.

PROOF. By Propositions 3.1 and 3.2, the set $\{{}_F\chi_1\,\chi\,\psi:\chi\in S \text{ and } \psi\in S_l\}$ is a group of characters of G_F. The order of this group is 32. Hence ${}_F\chi_1\,h$ takes values in $16\,\boldsymbol{Z}_2$, so h does as well.

Now Theorem 2.1 gives

COROLLARY 4.4. $\Delta_c(0,h)=\sum_{\chi\in S}\sum_{\psi\in S_l}\Delta_c(0,{}_F(\chi\psi))/2$ *lies in* $64\,\boldsymbol{Z}_2$.

In the next paragraphs we analyze the contributions $\bmod 64\,\boldsymbol{Z}_2[i]$ of various terms in the sum for $\Delta_c(0,h)$ in Corollary 4.4.

IV. 1. $\chi\in\{\chi_2,\chi_2^{-1}\}$ **and** $\psi\in\{\psi_1,\psi_l\}$.

By Corollary 2.2 and our normalization of d and c we have

$$\begin{aligned}(4.1)\qquad &\Delta_c(0,{}_F\chi_2\,\psi)+\Delta_c(0,{}_F\chi_2^{-1}\psi)\\ &=(1-{}_F\chi_2\,\psi(d))L(0,{}_F\chi_2\,\psi)+(1-{}_F\chi_2^{-1}\psi(d))L(0,{}_F\chi_2^{-1}\psi)\\ &=(1-i\psi(d))L(0,{}_F\chi_2\,\psi)+(1+i\psi(d))L(0,{}_F\chi_2^{-1}\psi).\end{aligned}$$

Now $\mathrm{Ind}_{K/\boldsymbol{Q}}\chi_2=\mathrm{Ind}_{K/\boldsymbol{Q}}\chi_2^{-1}$, and F is fixed by $\mathrm{Gal}(K/\boldsymbol{Q})$. It follows that $L(0,{}_F\chi_2\,\psi)=L(0,{}_F\chi_2^{-1}\psi)$. Thus the sum in (4.1) equals $2L(0,{}_F\chi_2\,\psi)$. If $\psi=\psi_l$ then $\chi_2\,\psi_l$ is primitive of conductor F, so $L(0,{}_F(\chi_2\,\psi_l))=L(0,\chi_2\,\psi_l)$, and this is equal to $L(0,V\otimes t_l)$ by Lemma 4.1. Suppose now that $\psi=\psi_1$. Then

$$(4.2)\qquad L(0,{}_F(\chi_2\,\psi_1))=(1-\chi_2(P_l))(1-\chi_2(\bar{P}_l))L(0,\chi_2).$$

By Proposition 3.2, $\chi_2^2(P_l)=\left(\frac{l}{q}\right)=-1$, so $\chi_2(P_l)$ is a fourth root of unity. Since the nontrivial element of $\mathrm{Gal}(K/\boldsymbol{Q})$ takes χ_2 to χ_2^{-1}, we have $\chi_2(\bar{P}_l)=\chi_2(P_l)^{-1}$. Thus (4.2) is simply

$$L(0,{}_F(\chi_2\,\psi_1))=2L(0,\chi_2)=2L(0,V)$$

where the second equality is from Lemma 4.1. We conclude that

$$(4.3)\qquad \sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\ \sum_{\psi\in\{\psi_1,\psi_l\}}\Delta_c(0,{}_F(\chi\psi))/2=L(0,V\otimes t_l)+2L(0,V).$$

IV. 2. $\chi\in\{\chi_3,\chi_3'\}$ **and** $\psi\in\{\psi_1,\psi_l',\psi_l'',\psi_l\}$.

These terms contribute nothing to $\Delta_c(0, h)$ mod $64\, Z_2[i]$ because of

LEMMA 4.5. *For χ and ψ as in* IV. 2,

$$\Delta_c(0, {}_F(\chi\psi)) \equiv 0 \quad mod\ 128\, Z_2.$$

PROOF. Let L be the quadratic extension of K corresponding to $\chi\psi$ by class field theory. We have $L(s, \chi\psi) = \zeta_L(s)/\zeta_K(s)$, where $\zeta_k(s)$ is the zeta function of the number field k. One has

$$\lim_{s\to 0} s^{-r(k)}\zeta_k(s) = -h_k \operatorname{Reg}(k)/w_k$$

where $r(k) = r_1(k) + r_2(k) - 1$ is the order of vanishing of $\zeta_k(s)$ at $s=0$, h_k is the class number of k, $\operatorname{Reg}(k)$ is the regulator of k, and w_k is the number of roots of unity in k. Since $\chi\psi$ is an odd character, the functional equation of $L(s, \chi\psi)$ shows $L(s, \chi\psi)$ does not vanish at $s=0$. Hence

$$L(0, \chi\psi) = \frac{h_L \operatorname{Reg}(L) w_K}{h_K \operatorname{Reg}(K) w_L}. \tag{4.4}$$

Let f' (resp. f'') be the conductor of the primitive character χ (resp. ψ). We have

$$f' = \begin{cases} P_p P_r P_q & \text{if} \quad \chi = \chi_3, \\ P_p P_r \bar{P}_q & \text{if} \quad \chi = \chi_3' \end{cases}$$

and $\operatorname{cond}(\chi\psi) = f'f''$. Thus L/K ramifies over odd primes of K, and L is not Galois over Q. Since $\chi\psi$ is an odd character, L is totally complex, with $r(L) = r(K) = 1$. It follows that $w_L = w_K = 2$, the units of L and K are equal, and $\operatorname{Reg}(L) = 2\operatorname{Reg}(K)$. Now (4.4) becomes

$$L(0, \chi\psi) = 2h_L/h_K. \tag{4.5}$$

Let ∂ be the number of prime ideals dividing f''. The class number of K is odd, and L/K is quadratic and ramified over P_∞ and $\bar{P}_\infty$ and over the $3+\partial$ prime divisors of $\operatorname{cond}(\chi\psi) = f'f''$. Since $\mathcal{O}_K^*$ has two generators, it follows from genus theory that $2^{5+\partial}/2^2 = 2^{3+\partial}$ divides the integer $2h_L/h_K = L(0, \chi\psi)$. Now by Corollary 2.2,

$$\begin{aligned}\Delta_c(0, {}_F(\chi\psi)) &= (1-\chi\psi(d))L(0, {}_F(\chi\psi)) \\ &= (1-\chi\psi(d)) \prod_{Q|F/\operatorname{cond}(\chi\psi)} (1-\chi\psi(Q))L(0, \chi\psi).\end{aligned}$$

Here $\chi\psi(d)$ and $\chi\psi(Q)$ for $Q|F/\operatorname{cond}(\chi\psi)$ are in $\{\pm 1\}$. Since there are

$6-(3+\partial)$ primes Q dividing $F/\mathrm{cond}(\chi\psi)$, and $2^{3+\partial}$ divides $L(0,\chi\psi)\in \boldsymbol{Z}$, we conclude that 2^7 divides $\Delta_c(0, {}_F(\chi\psi))\in \boldsymbol{Z}$. This proves Lemma 4.5.

IV. 3. $\chi\in\{\chi_1, \mu_q\ \chi_1\}$ **and** $\psi\in\{\psi_1, \psi_l', \psi_l'', \psi_l\}$.

These terms contribute nothing to $\Delta_c(0,h)$. For later use, we prove a slightly more general result.

LEMMA 4.6. *Suppose that* $\chi\in\{\chi_1, \mu_q\ \chi_1\}$ *and* $\psi\in\{\psi_1, \psi_l', \psi_l'', \psi_l\}$. *Let* $F_2|F$ *be an integral ideal divisible by q, by the conductor of the (primitive) character* $\chi\psi$, *and by at least one of the primes* P_l *or* $\bar{P}_l$. *Then* $\Delta_{c'}(0, {}_{F_2}(\chi\psi))=0$ *if* $c'\in\{c, c^{-1}\}$.

PROOF. From Corollaries 3.5 and 2.2 we have $\chi_1(c)=\mu_q(c)=-1$ and $\psi(c)=1$. Hence

$$\Delta_{c'}(0, {}_{F_2}(\chi_1\ \mu_q\ \psi))=(1-\chi_1\ \mu_q\ \psi(c'))L(0, {}_{F_2}(\chi_1\ \mu_q\ \psi))=0.$$

We are thus reduced to considering the case in which $\chi=\chi_1$. We will need the following table.

TABLE 4.7. *Some character values* $\beta(P)$.

$\beta \backslash P$	P_q	$\bar{P}_q$	P_l	$\bar{P}_l$
χ_1	-1	-1	1	1
ψ_l'	± 1	∓ 1	0	?
ψ_l''	∓ 1	± 1	?	0
ψ_l	-1	-1	0	0

To establish this table, first note that by Proposition 3.1, χ_1 corresponds to the quadratic extension $K(\sqrt{-p})/K$. Since P_q, $\bar{P}_q$, P_l and $\bar{P}_l$ are of degree 1, and $-\left(\frac{q}{p}\right)=\left(\frac{l}{p}\right)=1$, the first row in Table 4.7 is as indicated. The last row of the table is clear from $\psi_l=t_l\circ\mathrm{Norm}_{K/Q}$, where $t_l(q)=\left(\frac{l}{q}\right)=-1$ by assumption. Now $\psi_l=\psi_l'\circ\psi_l''$, and $\mathrm{Gal}(K/\boldsymbol{Q})$ permutes the elements of $\{\psi_l', \psi_l''\}$, $\{P_q, \bar{P}_q\}$ and $\{P_l, \bar{P}_l\}$. Hence the middle two rows of Table 4.7 follow from the last one together with the fact that ψ_l'

(resp. ϕ_l'') has conductor P_l (resp. $\bar{P}_l$).

From Table 4.7 we see that if $\phi\in\{\phi_l', \phi_l'', \phi_l\}$ then there is a prime $P\in\{P_q, \bar{P}_q\}$ so $1-\chi_1\ \phi(P)=0$. Hence since $q\mathcal{O}_K=P_q\bar{P}_q$ divides F_2 by assumption, we conclude that $L(0, {}_{F_2}(\chi_1\ \phi))=0$ for these ϕ, and hence also that $\Delta_{c'}(0, {}_{F_2}(\chi_1\ \phi))=0$ in view of Corollary 2.2. The only remaining case is when $\chi=\chi_1$ and $\phi=\phi_1$. Then if $P\in\{P_l, \bar{P}_l\}$ we have $\chi_1\phi_1(P)=\chi_1(P)=1$ by Table 4.7. Since one of P_l or $\bar{P}_l$ divides F_2 by assumption, it follows that $L(0, {}_{F_2}(\chi_1\ \phi_1))$ and $\Delta_{c'}(0, {}_{F_2}(\chi_1\ \phi_1))$ are 0.

IV. 4. $\chi\in\{\chi_q, \chi_q^{-1}\}$ **and** $\phi\in\{\phi_1, \phi_l', \phi_l'', \phi_l\}$.

These terms contribute nothing to $\Delta_c(0, h)$ mod $64\,Z_2[i]$ because of

PROPOSITION 4.8.

$$\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\ \sum_{\phi\in\{\phi_1,\phi_l',\phi_l'',\phi_l\}} \Delta_c(0, {}_F(\chi\phi))\equiv 0 \qquad mod\ 128\,Z_2[i].$$

The following lemma will be used also in Section IV. 5.

LEMMA 4.9. *The number* $L(0, \chi_q)$ *lies in* $4\,Z_2[i]$ *but not in* $4(1+i)Z_2[i]$.

PROOF. First suppose $q\equiv 1$ mod 8. By Proposition 3.2 we have ${}_{pq}\chi_q=(\gamma t_p)\circ\mathrm{Norm}_{K/Q}$, where γ is a primitive even quartic Dirichlet character of conductor q and t_p is the Legendre symbol mod p. From this and $K=Q(\sqrt{pr})$ we have $\mathrm{Ind}_{K/Q}\chi_q=\gamma t_p+\gamma t_r$, where γt_p and γt_r are identified with primitive Dirichlet characters of conductors qp and qr, respectively. Hence $L(0, \chi_q)=L(0, \gamma t_p)\cdot L(0, \gamma t_r)$. If we show $L(0, \gamma t_p)$ is in $2\,Z_2[i]$ but not in $2(1+i)Z_2[i]$, then the same will be true of $L(0, \gamma t_r)$ by symmetry, and Lemma 4.9 will be proved when $q\equiv 1$ mod 8.

It is well known (cf. [4, Section 1]) that

$$L(0, z)=\sum_{j=1}^{m} z(j)\left(\frac{1}{2}-\frac{j}{m}\right)$$

if z is a (possibly imprimitive) Dirichlet character of conductor m. We will write this as

$$L(0, z)=\partial(z)\phi(m)/2-\sum_{j=1}^{m} z(j)j/m \tag{4.6}$$

where $\partial(z)=1$ if z is the trivial character mod m and $\partial(z)=0$ otherwise, and $\phi(m)$ is Euler's phi function. From this one has the following:

If z is an odd character and m is odd, then

$$(4.7)\qquad L(0, z) = -\sum_{j=1}^{(m-1)/2} z(j)(j-(m-j))/m = \sum_{j=1}^{(m-1)/2} z(j)(m-2j)/m.$$

Consider now the function

$$\epsilon = (\gamma - {}_qt_1)\cdot(t_p + {}_pt_1) = \gamma t_p + \gamma_p t_1 - {}_{qp}t_p - {}_{qp}t_1$$

on $(Z/qp)^*$, where t_1 denotes the trivial Dirichlet character of conductor 1. The values of ϵ are in $2(1+i)Z_2[i]$. Hence (4.6) gives

$$(4.8)\qquad \begin{aligned} L(0, \epsilon) &= L(0, \gamma t_p) + L(0, \gamma_p t_1) - L(0, {}_{qp}t_p) - L(0, {}_{qp}t_1) \\ &= -\phi(qp)/2 - \sum_{j=1}^{m} \epsilon(j)j/m \\ &\equiv 0 \mod 2(1+i)Z_2[i]. \end{aligned}$$

Since $\gamma_p t_1$ and ${}_{qp}t_1$ are even characters of conductor qp, we have

$$L(0, \gamma_p t_1) = L(0, {}_{qp}t_1) = 0.$$

Hence (4.8) gives

$$(4.9)\qquad L(0, \gamma t_p) \equiv L(0, {}_{qp}t_p) \mod 2(1+i)Z_2[i].$$

Here

$$\begin{aligned} L(0, {}_{qp}t_p) &= (1 - t_p(q))L(0, t_p) \\ &= \left(1 - \left(\frac{q}{p}\right)\right) \sum_{j=1}^{(p-1)/2} t_p(j)(p-2j)/p \\ &\equiv 2 \mod 4Z_2 \end{aligned}$$

since $\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right) = -1$, t_p is an odd character of conductor p, and $p \equiv 3 \mod 4$. Now (4.9) shows $L(0, \gamma t_p)$ is in $2Z_2[i]$ but not $2(1+i)Z_2[i]$, so by our earlier remarks, we have proved Lemma 4.9 when $q \equiv 1 \mod 8$.

Now suppose that $q \equiv 5 \mod 8$. By Proposition 3.2 we have $\chi_q = \lambda \circ \mathrm{Norm}_{K/Q}$ where λ is a primitive odd Dirichlet character of conductor q. Hence in this case we have $\mathrm{Ind}_{K/Q}\chi_q = \lambda + \lambda t_{pr}$ so

$$(4.10)\qquad L(0, \chi_q) = L(0, \lambda)\cdot L(0, \lambda t_{pr}).$$

Now (4.7) gives

$$(4.11) \qquad L(0,\lambda)=\sum_{j=1}^{(q-1)/2}\lambda(j)(q-2j)/q \equiv \sum_{i=1}^{(q-1)/2}\lambda(j) \mod 2\,Z_2[i].$$

Since $\lambda(-j)=-\lambda(j)$, exactly half of the integers j in the interval $[1, (q-1)/2]$ have $\lambda(j)=\pm 1$, while $\lambda(j)=\pm i$ for the remaining j. Since $(q-1)/4$ is odd, we conclude from (4.11) that

$$(4.12) \qquad L(0,\lambda)\equiv 1+i \mod 2\,Z_2[i].$$

Consider now the function

$$\epsilon=\lambda\ {}_{pr}t_1+\lambda\ {}_{pr}t_p+\lambda\ {}_{pr}t_r+\lambda t_{pr}$$

on $(Z/prq)^*$. This function takes values in $4\,Z_2[i]$, so (4.6) gives

$$(4.13) \qquad \begin{aligned} L(0,\epsilon)&=L(0,\lambda\ {}_{pr}t_1)+L(0,\lambda\ {}_{pr}t_p)+L(0,\lambda\ {}_{pr}t_r)+L(0,\lambda t_{pr})\\ &=\sum_{j=1}^{prq}-\epsilon(j)j/(prq)\\ &\equiv 0 \mod 4\,Z_2[i]. \end{aligned}$$

Since $\lambda\ {}_{pr}t_p$ and $\lambda\ {}_{pr}t_r$ are even Dirichlet characters, we have $L(0,\lambda\ {}_{pr}t_p)= L(0,\lambda\ {}_{pr}t_r)=0$. Thus (4.13) shows

$$(4.14) \qquad L(0,\lambda t_{pr})\equiv -L(0,\lambda\ {}_{pr}t_1) \mod 4\,Z_2[i].$$

Now

$$(4.15) \qquad L(0,\lambda\ {}_{pr}t_1)=(1-\lambda(p))(1-\lambda(r))L(0,\lambda).$$

Since λ^2 is the Legendre symbol mod q, both $\lambda(p)$ and $\lambda(r)$ are fourth roots of unity. Hence (4.15) and (4.12) show $L(0,\lambda\ {}_{pr}t_1)$ lies in $(1+i)^3\,Z_2[i]$ but not in $4\,Z_2[i]$. Hence (4.14), (4.12) and (4.10) show $L(0,\chi_q)$ is in $4\,Z_2[i]$ but not $4(1+i)Z_2[i]$, so Lemma 4.9 is proved.

LEMMA 4.10. *Let $F'=P_q\bar{P}_qP_l\bar{P}_l$, and suppose $\chi\in\{\chi_q,\chi_q^{-1}\}$ and $\psi\in\{\psi_1,\psi_l',\psi_l'',\psi_l\}$. Then $\chi\psi(P_p)$ and $\chi\psi(P_r)$ are primitive fourth roots of unity, so $\chi\psi(P_pP_r)=\pm 1$. One has*

$$\Delta_c(0,{}_F(\chi\psi))=\begin{cases}2\Delta_c(0,{}_{F'}(\chi\psi)) & \text{if } \chi\psi(P_pP_r)=1,\\ -2\chi\psi(P_p)\Delta_c(0,{}_{F'}(\chi\psi)) & \text{if } \chi\psi(P_pP_r)=-1.\end{cases}$$

PROOF. By Propositions 3.1 and 3.2, the primitive character associated to $(\chi\phi)^2$ is $\mu_q=t_q\circ\mathrm{Norm}_{K/Q}$. Now $\mu_q(P_p)=t_q(p)=\left(\frac{p}{q}\right)=-1$, and similarly $\mu_q(P_r)=-1$. Hence $\chi\phi(P_p)$ and $\chi\phi(P_r)$ are primitive fourth roots of unity, so $\chi\phi(P_pP_r)=\pm1$. Since $F=F'P_pP_r$, we have

$$\Delta_c(0, {}_F(\chi\phi))=(1-\chi\phi(P_p))(1-\chi\phi(P_r))\Delta_c(0, {}_{F'}(\chi\phi))$$

so the last statement in the Lemma is clear.

LEMMA 4.11. *Define $\partial=0$ if $\chi_q(P_pP_r)=1$ and let $\partial=1$ otherwise. Let $F'=P_q\bar{P}_qP_l\bar{P}_l$ and let T be the following function on $G_{F'}$:*

$$T=\sum_{\chi\in\{\chi_q,\chi_q^{-1},\chi_1,\chi_1\mu_q\}}\ \sum_{\phi\in\{\phi_1,\phi'_l,\phi''_l,\phi_l\}} 2(-(\chi\phi)(P_p))^{\partial}{}_{F'}(\chi\phi).$$

Here $(-(\chi\phi)(P_p))^{\partial}$ is in $\{\pm1, \pm i\}$, and this number is 1 if $\partial=0$. Then $\Delta_c(0, T)$ lies in $128\,Z_2[i]$.

PROOF. As χ ranges over $\{\chi_q, \chi_q^{-1}, \chi_1, \chi_1\,\mu_q\}$ and ϕ ranges over $\{\phi_1, \phi'_l, \phi''_l, \phi_l\}$, the character ${}_{F'}(\chi\phi\chi_1)$ ranges over a group of characters of $G_{F'}$, this group having order 16. Hence the function T' which is the sum of these ${}_{F'}(\chi\phi\chi_1)$ takes values in $16\,Z$. Suppose $\alpha\in G_{F'}$. We have $T(\alpha)=2\chi_1(\alpha)^{-1}T'(\alpha)$ if $\partial=0$. If $\partial=1$ then $T(\alpha)=-2\chi_1(\beta)^{-1}T'(\beta)$, where $\beta=[P_p]\alpha$ and $[P_p]$ denotes the image of P_p in $G_{F'}$. Thus T takes values in $32\,Z_2[i]$ in all cases. Hence Lemma 4.11 follows from Theorem 2.1.

LEMMA 4.12. *$\phi'_l(P_pP_r)=\phi''_l(P_pP_r)=\pm1$ and $\phi_1(P_pP_r)=\phi_l(P_pP_r)=1$.*

PROOF. We have $\phi_l(P_pP_r)=t_l(pr)=\left(\frac{pr}{l}\right)=1$. Since $\phi_l=\phi'_l\ \phi''_l$ and ϕ'_l and ϕ''_l are quadratic, the lemma follows.

To prove Proposition 4.8, we now distinguish two cases.

Case 1. $\phi'_l(P_pP_r)=1$.

Suppose $\phi\in\{\phi_1, \phi'_l, \phi''_l, \phi_l\}$. Since $\phi'_l(P_pP_r)=1$ by hypothesis, $\phi(P_pP_r)=1$ by Proposition 3.1. Hence $\chi_q\ \phi(P_pP_r)=\chi_q^{-1}\phi(P_pP_r)=\chi_q(P_pP_r)=\pm1$. Now Lemma 4.10 gives $\Delta_c(0, {}_F(\chi\phi))=2(-\chi\phi(P_p))^{\partial}\Delta_c(0, {}_{F'}(\chi\phi))$ if $\chi\in\{\chi_q, \chi_q^{-1}\}$. Thus

$$\Big\{\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\ \sum_{\phi\in\{\phi_1,\phi'_l,\phi''_l,\phi_l\}}\Delta_c(0, {}_F(\chi\phi))\Big\}-\Delta_c(0, T)$$
$$=\sum_{\chi\in\{\chi_1,\chi_1\mu_q\}}\ \sum_{\phi\in\{\phi_1,\phi'_l,\phi''_l,\phi_l\}} -2(-\chi\phi(P_p))^{\partial}\Delta_c(0, {}_{F'}(\chi\phi)).$$

By Lemma 4.6, each of the terms in the double sum on the right hand side is 0. Since $\Delta_c(0, T) \in 128\, Z_2[i]$ by Lemma 4.11, this implies Proposition 4.8 in case 1.

Case 2. $\psi_l'(P_pP_r) = -1$.

Suppose $\chi \in \{\chi_q, \chi_q^{-1}\}$. By Proposition 3.1, $\chi\psi_l(P_pP_r) = \chi\psi_1(P_pP_r) = \chi_q(P_pP_r)$ and $\chi\psi_l'(P_pP_r) = \chi\psi_l''(P_pP_r) = -\chi_q(P_pP_r)$. Hence Lemma 4.10 gives

$$\begin{aligned}
&\Big\{ \sum_{\chi \in \{\chi_q, \chi_q^{-1}\}} \sum_{\psi \in \{\psi_1, \psi_l', \psi_l'', \psi_l\}} \Delta_c(0, {}_F(\chi\psi))\Big\} - \Delta_c(0, T) \\
&= \sum_{\chi \in \{\chi_q, \chi_q^{-1}\}} \sum_{\psi \in \{\psi_l', \psi_l''\}} (-2+4\partial)(1+\chi\psi(P_p))\Delta_c(0, {}_{F'}(\chi\psi)) \\
&\quad - \sum_{\chi \in \{\chi_1, \chi_1\mu_q\}} \sum_{\psi \in \{\psi_1, \psi_l', \psi_l'', \psi_l\}} 2(-(\chi\psi)(P_p))^{\partial}\Delta_c(0, {}_{F'}(\chi\psi)).
\end{aligned}$$

By Lemma 4.6, the second double sum on the right hand side is identically 0. Since $\Delta_c(0, T) \in 128\, Z_2[i]$ by Lemma 4.11, we conclude that

$$\begin{aligned}
(4.16)\qquad &\sum_{\chi \in \{\chi_q, \chi_q^{-1}\}} \sum_{\psi \in \{\psi_1, \psi_l', \psi_l'', \psi_l\}} \Delta_c(0, {}_F(\chi\psi)) \\
&\equiv \sum_{\chi \in \{\chi_q, \chi_q^{-1}\}} \sum_{\psi \in \{\psi_l', \psi_l''\}} (-2+4\partial)(1+\chi\psi(P_p))\Delta_c(0, {}_{F'}(\chi\psi)) \mod 128\, Z_2[i].
\end{aligned}$$

Let $\mathrm{Gal}(K/Q) = \{1, \sigma\}$. Then $\chi_q^{\sigma} = \chi_q$, $(\psi_l')^{\sigma} = \psi_l''$, $F'^{\sigma} = F'$, $P_p^{\sigma} = P_p$ and $\psi_l'(c) = \psi_l''(c)$. Thus if $\chi \in \{\chi_q, \chi_q^{-1}\}$ we have

$$\begin{aligned}
(1+\chi\psi_l'(P_p))\Delta_c(0, {}_{F'}(\chi\psi_l')) &= (1+\chi\psi_l'(P_p))(1-\chi\psi_l'(c))L(0, {}_{F'}(\chi\psi_l')) \\
&= (1+\chi\psi_l''(P_p))(1-\chi\psi_l''(c))L(0, {}_{F'}(\chi\psi_l'')) \\
&= (1+\chi\psi_l''(P_p))\Delta_c(0, {}_{F'}(\chi\psi_l'')).
\end{aligned}$$

Thus (4.16) becomes

$$\begin{aligned}
(4.17)\qquad &\sum_{\chi \in \{\chi_q, \chi_q^{-1}\}} \sum_{\psi \in \{\psi_1, \psi_l', \psi_l'', \psi_l\}} \Delta_c(0, {}_F(\chi\psi)) \\
&\equiv \sum_{\chi \in \{\chi_q, \chi_q^{-1}\}} (-4+8\partial)(1+\chi\psi_l'(P_p))\Delta_c(0, {}_{F'}(\chi\psi_l')) \mod 128\, Z_2[i].
\end{aligned}$$

Let W be the function on $G_{F'}$ defined by

$$W = \sum_{\chi \in \{\chi_q, \chi_q^{-1}, \chi_1, \mu_q\chi_1\}} \sum_{\psi \in \{\psi_1, \psi_l'\}} (1+\chi\psi(P_p))_{F'}(\chi\psi).$$

Then W takes values in $8Z$, so $\Delta_c(0, W) \in 32\, Z_2[i]$ by Theorem 2.1. Hence we may subtract $(-4+8\partial)\Delta_c(0, W)$ from the right hand side of

(4.17) to have

$$\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\phi\in\{\phi_1,\phi_l',\phi_l'',\phi_l\}}\Delta_c(0,{}_F(\chi\phi))$$
$$\equiv\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}(4-8\partial)(1+\chi\phi_1(P_p))\Delta_c(0,{}_{F'}(\chi\phi_1))$$
$$+\sum_{\chi\in\{\chi_1,\chi_1\mu_q\}}\sum_{\phi\in\{\phi_1,\phi_l'\}}(4-8\partial)(1+\chi\phi(P_p))\Delta_c(0,{}_{F'}(\chi\phi)) \mod 128\,Z_2[i]. \tag{4.18}$$

By Lemma 4.6, each of the terms in the double sum on the right hand side of (4.18) is identically 0. Since ϕ_1 is the trivial character, we may now rewrite (4.18) as

$$\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\sum_{\phi\in\{\phi_1,\phi_l',\phi_l'',\phi_l\}}\Delta_c(0,{}_F(\chi\phi))$$
$$\equiv\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}(4-8\partial)(1+\chi(P_p))\Delta_c(0,{}_{F'}\chi) \mod 128\,Z_2[i]. \tag{4.19}$$

For $\chi\in\{\chi_q,\chi_q^{-1}\}$ we have

$$\Delta_c(0,{}_{F'}\chi)=(1-\chi(c))(1-\chi(P_l))(1-\chi(\bar{P}_l))L(0,\chi).$$

Since χ_q^{-1} is the conjugate of χ_q by the non-trivial automorphism of $\mathrm{Gal}(Q(i)/Q)$, we conclude that

$$\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}(4-8\partial)(1+\chi(P_p))\Delta_c(0,{}_{F'}\chi)=\mathrm{Trace}_{Q(i)/Q}(\beta) \tag{4.20}$$

where

$$\beta=(4-8\partial)(1+\chi_q(P_p))\Delta_c(0,{}_{F'}\chi_q)$$
$$=(4-8\partial)(1+\chi_q(P_p))(1-\chi_q(c))(1-\chi_q(P_l))(1-\chi_q(\bar{P}_l))L(0,\chi_q).$$

By Corollaries 3.5 and 2.2 and Proposition 3.2, each of $\chi_q(c)$, $\chi_q(P_l)$, $\chi_q(\bar{P}_l)$ and $\chi_q(P_p)$ are primitive fourth roots of unity. Since $L(0,\chi_q)$ lies in $4\,Z_2[i]$ by Lemma 4.9, β lies in $4\cdot(1+i)^4\cdot 4\,Z_2[i]=64\,Z_2[i]$. Hence $\mathrm{Trace}_{Q(i)/Q}(\beta)$ is in $128\,Z_2[i]$, so we are done by (4.20) and (4.19).

IV. 5. $\chi\in\{\chi_2,\chi_2^{-1}\}$ and $\phi\in\{\phi_l',\phi_l''\}$.

PROPOSITION 4.13.

$$\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\sum_{\phi\in\{\phi_l',\phi_l''\}}\Delta_c(0,{}_F(\chi\phi))\equiv 64 \quad mod\ 128\,Z_2[i].$$

The proof of this proposition requires several preliminary results, and will be completed after Lemma 4.22 below.

LEMMA 4.14.

$$\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\ \sum_{\phi\in\{\phi_l',\phi_l''\}} \Delta_c(0,{}_F(\chi\phi))=2\{L(0,{}_F(\chi_2\ \phi_l'))+L(0,{}_F(\chi_2^{-1}\phi_l'))\}.$$

PROOF. Let $\mathrm{Gal}(K/Q)=\{1,\sigma\}$. We have $\chi_2^\sigma=\chi_2^{-1}$, $(\phi_l')^\sigma=\phi_l''$ and $F=F^\sigma$. Hence $L(0,{}_F(\chi_2^{\pm1}\phi_l'))=L(0,{}_F(\chi_2^{\pm1}\phi_l')^\sigma)=L(0,{}_F(\chi_2^{\mp1}\phi_l''))$. Thus since $\phi_l'(c)=\phi_l''(c)=1$ and $\chi_2(c)$ is a primitive fourth root of unity, we have

$$\begin{aligned}&\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\ \sum_{\phi\in\{\phi_l',\phi_l''\}} \Delta_c(0,{}_F(\chi\phi))\\&=(1-\chi_2(c))L(0,{}_F(\chi_2\ \phi_l'))+(1-\chi_2(c))L(0,{}_F(\chi_2\ \phi_l''))\\&\quad+(1-\chi_2^{-1}(c))L(0,{}_F(\chi_2^{-1}\phi_l'))+(1-\chi_2^{-1}(c))L(0,{}_F(\chi_2^{-1}\phi_l''))\\&=(1-\chi_2(c)+1-\chi_2^{-1}(c))(L(0,{}_F(\chi_2\ \phi_l'))+L(0,{}_F(\chi_2^{-1}\phi_l')))\\&=2\{L(0,{}_F(\chi_2\ \phi_l'))+L(0,{}_F(\chi_2^{-1}\phi_l'))\}.\end{aligned}$$

COROLLARY 4.15. *There is a* $c'\in\{c,c^{-1}\}$ *so* $\chi_2^{\pm1}\phi_l'(c')=\chi_2^{\pm1}\phi_l'(\bar{P}_l)$. *Let* $F''=P_pP_r\bar{P}_qP_qP_l=F/\bar{P}_l$. *Then*

$$\begin{aligned}&\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\ \sum_{\phi\in\{\phi_l',\phi_l''\}} \Delta_c(0,{}_F(\chi\phi))\\&=2\{\Delta_{c'}(0,{}_{F''}(\chi_2\ \phi_l'))+\Delta_{c'}(0,{}_{F''}(\chi_2^{-1}\phi_l'))\}.\end{aligned}\tag{4.21}$$

PROOF. By Proposition 3.2 and Corollaries 3.5 and 2.2, $\chi_2(\bar{P}_l)$ and $\chi_2(c)$ are primitive fourth roots of unity. Since ϕ_l' is quadratic character, we can find a c' as in the Lemma. Since $\chi_2^{\pm1}\phi_l'$ is a primitive character of conductor F'', we have

$$\begin{aligned}L(0,{}_F(\chi_2^{\pm1}\phi_l'))&=(1-\chi_2^{\pm1}\phi_l'(\bar{P}_l))L(0,{}_{F''}(\chi_2^{\pm1}\phi_l'))\\&=(1-\chi_2^{\pm1}\phi_l'(c'))L(0,{}_{F''}(\chi_2^{\pm1}\phi_l'))\\&=\Delta_{c'}(0,{}_{F''}(\chi_2^{\pm1}\phi_l')).\end{aligned}$$

Hence (4.21) follows from Lemma 4.14.

LEMMA 4.16. *Let* F'' *be as in Corollary* 4.15, *and let* S *be the set of characters defined in Proposition* 3.2. *Define* T' *to be the following function on* $G_{F''}$:

$$T'=2\sum_{\chi\in S}\ \sum_{\phi\in\{\phi_1,\phi_l'\}} {}_{F''}(\chi\phi).$$

Then $\Delta_{c'}(0,T')\equiv0 \mod 128\,Z_2[i]$.

PROOF. By Propositions 3.2 and 3.1, ${}_{F''}(\chi_1)T'/2$ is a sum of the elements of a group of characters of $G_{F''}$, this group having order $2\sharp S=16$. Hence T' takes values in $32\,Z_2[i]$, so $\Delta_{c'}(0, T')\equiv 0 \bmod 128\,Z_2[i]$ by Theorem 2.1.

COROLLARY 4.17.

$$\begin{aligned}(4.22)\qquad &\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\ \sum_{\phi\in\{\phi_l',\phi_l''\}} \Delta_c(0, {}_F(\chi\phi))\\ &\equiv -2\{\Delta_{c'}(0, {}_{F''}(\chi_2))+\Delta_{c'}(0, {}_{F''}(\chi_2^{-1}))\}-2\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\ \sum_{\phi\in\{\phi_1,\phi_l'\}} \Delta_{c'}(0, {}_{F''}(\chi\phi))\\ &\qquad\qquad \mathit{mod}\ 128\,Z_2[i].\end{aligned}$$

PROOF. Subtract $\Delta_{c'}(0, T')\in 128\,Z_2[i]$ from the right hand side of (4.21) in Corollary 4.15. Since ${}_{F''}(\chi_2^{\pm 1}\phi_1)={}_{F''}(\chi_2^{\pm 1})$, one gets the terms on the right hand side of (4.22) in addition to the following terms:

$$(4.23)\qquad -2\sum_{\chi\in\{\chi_3,\chi_3'\}}\ \sum_{\phi\in\{\phi_1,\phi_l'\}} \Delta_{c'}(0, {}_{F''}(\chi\phi))$$

$$(4.24)\qquad -2\sum_{\chi\in\{\chi_1,\chi_1\mu_q\}}\ \sum_{\phi\in\{\phi_1,\phi_l'\}} \Delta_{c'}(0, {}_{F''}(\chi\phi)).$$

One shows that (4.23) lies in $128\,Z_2[i]$ by an argument very similar to that of Lemma 4.5. (The fact that one is now working with conductor $F''=F/\bar{P}_l$ rather than F is compensated by the fact that one is proving a congruence which is weaker by one power of two than that in Lemma 4.5.) By Lemma 4.6, the sum in (4.24) is identically zero.

LEMMA 4.18.

$$2\{\Delta_{c'}(0, {}_{F''}(\chi_2))+\Delta_{c'}(0, {}_{F''}(\chi_2^{-1}))\}\equiv 32(1+\phi_l'(\bar{P}_l))\quad \mathit{mod}\ 128\,Z_2[i].$$

PROOF. Since $\mathrm{Ind}_{K/Q}\chi_2=\mathrm{Ind}_{K/Q}\chi_2^{-1}$ has rational character, we have $L(0, \chi_2)=L(0, \chi_2^{-1})\in \boldsymbol{Q}$. The conductor of χ_2 is $P_q\bar{P}_qP_pP_r=F''/P_l$. Hence

$$\begin{aligned}(4.25)\qquad &\Delta_{c'}(0, {}_{F''}(\chi_2))+\Delta_{c'}(0, {}_{F''}(\chi_2^{-1}))\\ &=\{(1-\chi_2(c'))(1-\chi_2(P_l))+(1-\chi_2^{-1}(c'))(1-\chi_2^{-1}(P_l))\}L(0, \chi_2).\end{aligned}$$

Since $c'\in\{c, c^{-1}\}$ we have $\phi_l'(c')=1$. Hence $\chi_2(c')=\chi_2\phi_l'(c')=\chi_2\phi_l'(\bar{P}_l)$. Now $\chi_2(P_l)=\chi_2^{-1}(\bar{P}_l)$ is a primitive fourth root of unity by Proposition 3.2, and $\phi_l'(\bar{P}_l)=\pm 1$. On using these facts to simplify the right hand side of (4.25) we get

$$(4.26)\qquad \begin{aligned}&\Delta_{c'}(0, {}_{F''}(\chi_2))+\Delta_{c'}(0, {}_{F''}(\chi_2^{-1}))\\ &=\{1-\chi_2\ \psi_l'(\bar{P}_l)-\chi_2(P_l)+\chi_2(\bar{P}_lP_l)\psi_l'(\bar{P}_l)\\ &\quad +1-\chi_2^{-1}\psi_l'(\bar{P}_l)-\chi_2^{-1}(P_l)+\chi_2^{-1}(\bar{P}_lP_l)\psi_l'(\bar{P}_l)\}L(0, \chi_2)\\ &=\{2+2\psi_l'(\bar{P}_l)\}L(0, \chi_2).\end{aligned}$$

By Lemma 4.1, $L(0, \chi_2)$ is a rational integer which is exactly divisible by 8, so Lemma 4.18 follows from (4.26).

PROPOSITION 4.19.

$$2 \sum_{\chi\in\{\chi_q,\chi_q^{-1}\}} \sum_{\psi\in\{\psi_1,\psi_l'\}} \Delta_{c'}(0, {}_{F''}(\chi\psi))\equiv 32(1-\psi_l'(P_pP_r)) \quad mod\ 128\, Z_2[i].$$

To prove this proposition, we will need the following two lemmas, which are proved in exactly the same way as Lemmas 4.10 and 4.11.

LEMMA 4.20. *Let $F_3=P_q\bar{P}_qP_l$, and suppose $\chi\in\{\chi_q, \chi_q^{-1}\}$ and $\psi\in\{\psi_1, \psi_l'\}$. Then $\chi\psi(P_p)$ and $\chi\psi(P_r)$ are primitive fourth roots of unity, so $\chi\psi(P_pP_r)=\pm 1$. One has*

$$\Delta_{c'}(0, {}_{F''}(\chi\psi))=\begin{cases}2\Delta_{c'}(0, {}_{F_3}(\chi\psi)) & \text{if } \chi\psi(P_pP_r)=1,\\ -2\chi\psi(P_p)\Delta_{c'}(0, {}_{F_3}(\chi\psi)) & \text{if } \chi\psi(P_pP_r)=-1.\end{cases}$$

LEMMA 4.21. *Define $\partial'=0$ if $\chi_q\ \psi_l'(P_pP_r)=1$ and let $\partial'=1$ otherwise. Let $F_3=P_q\bar{P}_qP_l$ as in Lemma 4.20, and let T'' be the following function on G_{F_3}:*

$$T''=\sum_{\chi\in\{\chi_q,\chi_q^{-1},\chi_1,\chi_1\mu_q\}} \sum_{\psi\in\{\psi_1,\psi_l'\}} 4(-(\chi\psi)(P_p))^{\partial'}{}_{F_3}(\chi\psi).$$

Then $\Delta_{c'}(0, T'')$ lies in $128\, Z_2[i]$.

To prove Proposition 4.19, we now distinguish two cases.

Case 1. $\psi_l'(P_pP_r)=1$

Suppose $\chi\in\{\chi_q, \chi_q^{-1}\}$ and $\psi\in\{\psi_1, \psi_l'\}$. In Case 1, $\chi\psi(P_pP_r)=\chi_q(P_pP_r)=\pm 1$. Hence Lemma 4.20 gives $\Delta_{c'}(0, {}_{F''}(\chi\psi))=2(-\chi\psi(P_p))^{\partial'}\Delta_{c'}(0, {}_{F_3}(\chi\psi))$. Therefore

$$\begin{aligned}&\{2 \sum_{\chi\in\{\chi_q,\chi_q^{-1}\}} \sum_{\psi\in\{\psi_1,\psi_l'\}} \Delta_{c'}(0, {}_{F''}(\chi\psi))\}-\Delta_{c'}(0, T'')\\ &= \sum_{\chi\in\{\chi_1,\chi_1\mu_q\}} \sum_{\psi\in\{\psi_1,\psi_l'\}} -4(-(\chi\psi)(P_p))^{\partial'}\Delta_{c'}(0, {}_{F_3}(\chi\psi)).\end{aligned}$$

By Lemma 4.6, each of the terms in the double sum on the right hand side is 0. Since $\Delta_{c'}(0, T'')\in 128\,Z_2[i]$ by Lemma 4.21, this implies Proposition 4.19 in Case 1.

Case 2. $\phi_l'(P_pP_r)=-1$.

Suppose $\chi\in\{\chi_q, \chi_q^{-1}\}$. In Case 2, $\chi\phi_1(P_pP_r)=\chi_q(P_pP_r)$ and $\chi\phi_l'(P_pP_r)=-\chi_q(P_pP_r)$. Hence Lemma 4.20 gives

$$\Delta_{c'}(0, {}_{F''}(\chi\phi_l'))=2(-\chi\phi_l'(P_p))^{\partial'}\Delta_{c'}(0, {}_{F_3}(\chi\phi_l'))$$

and

$$\Delta_{c'}(0, {}_{F''}(\chi\phi_1))=2(-\chi\phi_1(P_p))^{\partial'+1}\Delta_{c'}(0, {}_{F_3}(\chi\phi_1)).$$

Thus

$$\begin{aligned}\{2 &\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\ \sum_{\phi\in\{\phi_1,\phi_l'\}}\Delta_{c'}(0, {}_{F''}(\chi\phi))\}-\Delta_{c'}(0, T'')\\ &=\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}(-4+8\partial')(1+\chi\phi_1(P_p))\Delta_{c'}(0, {}_{F_3}(\chi\phi_1))\\ &\quad-\sum_{\chi\in\{\chi_1,\chi_1\mu_q\}}\ \sum_{\phi\in\{\phi_1,\phi_l'\}}4(-(\chi\phi)(P_p)^{\partial'})\Delta_{c'}(0, {}_{F_3}(\chi\phi)).\end{aligned}$$

By Lemma 4.6, the second double sum on the right hand side is identically 0. Since $\Delta_{c'}(0, T'')\in 128\,Z_2[i]$ by Lemma 4.21, and ϕ_1 is the trivial character, we conclude that

$$\begin{aligned}(4.27)\qquad 2&\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}\ \sum_{\phi\in\{\phi_1,\phi_l'\}}\Delta_{c'}(0, {}_{F''}(\chi\phi))\\ &\equiv\sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}(-4+8\partial')(1+\chi(P_p))\Delta_{c'}(0, {}_{F_3}(\chi))\mod 128\,Z_2[i].\end{aligned}$$

Since χ_q^{-1} is the complex conjugate of χ_q, we have

$$(4.28)\qquad \sum_{\chi\in\{\chi_q,\chi_q^{-1}\}}(-4+8\partial')(1+\chi(P_p))\Delta_{c'}(0, {}_{F_3}(\chi))=\mathrm{Trace}_{Q(i)/Q}(\beta')$$

where

$$\begin{aligned}\beta'&=(-4+8\partial')(1+\chi_q(P_p))\Delta_{c'}(0, {}_{F_3}(\chi_q))\\ &=(-4+8\partial')(1+\chi_q(P_p))(1-\chi_q(c'))(1-\chi_q(P_l))L(0, \chi_q).\end{aligned}$$

Each of $\chi_q(P_p)$, $\chi_q(c')$ and $\chi_q(P_l)$ are primitive fourth roots of unity, and $L(0, \chi_q)$ lies in $4\,Z_2[i]$ but not in $4(1+i)Z_2[i]$ by Lemma 4.9. Hence β' lies in $4(1+i)^3 4\,Z_2[i]=32(1+i)Z_2[i]$ but not in $64\,Z_2[i]$. It follows that

$\mathrm{Trace}_{Q(i)/Q}(\beta')\equiv 64 \bmod 128\,Z_2$. Hence (4.27) and (4.28) show Proposition 4.19 in Case 2.

LEMMA 4.22. $\phi_l'(P_pP_r)=\phi_l'(\bar{P}_l)$.

PROOF. (Compare [3, p. 107-110].) By Proposition 3.1, $\{\phi_1, \phi_l', \phi_l'', \phi_l\}$ is the group of characters of $G_l^+/(G_l^+)^2$ when G_l^+ is the narrow ray class group of K of conductor $l\mathcal{O}_K=P_l\bar{P}_l$. Let L be the biquadratic extension of K which corresponds by class field theory to $G_l^+/(G_l^+)^2$. Then L is a Galois, non-abelian extension of Q of order 8. Since $\mathrm{Gal}(L/K)=(Z/2)\oplus(Z/2)$, $\mathrm{Gal}(L/Q)$ must be dihedral. The biquadratic subfield of L must be $Q(\sqrt{pr}, \sqrt{l})$ since $l\equiv 1 \bmod 4$. The places of Q ramified in L are those determined by p, r and l, and the ramification degree of each of these places is two. If $L/Q(\sqrt{l})$ were cyclic, then all the inertia groups of $\mathrm{Gal}(L/Q(\sqrt{l}))$ would be contained in $\mathrm{Gal}(L/Q(\sqrt{pr}, \sqrt{l}))$. But then $Q(\sqrt{pr}, \sqrt{l})/Q(\sqrt{l})$ would be unramified, which is not true. Hence $L/Q(\sqrt{l})$ is biquadratic, and so $L/Q(\sqrt{prl})$ must be cyclic and unramified.

Suppose $t=p$, r or l, and let $f(t)$ be the residue field degree of a place of L over t. If $t=p$ or r, then since P_t is fixed by $\mathrm{Gal}(K/Q)$ and has residue field degree 1, $f(t)$ equals 1 if $\phi_l'(P_t)=1$ and otherwise $f(t)=2$ and $\phi_l'(P_t)=-1$. If $t=l$, then $f(l)=1$ if $\phi_l'(\bar{P}_l)=1$ and otherwise $f(l)=2$ and $\phi_l'(\bar{P}_l)=-1$. Thus the equality $\phi_l'(P_pP_r)=\phi_l'(\bar{P}_l)$ which we wish to show is equivalent to

$$f(p)+f(r)+f(l)\equiv 1 \bmod 2. \tag{4.29}$$

The extension $L/Q(\sqrt{prl})$ is cyclic, quartic and unramified, and hence $\mathrm{Gal}(L/Q(\sqrt{prl}))$ is identified via the Artin map with a quotient $\mathrm{Cl}(Q(\sqrt{prl}))/H$ of the ideal class group $\mathrm{Cl}(Q(\sqrt{prl}))$ of $Q(\sqrt{prl})$. Let Q_t be the (ramified) prime of $Q(\sqrt{prl})$ over t for $t=p$, r or l. Since the primes over Q_t in L have residue field degree 1 or 2, we conclude that $f(t)=1$ if the ideal class $[Q_t]$ of Q_t lies in H, and otherwise $f(t)=2$ and $[Q_t]$ is not in H. Because $(\sqrt{prl})\mathcal{O}_{Q(\sqrt{prl})}$ is a principal ideal, we have $[Q_p]\cdot[Q_r]\cdot[Q_l]=1$, and this implies (4.29).

PROOF OF PROPOSITION 4.29. By Corollary 4.17, Lemma 4.18 and Proposition 4.19 we have

$$\sum_{\chi\in\{\chi_2,\chi_2^{-1}\}}\ \sum_{\phi\in\{\phi_l',\phi_l''\}} \Delta_c(0, {}_F(\chi\phi))$$

$$\equiv -32(1+\psi_l'(\bar{P}_l)+1-\psi_l'(P_pP_r)) \mod 128\,Z_2[i].$$

Hence Proposition 4.13 follows from Lemma 4.22.

IV. 6. Completion of the proof of Theorem 1.1.

Let h be as in Definition 4.2, so $\Delta_c(0, h)\in 64\,Z_2$ by Corollary 4.4. We compute $\Delta_c(0, h)$ via equation (4.3), Lemma 4.5, Lemma 4.6, Proposition 4.8 and Proposition 4.13. This leads to the congruence

$$L(0, V\otimes t_l)+2L(0, V)+32\equiv 0 \mod 64\,Z_2[i]. \tag{4.30}$$

By Lemma 4.1, $L(0, V\otimes t_l)$ and $L(0, V)$ are rational integers which are exactly divisible by 16 and 8, respectively. Hence (4.30) implies

$$L(0, V\otimes t_l)\equiv 2L(0, V) \mod 64\,Z$$

as claimed.

IV. 7. An example.

When $(p, r, q)=(3, 7, 5)$, one has $L(0, V)=-8$, so Theorem 1.1 gives $L(0, V\otimes t_l)\equiv -16 \mod 64\,Z$ for $l\in\mathcal{S}$. This example was treated in [3, Thm. 4.3.11] using Shintani's formulas.

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