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A construction of the perturbed solution of semilinear elliptic equation in the singularly perturbed domain

Dedicated to Professor Hiroshi Fujita on his 60th birthday

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§1 Introduction

In this paper we will deal with some inverse problem associated with the characterization of the structure of solutions of the semilinear elliptic equation $(1.1)_\zeta$ on a singularly perturbed domain (Figure 1) with Neumann boundary condition:

$$(1.1)_\zeta \qquad \begin{cases} \Delta v+f(v)=0 & \text{in } \Omega(\zeta), \\ \partial v/\partial \nu=0 & \text{on } \partial\Omega(\zeta). \end{cases}$$

Here $\Omega(\zeta)\subset \boldsymbol{R}^n$ is a bounded domain for each $\zeta>0$ and is expressed in the form $\Omega(\zeta)=D_1\cup D_2\cup Q(\zeta)$, where the varying portion $Q(\zeta)$ degenerates to a 1-dimensional line segment L as $\zeta\to 0$. Our aim, roughly speaking, is to construct a solution v_ζ of $(1.1)_\zeta$ whose asymptotic behavior as $\zeta\to 0$ is prescribed in a sense to be specified later.

Reaction diffusion equations on a singularly perturbed domain of the same type as $\Omega(\zeta)$ have been dealt with by several authors (cf. [5], [9], [10], [11], [14], [16], [18], [20]). Vegas [20] adopted $f(\lambda, v)=\lambda v-v^p$ ($p\geqq 2$: integer, $\lambda>0$: small) for the nonlinear term in $(1.1)_\zeta$ and investigated the transition phenomenon of the structure of solutions of $(1.1)_\zeta$ when the parameter $\zeta>0$ decreases to 0 and described a complete bifurcation diagram for sufficiently small $\lambda>0$. Hale and Vegas [5] constructed a perturbed solution v_ζ of (1.1) whose behavior satisfies

$$v_\zeta(x)\sim a_i \text{ in } D_i \ (i=1,2) \text{ for small } \zeta>0,$$

where $f(a_i)=0$, $f'(a_i)\neq 0$ $(i=1, 2)$ and $|f'(u)|$ is assumed to be bounded by some adequately small constant determined by the domain. In the above two cases ([5], [20]), $f'(v)$ is assumed to be small around the solution v_ζ, which is somewhat a strong hypothesis. Under such a hypothesis, v_ζ necessarily approaches some constant in each D_i $(i=1, 2)$ as $\zeta\to 0$ and the structure of solutions of $(1.1)_\zeta$ is rather simple. Moreover, in [5] and [20], the behavior of v_ζ on the varying portion $Q(\zeta)$ was not analyzed.

In [10] and [11], we have dealt with $(1.1)_\zeta$ in $\Omega(\zeta)$ for more general $f\in C^\infty(\boldsymbol{R})$ without any assumption on the bound of $\dfrac{\partial f}{\partial u}$ and considered the behaviors of solutions on $Q(\zeta)$. We have proved that any solution v_ζ of $(1.1)_\zeta$ for small $\zeta>0$ is approximated in the sense of "uniform convergence" by some triple of solutions (w_1, w_2, V) of the following system of equations:

$$(1.2)\qquad \begin{cases} \Delta w_i+f(w_i)=0 & \text{in } D_i, \\ \partial w_i/\partial\nu=0 & \text{on } \partial D_i, \end{cases} \quad (i=1, 2)$$

$$(1.3)\qquad \begin{cases} d^2V/dz^2+f(V)=0 & z\in L, \\ V|_{\partial D_i\cap\partial L}=w_i|_{\partial D_i\cap\partial L} & (i=1, 2), \end{cases}$$

where z is the canonical parameter on L ([10; Th. 2 and Th. 3], [11; Th. 2]). We state this characterization theorem in a revised form in Proposition 3.1, because we make full use of it in this paper. Along with the above theorem, we have given an example of a nonlinear term f for which there exist three solutions $v_\zeta^{(0)}<v_\zeta^{(1)}<v_\zeta^{(2)}$ such that all $v_\zeta^{(j)}$ $(0\leqq j\leqq 2)$ approach the same constant in each D_i as $\zeta\to 0$ while their behaviors on $Q(\zeta)$ up to the stability are totally different from one another ([10; Th. 4]). In this case, w_1 and w_2 are constant stable solutions of (1.2). We have also proved, by the aid of Matano's method [16], that for any given stable solution (w_1, w_2) of (1.2), there exists a stable solution v_ζ of $(1.1)_\zeta$ which satisfies $v_\zeta(x)\sim w_i(x)$ in D_i $(i=1, 2)$ ([10; Th. 1], [11; Th. 1]). But in this case, we have not taken into consideration the behavior of v_ζ on $Q(\zeta)$. Therefore we have not yet constructed a solution of $(1.1)_\zeta$ which not only satisfies the above conditions in $D_1\cup D_2$ but also is approximated on $Q(\zeta)$ by a solution of (1.3) that is chosen arbitrarily. In order to study such a problem systematically, in this paper we will consider a general inverse problem associated with the characterization of $(1.1)_\zeta$ by (1.2) and (1.3). More precisely, for any given triple of solutions $(w_1, w_2,$

V) of the system of equations (1.2) and (1.3) without any assumption on the stability of w_i and V in (1.2) and (1.3) respectively, we will seek for a family $\{v_\zeta\}_{0<\zeta<\zeta_*}$ such that each $v_\zeta \in C^\infty(\overline{\Omega(\zeta)})$ is a solution of $(1.1)_\zeta$ and satisfies the following asymptotic behaviors when $\zeta \to 0$.

$$v_\zeta \sim w_i \text{ in } D_i \ (i=1, 2), \ v_\zeta \sim V \text{ in } Q(\zeta).$$

We will give an affirmative answer to the above problem under the condition of non-degeneracy of (w_1, w_2, V) (cf. (II. 5), (II. 6)) by a rather direct method. Thus, from this and the results in [11], we see that the equation $(1.1)_\zeta$ for small $\zeta>0$ is usually equivalent to the system of equations (1.2) and (1.3). Our method is rather constructive. We reduce the equation $(1.1)_\zeta$ to a finite dimensional problem by using the eigenfunctions of the linearized problem of $(1.1)_\zeta$ around the approximate solution. In this procedure, we have a difficulty that some eigenfunctions behave in a highly singular manner when $\zeta \to 0$ and this phenomena is associated with the partial degeneration of the domain. Thus we need some elaborate estimates of these singular behaviors. For this, we rely on our results in [12], thereby we can obtain a well reduced equation. Our method can be applied to more general reaction-diffusion equations in $\Omega(\zeta)$. The result in this paper was published in [13] with a brief idea of the proof.

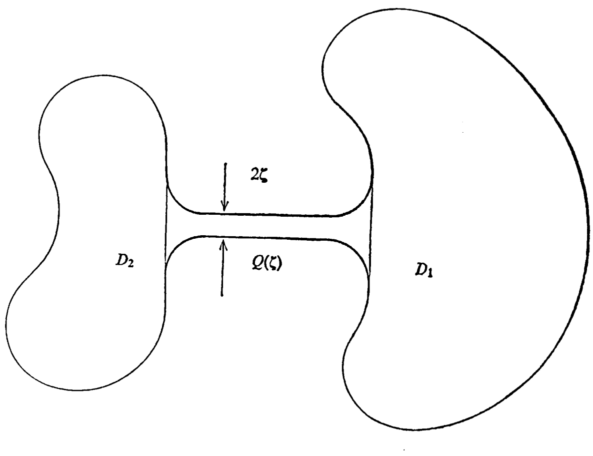

Figure 1.

§ 2 Formulation and Main Theorem

In this section, we present the main result of this paper. We first establish the necessary setting in which the problem is formulated, i.e., the singularly perturbed domain $\Omega(\zeta)$ such as in Figure 1, the nonlinear term f and the given data (w_1, w_2, V) and the condition of non-degeneracy.

We set the domain $\Omega(\zeta)$ in the following form:

$$\Omega(\zeta)=D_1\cup D_2\cup Q(\zeta)$$

where D_i $(i=1,2)$ and $Q(\zeta)$ are defined in (II. 1) and (II. 2) below, where $x'=(x_2, x_3, \cdots, x_n)\in \boldsymbol{R}^{n-1}$.

(II. 1) D_1 and D_2 are bounded domains in $\boldsymbol{R}^n$ satisfying $\bar{D}_1\cap\bar{D}_2=\emptyset$, each D_i has a smooth boundary ∂D_i and the following conditions hold for some positive constant $\zeta_*>0$.

$$\bar{D}_1\cap\{x=(x_1, x')\in \boldsymbol{R}^n|\ x_1\leqq 1,\ |x'|<3\zeta_*\}=\{(1, x')\in \boldsymbol{R}^n|\ |x'|<3\zeta_*\},$$
$$\bar{D}_2\cap\{x=(x_1, x')\in \boldsymbol{R}^n|\ x_1\geqq -1,\ |x'|<3\zeta_*\}=\{(-1, x')\in \boldsymbol{R}^n|\ |x'|<3\zeta_*\}.$$

(II. 2) $Q(\zeta)=R_1(\zeta)\cup R_2(\zeta)\cup \Gamma(\zeta)$,

$$R_1(\zeta)=\{(x_1, x')\in \boldsymbol{R}^n|\ 1-2\zeta<x_1\leqq 1,\ |x'|<\zeta\rho((x_1-1)/\zeta)\},$$
$$R_2(\zeta)=\{(x_1, x')\in \boldsymbol{R}^n|\ -1\leqq x_1<-1+2\zeta,\ |x'|<\zeta\rho((-1-x_1)/\zeta)\},$$
$$\Gamma(\zeta)=\{(x_1, x')\in \boldsymbol{R}^n|\ -1+2\zeta\leqq x_1\leqq 1-2\zeta,\ |x'|<\zeta\},$$

where $\rho\in C^0((-2, 0])\cap C^\infty((-2, 0))$ is a positive function such that $\rho(0)=2$, $\rho(s)=1$ for $s\in(-2, -1)$, $d\rho/ds>0$ for $s\in(-1, 0)$ and the inverse function $\rho^{-1}: (1, 2)\longrightarrow(-1, 0)$ satisfies $\lim_{\xi\uparrow 2-0}\dfrac{d^k\rho^{-1}}{d\xi^k}(\xi)=0$ for any positive integer $k\geqq 1$. Hereafter we put two points $p_1=(1, 0, \cdots, 0)$, $p_2=(-1, 0, \cdots, 0)$, and the set $L=\{(z, 0, \cdots, 0)\in \boldsymbol{R}^n|\ -1<z<1\}$.

We impose the following conditions.

(II. 3) $f\in C^\infty(\boldsymbol{R})$, $\limsup_{\xi\to+\infty} f(\xi)<0$, $\liminf_{\xi\to-\infty} f(\xi)>0$.

(II. 4) There exists a system of solutions (w_1, w_2, V) of (1.2) and (1.3) in $C^\infty(\bar{D}_1)\times C^\infty(\bar{D}_2)\times C^\infty([-1, 1])$.

DEFINITION 1. For the above solutions (w_1, w_2, V) in (II. 4), we denote by $\{\omega_k\}_{k=1}^\infty$ and $\{\lambda_k\}_{k=1}^\infty$, respectively, the system of the eigenvalues arranged in increasing order (counting multiplicity) of the following eigenvalue problems (2.1) and (2.2):

$$\text{(2.1)} \qquad \begin{cases} \Delta\phi+f'(w)\phi+\omega\phi=0 & \text{in } D_1\cup D_2, \\ \partial\phi/\partial\nu=0 & \text{on } \partial D_1\cup\partial D_2, \end{cases}$$

where $w(x)=\begin{cases} w_1(x) & \text{for } x\in D_1 \\ w_2(x) & \text{for } x\in D_2 \end{cases}$, and

$$\text{(2.2)} \qquad \begin{cases} \dfrac{d^2S}{dz^2}+f'(V)S+\lambda S=0 & -1<z<1, \\ S(1)=S(-1)=0. \end{cases}$$

We assume the following non-degeneracy condition of (w_1, w_2, V).

$$\text{(II. 5)} \qquad \{\omega_k\}_{k=1}^{\infty}\cup\{\lambda_k\}_{k=1}^{\infty}\not\ni 0$$

$$\text{(II. 6)} \qquad \{\omega_k\}_{k=1}^{\infty}\cap\{\lambda_k\}_{k=1}^{\infty}=\varnothing.$$

Now we present the main result of this paper.

THEOREM. *Assume* $n\geqq 3$ *and* (II. 1)～(II. 6). *Then, for any* $\zeta\in(0, \zeta_*)$, *there exists a solution* v_ζ *of* $(1.1)_\zeta$ *such that*

$$\text{(2.3)} \qquad \lim_{\zeta\to 0}\sup_{x\in D_1\cup D_2}|v_\zeta(x)-w(x)|=0,$$

$$\text{(2.4)} \qquad \lim_{\zeta\to 0}\sup_{x\in Q(\zeta)}|v_\zeta(x_1, x')-V(x_1)|=0.$$

REMARK 2.1. This theorem can be regarded as a much improved version of the results in [5; Th. 2.3], [10; Th. 1] and [11; Th. 1] because we deal with the case where w_1 and w_2 are not necessarily constant functions and not necessarily stable in (1.2) and also because we obtain the behavior of the solution v_ζ in the infinitesimal remnant part $Q(\zeta)$ ($\zeta>0$ small) besides its behavior in $D_1\cup D_2$. We can also prove, with the aid of Proposition 3.2, that the limit of the set of the linearized eigenvalues of v_ζ (as $\zeta\to 0$) coincides with $\{\omega_k\}_{k=1}^{\infty}\cup\{\lambda_k\}_{k=1}^{\infty}$.

REMARK 2.2. From (II. 3), if we put $\overline{M}=\sup\{\xi\in\boldsymbol{R}|\ f(\xi)=0\}$, $\underline{M}=\inf\{\xi\in\boldsymbol{R}|\ f(\xi)=0\}$, then any solution v of $(1.1)_\zeta$ (if it exists) satisfies the inequality $\underline{M}\leqq v(x)\leqq\overline{M}$ for $x\in\Omega(\zeta)$. Therefore, to prove Theorem, we have only to construct the solution of the equation given by replacing f in (1.1) by $f_*\in C^\infty(\boldsymbol{R})$ such that $f_*(\xi)>0$ for $\xi<\underline{M}$, $f_*(\xi)<0$ for $\xi>\overline{M}$ and

$$f_*(\xi)=\begin{cases} f(\overline{M}+1) & \text{for } \xi\in[\overline{M}+2,\infty) \\ f(\xi) & \text{for } \xi\in[\underline{M}-1,\overline{M}+1] \\ f(\underline{M}-1) & \text{for } \xi\in(-\infty,\underline{M}-2]. \end{cases}$$

In view of this, the proof of Theorem, we assume without loss of generality that $\partial f/\partial\xi$ has compact support in $\boldsymbol{R}$; in particular f and $\partial f/\partial\xi$ are bounded in $\boldsymbol{R}$.

REMARK 2.3. The condition (II. 6) in Theorem can be weakened to the following condition:

(II. 6)′ $\{\omega_k\}_{k=1}^{\infty}\cap\{\lambda_k\}_{k=1}^{q}=\emptyset$, $\{\omega_k\}_{k=1}^{q}\cap\{\lambda_k\}_{k=1}^{\infty}=\emptyset$
for some natural number q such that

$$\min(\omega_{q+1},\lambda_{q+1})\geqq 2\max_{\xi\in[\underline{M}-2,\overline{M}+2]}|f'(\xi)|+4 \quad\text{and}\quad \omega_{q+1}>\omega_q.$$

In fact, we use (II. 6) to apply Proposition 3.2 where (II. 6) exactly corresponds to the condition (III. 1). But in the proof of Theorem, we only use certain eigenfunctions up to some finite number q (i.e., $\{\psi_{k,\zeta}\}_{k=1}^{q}$ and $\{\phi_{k,\zeta}\}_{k=1}^{q}$), and then for this purpose only, (III. 1) is not necessary to hold completely in Proposition 3.2 and so we can replace (II. 6) by a weaker condition (II. 6)′. See also Remark 3.3.

§ 3 Preliminaries

In this section we state some of the results obtained in [10], [11] in a revised form and one in [12]. We will use them in this work. As stated in § 1, Proposition 3.1 is a characterization theorem for solutions of a semilinear elliptic equation on $\Omega(\zeta)$. Proposition 3.2 will give an elaborate characterization of the eigenfunctions of the linearized problem around the approximate solution which enable us to carry out a direct constructive method in the proof of Theorem. We do not give the proofs of the propositions because the proof of Proposition 3.1 can be carried out by the same arguments as in [10] and [11] except for some inessential changes and Proposition 3.2 is the same as Theorem 4 in [12].

We consider the following equation:

$$\text{(3.1)}\qquad \begin{cases} \Delta v+f_\zeta(x,v)=H_\zeta(x) & \text{in } \Omega(\zeta), \\ \dfrac{\partial v}{\partial\nu}=0 & \text{on } \partial\Omega(\zeta). \end{cases}$$

Here the nonlinear term $f_\zeta \in C^\infty(\overline{\Omega(\zeta)} \times \boldsymbol{R})$ is given as follows:

$$f_\zeta(x, \xi) = h_\zeta(x) g(\xi) \qquad (0 < \zeta < \zeta_*),$$

where h_ζ, H_ζ and g satisfy the following conditions:

(i) $g \in C^\infty(\boldsymbol{R}),\ \limsup_{\xi\to\infty} g(\xi) < 0,\ \liminf_{\xi\to-\infty} g(\xi) > 0,$

(ii) $h_\zeta, H_\zeta \in C^\infty(\overline{\Omega(\zeta)})$ and there exist a sequence of positive values $\{\zeta_m\}_{m=1}^\infty$ and $h, H \in C^\infty(\overline{D_1 \cup D_2})$, $\bar{h}, \bar{H} \in C^\infty([-1, 1])$ such that

$$\begin{aligned}
&h_\zeta(x) > 0 \quad \text{in } \overline{\Omega(\zeta)}, \quad \lim_{m\to\infty} \zeta_m = 0\\
&\lim_{m\to\infty} \sup_{x\in D_1\cup D_2} |h_{\zeta_m}(x) - h(x)| = 0 \qquad (i=1, 2),\\
&\lim_{m\to\infty} \sup_{x\in Q(\zeta_m)} |h_{\zeta_m}(x_1, x') - \bar{h}(x_1)| = 0,\\
&\lim_{m\to\infty} \sup_{x\in D_1\cup D_2} |H_{\zeta_m}(x) - H(x)| = 0 \qquad (i=1, 2),\\
&\lim_{m\to\infty} \sup_{x\in Q(\zeta_m)} |H_{\zeta_m}(x_1, x') - \bar{H}(x_1)| = 0.
\end{aligned}$$

Note that $\bar{h}(1) = h(p_1)$, $\bar{H}(1) = H(p_1)$, $\bar{h}(-1) = h(p_2)$, $\bar{H}(-1) = H(p_2)$ necessarily hold from (ii).

In the above situation, we have the following.

PROPOSITION 3.1. *Assume* (i), (ii) *and* $n \geqq 3$. *For each* $\zeta \in (0, \zeta_*)$, *let* v_ζ *be any solution of* (3.1). *Then there exist a subsequence* $\{\sigma_m\}_{m=1}^\infty \subset \{\zeta_m\}_{m=1}^\infty$ *and functions* $w \in C^\infty(\overline{D_1 \cup D_2})$ *and* $V \in C^\infty([-1, 1])$ *such that the following conditions are satisfied*:

$$\begin{cases} \Delta w + h(x) g(w) = H(x) & \text{in } D_1 \cup D_2,\\ \partial w/\partial\nu = 0 & \text{on } \partial D_1 \cup \partial D_2, \end{cases}$$

$$\begin{cases} d^2V/dz^2 + \bar{h}(z) g(V) = \bar{H}(z) \quad \text{for } z \in (-1, 1),\\ V(1) = w(p_1), \quad V(-1) = w(p_2), \end{cases}$$

$$\begin{aligned}
&\lim_{m\to\infty} \sup_{x\in D_1\cup D_2} |v_{\sigma_m}(x) - w(x)| = 0,\\
&\lim_{m\to\infty} \sup_{x\in Q(\sigma_m)} |v_{\sigma_m}(x_1, x') - V(x_1)| = 0.
\end{aligned}$$

REMARK 3.1. In the case where $h_\zeta \equiv 1$ and $H_\zeta \equiv 0$, Proposition 3.1 is reduced to Theorem 2 in [11].

REMARK 3.2. If there exists an a-priori bound for v_ζ such that

$|v_\zeta(x)|\leqq K$ in $\Omega(\zeta)$ $(0<\zeta<\zeta_*)$, the inequalities in (i) need not hold, for we can modify $g(\xi)$ for $\xi\in\boldsymbol{R}$ with $|\xi|\geqq K$.

We consider the following eigenvalue problem.

$$(3.2)\qquad \begin{cases} \Delta\Phi+h_\zeta^*(x)\Phi+\mu\Phi=0 & \text{in } \Omega(\zeta), \\ \dfrac{\partial\Phi}{\partial\nu}=0 & \text{on } \partial\Omega(\zeta), \end{cases}$$

where $h^*\in C^\infty(\overline{D_1\cup D_2})$ and $\bar{h}^*\in C^\infty([-1,1])$ are such that

$$(3.3)\qquad \begin{cases} \lim\limits_{\zeta\to 0}\sup\limits_{x\in D_1\cup D_2}|h_\zeta^*(x)-h^*(x)|=0, \\ \lim\limits_{\zeta\to 0}\sup\limits_{x\in Q(\zeta)}|h_\zeta^*(x_1,x')-\bar{h}^*(x_1)|=0. \end{cases}$$

DEFINITION 2. Let $\{\mu_k(\zeta)\}_{k=1}^\infty$ and $\{\Phi_{k,\zeta}\}_{k=1}^\infty$ be the eigenvalues arranged in increasing order (counting multiplicity) and the complete system of the orthonormalized eigenfunctions of the eigenvalue problem (3.2).

DEFINITION 3. Let $\{\omega_k\}_{k=1}^\infty$ and $\{\lambda_k\}_{k=1}^\infty$, respectively, be the sequences of the eigenvalues of (3.4) and (3.5) arranged in increasing order (counting multiplicity),

$$(3.4)\qquad \begin{cases} \Delta\phi+h^*(x)\phi+\omega\phi=0 & \text{in } D_1\cup D_2, \\ \dfrac{\partial\phi}{\partial\nu}=0 & \text{on } \partial D_1\cup\partial D_2, \end{cases}$$

$$(3.5)\qquad \begin{cases} d^2S/dz^2+\bar{h}^*(z)S+\lambda S=0, \quad -1<z<1, \\ S(1)=S(-1)=0. \end{cases}$$

We assume the following condition.

$$(\text{III. }1)\qquad \{\omega_k\}_{k=1}^\infty\cap\{\lambda_k\}_{k=1}^\infty=\emptyset.$$

PROPOSITION 3.2. *Assume* (III. 1) *and* $n\geqq 3$. *The sets* $\{\mu_k(\zeta)\}_{k=1}^\infty$ *and* $\{\Phi_{k,\zeta}\}_{k=1}^\infty$ *are separated as follows*:

$$\begin{cases} \{\mu_k(\zeta)\}_{k=1}^\infty=\{\omega_k(\zeta)\}_{k=1}^\infty\cup\{\lambda_k(\zeta)\}_{k=1}^\infty \\ \{\Phi_{k,\zeta}\}_{k=1}^\infty=\{\phi_{k,\zeta}\}_{k=1}^\infty\cup\{\psi_{k,\zeta}\}_{k=1}^\infty, \end{cases}$$

where

(3.6) $$\lim_{\zeta\to 0}\omega_k(\zeta)=\omega_k,\ \lim_{\zeta\to 0}\lambda_k(\zeta)=\lambda_k.$$

For any sequence of positive values $\{\zeta_m\}_{m=1}^{\infty}$ *such that* $\lim_{m\to\infty}\zeta_m=0$ *there exists a subsequence* $\{\sigma_m\}_{m=1}^{\infty}\subset\{\zeta_m\}_{m=1}^{\infty}$ *and the complete system of the eigenfunctions* $\{\phi_k\}_{k=1}^{\infty}\subset C^{\infty}(\overline{D_1\cup D_2})$ *of* (3.4) *and* $\{S_k\}_{k=1}^{\infty}$ *of* (3.5) *respectively such that* $(\phi_k\cdot\phi_m)_{L^2(D_1\cup D_2)}=\delta_{k,m}$ $(S_k\cdot S_m)_{L^2((-1,1))}=\delta_{k,m}$ *for* $k, m\geqq 1$ *and that the following conditions are satisfied for each* $k\geqq 1$:

(3.7) $$\begin{cases}\lim_{m\to\infty}\sup_{x\in D_1\cup D_2}|\phi_{k,\sigma_m}(x)-\phi_k(x)|=0,\\ \lim_{m\to\infty}\sup_{x\in Q(\sigma_m)}|\phi_{k,\sigma_m}(x_1,x')-V_k(x_1)|=0.\end{cases}$$

(3.8) $$\begin{cases}\lim_{m\to\infty}\sup_{x\in Q(\sigma_m)}|d_{n-1}^{1/2}\sigma_m^{(n-1)/2}\phi_{k,\sigma_m}(x_1,x')-S_k(x_1)|=0,\\ \text{or}\\ \lim_{m\to\infty}\sup_{x\in Q(\sigma_m)}|d_{n-1}^{1/2}\sigma_m^{(n-1)/2}\phi_{k,\sigma_m}(x_1,x')+S_k(x_1)|=0,\end{cases}$$

where $d_{n-1}=\dfrac{\pi^{(n-1)/2}}{\Gamma((n+1)/2)}$ *and* V_k *is the unique solution of the following two point boundary value problem* (3.9) *for each* $k\geqq 1$:

(3.9) $$\begin{cases}d^2V/dz^2+\bar{h}^*(z)V+\omega_kV=0, & -1<z<1,\\ V(1)=\phi_k(p_1),\ V(-1)=\phi_k(p_2).\end{cases}$$

For any $k\geqq 1$, *there exists a constant* $\eta_*(k)>0$ *such that*

(3.10) $$\begin{cases}0<\liminf_{\zeta\to 0}\inf_{x\in R_i(\zeta)\cup\Sigma_i(3\zeta)}\zeta^{(n-3)/2}|\phi_{k,\zeta}(x)|\\ \leqq\limsup_{\zeta\to 0}\sup_{x\in R_i(\zeta)\cup\Sigma_i(3\zeta)}\zeta^{(n-3)/2}|\phi_{k,\zeta}(x)|<+\infty,\end{cases}$$

(3.11) $$\begin{cases}0<\liminf_{\zeta\to 0}\inf_{x\in\Sigma_i(\eta)\setminus\Sigma_i(3\zeta)}\zeta^{-(n-1)/2}|x-p_i|^{n-2}|\phi_{k,\zeta}(x)|\\ \leqq\limsup_{\zeta\to 0}\sup_{x\in\Sigma_i(\eta)\setminus\Sigma_i(3\zeta)}\zeta^{-(n-1)/2}|x-p_i|^{n-2}|\phi_{k,\zeta}(x)|<+\infty,\end{cases}$$

(3.12) $$\begin{cases}0<\liminf_{\zeta\to 0}\sup_{x\in D_i\setminus\Sigma_i(\eta)}\zeta^{-(n-1)/2}|\phi_{k,\zeta}(x)|\\ \leqq\limsup_{\zeta\to 0}\sup_{x\in D_i\setminus\Sigma_i(\eta)}\zeta^{-(n-1)/2}|\phi_{k,\zeta}(x)|<+\infty,\end{cases}$$

(3.13) $$\begin{cases}0<\liminf_{\zeta\to 0}\zeta^{-(n-1)/2}\|\phi_{k,\zeta}\|_{L^1(\Omega(\zeta))}\\ \leqq\limsup_{\zeta\to 0}\zeta^{-(n-1)/2}\|\phi_{k,\zeta}\|_{L^1(\Omega(\zeta))}<+\infty,\end{cases}$$

$$\lim_{\zeta\to 0} d_{n-1}^{-1/2}\zeta^{-(n-1)/2}\|\phi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}=1 \tag{3.14}$$

for any $\eta\in(0,\eta_*(k))$ *and* $i=1,2$.

REMARK 3.3. If we assume the following condition (III. 1)′ in place of (III. 1), we also have such results as (3.7)$\sim$(3.14) for $k=1,2,\cdots,q$.

$$\{\omega_k\}_{k=1}^\infty\cap\{\lambda_k\}_{k=1}^q=\emptyset,\quad \{\omega_k\}_{k=1}^q\cap\{\lambda_k\}_{k=1}^\infty=\emptyset, \tag{III. 1)′}$$

where q is a natural number such that $\omega_{q+1}>\omega_q$.

§ 4 Construction of the approximate solution

To construct an approximate solution $A_\zeta\in C^\infty(\overline{\Omega(\zeta)})$, we prepare the following function $\iota_\zeta\in C^\infty(\boldsymbol{R})$ such that $0\leqq\iota_\zeta(z)\leqq 1$ for $z\in\boldsymbol{R}$ and $\iota_\zeta(z)=1$ for $z\in[3\zeta,\infty)$, $\iota_\zeta(z)=0$ for $z\in(-\infty,2\zeta]$. We define $\tilde{A}_\zeta\in C^\infty(\overline{\Omega(\zeta)})$ as

$$\tilde{A}_\zeta(x)=\begin{cases} V(x_1) & x\in\Gamma(\zeta)\cap\{-1+3\zeta\leqq x_1\leqq 1-3\zeta\}\\ \iota_\zeta(1-x_1)V(x_1)+(1-\iota_\zeta(1-x_1))V(1) & x\in\Gamma(\zeta)\cap\{1-3\zeta<x_1\leqq 1-2\zeta\}\\ \iota_\zeta(x_1-1)V(x_1)+(1-\iota_\zeta(x_1-1))V(-1) & x\in\Gamma(\zeta)\cap\{2\zeta-1\leqq x_1<3\zeta-1\}\\ w_i(p_i) & x\in R_i(\zeta)\cup\Sigma_i(2\zeta)\\ & (i=1,2)\\ \iota_\zeta(|x-p_i|)w_i(x)+(1-\iota_\zeta(|x-p_i|))w_i(p_i) & x\in\Sigma_i(3\zeta)\backslash\Sigma_i(2\zeta)\\ & (i=1,2)\\ w_i(x) & x\in D_i\backslash\Sigma_i(3\zeta)\\ & (i=1,2).\end{cases}$$

We can easily check

$$\lim_{\zeta\to 0}\sup_{x\in D_i}|\tilde{A}_\zeta(x)-w_i(x)|=0\qquad(i=1,2),$$
$$\lim_{\zeta\to 0}\sup_{x\in Q(\zeta)}|\tilde{A}_\zeta(x_1,x')-V(x_1)|=0.$$

We define the approximate solution $A_\zeta\in C^\infty(\overline{\Omega(\zeta)})$ by the unique solution of the equation

$$\begin{cases}\Delta A_\zeta-A_\zeta+\tilde{A}_\zeta+f(\tilde{A}_\zeta)=0 & \text{in } \Omega(\zeta),\\ \partial A_\zeta/\partial\nu=0 & \text{on } \partial\Omega(\zeta).\end{cases}\tag{4.1}$$

Then we have the following.

LEMMA 4.1.

$$(4.2)\qquad \begin{cases} \lim_{\zeta\to 0}\sup_{x\in D_i}|A_\zeta(x)-w_i(x)|=0 \quad (i=1,2), \\ \lim_{\zeta\to 0}\sup_{x\in Q(\zeta)}|A_\zeta(x_1,x')-V(x_1)|=0, \end{cases}$$

$$(4.3)\qquad \lim_{\zeta\to 0}\sup_{x\in\Omega(\zeta)}|\Delta A_\zeta(x)+f(A_\zeta(x))|=0.$$

PROOF OF LEMMA 4.1. Define the constants K_1, K_2, as follows: $K_1=\sup\{|\xi| \mid \xi\in \boldsymbol{R},\ f(\xi)=0\}$, $K_2=\sup_{|\xi|\leqq K_1}(|\xi|+|f(\xi)|)$. Then, from (4.1), we have the following estimate:

$$-K_2\leqq A_\zeta(x)\leqq K_2 \quad \text{for any } x\in\Omega(\zeta)\ (0<\zeta<\zeta_*).$$

Therefore we can apply Proposition 3.1, to see that for any sequence of positive values $\{\zeta_m\}_{m=1}^\infty$ such that $\lim_{m\to\infty}\zeta_m=0$, there exist a subsequence $\{\sigma_m\}_{m=1}^\infty\subset\{\zeta_m\}_{m=1}^\infty$ and $A\in C^\infty(\overline{D_1\cup D_2})$, $\bar{A}\in C^\infty([-1,1])$ such that

$$\lim_{m\to\infty}\sup_{x\in D_1\cup D_2}|A_{\sigma_m}(x)-A(x)|=0,\quad \lim_{m\to\infty}\sup_{x\in Q(\sigma_m)}|A_{\sigma_m}(x_1,x')-\bar{A}(x_1)|=0$$

$$(4.4)\qquad \begin{cases} \Delta A-A+w+f(w)=0 & \text{in } D_1\cup D_2, \\ \partial A/\partial\nu=0 & \text{on } \partial D_1\cup\partial D_2, \end{cases}$$

$$(4.5)\qquad \begin{cases} d^2\bar{A}/dz^2-\bar{A}+V+f(V)=0 \quad -1<z<1, \\ \bar{A}(1)=A(p_1),\ \bar{A}(-1)=A(p_2). \end{cases}$$

By the equation (1.2) and (4.4), we have

$$\Delta(w-A)-(w-A)=0 \ \text{ in } D_1\cup D_2,\quad \partial(w-A)/\partial\nu=0 \ \text{ on } \partial D_1\cup\partial D_2$$

and by the maximum principle, we conclude $w\equiv A$ in $D_1\cup D_2$. Thus $\bar{A}(1)=A(p_1)=w(p_1)=V(1)$ and $\bar{A}(-1)=A(p_2)=w(p_2)=V(-1)$ are satisfied and then from (1.3) and (4.5), we have

$$\frac{d^2}{dz^2}(V-\bar{A})-(V-\bar{A})=0 \text{ in } (-1,1),\ V(1)=\bar{A}(1),\ V(-1)=\bar{A}(-1)$$

and we conclude that $\bar{A}\equiv V$ for $-1<z<1$. Therefore by the arbitrariness of $\{\zeta_m\}_{m=1}^\infty$, we conclude (4.2). We can easily deduce (4.3) by applying (4.2) in (4.1) and we conclude Lemma 4.1.

§ 5 Proof of Theorem

As we stated in Remark 2.1, we assume without loss of generality that $f(\xi)$ and $\partial f(\xi)/\partial\xi$ are bounded in $\boldsymbol{R}$. It is easy to see that for $h_\zeta(x)\equiv f'(A_\zeta(x))$, $h(x)\equiv f'(w(x))$, $\bar{h}(z)\equiv f'(V(z))$, the condition (3.3) holds from Lemma 4.1, that the eigenvalues $\{\omega_k\}_{k=1}^\infty$ and $\{\lambda_k\}_{k=1}^\infty$ in Definition 1 and Definition 3 coincide with each other and that the conditions (III. 1) follows from (II. 5). Thus let $\{\mu_k(\zeta)\}_{k=1}^\infty$, $\{\Phi_{k,\zeta}\}_{k=1}^\infty$ and $\{\omega_k\}_{k=1}^\infty$, $\{\lambda_k\}_{k=1}^\infty$ be those fixed in Definition 2 and Definition 3, respectively, with $h_\zeta^*, h^*, \bar{h}^*$ replaced by $h_\zeta, h, \bar{h}$. Then we can use Proposition 3.2. Therefore we can decompose $\{\mu_k(\zeta)\}_{k=1}^\infty$ and $\{\Phi_{k,\zeta}\}_{k=1}^\infty$ of the linearized eigenvalue problem

$$(5.1)\qquad \begin{cases} \Delta\Phi+f'(A_\zeta)\Phi+\mu\Phi=0 & \text{in } \Omega(\zeta), \\ \partial\Phi/\partial\nu=0 & \text{on } \partial\Omega(\zeta), \end{cases}$$

as

$$\{\mu_k(\zeta)\}_{k=1}^\infty=\{\omega_k(\zeta)\}_{k=1}^\infty\cup\{\lambda_k(\zeta)\}_{k=1}^\infty,$$

$$\{\Phi_{k,\zeta}\}_{k=1}^\infty=\{\phi_{k,\zeta}\}_{k=1}^\infty\cup\{\psi_{k,\zeta}\}_{k=1}^\infty,$$

where

$$(5.2)\qquad \lim_{\zeta\to 0}\omega_k(\zeta)=\omega_k,\quad \lim_{\zeta\to 0}\lambda_k(\zeta)=\lambda_k \quad (k\geqq 1).$$

Now we define the function space as follows. $X(\zeta)$ is the Hilbert space $H^1(\Omega(\zeta))$ which is endowed with the following inner product:

$$(\Phi\cdot\Psi)_{X(\zeta)}=\int_{\Omega(\zeta)}(\nabla\Phi\nabla\Psi-f'(A_\zeta)\Phi\Psi+M\Phi\Psi)dx \quad (M=\sup_{\xi\in R}|f'(\xi)|+1).$$

We decompose $X(\zeta)$ as $X(\zeta)=X_1(\zeta)\oplus X_2(\zeta)$, where

$$X_1(\zeta)=\text{L.h.}\,[\{\phi_{k,\zeta}\}_{k=1}^q\cup\{\psi_{k,\zeta}\}_{k=1}^q],$$
$$X_2(\zeta)=\text{L.h.}\,[\{\phi_{k,\zeta}\}_{k=q+1}^\infty\cup\{\psi_{k,\zeta}\}_{k=q+1}^\infty],$$

where L.h. $[W]$ $(\subset X(\zeta))$ is the closed subspace of $X(\zeta)$ spanned by the set $W\subset X(\zeta)$ and the integer q is to be determined later.

We define a projection operator, defined in $L^2(\Omega(\zeta))$ and mapping $X(\zeta)$ onto $X_1(\zeta)$, as follows:

$$P_\zeta\Phi(x)=\sum_{k=1}^q\left(\left(\int_{\Omega(\zeta)}\phi_{k,\zeta}\Phi dx\right)\phi_{k,\zeta}(x)+\left(\int_{\Omega(\zeta)}\psi_{k,\zeta}\Phi dx\right)\psi_{k,\zeta}(x)\right).$$

LEMMA 5.1. *For any natural number $q\geqq 1$ such that $\omega_{q+1}>\omega_q$ there* *exists a constant* $d(q)>0$ *which is independent of* ζ *and satisfies*

$$(5.3)\qquad \|P_\zeta \Phi\|_{L^\infty(\Omega(\zeta))} \leqq d(q)\|\Phi\|_{L^\infty(\Omega(\zeta))} \quad \text{for } \Phi\in L^\infty(\Omega(\zeta)) \ (0<\zeta<\zeta_*).$$

For any sequence of positive values $\{\zeta_m\}_{m=1}^\infty$ *such that* $\lim_{m\to\infty}\zeta_m=0$, *there exist a subsequence* $\{\sigma_m\}_{m=1}^\infty\subset\{\zeta_m\}_{m=1}^\infty$ *and functions* $\{W_k\}_{k=1}^q\subset\Big(C^\infty\Big(\bigcup_{i=1}^2(\bar{D}_i\backslash\{p_i\})\Big)\cap L^1(D_1\cup D_2)\Big)$ *such that*

$$(5.4)\quad \lim_{m\to\infty}\sup_{x\in Q(\sigma_m)}|d_{n-1}^{1/2}\sigma_m^{(n-1)/2}\phi_{k,\sigma_m}(x_1,x')-S_k(x_1)|=0,$$

$$(5.5)\quad \lim_{m\to\infty}\phi_{k,\sigma_m}/(d_{n-1}^{1/2}\sigma_m^{(n-1)/2})=W_k \text{ in } C^\infty(\overline{(D_1\backslash\Sigma_1(\eta))\cup(D_2\backslash\Sigma_2(\eta))}) \text{ for any } \eta>0$$

and that the following conditions hold:

$$(5.6)\quad \lim_{m\to\infty}\sup_{x\in D_1\cup D_2}\Big|(P_{\zeta_m}\Phi_{\zeta_m})(x)-\sum_{k=1}^q\Big(\int_{D_1\cup D_2}\phi_k\Phi dx\Big)\phi_k(x)\Big|=0,$$

$$(5.7)\quad \lim_{m\to\infty}\sup_{x\in Q(\sigma_m)}\Big|(P_{\sigma_m}\Phi_{\sigma_m})(x_1,x') - \sum_{k=1}^q\Big(\Big(\int_{-1}^1 S_k\bar{\Phi}dz+\int_{D_1\cup D_2}W_k\Phi dx\Big)S_k(x_1)+\Big(\int_{D_1\cup D_2}\phi_k\Phi dx\Big)V_k(x_1)\Big)\Big|=0$$

for any functions $\Phi_\zeta\in C^0(\overline{\Omega(\zeta)})$ $(\zeta\in(0,\zeta_*))$, $\Phi\in C^0(\overline{D_1\cup D_2})$ *and* $\bar{\Phi}\in C^0([-1,1])$ *such that*

$$\lim_{m\to\infty}\sup_{x\in D_1\cup D_2}|\Phi_{\zeta_m}(x)-\Phi(x)|=0,\qquad \lim_{m\to\infty}\sup_{x\in Q(\zeta_m)}|\Phi_{\zeta_m}(x_1,x')-\bar{\Phi}(x_1)|=0.$$

Here $\{\phi_k\}_{k=1}^\infty\subset C^\infty(\overline{D_1\cup D_2})$, $\{V_k\}_{k=1}^\infty$, $\{S_k\}_{k=1}^\infty\subset C^\infty([-1,1])$ *are those functions given in Proposition* 3.2 *for* $\{\zeta_m\}_{m=1}^\infty$ (*cf.* (3.7), (3.9)).

REMARK 5.1. (5.6) in Lemma 5.1 is independent of the choice of the orthonormal systems of eigenfunctions $\{\phi_k\}_{k=1}^q$.

PROOF OF LEMMA 5.1. By using the definition, we have

$$\|P_\zeta\Phi\|_{L^\infty(\Omega(\zeta))}\leqq\sum_{k=1}^q\{\|\Phi\|_{L^\infty(\Omega(\zeta))}\|\phi_{k,\zeta}\|_{L^1(\Omega(\zeta))}\|\phi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))} + \|\Phi\|_{L^\infty(\Omega(\zeta))}\|\psi_{k,\zeta}\|_{L^1(\Omega(\zeta))}\|\psi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}\}.$$

By Proposition 3.2, there exist $\zeta_*'>0$, $d'(q)>0$ such that

(5.8) $$\sup_{x\in\Omega(\zeta)} \|\phi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))} \leqq d'(q),$$

(5.9) $$\|\phi_{k,\zeta}\|_{L^\infty(Q(\zeta))} \sim O(\zeta^{-(n-1)/2}),$$

(5.10) $$\|\phi_{k,\zeta}\|_{L^1(Q(\zeta))} \sim O(\zeta^{(n-1)/2})$$

for any $\zeta\in(0,\zeta'_*)$, $1\leqq k\leqq q$. By applying these estimates to the first inequality, we conclude (5.3). By Proposition 3.2, for any subsequence $\{\sigma'_m\}_{m=1}^\infty\subset\{\zeta_m\}_{m=1}^\infty$, we can take a subsequence $\{\sigma_m\}_{m=1}^\infty\subset\{\sigma'_m\}_{m=1}^\infty$ and $\{\phi_k\}_{k=1}^\infty\subset C^\infty(\overline{D_1\cup D_2})$, $\{S_k\}_{k=1}^\infty$, $\{V_k\}_{k=1}^\infty\subset C^\infty([-1,1])$ in Proposition 3.2.

(5.11) $$\begin{aligned}&(P_\zeta\Phi_\zeta)(x)-\Big(\sum_{k=1}^q\int_{D_1\cup D_2}\phi_k\Phi dx\Big)\phi_k(x)\\&=\sum_{k=1}^q\Big\{\Big(\int_{D_1\cup D_2}\phi_{k,\zeta}\Phi_\zeta dx\Big)\phi_{k,\zeta}(x)-\Big(\int_{D_1\cup D_2}\phi_k\Phi dx\Big)\phi_k(x)\Big\}\\&\quad+\sum_{k=1}^q\Big(\int_{Q(\zeta)}\phi_{k,\zeta}\Phi_\zeta dx\Big)\phi_{k,\zeta}(x)+\sum_{k=1}^q\Big(\int_{\Omega(\zeta)}\phi_{k,\zeta}\Phi_\zeta dx\Big)\phi_{k,\zeta}(x).\end{aligned}$$

We put $\zeta=\sigma_m$ and let $m\to\infty$ in (5.11). From the properties (3.7), (5.8) and $\lim_{\zeta\to0}\mathrm{Vol}(Q(\zeta))=0$, the first and second terms tend to 0 as $m\to\infty$, uniformly in $D_1\cup D_2$. From (3.10)$\sim$(3.12), it follows that

(5.12) $$\|\phi_{k,\zeta}\|_{L^\infty(D_1\cup D_2)} \sim O(\zeta^{-(n-3)/2}),$$

and then the third term in (5.11) tends to 0 as $m\to\infty$, uniformly in $D_1\cup D_2$. Thus by the arbitrariness of the choice of $\{\sigma'_m\}_{m=1}^\infty$, we conclude (5.6). By (3.11) and (3.12), there exist positive constants $d''(q)>0$ and $\zeta''_*>0$ such that

(5.13) $$\frac{|\phi_{k,\zeta}(x)|}{d_{n-1}^{1/2}\zeta^{(n-1)/2}}\leqq d''(q)\Big(1+\frac{1}{|x-p_i|^{n-2}}\Big)\quad\text{for } x\in\bigcup_{i=1}^2(D_i\backslash\Sigma_i(3\zeta)),$$

for $\zeta\in(0,\zeta''_*)$, $1\leqq k\leqq q$. Applying some a-priori estimates of elliptic equations (cf. [12; Prop. 3.2]), repeatedly to $\phi_{k,\zeta}/(d_{n-1}^{1/2}\zeta^{(n-1)/2})$ $(\zeta>0)$, we obtain the compactness of these functions by a diagonal argument. Then we get a subsequence $\{\sigma_m\}_{m=1}^\infty\subset\{\sigma'_m\}_{m=1}^\infty$ and $W_k\in C^\infty\Big(\bigcup_{i=1}^2(\bar{D}_i\backslash\{p_i\})\Big)$ satisfying (5.5) and

$$|W_k(x)|\leqq d''(q)\Big(1+\frac{1}{|x-p_i|^{n-2}}\Big)\quad\text{for any } x\in D_1\cup D_2.$$

$$(5.14)\quad \begin{aligned}&\Big|(P_\zeta\Phi_\zeta)(x_1,x')\\&-\sum_{k=1}^{q}\Big\{\Big(\int_{-1}^{1}S_k\bar{\Phi}dz+\int_{D_1\cup D_2}W_k\Phi dx\Big)S_k(x_1)+\Big(\int_{D_1\cup D_2}\phi_k\Phi dx\Big)V(x_1)\Big\}\Big|\\&\leqq\sum_{k=1}^{q}\Big|\Big(\int_{Q(\zeta)}\psi_{k,\zeta}\Phi_\zeta dx\Big)\psi_{k,\zeta}(x)-\Big(\int_{-1}^{1}S_k\bar{\Phi}dz\Big)S_k(x_1)\Big|\\&+\sum_{k=1}^{q}\Big|\Big(\int_{D_1\cup D_2}\phi_{k,\zeta}\Phi_\zeta dx\Big)\phi_{k,\zeta}(x)-\Big(\int_{D_1\cup D_2}\phi_k\Phi dx\Big)V_k(x_1)\Big|\\&+\sum_{k=1}^{q}\Big|\Big(\int_{D_1\cup D_2}\psi_{k,\zeta}\Phi_\zeta dx\Big)\psi_{k,\zeta}(x)-\Big(\int_{D_1\cup D_2}W_k\Phi dx\Big)S_k(x_1)\Big|\\&+\sum_{k=1}^{q}\Big|\Big(\int_{Q(\zeta)}\phi_{k,\zeta}\Phi_\zeta dx\Big)\phi_{k,\zeta}(x)\Big|.\end{aligned}$$

Similarly, put $\zeta=\sigma_m$ and let $m\to\infty$ in (5.14). By (3.7), (3.8), (5.8) and $\lim_{\zeta\to 0}\mathrm{Vol}(Q(\zeta))=0$, the supremum of the first, second and forth terms of the right side of (5.14) in $Q(\sigma_m)$ tend to 0 as $m\to\infty$. On the other hand, from (5.5) and the Lebesgue's convergence theorem, we have

$$\lim_{m\to\infty}\int_{D_1\cup D_2}\frac{\psi_{k,\sigma_m}(x)}{d_{n-1}^{1/2}\sigma_m^{(n-1)/2}}\Phi_{\sigma_m}(x)dx=\int_{D_1\cup D_2}W_k(x)\Phi(x)dx.$$

Then from (3.8), the supremum of the third term of the right side of (5.14) in $Q(\sigma_m)$ tends to 0. By this, we conclude (5.7) and complete the proof of Lemma 5.1.

We apply the Lyapunov-Schmidt procedure to seek for the solution v_ζ of $(1.1)_\zeta$ in the following form:

$$(5.15)\qquad v_\zeta(x)=A_\zeta(x)+\Phi_\zeta^{(1)}(x)+\Phi_\zeta^{(2)}(x),$$

where $\Phi_\zeta^{(i)}\in X_i(\zeta)$ $(i=1,2)$.

We project the equation (1.1) to the subspaces $X_1(\zeta)$ and $X_2(\zeta)$ respectively, according to (5.15), i.e.,

$$(5.16)\quad \begin{aligned}&\Delta\Phi_\zeta^{(1)}+f'(A_\zeta)\Phi_\zeta^{(1)}\\&+P_\zeta\Big(f(A_\zeta+\Phi_\zeta^{(1)}+\Phi_\zeta^{(2)})-f(A_\zeta)-f'(A_\zeta)(\Phi_\zeta^{(1)}+\Phi_\zeta^{(2)})+g_\zeta\Big)=0\quad\text{in }\Omega(\zeta),\end{aligned}$$

$$(5.17)\quad \begin{cases}\Delta\Phi_\zeta^{(2)}+f'(A_\zeta)\Phi_\zeta^{(2)}\\ +(I-P_\zeta)\Big(f(A_\zeta+\Phi_\zeta^{(1)}+\Phi_\zeta^{(2)})-f(A_\zeta)-f'(A_\zeta)(\Phi_\zeta^{(1)}+\Phi_\zeta^{(2)})+g_\zeta\Big)=0\\ \qquad\qquad\text{in }\Omega(\zeta),\\ \partial\Phi_\zeta^{(2)}/\partial\nu=0\qquad\text{on }\partial\Omega(\zeta),\end{cases}$$

where $g_\zeta(x)=\Delta A_\zeta(x)+f(A_\zeta(x))$.
From Lemma 4.1, we have $\lim_{\zeta\to 0}\sup_{x\in\Omega(\zeta)}|g_\zeta(x)|=0$.

We parametrize $\Phi_\zeta^{(1)}$ as follows (and we denote it by $\Phi_{T,\zeta}^{(1)}$):

$$\Phi_{T,\zeta}^{(1)}(x)=\sum_{k=1}^{q}(\tau_k\phi_{k,\zeta}(x)+\tau_{q+k}\tilde{\phi}_{k,\zeta}(x))\in X_1(\zeta),$$

where $\tilde{\phi}_{k,\zeta}(x)=\phi_{k,\zeta}(x)/\|\phi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}$ and $T=(\tau_1,\tau_2,\cdots,\tau_{2q})$. We easily see from Proposition 3.2, that there exists a positive constants $\bar{\zeta}_*\in(0,\zeta_*)$ and $\bar{d}(q)>0$ such that

$$0<1/\bar{d}(q)\leqq\|\Phi_{T,\zeta}^{(1)}\|_{L^\infty(\Omega(\zeta))}/|T|\leqq\bar{d}(q)\qquad(0<\zeta<\bar{\zeta}_*),$$

where $|T|=\left(\sum_{k=1}^{2q}|\tau_k|^2\right)^{1/2}$. By this we can use T as the parameter in place of $\Phi_{T,\zeta}^{(1)}$.

Fix q large so that

$$(5.18)\qquad\left(\begin{array}{l}\min(\omega_{q+1},\lambda_{q+1})\geqq 2\sup_{\xi\in R}|f'(\xi)|+4\\ \omega_{q+1}>\omega_q.\end{array}\right.$$

First we seek for the solution of (5.17) for given $\Phi_{T,\zeta}^{(1)}\in X_1(\zeta)$.

LEMMA 5.2. *There exists a constant $\delta_0>0$ such that for any $\Phi_{T,\zeta}^{(1)}\in X_1(\zeta)$ $(|T|\leqq\delta_0,\ \zeta\in(0,\delta_0))$, there exists a unique solution $\Phi_{T,\zeta}^{(2)}\in X_2(\zeta)$ which satisfies*

$$(5.19)\qquad\lim_{\delta\to 0}\sup_{0<\zeta<\delta,|T|\leqq\delta}\|\Phi_{T,\zeta}^{(2)}\|_{L^2(\Omega(\zeta))}=0.$$

PROOF OF LEMMA 5.2. We define the functional J_ζ on $X(\zeta)$ by

$$J_\zeta(u)=\int_{\Omega(\zeta)}\left(\frac{1}{2}|\nabla u(x)|^2-\int_{A_\zeta(x)}^{u(x)}f(\xi)d\xi\right)dx\qquad\text{for } u\in X(\zeta).$$

We will find the solution of the equation (5.17) as the minimizer of the functional $J_\zeta(A_\zeta+\Phi_\zeta^{(1)}+\Phi_\zeta^{(2)})$ in $\Phi_\zeta^{(2)}\in X_2(\zeta)$ for small $\Phi_\zeta^{(1)}\in X_1(\zeta)$. By a simple calculation, we have,

$$(5.20)\qquad J_\zeta(A_\zeta+\Phi_\zeta^{(1)}+\Phi_\zeta^{(2)})-J_\zeta(A_\zeta+\Phi_\zeta^{(1)})$$
$$=\int_{\Omega(\zeta)}\left(\frac{1}{2}|\nabla\Phi_\zeta^{(2)}|^2-\int_{A_\zeta+\Phi_\zeta^{(1)}}^{A_\zeta+\Phi_\zeta^{(1)}+\Phi_\zeta^{(2)}}(f(\xi)-f(A_\zeta+\Phi_\zeta^{(1)}))d\xi\right.$$
$$\left.-\Phi_\zeta^{(2)}(\Delta\Phi_\zeta^{(1)}+f(A_\zeta+\Phi_\zeta^{(1)})-f(A_\zeta)+g_\zeta)\right)dx$$

$$\geqq \int_{\Omega(\zeta)} \left\{ \frac{1}{2} |\nabla \Phi_\zeta^{(2)}|^2 - \frac{1}{2} \sup_{\xi \in R} |f'(\xi)| |\Phi_\zeta^{(2)}|^2 - \frac{1}{4} |\Phi_\zeta^{(2)}|^2 \right.$$
$$\left. - |\Delta \Phi_\zeta^{(1)} + f(A_\zeta + \Phi_\zeta^{(1)}) - f(A_\zeta) + g_\zeta|^2 \right\} dx$$
$$\geqq \|\Phi_\zeta^{(2)}\|_{L^2(\Omega(\zeta))}^2 - \|\Delta \Phi_\zeta^{(1)} + f(A_\zeta + \Phi_\zeta^{(1)}) - f(A_\zeta) + g_\zeta\|_{L^2(\Omega(\zeta))}^2.$$

Thus for fixed $\Phi_\zeta^{(1)} = \Phi_{T,\zeta}^{(1)}$, we can find, by (5.20), a minimizer $\Phi_{T,\zeta}^{(2)}$ of $J_\zeta(A_\zeta + \Phi_{T,\zeta}^{(1)} + \Phi_\zeta^{(2)})$ in $X_2(\zeta)$. By $\lim_{\zeta \to 0} \sup_{x \in \Omega(\zeta)} |g_\zeta(x)| = 0$ and the relation between $\Phi_{T,\zeta}^{(1)}$ and $T = (\tau_1, \cdots, \tau_{2q})$, and $\Delta \Phi_{T,\zeta}^{(1)} = -\sum_{k=1}^{q} (\tau_k \omega_k(\zeta) \phi_{k,\zeta} + \tau_{q+k} \lambda_k(\zeta) \tilde{\phi}_{k,\zeta})$, we can easily deduce (5.19) for any minimizer $\Phi_{T,\zeta}^{(2)}$. If there exist two solutions $\Phi_{T,\zeta}^{(2)}$, $\bar{\Phi}_{T,\zeta}^{(2)}$ of (5.17) for fixed $\Phi_{T,\zeta}^{(1)}$, we deduce an equality for the difference of two solutions by (5.17) and integrate it over $\Omega(\zeta)$ after multiplying it by $\Phi_{T,\zeta}^{(2)} - \bar{\Phi}_{T,\zeta}^{(2)}$ and we have

$$\int_{\Omega(\zeta)} \Big(|\nabla (\Phi_{T,\zeta}^{(2)} - \bar{\Phi}_{T,\zeta}^{(2)})|^2$$
$$- (f(A_\zeta + \Phi_{T,\zeta}^{(1)} + \Phi_{T,\zeta}^{(2)}) - f(A_\zeta + \Phi_{T,\zeta}^{(1)} + \bar{\Phi}_{T,\zeta}^{(2)}))(\Phi_{T,\zeta}^{(2)} - \bar{\Phi}_{T,\zeta}^{(2)}) \Big) dx = 0.$$

In view of (5.18), let $\delta_0 > 0$ be a constant such that

$$\min(\omega_{q+1}(\zeta),\ \lambda_{q+1}(\zeta)) \geqq 2 \sup_{\xi \in R} |f'(\xi)| + 2 \qquad \text{for any } \zeta \in (0, \delta_0).$$

Then we conclude $\Phi_{T,\zeta}^{(2)} \equiv \bar{\Phi}_{T,\zeta}^{(2)}$ in $\Omega(\zeta)$. Thus we complete the proof of Lemma 5.2.

LEMMA 5.3.

$$\lim_{\delta \to 0} \sup_{|T| \leqq \delta, \zeta \in (0,\delta)} \|\Phi_{T,\zeta}^{(2)}\|_{L^\infty(\Omega(\zeta))} = 0.$$

PROOF OF LEMMA 5.3. (First step) We will prove

$$\limsup_{\zeta \to 0} \sup_{|T| \leqq \delta} \|\Phi_{T,\zeta}^{(2)}\|_{L^\infty(\Omega(\zeta))} < +\infty \qquad \text{for any } \delta > 0.$$

Assume that there exist $\delta > 0$ and $\{T_m\}_{m=1}^\infty$, $\{\zeta_m\}_{m=1}^\infty$ such that

$$|T_m| \leqq \delta,\ \lim_{m \to \infty} \zeta_m = 0 \quad \text{and} \quad \lim_{m \to \infty} \|\Phi_{T_m,\zeta_m}^{(2)}\|_{L^\infty(\Omega(\zeta_m))} = \infty.$$

Put $\tilde{\Phi}_{T,\zeta}^{(2)}(x) = \Phi_{T,\zeta}^{(2)}(x) / \|\Phi_{T,\zeta}^{(2)}\|_{L^\infty(\Omega(\zeta))}$. We investigate the asymptotic behavior of $\tilde{\Phi}_{T_m,\zeta_m}^{(2)}$. From (5.17), $\tilde{\Phi}_{T,\zeta}^{(2)}$ satisfies.

(5.21)
$$\Delta\tilde{\Phi}^{(2)}_{T,\zeta}+f'(A_\zeta)\tilde{\Phi}^{(2)}_{T,\zeta}+(I-P_\zeta)\left\{\frac{f(A_\zeta+\Phi^{(1)}_{T,\zeta}+\Phi^{(2)}_{T,\zeta})-f(A_\zeta)}{a_{T,\zeta}}-f'(A_\zeta)\left(\frac{\Phi^{(1)}_{T,\zeta}}{a_{T,\zeta}}+\tilde{\Phi}^{(2)}_{T,\zeta}\right)+\frac{g_\zeta}{a_{T,\zeta}}\right\}=0 \quad \text{in } \Omega(\zeta),$$

(5.22)
$$\partial\tilde{\Phi}^{(2)}_{T,\zeta}/\partial\nu=0 \quad \text{on } \partial\Omega(\zeta),$$

where $a_{T,\zeta}=\|\Phi^{(2)}_{T,\zeta}\|_{L^\infty(\Omega(\zeta))}$.
We can not apply Proposition 3.1 directly to (5.21) and (5.22) because (5.21) contains the integral operator P_ζ, but by the good properties (5.2), (5.5), (5.6), we can prove a result similar to Proposition 3.1 as for (5.21) and (5.22) by recovering the arguments in [10] and [11] by using $\lim_{m\to\infty} a_{T_m,\zeta_m}=\infty$ and $\|\tilde{\Phi}^{(2)}_{T,\zeta}\|_{L^\infty(\Omega(\zeta))}=1$. There exist subsequence $\{m(j)\}_{j=1}^\infty$, $\Phi\in C^\infty(\overline{D_1\cup D_2})$, $\bar{\Phi}\in C^\infty([-1,1])$ such that $\Phi(p_1)=\bar{\Phi}(1)$, $\Phi(p_2)=\bar{\Phi}(-1)$ and that

(5.23)
$$\begin{cases}\lim_{j\to\infty}\sup_{x\in D_1\cup D_2}|\tilde{\Phi}^{(2)}_{T_{m(j)},\zeta_{m(j)}}(x)-\Phi(x)|=0\\ \lim_{j\to\infty}\sup_{x\in Q(\zeta_{m(j)})}|\tilde{\Phi}^{(2)}_{T_{m(j)},\zeta_{m(j)}}(x_1,x')-\bar{\Phi}(x)|=0.\end{cases}$$

By $\lim_{m\to\infty}\|\tilde{\Phi}^{(2)}_{T_m,\zeta_m}\|_{L^2(\Omega(\zeta_m))}=0$, we see $\Phi\equiv 0$ in $D_1\cup D_2$. We consider the limitation $\zeta\to 0$ in (5.21) and (5.22). From Lemma 5.1 there exist subsequence of $\{m(j)\}_{j=1}^\infty$ (which we also denote by $\{m(j)\}_{j=1}^\infty$) and $\bar{\Phi}\in C^\infty([-1,1])$ satisfying the following equation: Putting $\zeta=\zeta_{m(j)}$, $T=T_{m(j)}$, and letting $j\to\infty$ in (5.21) and (5.22), we have

(5.24)
$$\begin{cases}d^2\bar{\Phi}/dz^2+f'(V)\bar{\Phi}+\left\{-f'(V)\bar{\Phi}+\sum_{k=1}^{q}\left(\int_{-1}^{1}S_kf'(V)\bar{\Phi}dz\right)S_k\right\}=0, \ -1<z<1\\ \bar{\Phi}(1)=\bar{\Phi}(-1)=0.\end{cases}$$

On the other hand, we have, for any k $(1\leqq k\leqq q)$,

$$\int_{\Omega(\zeta_{m(j)})}\tilde{\Phi}^{(2)}_{T_{m(j)},\zeta_{m(j)}}d_{n-1}^{-1/2}\zeta_{m(j)}^{-(n-1)/2}\psi_{k,\zeta_{m(j)}}dx=0.$$

Let $j\to\infty$ in the above equation. By using (5.3), (5.19) and the convergence of $\tilde{\Phi}^{(2)}_{T_{m(j)},\zeta_{m(j)}}$, we see $(\bar{\Phi}\cdot S_k)_{L^2([-1,1])}=0$ for any k such that $1\leqq k\leqq q$. Multiply (5.24) by $\bar{\Phi}$ and integrate over $[-1,1]$ and then we get $\int_{-1}^{1}|\partial\bar{\Phi}/\partial z|^2dz=0$ and then from $\bar{\Phi}(\pm 1)=0$, we conclude $\bar{\Phi}\equiv 0$ in $[-1,1]$.

But from $\|\tilde{\Phi}^{(2)}_{T,\zeta}\|_{L^\infty(\Omega(\zeta))}=1$ and (5.23), it follows that $\max(\|\Phi\|_{L^\infty(D_1\cup D_2)}, \|\bar{\Phi}\|_{L^\infty([-1,1])})=1$. This is a contradiction and we complete the first step.

(Second step) We will prove the conclusion. Assume the contrary, i.e., there exist sequences $\{T_m\}_{m=1}^\infty$, $\{\zeta_m\}_{m=1}^\infty$ and a constant $d>0$ such that

$$(5.25)\qquad \begin{cases} \lim_{m\to\infty}|T_m|=0, \quad \lim_{m\to\infty}\zeta_m=0, \\ 0<1/d\leqq\|\Phi^{(2)}_{T_m,\zeta_m}\|_{L^\infty(\Omega(\zeta_m))}\leqq d \quad (m\geqq 1). \end{cases}$$

In the same manner as in the first step, we can apply an analogue of Proposition 3.1 in (5.17) with (5.25) for $\Phi^{(2)}_\zeta=\Phi^{(2)}_{T_m,\zeta_m}$ and we obtain a subsequence $\{m(j)\}_{j=1}^\infty$ and functions $\Phi\in C^\infty(\overline{D_1\cup D_2})$, $\bar{\Phi}\in C^\infty([-1,1])$ such that

$$(5.26)\qquad \begin{cases} \lim_{j\to\infty}\sup_{x\in D_1\cup D_2}|\Phi^{(2)}_{T_{m(j)},\zeta_{m(j)}}(x)-\Phi(x)|=0, \\ \lim_{j\to\infty}\sup_{x\in Q(\zeta_{m(j)})}|\Phi^{(2)}_{T_{m(j)},\zeta_{m(j)}}(x_1,x')-\bar{\Phi}(x_1)|=0, \\ \Phi(p_1)=\bar{\Phi}(1), \quad \Phi(p_2)=\bar{\Phi}(-1). \end{cases}$$

Applying Lemma 5.1 again, we obtain the following equation satisfied by Φ.

$$(5.27)\qquad \begin{cases} \Delta\Phi+f'(w)\Phi+(I-P_0)\Big(f(w+\Phi)-f(w)-f'(w)\Phi\Big)=0 & \text{in } D_1\cup D_2 \\ \partial\Phi/\partial\nu=0 \quad \text{on } \partial D_1\cup\partial D_2, \end{cases}$$

where $P_0\Psi(x)=\sum_{k=1}^{q}\left(\int_{D_1\cup D_2}\phi_k\Psi dx\right)\phi_k(x)$ for $x\in D_1\cup D_2$. Multiplying (5.27) by Φ and integrating over $D_1\cup D_2$, we get

$$\int_{D_1\cup D_2}\Big(|\nabla\Phi|^2-(f(w+\Phi)-f(w))\Phi\Big)dx=0.$$

Therefore

$$\int_{D_1\cup D_2}\Big(|\nabla\Phi|^2-\sup_{\xi\in R}|f'(\xi)||\Phi|^2\Big)dx\leqq 0.$$

But, by (5.2), (5.18) and (3.7), (5.26), we have $\omega_{q+1}\geqq 2\sup_{\xi\in R}|f'(\xi)|+4$ and $(\Phi\cdot\phi_k)_{L^2(D_1\cup D_2)}=0$ $(1\leqq k\leqq q)$. Hence we conclude $\Phi\equiv 0$. Simultaneously we also have the following equation by Lemma 5.1 with $\Phi\equiv 0$.

$$(5.28)\qquad \begin{cases} d^2\bar{\Phi}/dz^2+(f(V+\bar{\Phi})-f(V))\bar{\Phi}=0, \quad -1<z<1 \\ \bar{\Phi}(1)=\bar{\Phi}(-1)=0. \end{cases}$$

To deduce (5.28), we have used $\int_{-1}^{1} S_k\bar{\Phi}dz=0$ $(1\leqq k\leqq q)$ which can be proved as in the first step. Similarly, multiplying (5.28) by $\bar{\Phi}$ and integrating over $[-1, 1]$, we have

$$\int_{-1}^{1}\left(|d\bar{\Phi}/dz|^2-\sup_{\xi\in R}|f'(\xi)||\bar{\Phi}|^2\right)dz\leqq 0$$

and by using $\lambda_{q+1}\geqq 2\sup_{\xi\in R}|f'(\xi)|+4$, we conclude $\bar{\Phi}\equiv 0$. This contradicts the fact

$$\max\{\|\Phi\|_{L^\infty(D_1\cup D_2)}, \|\bar{\Phi}\|_{L^\infty([-1,1])}\}\geqq 1/d \qquad \text{(cf. (5.25))}.$$

We complete the proof of Lemma 5.3.

LEMMA 5.4.

$$\lim_{\delta\to 0}\sup_{|T|\leqq\delta,\zeta\in(0,\delta)}\left\|\frac{\partial\Phi_{T,\zeta}^{(2)}}{\partial\tau_k}\right\|_{L^\infty(\Omega(\zeta))}=0 \quad (1\leqq k\leqq 2q).$$

Differentiating (5.17) by τ_k, we have the following equation which $\frac{\partial\Phi_{T,\zeta}^{(2)}}{\partial\tau_k}$ satisfies.

$$(5.29)\quad \begin{cases} \Delta\left(\dfrac{\partial\Phi_{T,\zeta}^{(2)}}{\partial\tau_k}\right)+f'(A_\zeta)\dfrac{\partial\Phi_{T,\zeta}^{(2)}}{\partial\tau_k} \\ +(I-P_\zeta)\left\{\left(f'(A_\zeta+\Phi_{T,\zeta}^{(1)}+\Phi_{T,\zeta}^{(2)})-f'(A_\zeta)\right)\left(\dfrac{\partial\Phi_{T,\zeta}^{(1)}}{\partial\tau_k}+\dfrac{\partial\Phi_{T,\zeta}^{(2)}}{\partial\tau_k}\right)\right\}=0 \text{ in } \Omega(\zeta) \\ \dfrac{\partial}{\partial\nu}\left(\dfrac{\partial\Phi_{T,\zeta}^{(2)}}{\partial\tau_k}\right)=0 \qquad \text{on } \partial\Omega(\zeta). \end{cases}$$

We can carry out the completely same procedure for $\frac{\partial\Phi_{T,\zeta}^{(2)}}{\partial\tau_k}$ as the first and the second steps in Lemma 5.3. Therefore we omit the proof.

By the properties in Lemma 5.2 and Lemma 5.3, we can reduce the equation to a finite dimensional problem by multiplying (5.16) by $\phi_{k,\zeta}$ and $\|\phi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}\phi_{k,\zeta}$ $(1\leqq k\leqq q)$ and we obtain the following equation.

$$(5.30)\qquad E_\zeta(T)=\begin{pmatrix} F_{1,\zeta}(T) \\ \vdots \\ F_{2q,\zeta}(T)\end{pmatrix}=\begin{pmatrix} 0 \\ \vdots \\ 0\end{pmatrix}$$

where we put, for $1\leqq k\leqq q$,

$$F_{k,\zeta}(T)=-\omega_k(\zeta)\tau_k+\int_{\Omega(\zeta)}\phi_{k,\zeta}(x)G_{T,\zeta}(x)dx$$
$$F_{q+k,\zeta}(T)=-\lambda_k(\zeta)\tau_{q+k}+\int_{\Omega(\zeta)}\|\psi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}\psi_{k,\zeta}(x)G_{T,\zeta}(x)dx$$
$$G_{T,\zeta}=f(A_\zeta+\Phi^{(1)}_{T,\zeta}+\Phi^{(2)}_{T,\zeta})-f(A_\zeta)-f'(A_\zeta)(\Phi^{(1)}_{T,\zeta}+\Phi^{(2)}_{T,\zeta})+g_\zeta.$$

From Lemma 5.3 and $\lim_{\zeta\to 0}\sup_{x\in\Omega(\zeta)}|g_\zeta(x)|=0$, it is easy to see that

$$\lim_{\delta\to 0}\sup_{\zeta\in(0,\delta),|T|\leqq\delta}\Big(\sup_{x\in\Omega(\zeta)}|G_{T,\zeta}(x)|\Big)=0. \tag{5.31}$$

Then by using (3.12) and (3.13), we also have

$$\lim_{\delta\to 0}\sup_{\zeta\in(0,\delta),|T|\leqq\delta}|E_\zeta(T)|=0. \tag{5.32}$$

To apply the standard inverse function theorem of finite dimensional type, we investigate the Jacobian matrix of $E_\zeta(T)$ as follows:

$$\frac{\partial F_{i,\zeta}}{\partial\tau_j}(T)=-\delta_{i,j}\omega_i(\zeta)+\int_{\Omega(\zeta)}\phi_{k,\zeta}(x)\frac{\partial G_{T,\zeta}}{\partial\tau_j}(x)dx,$$
$$\frac{\partial F_{q+i,\zeta}}{\partial\tau_j}(T)=-\delta_{q+i,j}\lambda_i(\zeta)+\int_{\Omega(\zeta)}\|\psi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}\psi_{k,\zeta}(x)\frac{\partial G_{T,\zeta}}{\partial\tau_j}(x)dx,$$
$$\frac{\partial G_{T,\zeta}}{\partial\tau_j}=(f'(A_\zeta+\Phi^{(1)}_{T,\zeta}+\Phi^{(2)}_{T,\zeta})-f'(A_\zeta))\Big(\phi_{j,\zeta}+\frac{\partial\Phi^{(2)}_{T,\zeta}}{\partial\tau_j}\Big)\quad(1\leqq j\leqq q),$$
$$\frac{\partial G_{T,\zeta}}{\partial\tau_{q+j}}=(f'(A_\zeta+\Phi^{(1)}_{T,\zeta}+\Phi^{(2)}_{T,\zeta})-f'(A_\zeta))\Big(\tilde{\psi}_{j,\zeta}+\frac{\partial\Phi^{(2)}_{T,\zeta}}{\partial\tau_{q+j}}\Big)\quad(1\leqq j\leqq q),$$

$$\left|\left(\frac{\partial F_{i,\zeta}}{\partial\tau_j}(0)\right)_{1\leqq i,j\leqq 2q}+\begin{pmatrix}\omega_1(\zeta) & & & & & 0\\ & \omega_2(\zeta) & & & & \\ & & \ddots & & & \\ & & & \omega_q(\zeta) & & \\ & & & & \lambda_1(\zeta) & \\ 0 & & & & & \ddots \\ & & & & & & \lambda_q(\zeta)\end{pmatrix}\right| \tag{5.33}$$
$$\leqq c\sup_{x\in\Omega(\zeta)}\left|\frac{\partial G_{T,\zeta}}{\partial\tau_j}\right|\max_{1\leqq k\leqq q}\Big(\|\phi_{k,\zeta}\|_{L^1(\Omega(\zeta))},\quad\|\psi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}\|\psi_{k,\zeta}\|_{L^1(\Omega(\zeta))}\Big).$$

By Lemma 5.3, Lemma 5.4 and Proposition 3.2, we have

$$\lim_{\delta\to 0}\sup_{\zeta\in(0,\delta),|T|\leqq\delta}\Big(\sup_{x\in\Omega(\zeta)}\left|\frac{\partial G_{T,\zeta}}{\partial\tau_k}(x)\right|\Big)=0\quad(1\leqq k\leqq 2q).$$

Then by (5.2) and (5.33), we have

$$\lim_{\zeta\to 0}\sup_{\zeta\in(0,\delta),|T|\leqq\delta}\left|\frac{\partial E_\zeta}{\partial T}(T)+\begin{pmatrix}\omega_1 & & & & & 0\\ & \omega_2 & & & & \\ & & \ddots\ \omega_q & & & \\ & & & \lambda_1 & & \\ 0 & & & & \ddots & \\ & & & & & \lambda_q\end{pmatrix}\right|=0.$$

Therefore the Jacobian matrix $\frac{\partial E_\zeta}{\partial T}(T)$ is approximated by the non-singular matrix

$$-\begin{pmatrix}\omega_1 & & & & & 0\\ & \omega_2 & & & & \\ & & \ddots\ \omega_q & & & \\ & & & \lambda_1 & & \\ 0 & & & & \ddots & \\ & & & & & \lambda_q\end{pmatrix}\qquad \text{(cf. (II. 5))}.$$

Therefore by the inverse function theorem with (5.32), there exist constants $\delta_1>0$ and $\delta_2>0$ such that the equation (5.30) has a unique solution $|T_\zeta|\leqq\delta_1$ for any $\zeta\in(0,\delta_2)$. Thus we get a solution v_ζ of $(1.1)_\zeta$ by putting

$$v_\zeta(x)=A_\zeta(x)+\Phi^{(1)}_{T_\zeta,\zeta}(x)+\Phi^{(2)}_{T_\zeta,\zeta}(x).$$

Moreover, by the procedure of the construction, it is easy to see that (2.3) and (2.4) hold.

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