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*On the pseudodifferential operators with real analytic symbols and their applications**)

By TRAN Duc Van

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In [2] Dubinskii presented a concept of pseudodifferential operators with constant symbols analytic in an arbitrary domain $G\subset \boldsymbol{R}^N$ and gave various applications to mathematical physics. The basis of these applications is a nonformal algebra of differential operators of infinite order (DOIO) as operators acting invariantly and continuously in the corresponding Sobolev spaces of infinite order.

The need to study pseudodifferential operators with analytic symbol having singularities arises even in the simplest problems of mathematical physics [2]. To illustrate the idea on the use of pseudo-differential operators with analytic symbols (ΨDOAS) we consider the Dirichlet problem for the Laplace equation

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial x^2}=0, \qquad t>0,\ x\in \boldsymbol{R}^1, \tag{0.1}$$

$$u(0, x)=\varphi(x), \qquad |u(t, x)|\leq M<+\infty, \tag{0.2}$$

where $\varphi(x)\in L_2(\boldsymbol{R}^1)$.

For solving this problem we put $D\Longleftrightarrow\frac{1}{i}\partial/\partial x$ and regarding D as a real parameter, we find the solution of the ordinary differential equation

$$\frac{d^2U}{dt^2}-D^2U=0, \qquad U(0, D)=1, \qquad |U(t, D)|\leq M.$$

It is easy to see that $U(t, D)=\exp(-t|D|)$ and, consequently, a formal solution of the problem (0.1)-(0.2) is presented by the formula

$$u(t, x)=U(t, D)\varphi(x)=\exp(-t|D|)\varphi(x), \tag{0.3}$$

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where D now stands for the operator of differentiation, i.e. $D=\frac{1}{i}\partial/\partial x$.

The operator $\exp(-t|D|)$ has the analytic symbol $\exp(-t|\xi|)$ which has a singularity at $\xi=0$. At that time, the formula (0.3) has meaning if we put

$$|D|\varphi(x)=D\varphi_+(x)-D\varphi_-(x),$$

where

$$\varphi_\pm(x)=\frac{1}{2\pi}\int_{R^1_\pm}\hat{\varphi}(\xi)e^{ix\xi}d\xi, \quad R^1_+=\{x>0\}, \quad R^1_-=\{x<0\}, \tag{0.4}$$

and $\hat{\varphi}$ denotes the Fourier transform of φ. Then the operator $\exp(-t|D|)$ acts on $\varphi(x)$ by the formula

$$\exp(-t|D|)\varphi(x)=\exp(-tD)\varphi_+(x)+\exp(tD)\varphi_-(x),$$

that is,

$$\exp(-t|D|)\varphi(x)=\varphi_+(x+it)+\varphi_-(x-it).$$

By the same token the solution of the problem (0.1)-(0.2) is written in the form

$$u(t, x)=\varphi_+(x+it)+\varphi_-(x-it),$$

where $\varphi_\pm(x)$ are defined by (0.4). Substituting these expressions (0.4) in the last formula we obtain the classical Poisson integral

$$u(t, x)=\frac{t}{\pi}\int_{R^1}\frac{\varphi(y)dy}{t^2+(x-y)^2}.$$

Thus, the solvability of (0.1)-(0.2) is established.

We note that the domain of analyticity of the symbol $\exp(-t|\xi|)$ consists of two components R^1_+ and R^1_-, which correspond two function spaces, where the operator $\exp(-t|D|)$ acts invariantly as an operator of translation on $\pm it$, that is as a differential operator of infinite order. Thus, in local the operator $\exp(-t|D|)$ is a differential operator of infinite order acting in a corresponding suitable function space and, in this case, DOIO is an instrument of investigation. But in the solution of problems, that are ill-posed in the sense of Hadamard-Petrowskii, the use of the ΨDOASs constitutes the very essence of the approach. It means that the problems which are incorrect in the classical sense are correct in the function spaces, where the corresponding ΨDOASs act invariantly and continuously. To illustrate this method we consider the Cauchy problem for the heat inverse equation

$$\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}=0, \qquad u(0, x)=\varphi(x), \ t>0, \ x\in \boldsymbol{R}^1. \tag{0.5}$$

We can find that

$$u(t, x)=\exp(-t\partial^2/\partial x^2)\varphi(x).$$

For any $t>0$ the operator $\exp(-t\partial^2/\partial x^2)$ is a ΨDO, the symbol of which is $a(t, \xi)=\exp(t\xi^2)$, $\xi\in \boldsymbol{R}^1$. According to Theorem 5.2 below the operator $\exp(-t\partial^2/\partial x^2)$ gives isomorphisms

$$\exp(-t\partial^2/\partial x^2) : W_{R^1}^{\pm\infty}(\boldsymbol{R}^1) \longrightarrow W_{R^1}^{\pm\infty}(\boldsymbol{R}^1),$$

where

$$W_{R^1}^{+\infty}(\boldsymbol{R}^1)=\{f(x), \hat{f}(\xi)\in A'(\boldsymbol{R}^1), \text{ supp} \hat{f} \text{ is compact}\}$$
$$W_{R^1}^{-\infty}(\boldsymbol{R}^1)=(W_{R^1}^{+\infty}(\boldsymbol{R}^1))^*,$$

(see the exact definitions of $W_{R^1}^{\pm\infty}(\boldsymbol{R}^1)$ in §2 below). We remark that for any initial function $\varphi(x)\in W_{R^1}^{\pm\infty}(\boldsymbol{R}^1)$ there exists one and only one solution of the Cauchy problem (0.5) in the sense of $W_{R^1}^{\pm\infty}(\boldsymbol{R}^1)$ (see §7). After some simple calculations one can get that for any $\varphi(x)\in W_{R^1}^{+\infty}(\boldsymbol{R}^1)$ the solution of the problem (0.5) is given by the formula

$$u(t, x)=\frac{1}{2\sqrt{\pi t}}\int_{R^1} e^{(i\xi)^2/4t}\varphi(x+i\xi)d\xi, \qquad t>0.$$

Thus, the technique of the ΨDOAS and the introduction of the spaces $W_G^{\pm\infty}(\boldsymbol{R}^N)$ (see §2.5) are the core of described above method: problems which are incorrect in the classical sense are correct in these spaces.

In this paper, we shall give new spaces of test functions and generalized functions $W_G^{\pm\infty}(\boldsymbol{R}^N)$ whose most important property is invariance of the basic space $W_G^{+\infty}(\boldsymbol{R}^N)$ (and hence also $W_G^{-\infty}(\boldsymbol{R}^N)$) under the action of a ΨDO having symbol analytic in G and apply to give some applications to problems of PDEs. In §1 we assemble some general qualitative properties of the analytic functionals. In §2-4 we investigate the test function space $W_R^{+\infty}(\boldsymbol{R}^N)$ and the space of generalized functions $W_R^{-\infty}(\boldsymbol{R}^N)$ in a neighborhood of zero and the properties of DOIOs. The structure theorem for generalized functions is established: Every generalized function $h(x)\in W_R^{-\infty}(\boldsymbol{R}^N)$ can be represented in the form

$$h(x)=A(D)(2\sqrt{\pi})^{-N}\exp(-x^2/4),$$

where $A(D)$ is a pseudodifferential operator with symbol $A(\xi)=\hat{h}(\xi)$. $\exp(\xi^2)$ analytic in S_R. § 5 is devoted to the construction of an algebra of the ΨDOASs associated to an arbitrary domain $G\subset \boldsymbol{R}^N_\xi$. In the sections 6-8 we shall give some applications to the Cauchy problems and boundary value problems. In particular, within the framework of the theory of generalized functions $W^{\pm\infty}_G(\boldsymbol{R}^N)$ any partial differential operator with constant coefficients also has a fundamental solution of the Cauchy problem.

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§ 1. The space of analytic functionals.

Let K be a compact set of $\boldsymbol{R}^N$ whose point is denoted by $x=(x_1, \cdots, x_N)$. Let $D^\alpha=D^{\alpha_1}\cdots D^{\alpha_N}$, $D_j=-i\partial/\partial x_j$, $j=1, \cdots, N$, $\alpha=(\alpha_1, \cdots, \alpha_N)$, $|\alpha|=\alpha_1+\cdots+\alpha_N$. We denote by $A[K]$ the space of all real analytic functions in some neighborhood of K. That is, if $\varphi\in A[K]$, then φ is a C^∞-function in a neighborhood of K and there are positive constants C and h such that

$$\sup_{x\in K}|D^\alpha\varphi(x)|\leq Ch^{|\alpha|}\alpha!. \tag{1.1}$$

We say $\varphi_j\to 0$ in $A[K]$ as $j\to\infty$ if there is a constant $h>0$ such that

$$\sup_{\substack{x\in K\\ \alpha}}\frac{|D^\alpha\varphi_j(x)|}{h^{|\alpha|}\alpha!}\longrightarrow 0 \qquad \text{as } j\to\infty. \tag{1.2}$$

DEFINITION 1.1 (cf. [9, 3, 10]). We denote by $A'[K]$ the strong dual space of $A[K]$ and call its elements *analytic functionals carried by* K.

THEOREM 1.2 (Paley-Wiener theorem, cf. [3, 10]). *Let* $u\in A'[K]$ *then the Fourier-Laplace transform*

$$\hat{u}(\zeta)=u(\exp(-i\langle\cdot, \zeta\rangle))$$

is an entire analytic function such that for every $\varepsilon>0$

$$|\hat{u}(\zeta)|\leq C_\varepsilon\exp(L|\eta|+\varepsilon|\xi|), \qquad \zeta=\xi+i\eta\in \boldsymbol{C}^N, \tag{1.3}$$

where $L=\sup_{x\in K}|x|$.

Conversely, if $F(\zeta)$ *is an entire function satisfying estimate* (1.3) *with some constant* $L\geq 0$, *then* $F(\zeta)$ *is the Fourier-Laplace transform of a unique element in* $A'[\bar{S}_L]$, *where* $\bar{S}_L=\{x\in \boldsymbol{R}^N,\ |x|\leq L\}$.

We can consider $A'[K_1]\subset A'[K_2]$ if $K_1\subset K_2$ and set

$$A'(\boldsymbol{R}^N)=\bigcup_{K\subset \boldsymbol{R}^N} A'[K].$$

Then we have the conclusion

$$\mathcal{E}'(\boldsymbol{R}^N)\subset\mathcal{E}^{\{M_p\}'}(\boldsymbol{R}^N)\subset\mathcal{E}^{(M_p)'}(\boldsymbol{R}^N)\subset A'(\boldsymbol{R}^N), \tag{1.4}$$

where $\mathcal{E}'(\boldsymbol{R}^N)$ is the space of tempered distributions, $\mathcal{E}^{\{M_p\}'}(\boldsymbol{R}^N)$. $(\mathcal{E}^{(M_p)'}(\boldsymbol{R}^N))$ is the space of ultra-distributions of Roumieu type (of Beurling type) of class M_p (for the theory of ultradistributions we refer the reader to [6, 1]).

REMARK 1.3. In all papers on ultradistributions (see, for example, [6, 4, 1]), the Denjoy-Carleman class of several variables is defined as

$$\sup|D^\alpha f|\leq C\cdot h^{|\alpha|}M_{|\alpha|},\qquad |\alpha|=0, 1, \cdots.$$

But it is a natural way to define the Denjoy-Carleman class of several variables as

$$\sup|D^\alpha f|\leq C\cdot h^{|\alpha|}M_\alpha,\qquad |\alpha|=0, 1, \cdots.$$

(In the case of class of analytic functions we have the estimate $\sup|D^\alpha f|\leq C\cdot h^{|\alpha|}\alpha!$). Then we may consider the space of ultradistributions $\mathcal{E}^{\{M_\alpha\}'}(\boldsymbol{R}^N)$ of Roumieu type and $\mathcal{E}^{(M_\alpha)'}(\boldsymbol{R}^N)$ of Beurling type respectively. Nevertheless, the main results of [6, 4, 1] remain valid for $\mathcal{E}^{\{M_\alpha\}'}(\boldsymbol{R}^N)$ and $\mathcal{E}^{(M_\alpha)'}(\boldsymbol{R}^N)$ by virtue of Lelong's theorem [8]: The class $C(M_\alpha)$ of functions of N real variables is quasianalytic if and only if the class $C(M_p)$ of functions of a single real variable is quasianalytic, where

$$M_p=\min_{|\alpha|=p} M_\alpha,\qquad p=0, 1, \cdots.$$

THEOREM 1.4 [12, 3]. *If* $u\in A'(\boldsymbol{R}^N)$ *then there is a smallest compact set* $K\subset \boldsymbol{R}^N$ *such that* $u\in A'[K]$; *it is called the support of* u.

If $K_1, \cdots, K_r$ *are compact subsets of* $\boldsymbol{R}^N$ *and* $u\in A'(K_1\cup\cdots\cup K_r)$, *then one can find* $u_j\in A'(K_j)$ *so that*

$$u=u_1+\cdots+u_r.$$

§ 2. The space of test functions in a neighborhood of zero.

Let $x\in \boldsymbol{R}_x^N$, $N\geq 1$ and $\xi\in \boldsymbol{R}_\xi^N$ be real variables, $0<R\leq\infty$ be a real number and let $S_R=\{\xi\in \boldsymbol{R}^N\colon |\xi|<R\}$. Assume that $f(x)\colon \boldsymbol{R}_x^N\to \boldsymbol{C}^1$, that is, $f(x)$ is a function defined on the whole Euclidean space $\boldsymbol{R}_x^N$ taking complex values, in general.

DEFINITION 2.1. The space of test functions $W_R^{+\infty}(\boldsymbol{R}^N)$ is the set of function $f(x)$ satisfying the following condition: f admits analytic continuation as an entire function to $\boldsymbol{C}^N$ and for each $\varepsilon>0$ there exist constants $r<R$ and C_ε such that

$$|f(x+iy)|\leq C_\varepsilon \exp(r|y|+\varepsilon|x|),\qquad x+iy=z\in \boldsymbol{C}^N. \tag{2.1}$$

From the Paley-Wiener theorem 1.2 it follows that

PROPOSITION 2.2. *A function $f(x)$ belongs to $W_R^{+\infty}(\boldsymbol{R}^N)$ if and only if its analytic continuation $f(z)$ is the Fourier-Laplace transform of an analytic functional u with support in S_R* (supp $u\subset S_R$), *that is*

$$\check{f}=u,\qquad \hat{u}=f.$$

We list here some examples of test functions. From the inclusion (1.4) it is easy to see, that they are: all functions in $H^\infty(S_R)=\{f(x)\in L_2(\boldsymbol{R}^N),\ \operatorname{supp}\hat{f}\subset S_R\}$ [2], all functions in $\mathcal{M}_{\nu p}$, $1\leq p\leq+\infty$, $\nu<R$ [11], all functions in $W^{+\infty}(S_R)$ [14], in particular, all polynomials $P(x)$, and exponential polynomials $\exp(i\lambda x)P(x)$, $\lambda\in S_R$, etc...

We introduce a topology in $W_R^{+\infty}(\boldsymbol{R}^N)$ as follows:

DEFINITION 2.3. A sequence $f_n(x)\in W_R^{+\infty}(\boldsymbol{R}^N)$ is said to converge to $f(x)\in W_R^{+\infty}(\boldsymbol{R}^N)$ if and only if: For each $\varepsilon>0$ there exists $r<R$ such that

$$\sup_{z\in CN}|f_n(z)-f(z)|\exp(-r|y|-\varepsilon|x|)\longrightarrow 0,\qquad n\to\infty.$$

Let $\{r_k\}$ be a sequence $r_k<r_{k+1}$, $k=1, 2, \cdots$ and $r_k\to R$. In this case $\bar{S}_r\subset S_R$. We define a space $W_{r_k}^{+\infty}(\boldsymbol{R}^N)$ as a set of entire functions f such that their analytic continuations on $\boldsymbol{C}^N$ admit the estimate: for each $\varepsilon>0$

$$|f(x+iy)|\leq C_\varepsilon \exp(r_k|y|+\varepsilon|x|).$$

It is not hard to see, that

$$W_R^{+\infty}(\boldsymbol{R}^N)=\varinjlim_{k\to\infty} W_{r_k}^{+\infty}(\boldsymbol{R}^N).$$

By virtue of the property of inductive limit [7] and the Paley-Wiener theorem 1.2 we have

PROPOSITION 2.4. *The sequence* $\{f_n\}$ *converges to* $f(x)$ *in* $W_R^{+\infty}(\boldsymbol{R}^N)$ *if and only if there exists a compact* $K\subset S_R$ *such that*

$$\check{f}_n \longrightarrow \check{f} \qquad in\ A'[K].$$

§ 3. The differential operators of infinite order (DOIO).

Let the function $A(\xi)$ be expanded into the Taylor series

$$A(\xi)=\sum_{|\alpha|=0}^{\infty} a_\alpha \xi^\alpha, \qquad \xi\in S_R,\ a_\alpha=D^\alpha A(0)/\alpha! \tag{3.1}$$

and (3.1) converges for $\xi\in S_R$. We now consider the action of DOIO

$$A(D)=\sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha \tag{3.2}$$

in the space $W_R^{+\infty}(\boldsymbol{R}^N)$. A main result of this section is the following theorem.

THEOREM 3.1. *A DOIO with the symbol as above acts invariantly and continuously in* $W_R^{+\infty}(\boldsymbol{R}^N)$.

PROOF. Let $f(x)$ be an arbitrary function in $W_R^{+\infty}(\boldsymbol{R}^N)$. We will show that in $W_R^{+\infty}(\boldsymbol{R}^N)$ there exists the limit

$$\lim_{n\to\infty} A_n(D)f(x), \qquad A_n(D)=\sum_{|\alpha|=0}^{n} a_\alpha D^\alpha.$$

For this purpose we define the analytic functional $g(\xi)\in A'[\bar{S}_r]$ by the formula

$$g(\xi)=A(\xi)\check{f}(\xi), \qquad \operatorname{supp}\check{f}\subset\bar{S}_r, \qquad r<R.$$

Since $A_n(\xi)\to A(\xi)$ in $A[\bar{S}_r]$ we have

$$\begin{aligned}\lim \langle A_n(\xi)\check{f}(\xi), \varphi(\xi)\rangle &= \lim\langle \check{f}(\xi), A_n(\xi)\varphi(\xi)\rangle\\ &=\langle \check{f}(\xi), A(\xi)\varphi(\xi)\rangle=\langle g(\xi), \varphi(\xi)\rangle, \quad \forall\varphi\in A[S_r].\end{aligned}$$

On the other hand, $\operatorname{supp} A_n(\xi)\check{f}(\xi)\subset\operatorname{supp}\check{f}(\xi)$, $n=1, 2, \cdots$. Hence

$$\lim_{n\to\infty} A_n(\xi)\check{f}(\xi)=g(\xi) \quad \text{in} \quad A'[\bar{S}_r].$$

From the Proposition 2.4 we obtain

$$\lim_{n\to\infty} A_n(D)f(x)=\hat{g}(x)\in W_R^{+\infty}(\boldsymbol{R}^N).$$

We put $A(D)f(x)=\hat{g}(x)$ by definition and the invariance of $A(D)$ in $W_R^{+\infty}(\boldsymbol{R}^N)$ is proved.

Let now $f_n(x)\to f(x)$ in $W_R^{+\infty}(\boldsymbol{R}^N)$. Then there exists a number $r<R$ such that $\check{f}_n\to\check{f}$ in $A'[\bar{S}_r]$. Consequently $A(\xi)\check{f}_n(\xi)\to A(\xi)\check{f}(\xi)$ in $A'[\bar{S}_r]$. Applying the Fourier transform we obtain the continuity of $A(D)$

$$\lim_{n\to\infty} A(D)f_n(x)=A(D)f(x). \qquad \text{Q.E.D.}$$

§ 4. The space of generalized functions $W_R^{-\infty}(\boldsymbol{R}^N)$.

4.1. The definition of $W_R^{-\infty}(\boldsymbol{R}^N)$.

We denote by $W_R^{-\infty}(\boldsymbol{R}^N)$ the space of all continuous linear functionals defined on $W_R^{+\infty}(\boldsymbol{R}^N)$. We call elements of $W_R^{-\infty}(\boldsymbol{R}^N)$ the *generalized functions*. The space $W_R^{-\infty}(\boldsymbol{R}^N)$ has all the standard properties; for example:

a) If $h\in W_R^{-\infty}(\boldsymbol{R}^N)$ then $\dfrac{\partial h}{\partial x_i}\in W_R^{-\infty}(\boldsymbol{R}^N)$ can be defined by

$$\left\langle \frac{\partial h}{\partial x_i}, \varphi\right\rangle=\left\langle h, -\frac{\partial}{\partial x_i}\varphi\right\rangle, \qquad \forall\varphi\in W_R^{+\infty}(\boldsymbol{R}^N).$$

b) If $h\in W_R^{-\infty}(\boldsymbol{R}^N)$ and f is a function such that $f\varphi\in W_R^{+\infty}(\boldsymbol{R}^N)$ for all $\varphi\in W_R^{+\infty}(\boldsymbol{R}^N)$, then we define the product fh by

$$\langle fh, \varphi\rangle=\langle h, f\varphi\rangle, \qquad \varphi\in W_R^{+\infty}(\boldsymbol{R}^N).$$

Let $h\in W_R^{-\infty}(\boldsymbol{R}^N)$ and let $A(D)$ be a DOIO whose symbol $A(\xi)$ is analytic in S_R. Then

$$\langle A(D)h(x), \varphi(x)\rangle\overset{\text{def}}{=}\langle h(x), A(-D)\varphi(x)\rangle, \qquad \varphi\in W_R^{+\infty}(\boldsymbol{R}^N).$$

This expression is well-defined, because $A(-D)\varphi\in W_R^{+\infty}(\boldsymbol{R}^N)$ for any test function $\varphi\in W_R^{+\infty}(\boldsymbol{R}^N)$.

As a consequence of Theorem 3.1 we have

THEOREM 4.1. *The space $W_R^{-\infty}(\boldsymbol{R}^N)$ is invariant under differential operators of infinite order whose symbols are analytic in S_R.*

From Theorems 3.1 and 4.1 we obtain

THEOREM 4.2. *The set of all DOIOs with symbols analytic in S_R constitutes an algebra of operators isomorphic to the algebra $A(S_R)$ of functions analytic in S_R. This isomorphism is defined by the correspondence $A(D)\leftrightarrow A(\xi)$:*

$$\alpha A(D)+\beta B(D) \longleftrightarrow \alpha A(\xi)+\beta B(\xi),$$
$$A(D)\cdot B(D) \longleftrightarrow A(\xi)\cdot B(\xi).$$

In particular, if $A^{-1}(\xi)$ is also analytic in S_R, then $B(D)=I/A(D)$ is inverse to $A(D)$.

4.2. Examples of generalized functions in $W_R^{-\infty}(\boldsymbol{R}^N)$.

Example 1. If $|h(x)|\leq \exp(-a|x|)$, $a>0$ then $h(x)$ determines a generalized function by the formula

$$\langle h, \varphi\rangle \overset{\text{def}}{=} \int_{R^N} h(x)\varphi(x)dx, \qquad \forall\varphi\in W_R^{+\infty}(\boldsymbol{R}^N). \tag{4.1}$$

A generalized function in $W_R^{-\infty}(\boldsymbol{R}^N)$ is called regular if it is represented in the form (4.1).

Example 2. The delta function $\delta(x)$ determines a singular generalized function over $W_R^{+\infty}(\boldsymbol{R}^N)$ by the formula

$$\langle\delta(x), \varphi(x)\rangle=\varphi(0).$$

This is well-defined because every $\varphi(x)\in W_R^{+\infty}(\boldsymbol{R}^N)$ is continuous.

4.3. The Fourier transformation in $W_R^{-\infty}(\boldsymbol{R}^N)$.

We introduce the Fourier transform $\hat{h}(\xi)$ in $W_R^{-\infty}(\boldsymbol{R}^N)$ by the formula

$$\langle\hat{h}(\xi), \hat{\varphi}(-\xi)\rangle=(2\pi)^N\langle h(x), \varphi(x)\rangle, \tag{4.2}$$

where $\varphi(x)\in W_R^{+\infty}(\boldsymbol{R}^N)$ is any test function and $\hat{\varphi}(\xi)$ is its classical Fourier transform. Since $\varphi(x)\in W_R^{+\infty}(\boldsymbol{R}^N)$, $\hat{\varphi}(-\xi)\in A'(\boldsymbol{R}^N)$ and there exists a number $r<R$, such that $\operatorname{supp}\hat{\varphi}(-\xi)\subset\bar{S}_r\subset S_R$. It is clear that $\hat{h}(\xi)$ is a continuous linear functional over $A'[\bar{S}_r]$, $\forall r<R$. Hence it follows that $\hat{h}(\xi)$ is an analytic function in S_R [9]. Because $e^{-ix\xi}\in W_R^{+\infty}(\boldsymbol{R}^N)$, $\xi\in S_R$ we have

$$\begin{aligned}\langle h(x), e^{-ix\xi}\rangle &=(2\pi)^N\langle\hat{h}(\xi'), \delta(\xi'-\xi)\rangle\\ &=(2\pi)^N\hat{h}(\xi),\end{aligned}$$

then

$$\hat{h}(\xi)=(2\pi)^{-N}\langle h(x), e^{-ix\xi}\rangle, \tag{4.3}$$

and, in particular, $\hat{\delta}(\xi)=1$. It is clear that the formula (4.3) defines the Fourier transform $\hat{h}(\xi)$ of a generalized function $h(x)\in W_R^{-\infty}(\boldsymbol{R}^N)$ in the same way as the classical Fourier transform, and $\hat{h}(\xi)$ is analytic in S_R.

We are going to prove a structure theorem of generalized functions in $W_R^{-\infty}(\boldsymbol{R}^N)$.

THEOREM 4.3. *Every generalized function $h(x)\in W_R^{-\infty}(\boldsymbol{R}^N)$ can be represented in the form*

$$h(x)=A(D)(2\sqrt{\pi})^{-N}\exp(-x^2/4), \tag{4.4}$$

where $A(D)$ is a pseudodifferential operator with symbol $A(\xi)=\hat{h}(\xi)\times\exp(\xi^2)$ analytic in S_R.

PROOF. Since $h(x)\in W_R^{-\infty}(\boldsymbol{R}^N)$ it follows that $\hat{h}(\xi)$ is a function analytic in S_R. Then we can write

$$\hat{h}(\xi)=\hat{h}(\xi)\cdot 1$$

and applying the inverse Fourier-transform to this equality we have

$$h(x)=B(D)\delta(x),$$

where $B(D)$ is a pseudodifferential operator with the symbol $B(\xi)=\hat{h}(\xi)$. Thus if we prove the following lemma, the proof of Theorem 4.3 will be accomplished.

LEMMA 4.4. *The delta function $\delta(x)$ over $W_R^{+\infty}(\boldsymbol{R}^N)$ is represented in the form*

$$\delta(x)=(2\sqrt{\pi})^{-N}\exp(-\Delta)\exp(-x^2/4), \tag{4.5}$$

where Δ is Laplace operator.

PROOF OF LEMMA 4.4. We consider the following Cauchy problem

$$\frac{\partial\varepsilon(t, x)}{\partial t}-\Delta\varepsilon(t, x)=0 \tag{4.6}$$

$$\varepsilon(t, x)\,|_{t=0}=\delta(x)\in W_R^{-\infty}(\boldsymbol{R}^N). \tag{4.7}$$

The existence and uniqueness of $\varepsilon(t, x)$ follow from Theorem 7.2 below. Then we have

$$\varepsilon(t, x)=\exp(t\Delta)\delta(x),$$

in particular

$$\varepsilon(1, x)=\exp(\Delta)\delta(x). \tag{4.8}$$

By virtue of Theorem 4.2 or by Theorem 6.1 below from (4.8) we obtain that

$$\delta(x)=\exp(-\Delta)\varepsilon(1, x). \tag{4.9}$$

Now we consider the generalized function $\varepsilon(t, x)$. The action of this generalized function of $\varphi(x)\in W_R^{+\infty}(\boldsymbol{R}^N)$, $0<R\leq+\infty$, is defined by

$$\langle\varepsilon(t, x), \varphi(x)\rangle=\langle\delta(x), \exp(-t\Delta)\varphi(x)\rangle, \quad \varphi\in W_R^{+\infty}(\boldsymbol{R}^N). \tag{4.10}$$

We shall calculate the expression $\exp(-t\Delta)\varphi(x)$. Let $\varphi\in W_R^{+\infty}(\boldsymbol{R}^1)$ (For simplicity we consider the case $N=1$.). Then it follows from Theorem 3.1 that the function

$$u(t, x)=\exp\left(-t\frac{d^2}{dx^2}\right)\varphi(x)$$

belongs to $W_R^{+\infty}(\boldsymbol{R}^1)$ for $t>0$ then (4.10) is well-defined. We claim that $u(t, x)$ can be written in the form (Poisson representation)

$$u(t, x)=\frac{1}{2\sqrt{\pi t}}\int_{R^1}e^{-s^2/4t}\varphi(x-s)ds. \tag{4.11}$$

Indeed, since $\varphi(x)$ is an entire function,

$$\begin{aligned}\frac{1}{2\sqrt{\pi t}}\int_{R^1}e^{-s^2/4t}\varphi(x-s)ds&=\frac{1}{2\sqrt{\pi t}}\int_{R^1}e^{-s^2/4t}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^n\varphi(x)}{dx^n}(-s)^n ds\\&=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^n\varphi(x)}{dx^n}\frac{1}{2\sqrt{\pi t}}\int_{R^1}e^{-s^2/4t}(-s)^n ds.\end{aligned} \tag{4.12}$$

Next, by the substitution $s=2\sqrt{t}\eta$ we find

$$\begin{aligned}\frac{1}{2\sqrt{\pi t}}\int_{R^1}e^{-s^2/4t}(-s)^n ds&=\frac{1}{\sqrt{\pi}}(-2\sqrt{t})^n\int_{R^1}e^{-\eta^2}\eta^n d\eta\\&=\begin{cases}0, & n=2m+1,\\ \dfrac{1}{\sqrt{\pi}}(4t)^m\dfrac{(2m-1)!\sqrt{\pi}}{2^{2m-1}(m-1)!}.\end{cases}\end{aligned}$$

Then we derive directly from the last formula and (4.12) that

$$\frac{1}{2\sqrt{\pi t}}\int_{R^1} e^{-s^2/4t}\varphi(x-s)\,ds=\sum_{m=0}^{\infty}\frac{1}{m!}\,\frac{d^{2m}\varphi(x)}{dx^{2m}}(t)^m$$
$$=\exp\left(-t\frac{d^2}{dx^2}\right)\varphi(x),$$

as required.

In accordance with the formula (4.10) we find that

$$\langle\varepsilon(t,x),\varphi(x)\rangle=\frac{1}{(2\sqrt{\pi t}\,)^N}\int_{R^N} e^{-s^2/4t}\varphi(-s)\,ds.$$

It follows that $\varepsilon(t,x)$ is a regular generalized function determined by the function $-\frac{1}{2(\sqrt{\pi t}\,)^N}e^{-x^2/4t}$. In particular, by setting $t=1$ and from (4.9) we obtain

$$\delta(x)=\exp(-\Delta)\frac{1}{(2\sqrt{\pi}\,)^N}e^{-x^2/4}.$$

Hence we have Lemma 4.4. Q.E.D.

§ 5. The algebra of pseudodifferential operators with analytic symbol.

Let $G\subset \boldsymbol{R}^N$ be some domain, that is an open set in $\boldsymbol{R}^N_\xi$.

DEFINITION 5.1. The space of test functions $W_G^{+\infty}(\boldsymbol{R}^N)$ is the set of functions $f(x)$ satisfying the following conditions:

i) $f(x)$ admits analytic continuation as an entire function to $\boldsymbol{C}^N$ and

ii) $f(\zeta)$ is the Fourier-Laplace transform of an analytic functional $u\in A'(\boldsymbol{R}^N)$, with $\operatorname{supp} u\subset K\subset G$, where K is a compact.

We now turn to a description of the structure of $W_G^{+\infty}(\boldsymbol{R}^N)$. We denote by $S_R(\lambda)=\{\xi\in \boldsymbol{R}^N\colon |\xi-\lambda|<R\}$ a sphere centred at λ completely contained in G and, in accordance with the definition of $W_G^{+\infty}(\boldsymbol{R}^N)$ we set

$$W_{R,\lambda}^{+\infty}(\boldsymbol{R}^N)=\{f(x)\colon f(\zeta)=\hat{u}(\zeta),\ u\in A'(\boldsymbol{R}^N),\ \operatorname{supp} u\subset S_R(\lambda)\}.$$

Then we have that $f(x)\in W_{R,\lambda}^{+\infty}(\boldsymbol{R}^N)$ if and only if $\exp(-i\lambda x)f(x)\in W_R^{+\infty}(\boldsymbol{R}^N)$. Hence one can write symbolically

$$W_{R,\lambda}^{+\infty}(\boldsymbol{R}^N)=\exp(i\lambda x)\,W_R^{+\infty}(\boldsymbol{R}^N),$$

where $W_R^{+\infty}(\boldsymbol{R}^N)$ is the space of test functions in the neighborhood of zero constructed in § 2.

Further it is not hard to show that any function $f(x) \in W_G^{+\infty}(\boldsymbol{R}^N)$ can be represented in the form

$$f(x)=\sum_{k\in I} u_{\lambda_k}(x),$$

where $u_{\lambda_k}(x) \in W_{R_K,\lambda_k}^{+\infty}(\boldsymbol{R}^N)$, $S_{R_k}(\lambda_k) \subset G$ and I is a finite set of indices. Indeed, if $f \in W_G^{+\infty}(\boldsymbol{R}^N)$, then $\check{f} \in A'(\boldsymbol{R}^N)$ and $\operatorname{supp} \check{f} = K \subset G$. We can choose the compact subsets K_k such that $K=\cup K_k$ and $K_k \subset S_{R_k}$, $\lambda_k, k \in I$. By Theorem 1.4 we have $\check{f}=\sum_{k\in I} \check{u}_{\lambda_k}$, where $\check{u}_{\lambda_k} \in A'[K_k]$. Consequently $f=\sum_{k\in I} u_{\lambda_k}$, where $u_{\lambda_k} \in W_{R_k,\lambda_k}^{+\infty}(\boldsymbol{R}^N)$.

Definition 5.2. A sequence $\{f_n(x)\}$ is said to converge to $f(x)$ in $W_G^{+\infty}(\boldsymbol{R}^N)$ if there exists a compact $L \subset G$ such that

$$\check{f}_n \longrightarrow \check{f} \qquad \text{in } A'[L].$$

Let now $A(\xi)$ be an arbitrary complex-valued function which is analytic in G. It is possible to choose $S_{R_k}(\lambda_k)$ so that in each $S_{R_k}(\lambda_k)$ the function $A(\xi)$ can be expanded in the Taylor series

$$A(\xi)=\sum_{|\alpha|=0}^{\infty} a_\alpha(\lambda_k)(\xi-\lambda_k)^\alpha, \qquad k \in I,\ \xi \in S_{R_k}(\lambda_k).$$

For any function $u(x) \in W_G^{+\infty}(\boldsymbol{R}^N)$ we have

$$u(x)=\sum_{k\in I} u_{\lambda_k}(x),$$

and define the action of $A(D)$ on $u(x)$ by the formula

$$A(D)u(x) \overset{\text{def}}{=} \sum_{k\in I} \sum_{|\alpha|=0}^{\infty} a_\alpha(\lambda_k)(D-\lambda_k I)^\alpha u_{\lambda_k}(x),$$

and by Theorem 3.1 $A(D)u(x)$ is again a function in $W_G^{+\infty}(\boldsymbol{R}^N)$. Moreover, it can be shown by using Fourier-Laplace transform that this definition does not depend on the number of representing of function $u(x)$, i.e. the action is well-defined.

We now define by $(W_G^{+\infty}(\boldsymbol{R}^N))^*$ the space of all continuous linear functionals on $(W_G^{+\infty}(\boldsymbol{R}^N))$ and we set $W_G^{-\infty}(\boldsymbol{R}^N) \equiv (W_{-G}^{+\infty}(\boldsymbol{R}^N))^*$. Let $A(\xi)$ be a function analytic in G. We assign to it a pseudodifferential operator $A(D)$ acting in $W_G^{-\infty}(\boldsymbol{R}^N)$ according to the formula

$$\langle A(D)h(x),\ \varphi(x)\rangle \overset{\text{def}}{=} \langle h(x),\ A(-D)\varphi(x)\rangle,$$

where $h(x)\in W_G^{-\infty}(\boldsymbol{R}^N)$, $\varphi(x)\in W_G^{+\infty}(\boldsymbol{R}^N)$. The definition is correct because for $\varphi(x)\in W_G^{+\infty}(\boldsymbol{R}^N)$, $A(-D)\varphi\in W_G^{+\infty}(\boldsymbol{R}^N)$. Applying Theorem 4.2 we obtain the following results.

THEOREM 5.1. *A ΨDO $A(D)$ with symbol $A(\xi)$ analytic in G acts invariantly and continuously in $W_G^{+\infty}(\boldsymbol{R}^N)$ (and hence also in $W_G^{-\infty}(\boldsymbol{R}^N)$).*

THEOREM 5.2. *The set of operators $A(D)$ with symbols $A(\xi)$ analytic in a domain G and defined on $W_G^{\pm\infty}(\boldsymbol{R}^N)$ forms an operator algebra which is isomorphic to the algebra of functions analytic in G. This isomorphism is defined by the correspondence $A(D)\leftrightarrow A(\xi)$. Here*

$$\alpha A(D)\pm\beta B(D) \longleftrightarrow \alpha A(\xi)\pm\beta B(\xi),$$
$$A(D)\cdot B(D) \longleftrightarrow A(\xi)\cdot B(\xi).$$

In particular, if $A^{-1}(\xi)$ is also analytic in G, then $B(D)=I/A(D)$ is the operator inverse to $A(D)$. For any function $A(\xi)$ analytic in G the maps

$$A(D):\ W_G^{\pm\infty}(\boldsymbol{R}^N) \longrightarrow W_G^{\pm\infty}(\boldsymbol{R}^N)$$

are continuous. Here "+" corresponds to "+" and "−" to "−".

§ 6. Application to pseudodifferential equations.

Let $A(D)$ be a pseudodifferential operator with analytic symbol $A(\xi)$. We consider the equation

$$A(D)u(x)=h(x), \qquad x\in\boldsymbol{R}^n. \tag{6.1}$$

The results of § 5 have the following consequence.

THEOREM 6.1. *Suppose that $A(\xi)$ and $A^{-1}(\xi)$ are analytic in some domain $G\subset\boldsymbol{R}_\xi^N$. Then (6.1) for any $h(x)\in W_G^{\pm\infty}(\boldsymbol{R}^N)$ has the unique solution*

$$u(x)=\frac{1}{A(D)}h(x). \tag{6.2}$$

Example 1. We consider the Helmholtz equation (ω is a complex parameter)

$$\Delta u(x)+\omega^2 u=h(x), \qquad x\in\boldsymbol{R}^N. \tag{6.3}$$

Note that the symbol of the Helmholtz operator and of the inverse operator $I/(\Delta+\omega^2 I)$ are analytic for $\xi^2\neq\omega^2$, $\xi\in\boldsymbol{R}^N$. Hence, the whole space $\boldsymbol{R}_\xi^N$ is the common domain of analyticity of the symbols of both operators, except when ω is real. Consequently, the space $W_{R^N}^{+\infty}(\boldsymbol{R}^N)$ consists of all function $h(x)$ that admit an analytic continuation as entire functions and are the Fourier-Laplace transform of $\check{h}\in A'(\boldsymbol{R}^N)$. For any such function we find that the solution of (6.3) is given by

$$u(x)=\frac{I}{\Delta+\omega^2 I}h(x). \tag{6.4}$$

But when ω is real, then $G=\boldsymbol{R}_\xi^N\diagdown S$, where S is the sphere $\xi^2=\omega^2$. Hence in this case (6.4) yields a solution of (6.3) for any function $h(x)$, $\check{h}\subset A'(\boldsymbol{R}^N)$, $\operatorname{supp}\check{h}\subset G$.

Example 2. We consider the problem of the existence of a fundamental solution for the operator $A(D)$, that is, the solvability of the equation

$$A(D)\xi(x)=\delta(x), \qquad x\in\boldsymbol{R}^N.$$

Suppose that the domain of analyticity of the symbols $A(\xi)$ and $A^{-1}(\xi)$ is non-empty, $G\neq\varnothing$. Then clearly $\delta(x)\in W_G^{-\infty}(\boldsymbol{R}^N)$, therefore, the fundamental solution $\xi(x)$ exists as a functional on $W_G^{+\infty}(\boldsymbol{R}^N)$ and is given by

$$\xi(x)=\frac{1}{A(D)}\delta(x).$$

Example 3. We consider the equation with a shift

$$u(x+1)+u(x-1)=h(x), \qquad x\in\boldsymbol{R}. \tag{6.5}$$

Using Taylor's formula, we can write this as a differential equation of infinite order. We set

$$2\cosh\left(\frac{d}{dx}\right)u(x)=h(x), \qquad x\in\boldsymbol{R}^1. \tag{6.6}$$

Since $\cosh(i\xi)\neq 0$ for $\xi\neq\pi/2+k\pi$, $k=0,\pm1,\cdots$, we find that (6.6) or, what is the same, (6.5) for any $h(x)\in W_G^{\pm\infty}(\boldsymbol{R}^1)$, where $G=\boldsymbol{R}^1\diagdown\{\pi/2+k\pi\}$, has a unique solution

$$u(x)=\frac{1}{2}\operatorname{sech}\left(\frac{d}{dx}\right)h(x).$$

Let us now discuss the equation (6.1) when the symbol $A(\xi)$ has singularities in G, $h(x)\in W_G^{+\infty}(\boldsymbol{R}^N)$. Applying the Fourier transform to both parts of (6.1) we obtain

$$A(\xi)\hat{u}(\xi)=\hat{h}(\xi), \qquad \xi\in \boldsymbol{R}_\xi^N.$$

Let $u(x)\in W_G^{+\infty}(\boldsymbol{R}^N)$ be an arbitrary solution of (6.1). Then

$$\operatorname{sup}\hat{u}(\xi)\subset \operatorname{supp}\hat{h}(\xi)\cup\{\xi\in G,\ A(\xi)=0\}.$$

Hence, if $\hat{h}(\xi)=0$, $\operatorname{supp}\hat{u}(\xi)\subset\{\xi\in G,\ A(\xi)=0\}$. We denote the set $\{\xi\in G,\ A(\xi)=0\}$ by $\boldsymbol{O}(A, G)$.

We recall that a linear continuous operator $\phi: X\to Y$ is called Fredholm operator if $\dim\operatorname{Ker}\phi<+\infty$, $\dim\operatorname{Coker}\phi<+\infty$ and $\operatorname{Im}\phi$ is closed in Y. Clearly, $\operatorname{Im}A(D)=W_G^{+\infty}(\boldsymbol{R}^N)$ for any $A(D)$ with symbol analytic in G. Therefore, these operators are Fredholm if and only if $\dim\operatorname{Ker}A(D)<+\infty$.

PROPOSITION 6.2. *If the operator $A(D)$ with symbol analytic in G is Fredholm operator in $W_G^{+\infty}(\boldsymbol{R}^N)$, then the set $\boldsymbol{O}(A, G)$ is finite.*

PROOF. Let $A(\xi)$ have an infinite number of different zeros: $\xi_1, \xi_2, \cdots$ in G. Then the functions $\delta(\xi-\xi_1)$, $\delta(\xi-\xi_2)$, $\cdots$ are linearly independent. Because $A(\xi_j)=0$, then $\langle A(\xi)\delta(\xi-\xi_j),\ \varphi(\xi)\rangle=A(\xi_j)\varphi(\xi_j)=0$, $\forall\varphi\in A(G)$. Consequently, $F^{-1}(\delta(\xi-\xi_i))$ is the solution of the equation $A(D)u(x)=0$ and, thus, $\dim\operatorname{Ker}A(D)=\infty$. Q.E.D.

The inverse to Proposition 6.2 statement is valid only for the case $N=1$.

PROPOSITION 6.3. *Let $N=1$, $G\subset\boldsymbol{R}^1$. Let the symbol $A(\xi)$ have a finite set of zeros in G. Then $A(D)$ is Fredholm operator.*

PROOF. It is sufficient to prove Proposition 6.3 in the case $A(\xi)$ has only one zero of order n in G. For simplicity, let $A(0)=0$ and $0\in G$. Then, if $S(\xi)$ in a solution of the equation

$$A(\xi)S(\xi)=0,$$

it follows that $\operatorname{supp}S(\xi)=\{0\}$. Consequently, $S(\xi)$ is represented in the form

$$S(\xi)=\sum_{i=0}^{n-1}a_i\delta^{(i)}(\xi), \qquad a_i\in\boldsymbol{C}^1.$$

Thus we have $\dim \operatorname{Ker} A(D)<+\infty$. Q.E.D.

For the case $N\geq 2$ the Proposition 6.3 becomes false. We consider the counterexample constructed in [16, p. 112]. Let $N=2$ and $A(\xi)=\frac{1}{2}(\xi_1^2+\xi_2^2)$. It is clear that 0 is a unique zero of $A(\xi)$. Then for all $n\geq 2$ there exist numbers $c_\alpha\neq 0$ so that

$$S(\xi)=\sum_{|\alpha|=n} c_\alpha \delta^{(\alpha)}(\xi)$$

will be solutions of the equation $A(\xi)S(\xi)=0$ (For the details, see [16].).

§ 7. The Cauchy problems.

7.1. The Cauchy problems in the space of functions valued in $W_G^{+\infty}(\boldsymbol{R}^N)$. In the space $\boldsymbol{R}^{N+1}$ of the variables $t\in\boldsymbol{R}^1$ and $x\in\boldsymbol{R}^N$ we study the Cauchy problem for any system of partial differential equations of the form

$$\frac{\partial^{m_j}u_j(t,x)}{\partial t^{m_j}}+\sum_{k=1}^{l} A_{jk}(t,\partial/\partial t,D)u_k(t,x)=h_j(t,x); \tag{7.1}$$

$$\partial^k u_j(0,x)/\partial t^k=\varphi_{kj}(x),\qquad k=0,1,\cdots,m_j-1,\ j-1,\cdots,l, \tag{7.2}$$

where

$$A_{jk}(t,\partial/\partial t,D)=\sum_{i=0}^{m_j-1} A_{ijk}(t,D)\frac{\partial^i}{\partial t^i},$$

the $m_j>0$ are integers, the $A_{ijk}(t,D)$ are arbitrary pseudodifferential operators, and for each t the symbols $A_{ijk}(t,\xi)$ are analytic functions of ξ in some domain $G\subset\boldsymbol{R}_\xi^N$ which depend continuously on $t\in\boldsymbol{R}^1$.

We denote by $W_G^{\pm\infty,l}(\boldsymbol{R}^N)$ the space of vector-valued functions $u(x)=(u_1(x),\cdots,u_l(x))$, $u_i(x)\in W_G^{\pm\infty}(\boldsymbol{R}^N)$, $i=1,\cdots,l$ and we denote by $C^{k_1,\cdots,k_l}(\boldsymbol{R}^1, W_G^{\pm\infty,l}(\boldsymbol{R}^N))$, the space of vector-valued functions $u(t,x)=(u,(t,x),\cdots,u_l(t,x))$ which for each $t\in\boldsymbol{R}^1$ are vector-valued functions in $W_G^{\pm\infty,l}(\boldsymbol{R}^N)$ and $u_i(t,x)$ depends continuously on t together with its derivatives through order k_i $(i=1,\cdots,l)$.

THEOREM 7.1. *Let* $\varphi_k\in W_G^{\pm\infty,l}(\boldsymbol{R}^N)$ *and* $h(t,x)\in C^{0,\cdots,0}(\boldsymbol{R}^1, W_G^{\pm\infty,l}(G))$ *be arbitrary functions. Then there exists a unique solution of the Cauchy problem* (7.1)-(7.2) *in the space* $C^{m_1,\cdots,m_l}(\boldsymbol{R}^1, W_G^{\pm\infty,l}(\boldsymbol{R}^N))$.

PROOF. We use the method analogous to that of [2] except for the uniqueness of a solution. For simplicity of the exposition we consider the case $l=1$, $m_1=m$. First of all, we observe that by Duhamel's principle it suffices to consider the case $h(t, x)\equiv 0$. Thus, we are looking for the solution of the problem

$$L(\partial/\partial t, D)u\equiv\frac{\partial^m u}{\partial t^m}+\sum_{k=0}^{m-1} A_k(t, D)\frac{\partial^k u}{\partial t^k}=0 \tag{7.3}$$

$$\frac{\partial^k u(0, x)}{\partial t^k}=\varphi_k(x), \qquad k=0, 1, \cdots, m-1, \tag{7.4}$$

where $\varphi_k(x) \in W_G^{+\infty}(\boldsymbol{R}^N)$. With this aim we set formally $D\leftrightarrow\xi$, $(\xi=(\xi_1, \cdots, \xi_N))$ and solve the family of Cauchy problems for the ordinary differential equation

$$L(\partial/\partial t, \xi)u_i(t, \xi)=0,$$
$$u_i^{(k)}(0, \xi)=\delta_{ik} \qquad (0\leq k, i\leq m-1),$$

where δ_{ik} is the Kronecker symbol and ξ is a real parameter.

Since the $A_k(t, \xi)$ depend analytically on ξ in G, each $u_i(t, \xi)$ is an analytic function of ξ in G. We assign to each such "basic" solution $u_i(t, \xi)$ a DOIO $U_i(t, D)$, whose action is continuous in $W_G^{+\infty}(\boldsymbol{R}^N)$, in accordance with § 5. Clearly, the formula

$$u(t, x)=\sum_{i=0}^{m} U_i(t, D)\varphi_i(x)$$

then determines the required solution.

To prove the uniqueness of the solutions of the problem (7.1)-(7.2) we note that the Fourier-Laplace x-transform $\hat{u}(t, \xi)$ of the solution $u(t, x)$ is a solution of the Cauchy problem for the system of ordinary differential equations valued in the space of hyperfunctions and is therefore unique. Hence, so is $u(t, x)$. This completes the proof. Q.E.D.

7.2. The Cauchy problem in the space $W_G^{-\infty}(\boldsymbol{R}^N)$ and its fundamental solution.

THEOREM 7.2. *Let* $\varphi_k\in W_G^{-\infty,l}(\boldsymbol{R}^N)$ *and* $h(t, x)\in C^{0,\cdots,0}(\boldsymbol{R}^1, W_G^{-\infty,l}(\boldsymbol{R}^N))$ *be arbitrary functions. Then there exists a unique solution of the Cauchy problem* (7.1)-(7.2) *in the space* $C^{m_1,\cdots,m_l}(\boldsymbol{R}^1, W_G^{-\infty,l}(\boldsymbol{R}^N))$.

The proof of this theorem is analogous of the one of Theorem 5.2

in [2] and we omit it.

We now consider the problem of the existence of a fundamental solution of the Cauchy problem. The fundamental solution of the Cauchy problem for the operator $L(\partial/\partial t, D)$ (7.3) is a function $\varepsilon(t, x)$ that solves the problem

$$L(\partial/\partial t, D)\varepsilon(t, x)=0, \quad t\neq 0,$$
$$\varepsilon(0, x)=0, \cdots, \varepsilon^{(m-2)}(0, x)=0, \quad \varepsilon^{(m-1)}(0, x)=\delta(x).$$

If we know the fundamental solution of Cauchy problem we can obtain (at least formally) a solution of the Cauchy problem with arbitrary initial conditions $\varphi_k(x)$, $0\leq k\leq m-1$, by means of the operations of differentiation and convolution. Since $\delta(x)\in W_G^{-\infty}(\boldsymbol{R}^N)$ we have from Theorem 7.2:

COROLLARY 7.3. *For any operator $L(\partial/\partial t, D)$ of the form* (7.3) *with symbols $A_k(t, \xi)$ analytic in G the Cauchy problem has a unique fundamental solution valued in $W_G^{-\infty}(\boldsymbol{R}^N)$.*

7.3. The Cauchy problem for ordinary pseudodifferential equations.

Let $A(D)$ be a pseudodifferential operator with the symbol $A(\xi)$ analytic in $G\subset \boldsymbol{R}^1$ which has a finite number of different zeros in G.

THEOREM 7.4. *Let $\xi_1, \xi_2, \cdots, \xi_m$ be the zeros of $A(\xi)$ in G with their multiplicity $n_1, \cdots, n_m$ respectively. Then there exists a unique solution $u(x)\in W_G^{+\infty}(\boldsymbol{R}^1)$ of the Cauchy problem*

$$A(D)u(x)=h(x), \qquad h(x)\in W_G^{+\infty}(\boldsymbol{R}^1), \tag{7.5}$$

$$u^{(i)}(x_0)=c_i, \quad c_i\in \boldsymbol{C}^1, \quad i=0, 1, \cdots, n-1, \; n=n_1+\cdots+n_m. \tag{7.6}$$

PROOF. By virtue of Proposition 6.3 it follows that any solution of (7.5) is represented by the formula

$$u(x)=u_0(x)+\sum_{k=0}^{m}\sum_{j=0}^{n_k-1}\lambda_{kj}e^{ix\xi_k}x^j,$$

where $u_0(x)$ is a solution of (7.5), λ_{kj} are arbitrary numbers. We will show that if $u(x)$ satisfies (7.6) then λ_{kj} are uniquely defined. In fact we rewrite (7.6) in the form

$$\begin{bmatrix} \varphi_1(x_0) & \cdots & \varphi_n(x_0) \\ \varphi_1^{(1)}(x_0) & \cdots & \varphi_n^{(1)}(x_0) \\ \cdots & \cdots & \cdots \\ \varphi_1^{(n-1)}(x_0) & \cdots & \varphi_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} c_1 - u_0(x_0) \\ c_2 - u_0^{(1)}(x_0) \\ \cdots \\ c_n - u_0^{(n-1)}(x_0) \end{bmatrix} \tag{7.7}$$

where we denote

$$\begin{gathered} \varphi_1(x) = e^{ix\xi_1}, \cdots, \varphi_{n_1}(x) = e^{i\xi_1 x} x^{n_1-1}, \\ \varphi_{n_1+1}(x) = e^{ix\xi_2}, \cdots, \varphi_{n_1+n_2}(x) = e^{ix\xi_2} x^{n_2-1}, \\ \cdots\cdots\cdots \\ \varphi_{n_1+n_2+\cdots+n_{m-1}+1}(x) = e^{ix\xi_m}, \cdots, \varphi_n(x) = e^{ix\xi_m} x^{n_m-1}, \\ \lambda_1 = \lambda_{10}, \lambda_2 = \lambda_{11}, \cdots, \lambda_{n_1} = \lambda_{1,n_1-1}, \cdots, \lambda_n = \lambda_{m,n_m-1}. \end{gathered}$$

Clearly,

$$\det \begin{bmatrix} \varphi_1(x_0) & \cdots & \varphi_n(x_0) \\ \varphi_1^{(1)}(x_0) & \cdots & \varphi_n^{(1)}(x_0) \\ \cdots & \cdots & \cdots \\ \varphi_1^{(n-1)}(x_0) & \cdots & \varphi_n^{(n-1)}(x_0) \end{bmatrix} \neq 0,$$

then (7.7) has a unique solution $(\lambda_1, \cdots, \lambda_n)$. Q.E.D.

§ 8. The boundary-value problems.

In the strip $t \in (a, b)$, $x \in \boldsymbol{R}^N$, we consider the boundary-value problem

$$\frac{\partial^m u(t, x)}{\partial t^m} + \sum_{j=0}^{m-1} A_j(t, D) \frac{\partial^j u(t, x)}{\partial t^j} = 0, \tag{8.1}$$

$$\sum_{s=0}^{m-1} b_{js} \frac{\partial^s u(a, x)}{\partial t^s} = \varphi_j(x), \qquad j = 0, 1, \cdots, m-n-1, \ n \leq m, \tag{8.2}$$

$$\sum_{s=0}^{m-1} b_{js} \frac{\partial^s u(b, x)}{\partial t^s} = \psi_j(x), \qquad j = m-n, \cdots, m-1 \tag{8.3}$$

where $A_j(t, D)$ are ΨDOs with the symbols $A_j(t, \xi)$ analytic in some domain $G \subset R^N$ and depending continuously on $t \in \boldsymbol{R}^1$, $\|b_{js}\|$ is the non-singular numerical matrix.

In this section we will show that the problem (8.1)-(8.3) may be reduced to the Cauchy problem for the system of pseudodifferential equations with the initial conditions which are, in their turn, solutions of some systems of differential equations of infinite order.

We consider the family of Cauchy problems for the systems of ordinary differential equations with a real parameter $\xi \in G$:

$$\frac{\partial^m U_{0k}(t,\xi)}{\partial t^m}+\sum_{j=0}^{m-1} A_j(t,\xi)\frac{\partial^j U_{0k}(t,\xi)}{\partial t^j}=0, \tag{8.4}$$

$$V_{jk}(t,\xi)=\sum_{s=0}^{m-1} b_{js}\frac{\partial^s U_{0k}(t,\xi)}{\partial t^s}, \tag{8.5}$$

$$V_{jk}(a,\xi)=\delta_{jk}, \tag{8.6}$$

where $j, k=0, 1, \cdots, m-1$. These problems have unique solutions U_{0k}, V_{jk} depending analytically on ξ [5]. Further, we constitute the equations

$$\sum_{s=0}^{m-1} V_{js}(b, D)\varphi_s(x)=\psi_j(x), \qquad j=m-n, \cdots, m-1, \tag{8.7}$$

where $V_{js}(b, D)$ are ΨDO with symbols $V_{js}(b,\xi)$. We rewrite (8.7) in the form

$$\sum_{s=m-n}^{m-1} V_{js}(b, D)\varphi_s(x)=\psi_j(x)-\sum_{s=0}^{m-n-1} V_{js}(b, D)\varphi_s(x). \tag{8.8}$$

Then (8.8) is a system of n pseudodifferential equations with analytic symbols, where $\varphi_s(x)$, $s=m-n, \cdots, m-1$ are unknown functions.

THEOREM 8.1. *Let φ_j, $j=0, 1, \cdots, m-n-1$, ψ_j, $j=m-n, \cdots, m-1$ belongs to space $W_G^{+\infty}(\boldsymbol{R}^N)$. Then the solution of the problem* (8.1)-(8.3) *can be represented in the form*

$$u(t, x)=\sum_{k=0}^{m-1} U_{0k}(t, D)\varphi_k(x) \tag{8.9}$$

and belongs to the space $C^m(\boldsymbol{R}^1, W_G^{+\infty}(\boldsymbol{R}^N))$, where φ_k, $k=m-n, \cdots, m-1$ are solutions of the system (8.8). *Conversely, if $u(t,x)$ is a solution of the problem* (8.1)-(8.3) *then*

$$\varphi_s=\sum_{k=0}^{m-1} b_{sk}\frac{\partial^k u(a, x)}{\partial t^k}, \qquad s=m-n, \cdots, m-1$$

are solutions of the system (8.8).

PROOF. The fact that (8.9) satisfies (8.1) and (8.2) is a consequence of (8.4) and (8.5). We will show that (8.9) satisfies (8.3). Indeed using (8.7) we have for $j=m-n, \cdots, m-1$

$$\sum_{s=0}^{m-1} b_{js}\frac{\partial^s u(b, x)}{\partial t^s} = \sum_{k=0}^{m-1}\sum_{s=0}^{m-1} b_{js}\frac{\partial^s U_{0k}(b, D)}{\partial t^s}\cdot\varphi_k(x)$$
$$= \sum_{s=0}^{m-1} V_{js}(b, D)\varphi_s(x) = \psi_j(x).$$

To prove of the second part of Theorem 8.1 we will show that

$$\sum_{s=0}^{m-1} V_{js}(b, D)\Big(\sum_{k=0}^{m-1} b_{sk}\frac{\partial^k u(a, x)}{\partial t^k}\Big) = \psi_j(x), \qquad j = m-n, \cdots, m-1. \tag{8.10}$$

But (8.10) follows from the fact that

$$u(t, x) = \sum_{s=0}^{m-1} U_{0s}(t, D)\Big(\sum_{k=0}^{m-1} b_{sk}\frac{\partial^k u(a, x)}{\partial t^k}\Big).$$

This completes the proof.

REMARK. Using the results of this paper we can give the approximate methods of solving problems of mathematical physics based on the techniques of DOIOs. For the function space $W^{+\infty}(G)$ constructed in [15] some approximate methods are considered in [13, 14].

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Institute of Mathematics
Box 631, Bo Ho
Hanoi
Vietnam