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Radial solutions for $\Delta u+|x|^l|u|^{p-1}u=0$ *on the unit ball in* R^n

Dedicated to Professor Hiroshi Fujita on his 60th birthday

By Ken'ichi NAGASAKI

§ 1. Introduction

We are concerned with the radial, i.e. spherically symmetric, solutions of the nonlinear boundary value problem in the unit ball $\Omega=\{x||x|<1\}$ in $\boldsymbol{R}^n$:

$$\text{(P)} \qquad \Delta u+|x|^l|u|^{p-1}u=0 \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega,$$

where $n\geq 2$, $l\geq 0$ and $p>1$.

The radial solution $u=u(r)$ of (P), where $r=|x|$, can be obtained by solving the following ordinary differential equation:

$$\text{(RP)} \qquad \begin{aligned} &(r^{n-1}u')'+r^{l+n-1}|u|^{p-1}u=0 \qquad \text{for } r\in(0,1), \\ &u'(0)=0 \text{ and } u(1)=0. \end{aligned}$$

The equation of the form (RP) with $l=0$ has been known as the Lane-Emden equation in astrophysics from the beginning of this century, where u represents the density of stars. In 1973, Henon [4] proposed the equation with $l\neq 0$ to describe the spherical stellar structure and studied its stability through numerical computation. Moreover the case $n\geq 3$, $l=0$ and $p=(n+2)/(n-2)$ is relevant to Yamabe's problem in differential geometry.

We will review briefly some results obtained up to now for the case $l=0$. First the problem (P) with $n\geq 3$ has a positive solution for $p\in(1,(n+2)/(n-2))$, and, to the contrary, has no nontrivial solution for $p\in[(n+2)/(n-2),\infty)$ ([12]). Next, the positive solution of (P) is unique. In fact any positive solution of (P) must be radial when $l=0$. Further the positive solution of (RP) is unique in this case ([4]).

As for our problem $l\geq 0$ with $n\geq 3$, Ni [9] showed the existence of a positive radial solution of (P) for $p\in(1,(n+2+2l)/(n-2))$, applying the mountain pass lemma, but did not get the uniqueness. On the contrary, when Ω is an annulus $\{x \mid 0<a<|x|<b\}$ instead of the ball, the unique existence of a positive radial solution of (P) with $l=0$ has been established for $p\in(1,(n+2)/(n-2)]$ in the case $n\geq 3$ and for $p\in(1,\infty)$ in the case $n=2$ by Ni [10]. Later his result was improved as follows. Namely, in such an Ω, for each $k\in N$, the problem (P) with $l\in \boldsymbol{R}$ and $p\in(1,\infty)$ has at most one radial solution $u=u(r)$ which has exactly $(k-1)$ zeros in (a,b) and whose derivative at $r=a$ is positive ([11]). There the Sturm's comparison and separation theorem were made use of.

Our main aim of this paper is to study similar problems to Ni-Nussbaum's in [11] and determine the structure of radial solutions for (P) when Ω is the unit ball. The results read as follows.

THEOREM 1. *When $n\geq 3$ and $p\in(1,(n+2+2l)/(n-2))$, for each $k\in N$ there exists a unique radial solution $u=u(r)$ of* (P), *such that $u(0)$ is positive and $u(r)$ has exactly $(k-1)$ zeros in $(0,1)$.*

When $p\in[(n+2+2l)/(n-2),\infty)$, there exists no radial solution of (P) *except for the trivial one.*

THEOREM 2. *When $n=2$, for each $k=N$ there exists a unique radial solution $u=u(r)$ of* (P), *such that $u(0)$ is positive and $u(r)$ has exactly $(k-1)$ zeros in $(0,1)$.*

Our proof will be accomplished by refining the phase plane analysis developed in Chandrasekhar [2] and Joseph-Lundgren [6]. However, we shall make a delicate use of the particular nonlinearity of the equation (RP).

Here it should be noted that Gidas-Ni-Nirenberg's theorem cannot be applied to the problem (P) when $l>0$. Hence the existence or nonexistence of nonradial solutions of (P), is open even for positive ones. To our knowledge the same question for sign-changing nonradial solutions is also open even for $l=0$. However, we are able to derive from Rellich's identity the stronger nonexistence result than the latter half of Theorem 1.

THEOREM 3. *If $p\in[(n+2+2l)/(n-2),\infty)$, there exists no solution of* (P) *other than the trivial one.*

Concluding this section, we note that our method applies to the following nonlinear eigenvalue problem in the unit ball Ω in $\boldsymbol{R}^n$:

$$(\mathrm{P}_\lambda) \qquad \Delta u+\lambda|x|^l e^u=0 \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega,$$

where $n\geq 2$, $l\geq 0$ and $\lambda\geq 0$. Actually, for $n\geq 2$ and $l=0$, Joseph-Lundgren [6] found that the number of radial solutions of (P_λ) varies curiously according to the value λ and n. Following their idea, we show

THEOREM A. (**1**) *When $n\geq 10+4l$, there is a unique radial solution of (P_λ) for $\lambda\in(0, (n-2)(l+2))$, and there is no radial solution for $\lambda\in[(n-2)(l+2), \infty)$.*

(**2**) *When $n\in(2, 10+4l)$, there exists a positive value λ_1 such that (P_λ) has at least one radial solution for $\lambda\in(0, \lambda_1]$ and no solution for $\lambda\in(\lambda_1, \infty)$. Moreover, there exists a positive sequence $\{\lambda_j\}_{j=1}^{\infty}$ which has the following properties*:

1° *$\{\lambda_{2j-1}\}$ and $\{\lambda_{2j}\}$ converge monotonically to $(n-2)(l+2)$ respectively from above and from below,*

2° *there are exactly j solutions of (P_j) for $\lambda=\lambda_j$, $2j$ solutions for $\lambda\in(\lambda_{2j+1}, \lambda_{2j-1})$ and $(2j+1)$ solutions for $\lambda\in(\lambda_{2j}, \lambda_{2j+2})$,*

and

3° *there are infinitely many solutions of (P_λ) for $\lambda=(n-2)(l+2)$.*

(**3**) *When $n=2$, there exists a positive number λ^* such that (P_λ) has two solutions for $\lambda\in(0, \lambda^*)$, one solution for $\lambda=\lambda^*$ and no solution for $\lambda\in(\lambda^*, \infty)$.*

The organization of this paper is as follows. In Section 2 we introduce the phase plane analysis and give the proof of Theorems 1 and 2. In Section 3 we establish Theorem 3 with the aid of Rellich's identity. In Appendix we deal with the problem (P_λ) and prove Theorem A.

§ 2. Proof of Theorems 1 and 2: Phase plane analysis

For the moment, we consider the initial value problem:

$$\text{(IVP)}\qquad \begin{aligned}&(r^{n-1}u')'+r^{l+n-1}|u|^{p-1}u=0 \qquad \text{for } r\in(0,1),\\ &u(0)=A \text{ and } u'(0)=0,\end{aligned}$$

for $A>0$, instead of the boundary value problem (RP). We may assume that A is positive without loss of generality, because the differential equation in (IVP) is odd with respect to u.

First, we introduce the change of variables:

$$(2.1)\qquad u(r)=Av(s) \quad \text{and} \quad r=Bs,$$

where B is determined in the following manner:

[Case I] when $p \in ((n+l)/(n-2), \infty)$, we put

$$B=\{(n-2-\tau)\tau A^{1-p}\}^{1/(2+l)} \quad \text{with} \quad \tau=(2+l)/(p-1),$$

[Case II] when $p=(n+l)/(n-2)$, we put

$$B=A^{(1-p)/(2+l)},$$

[Case III] when $p \in (1, (n+l)/(n-2))$, we put

$$B=\{(\tau+2-n)\tau A^{1-p}\}^{1/(2+l)}.$$

According to the cases I, II and III, the equation in (IVP) is transformed respectively into the equations

$$(2.2)_{\mathrm{I}} \qquad (s^{n-1}v')'+(n-\tau-2)\tau s^{l+n-1}|v|^{p-1}v=0,$$

$$(2.2)_{\mathrm{II}} \qquad (s^{n-1}v')'+s^{l+n-1}|v|^{p-1}v=0,$$

and

$$(2.2)_{\mathrm{III}} \qquad (s^{n-1}v')'+(\tau+2-n)\tau s^{l+n-1}|v|^{p-1}v=0,$$

while the initial conditions in (IVP) are transformed into

$$(2.3) \qquad v(0)=1 \quad \text{and} \quad v'(0)=0.$$

The boundary condition $u(1)=0$ corresponds to the condition

$$(2.4) \qquad v(B^{-1})=0.$$

Next, we make another change of variables due to R. Emden (Chandrasekhar [2, p. 90] e.g.):

$$(2.5) \qquad w(t)=s^{\tau}v(s) \quad \text{and} \quad s=e^{t},$$

which transforms the equations $(2.2)_{\mathrm{I}}$, $(2.2)_{\mathrm{II}}$ and $(2.2)_{\mathrm{III}}$ respectively into the equations

$$(2.6)_{\mathrm{I}} \qquad w''+(n-2\tau-2)w'+(n-\tau-2)\tau(|w|^{p-1}-1)w=0,$$

$$(2.6)_{\mathrm{II}} \qquad w''-(n-2)w'+|w|^{p-1}w=0$$

and

$$(2.6)_{\mathrm{III}} \qquad w''+(n-2\tau-2)w'+(\tau+2-n)\tau(|w|^{p-1}+1)w=0.$$

The initial conditions (2.3) are transformed into

(2.7) $$\lim_{t\to-\infty} e^{-\tau t}w(t)=1 \quad \text{and} \quad \lim_{t\to-\infty} e^{-t}\{e^{-\tau t}w(t)\}'=0.$$

To show the unique existence of a classical solution $w=w(t)$ of (2.6) with (2.7), we prepare the following

LEMMA 2.1. *Let real numbers* α, β *and* ε *be given with the relations*

(2.8) $$\alpha\geq\beta,\ \alpha>\varepsilon^{-1} \quad \text{and} \quad \varepsilon>0,$$

and let $f=f(t)$ *be a* C^1*-function on* $\boldsymbol{R}$ *satisfying*

(2.9) $$|f'(t)|\leq c_1|t|^{\varepsilon}$$

for some positive c_1 *when* $|t|$ *is small enough.*

Then the differential equation

(2.10) $$\varphi''-(\alpha+\beta)\varphi'+\alpha\beta\varphi+f(\varphi)=0$$

with the conditions

(2.11) $$\lim_{t\to-\infty} e^{-\alpha t}\varphi(t)=1 \quad \text{and} \quad \lim_{t\to-\infty} e^{-t}\{e^{-\alpha t}\varphi(t)\}'=0$$

has a unique classical solution $\varphi=\varphi(t)$ *for sufficiently small* t.

PROOF. Put $\psi(t)=e^{-\alpha t}\varphi(t)-1$, then we obtain from (2.10) the equation

(2.10)′ $$\psi''+(\alpha-\beta)\psi'+e^{-\alpha t}f(e^{\alpha t}(\psi+1))=0$$

and from (2.11) the conditions

(2.11)′ $$\lim_{t\to-\infty}\psi(t)=\lim_{t\to-\infty} e^{-t}\psi'(t)=0.$$

Under the assumptions (2.8) and (2.9), the equation (2.10)′ with (2.11)′ turns out to be equivalent to the integral equation

(2.12) $$\psi(t)=-\int_{-\infty}^{t} e^{(\beta-\alpha)\xi}d\xi\int_{-\infty}^{\xi} e^{-\beta\eta}f(e^{\alpha\eta}(\psi(\eta)+1))d\eta.$$

We take a Banach space $X=\{\psi=\psi(t)\,|\,\psi\in C(-\infty,-T],\ \lim_{t\to-\infty}\psi(t)=0\}$ with a supremum norm $\|\cdot\|$, where $T>0$ is a constant chosen later. We take an operator Φ on X such that for $\psi\in X$, $\Phi(\psi)$ is defined with the right-hand side of (2.12).

We shall derive two estimates concerning Φ on a closed subset $K=\{\psi=\psi(t)\,|\,\|\psi\|\leq 1\}$ of X. In what follows, C stands for a generic positive constant and T is a constant chosen large enough.

For $\psi\in K$, we have

$$\|\Phi(\psi)\|\leq\int_{-\infty}^{-T}e^{(\beta-\alpha)\xi}d\xi\int_{-\infty}^{\xi}e^{-\beta\eta}Ce^{(\varepsilon+1)\alpha\eta}d\eta\leq Ce^{-\alpha\varepsilon T}.$$

Further, because of the mean value theorem, we have for $\psi_1\in K$ and $\psi_2\in K$,

$$\|\Phi(\psi_1)-\Phi(\psi_2)\|\leq\int_{-\infty}^{-T}e^{(\beta-\alpha)\xi}d\xi\int_{-\infty}^{\xi}e^{-\beta\eta}Ce^{\alpha\eta}\|\psi_1-\psi_2\|e^{\alpha\varepsilon\eta}d\eta\leq Ce^{-\alpha\varepsilon T}\|\psi_1-\psi_2\|.$$

If T is chosen again so that $Ce^{-\alpha\varepsilon T}<1$, the operator Φ becomes a contraction on K. Hence there exists a fixed point $\psi^*=\psi^*(t)$ of Φ which is unique in K.

Extending the function $\psi^*=\psi^*(t)$ for $t>-T$ as a solution of (2.10)$'$, we obtain a classical solution of (2.10)$'$ with (2.11)$'$.

The uniqueness of the fixed point of Φ in X remains still to be proved, however it can be easily checked by the fact that every solution of (2.10)$'$ with (2.11)$'$ belongs to K for appropriately large T.

Lemma 2.1 with $\alpha=\tau$ and $\beta=\tau+2-n$ in all cases I, II, and III yields the unique existence of a solution $w=w(t)$ of (2.6) with (2.7).

For the solution $w=w(t)$, we put $z(t)=w'(t)$ and trace the orbit $\mathcal{O}=\{(w(t), z(t))\,|\,t>-\infty\}$ in (w, z)-plane. We deal with the case I in the first place. From $(2.6)_{\rm I}$ we have an autonomous system

$$\frac{d}{dt}\begin{pmatrix} w \\ z \end{pmatrix}=\begin{pmatrix} z \\ -(n-2-\tau)\tau(|w|^{p-1}-1)w-(n-2-2\tau)z \end{pmatrix}, \tag{2.13}$$

which has just three singular points $\mathrm{O}(0,0)$ and $\mathrm{S}_{\pm}(\pm1,0)$.

We examine here the behavior of the orbit of (2.13) near singular points, which plays an essential role in later arguments.

PROPOSITION 2.2. (**1**) *The orbit $\mathcal{O}$ tends to* O *along and below the line $z=\tau w$ $(w>0)$ as $t\to-\infty$. Moreover there exists no other orbit which tends to* O *in the right half-plane $\{(w,z)\,|\,w>0\}$ as $t\to-\infty$.*

(**2**) *There exists exactly one orbit which tends to* O *in the right half-plane as $t\to+\infty$. This orbit tends to* O *along $z=(\tau+2-n)w$.*

PROOF. From $(2.6)_{\rm I}$ and (2.7), we have

$$\lim_{t\to-\infty}w(t)=\lim_{t\to-\infty}\{z(t)-\tau w(t)\}=0,$$

noting that for t large enough,

$$w(t)>0$$

and

$$z(t)-\tau w(t)=-e^{-(n-2-\tau)t}\int_{-\infty}^{t}(n-2-\tau)\tau e^{(n-2-\tau)\eta}|w|^{p-1}w d\eta<0.$$

The former part of (**1**) follows easily from these relations.

The linearized system of (2.13) at O can be expressed as

$$\frac{d}{dt}\begin{pmatrix} h \\ k \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ (n-2-\tau)\tau & -(n-2-2\tau) \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}. \tag{2.14}$$

Since the eigenvalues of the matrix in (2.14) are τ and $\tau+2-n$, where $\tau>0$ and $\tau+2-n<0$, there exists exactly two orbit which tends to O as $t\to-\infty$: one along $z=\tau w$ from the right and the other along the same line from the left ([3] Chap. 15). This shows the latter part of (**1**).

The assertion (**2**) is proved in a similar manner.

PROPOSITION 2.3. *When* $p\in((n+2+2l)/(n-2),\infty)$, *every orbit of* (2.13) *near the singular points* $S_\pm$ *necessarily approaches* $S_\pm$ *as* $t\to+\infty$.

In contrast, when $p\in((n+l)/(n-2),(n+2+2l)/(n-2))$, *every orbit of* (2.13) *near* $S_\pm$ *approaches* $S_\pm$ *as* $t\to-\infty$.

PROOF. The linearized system of (2.13) at $S_\pm$ can be expressed as

$$\frac{d}{dt}\begin{pmatrix} h \\ k \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ -(n-2-\tau)(l+2) & -(n-2-2\tau) \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}. \tag{2.15}$$

Let μ_i $(i=1,2)$ denote the eigenvalues of the matrix in (2.15), then we have

$$\mu_1\mu_2=(n-2-\tau)(l+2)>0 \quad \text{and} \quad \mu_1+\mu_2=2\tau+2-n>(<)0$$

according to $p<(>)(n+2+2l)/(n-2)$ respectively. Hence, for $p\in((n+2+2l)/(n-2),\infty)$, the real parts of μ_i $(i=1,2)$ are both negative, and for $p\in((n+l)/(n-2),(n+2+2l)/(n-2))$, they are both positive. At this point the assertion is obvious ([2] Chap. 15).

In the preceding proof we have not cared about whether the eigenvalues μ_i $(i=1,2)$ are real or imaginary because it does not give any effect on our later analysis.

PROPOSITION 2.4. *The orbit $\mathcal{O}$ never meets with itself.*

It has no closed orbit, hence no limit cycle if $p\in(1,\infty)$ and $p\neq(n+2+2l)/(n-2)$.

PROOF. The first assertion is due to the general property of an orbit of autonomous system.

The existence of any closed orbit is excluded by Bendixon's theorem, because

$$\frac{\partial z}{\partial w}+\frac{\partial}{\partial z}\{-(n-2-\tau)(|w|^{p-1}-1)w-(n-2-2\tau)z\}=-(n-2-2\tau)\neq 0$$

for $p\neq(n+2+2l)/(n-2)$.

These propositions suggest that the behavior of $\mathcal{O}$ is quite different in the cases $p>(n+2+2l)/(n-2)$ and $p<(n+2+2l)/(n-2)$. Actually we have the following lemmas which describe three types of its behavior.

LEMMA 2.5. *When $p\in((n+2+2l)/(n-2),\infty)$, the orbit $\mathcal{O}$ never meets the z-axis. It approaches S_+ as $t\to+\infty$.* (cf. Figure 1.)

PROOF. Observing $\dfrac{dz}{dt}=\dfrac{dw}{dt}\cdot\dfrac{dz}{dw}=z\dfrac{dz}{dw}$, we obtain from (2.13) that

$$z\frac{dz}{dw}=-(n-2-2\tau)z-(n-2-\tau)\tau(|w|^{p-1}-1)w, \tag{2.16}$$

or

$$\begin{aligned}\frac{dz}{dw}&=-\frac{n-2-2\tau}{z}\left\{z-\frac{(n-2-\tau)\tau}{-(n-2-2\tau)}(|w|^{p-1}-1)w\right\}\\&\equiv-\frac{n-2-2\tau}{z}\{z-g(w)\}.\end{aligned} \tag{2.16$'$}$$

Let $\mathcal{K}$ and $\mathcal{L}$ denote the curve $z=g(w)$ and the line $z=\tau w$ in (w,z)-plane respectively, then divide the phase plane into four domains $\mathcal{D}_j$ $(j=1,2,3,4)$, where

$$\begin{aligned}&\mathcal{D}_1 : z>0 \text{ and } z>g(w),\\&\mathcal{D}_2 : z>0 \text{ and } z<g(w),\\&\mathcal{D}_3 : z<0 \text{ and } z>g(w),\\&\mathcal{D}_4 : z<0 \text{ and } z<g(w).\end{aligned}$$

From Proposition 2.2, the orbit $\mathcal{O}$ leaves O along and below $\mathcal{L}$ at $t=-\infty$ and goes right and upwards in $\mathcal{D}_2$. Crossing the curve $\mathcal{K}$

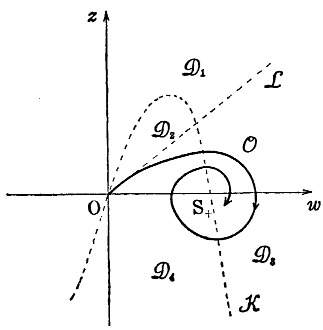

Figure 1.

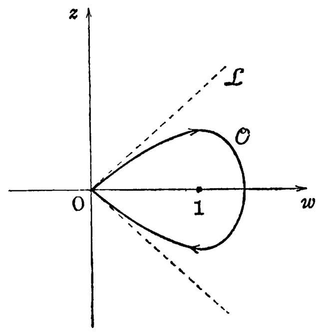

Figure 2.

horizontally, $\mathcal{O}$ goes right and downwards in $\mathcal{D}_1$.

At this point there are three possibilities concerning the behavior of $\mathcal{O}$, that is, an escape to infinity in $\mathcal{D}_1$, an approach to S_+ and a vertical cut of the w-axis. However, in the first possibility, z remains finite, while w and consequently $\left|\frac{dz}{dw}\right|$ tend to infinity from (2.16)′, which is a contradiction. In the second possibility, the assertion follows directly from Proposition 2.3. Therefore we have only to trace $\mathcal{O}$ after its vertical entrance into $\mathcal{D}_3$.

Going left and downwards in $\mathcal{D}_3$, $\mathcal{O}$ eventually meets the curve $\mathcal{K}$ and crosses it horizontally into $\mathcal{D}_4$. Otherwise, w being finite and $|z|$ tending to infinity, $\frac{dz}{dw}$ converges to $-(n-2-2\tau)$ from (2.16)′. This is a contradiction. Now, $\mathcal{O}$ goes left and upwards in $\mathcal{D}_4$, but it neither approaches O, nor crosses the z-axis. This will be shown below.

In fact, assume that $\mathcal{O}$ approaches O, and hence along the line $z=(\tau+2-n)w$ by Proposition 2.2. Then, take a point P_1 in the domain $\mathcal{E}$ enclosed by $\mathcal{O}$ and trace the orbit $\mathcal{O}_1$ starting from P_1 at $t=0$. $\mathcal{O}_1$ remains in $\mathcal{E}$ for any t, but it could approach neither O from Proposition 2.2 (2), nor S_+ from Proposition 2.3 when t tends to $-\infty$. This leads to a contradiction since $\mathcal{O}_1$ has no limit cycle.

Next, assume that $\mathcal{O}$ crosses the z-axis at $P_2(0, \zeta_2)$. Let us take a point $P_3(0, \zeta_3)$ with $\zeta_2<\zeta_3<0$ and trace back the orbit starting from P_3 at $t=0$, which yields the same contradiction.

As a result $\mathcal{O}$ must approach S_+ or cross the w-axis vertically at a point in $0<w<1$. Even in the latter case, a further trace of $\mathcal{O}$ shows that $\mathcal{O}$ eventually approaches S_+ without meeting the z-axis as t tends to infinity.

REMARK 2.6. We can conclude that any solution $u=u(r)$ of (IVP) with $p\in((n+2+2l)/(n-2), \infty)$ is positive for $r\in(0, \infty)$. Moreover, we have

$$u(r)=O(r^{-\tau}) \qquad \text{as } r\to+\infty.$$

LEMMA 2.7. *When* $p=(n+2+2l)/(n-2)$, $\mathcal{O}$ *forms a ring which starts from* O *along the line* $z=\tau w$ $(w>0)$ *and terminates at* O *along the line* $z=-\tau w$ $(w>0)$. (cf. Figure 2.)

PROOF. In this case the equation (2.16) is reduced to

$$z\frac{dz}{dw}=-\tau^2(|w|^{p-1}-1)w. \tag{2.17}$$

Integrating (2.17) and taking into account that (z, w) on $\mathcal{O}$ approaches $(0, 0)$ as $t\to-\infty$, we derive the equation of the curve formed by $\mathcal{O}$, namely

$$z^2+\tau^2\left(\frac{2}{p+1}|w|^{p+1}-w^2\right)=0. \tag{2.18}$$

The assertion follows obviously from (2.18).

REMARK 2.8. We obtain from (2.18),

$$\frac{dw}{dt}=\tau w\left(1-\frac{2}{p+1}w^{p-1}\right)^{1/2}, \tag{2.19}$$

where w and $z=\dfrac{dw}{dt}$ are assumed to be positive.

Owing to the condition (2.7), integration of (2.19) yields

$$1-\frac{2}{p+1}\{w(t)\}^{p-1}=\left\{\frac{2(p+1)-e^{(p-1)\tau t}}{e^{(p-1)\tau t}+2(p+1)}\right\}^2. \tag{2.20}$$

Converting the relation (2.20) into that of $u(r)$ through (2.1) and (2.5), we finally obtain the solution of (IVP) with $p=(n+2+2l)/(n-2)$:

$$u(r)=A(1+Dr^{l+2})^{-2/(p-1)},$$

where $D=A^{p-1}/\{(n-2)(n+l)\}$.

LEMMA 2.9. *When* $p\in((n+l)/(n-2),\ (n+2+2l)/(n-2))$, *the orbit* $\mathcal{O}$ *leaves spirally away from* O *as* t *increases, crossing the negative and the positive part of the* z*-axis alternately.* (cf. Figure 3.)

PROOF. We define $g(w)$, $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{D}_j$ $(j=1,2,3,4)$ by the same as before, and further $\mathcal{D}_j^{\pm}$ by

$$\mathcal{D}_j^{\pm}=\mathcal{D}_j\cap\{(w,z)\mid \pm w>0\}.$$

Similarly in the proof of Lemma 2.5, the orbit $\mathcal{O}$, starting from O, goes through $\mathcal{D}_1^+$ and $\mathcal{D}_2^+$ until it crosses the w-axis vertically. In $\mathcal{D}_4^+$, $\mathcal{O}$ goes left and downwards.

Assume that $\mathcal{O}$ enters into $\mathcal{D}_3^+$, then we shall see that it necessarily leaves $\mathcal{D}_3^+$ and enters into $\mathcal{D}_4^+$ again. Actually, if not so, it would tend to O along $z=(\tau+2-n)w$ or cross the w-axis at a point in $0<w<1$. However the first case is impossible from the same reasoning as in Lemma 2.5. Noting that S_+ is not an attractor at $t=+\infty$ from Proposition 2.3 and that $\mathcal{O}$ has no limit cycle, we also eliminate the possibility of the second case.

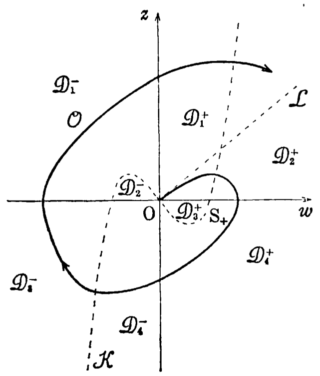

Figure 3.

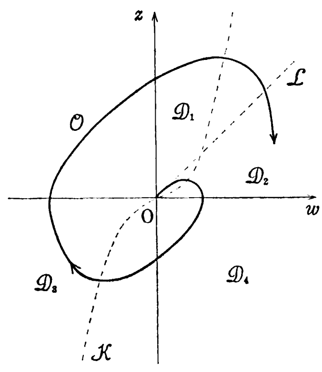

Figure 4.

Next, assume that $\mathcal{O}$ escapes to infinity in $\mathcal{D}_4^+$, then w remains finite and $\dfrac{dz}{dw}$ converges to $2\tau+2-n$ from (2.16)′. This is a contradiction.

Hence $\mathcal{O}$ meets the negative part of the z-axis and enters into $\mathcal{D}_4^-$.

A similar argument shows that, crossing $\mathcal{K}$ horizontally and then the negative part of the w-axis, $\mathcal{O}$ enters into $\mathcal{D}_1^-$ and meets the positive part of the z-axis.

In view of Proposition 2.4 the assertion is obvious at this point.

Hereafter we shall trace the orbit $\mathcal{O}$ in the cases II and III.

LEMMA 2.10. *When* $p=(n+l)/(n-2)$, *the orbit* $\mathcal{O}$ *behaves the same as in Lemma* 2.9. (cf. Figure 4.)

PROOF. From $(2.6)_{\mathrm{II}}$, we have

$$\frac{dz}{dw}=\frac{n-2}{z}\left\{z-\frac{1}{n-2}|w|^{p-1}w\right\}\equiv\frac{n-2}{z}\{z-g_1(w)\} \tag{2.21}$$

and

$$z(t)-\tau w(t)=z(t)-(n-2)w(t)=-\int_{-\infty}^{t}|w|^{p-1}wd\eta. \tag{2.22}$$

As before, defining the curve $\mathcal{K}$, the line $\mathcal{L}$ and the domains $\mathcal{D}_j$ $(j=1, 2, 3, 4)$ with $g_1(w)$ in place of $g(w)$, we begin to trace $\mathcal{O}$.

The orbit $\mathcal{O}$ starts from O along and below $\mathcal{L}$ according to (2.22). It goes out of $\mathcal{D}_1$ crossing $\mathcal{K}$. Now, the trace of $\mathcal{O}$ will be carried out more easily than in Lemma 2.9, since there are no domains such as $\mathcal{D}_3^+$ and $\mathcal{D}_2^-$. Hence we shall not repeat here.

LEMMA 2.11. *When* $p\in(1, (n+l)/(n-2))$, *the orbit* $\mathcal{O}$ *behaves the same as in Lemma* 2.9.

The proof of Lemma 2.11 will be done in the same manner. So we shall omit it here.

Now we are in the position to see how the trace of the orbit of $\mathcal{O}$ tells us about the radial solution of (P), that is, to complete

PROOF OF THEOREM 1. In terms of the changes of variables (2.1) and (2.5), the zeros of $u=u(r)$ corresponds to that of $w=w(t)$. Therefore, if the orbit $\mathcal{O}$ never meets the z-axis at a finite t, the solution of $u=u(r)$ of (IVP) never vanishes. This fact, together with Lemmas 2.5 and 2.7, proves the second part of the assertion.

On the other hand, in the cases that $\mathcal{O}$ goes across the z-axis at $t=t^*$, the solution $u=u(r)$ of (IVP) with A determined by the relation

$B=e^{-t^*}$ vanishes at $r=1$. Here it will be worthwhile to remark that there is a one-to-one correspondence between the points on the orbit $\mathcal{O}$ and the real numbers t through the condition (2.7).

Take the points P_k $(k=1, 2, 3, \cdots)$ at which the orbit $\mathcal{O}$ crosses the z-axis just k times after starting from O at $t=-\infty$, and determine the number t_k which corresponds to P_k. Next, choose the value A_k of A by the relation $B=e^{-t_k}$ and put $A=A_k$ in the problem (IVP). Then the above argument guarantees that the solution $u_k=u_k(r)$ of (IVP) with A_k has just $(k-1)$ zeros in $(0, 1)$. This yields the existence of a desired radial solution of (P). Meanwhile, the uniqueness of such a solution is reduced to that of the orbit $\mathcal{O}$, which has already been proved.

With the same procedure Theorem 2 will be proved, so we shall only examine the behavior of the orbit $\mathcal{O}$.

PROOF OF THEOREM 2. For the initial problem (IVP) we introduce the change of variables:

$$u(r)=Av(s) \quad \text{and} \quad r=\{\tau^2 A^{1-p}\}^{1/(l+2)}s \quad \text{with} \quad \tau=(l+2)/(p-1).$$

Successively we make another change of variables:

$$w(t)=s^{\tau}v(s) \quad \text{and} \quad s=e^t.$$

Accordingly, (IVP) is transformed into

$$w''-2\tau w'+\tau^2(|w|^{p-1}+1)w=0 \tag{2.23}$$

with

$$\lim_{t\to-\infty} e^{-\tau t}w(t)=1 \quad \text{and} \quad \lim_{t\to-\infty} e^{-t}\{e^{-\tau t}w(t)\}'=0. \tag{2.24}$$

The unique existence of the solution $w=w(t)$ of (2.23) with (2.24) follows from Lemma 2.1. Consequently, we obtain the function $z=z(t)$ and the orbit $\mathcal{O}$ as before.

Noting that the equivalent autonomous system to (2.23):

$$\frac{d}{dt}\begin{pmatrix} w \\ z \end{pmatrix}=\begin{pmatrix} z \\ -\tau^2(|w|^{p-1}+1)w+2\tau z \end{pmatrix}$$

has only a singular point O(0, 0) which is an attractor at $t=-\infty$, we get to the conclusion that the orbit $\mathcal{O}$ behaves the same as in the case III for $n\geq 3$.

§ 3. Proof of Theorem 3: A conclusion from Rellich's identity

Symmetry of positive solutions for the boundary value problem:

$$\Delta u+f(r, u)=0 \quad \text{in } \Omega=\{x \mid r=|x|<1\}\subset \mathbf{R}^n, \tag{3.1}$$

$$u=0 \quad \text{on } \partial\Omega, \tag{3.2}$$

where $f(r, u)$ is of class $C^1([0,1]\times \mathbf{R})$, has been investigated by Gidas-Ni-Nirenberg [4]. There it is shown that any positive $C^2(\bar{\Omega})$-solution $u=u(r)$ of (3.1) with (3.2) is radial provided that f is decreasing in r, which is not the case for (P). Hence it is open whether the solutions, especially the positive ones, of (P) are necessarily radial or not.

Nevertheless, we have a stronger nonexistence result than the second part of Theorem 1. To show this, we prepare the following identity, which follows directly from Rellich's one.

LEMMA 3.1. *Let u be a $C^2(\bar{\Omega})$-solution of the problem*

$$\Delta u+|x|^l f(u)=0 \quad \text{in } \Omega, \tag{3.3}$$

$$u=0 \quad \text{on } \partial\Omega, \tag{3.4}$$

where f is of class $C^1(\mathbf{R})$ and $l\geq 0$. Then we have

$$\frac{1}{2}\int_{\partial\Omega}\left|\frac{\partial u}{\partial \nu}\right|^2 dx=\int_{\Omega}\left\{(n+l)F(u)-\frac{n-2}{2}uf(u)\right\}|x|^l dx, \tag{3.5}$$

where ν denotes the outward unit normal on $\partial\Omega$ and $F(t)=\int_0^t f(s)ds$.

PROOF. For u in $C^2(\bar{\Omega})$, we have

$$\Delta u\sum_{k=1}^n x_k\left(\frac{\partial u}{\partial x_k}\right)=\sum_{j,k=1}^n\left[x_k\frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial x_j}\cdot\frac{\partial u}{\partial x_k}\right)-\frac{1}{2}x_k\frac{\partial}{\partial x_k}\left(\frac{\partial u}{\partial x_j}\right)^2\right]. \tag{3.6}$$

Integrating (3.6) in Ω, we can derive the Rellich's identity [13]

$$\int_{\Omega}\Delta u\sum_{k=1}^n x_k\left(\frac{\partial u}{\partial x_k}\right)dx=\int_{\partial\Omega}(x\cdot\nabla u)^2 dS-\frac{1}{2}\int_{\partial\Omega}|\nabla u|^2 dS+\frac{n-2}{2}\int_{\Omega}|\nabla u|^2 dx. \tag{3.7}$$

When u is a solution of (3.3) and (3.4), we have

$$\int_{\Omega}\Delta u\sum_{k=1}^n x_k\left(\frac{\partial u}{\partial x_k}\right)dx=-\int_{\Omega}\sum_{k=1}^n x_k|x|^l\frac{\partial}{\partial x_k}F(u)dx=(n+l)\int_{\Omega}F(u)|x|^l dx, \tag{3.8}$$

(3.9) $$\int_{\partial\Omega}(x\cdot\nabla u)^2dS-\frac{1}{2}\int_{\partial\Omega}|\nabla u|^2dS=\frac{1}{2}\int_{\partial\Omega}\left|\frac{\partial u}{d\nu}\right|^2dS$$

and

(3.10) $$\int_{\Omega}|\nabla u|^2dx=-\int_{\Omega}u\Delta u dx=\int_{\Omega}uf(u)|x|^l dx.$$

Substituting (3.8), (3.9) and (3.10) into (3.7), we obtain the identity (3.5).

The identity (3.5) plays a key role in the proof of Theorem 3. In fact, in the case $f(t)=|t|^{p-1}t$, (3.5) reads as

(3.11) $$\frac{1}{2}\int_{\partial\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2dx=\left(\frac{n+l}{p+1}-\frac{n-2}{2}\right)\int_{\Omega}|x|^l|u|^{p+1}dx,$$

where u is a C^2-solution of (P).

For $p\in((n+2+2l)/(n-2),\infty)$, the value $\dfrac{n+l}{p+1}-\dfrac{n-2}{2}$ is negative. Therefore, the identity (3.11) holds if and only if

$$\int_{\partial\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2dx=\int_{\Omega}|x|^l|u|^{p+1}dx=0,$$

which implies that u vanishes identically in Ω.

For $p=(n+2+2l)/(n-2)$, the value $\dfrac{n+l}{p+1}-\dfrac{n-2}{2}$ is equal to zero, which means from (3.11) that

(3.12) $$\frac{\partial u}{\partial\nu}=0 \quad \text{on } \partial\Omega.$$

Now consider the linear elliptic problem

$$\Delta v+|u|^{p-1}v=0 \quad \text{in } \Omega, \qquad v=0 \quad \text{on } \partial\Omega.$$

If a solution v of the above problem satisfies the condition

$$\frac{\partial v}{\partial\nu}=0 \quad \text{on } \partial\Omega,$$

then v vanishes identically in Ω. This fact holds by Calderón's uniqueness theorem and the unique continuation theorem for elliptic equations of second order. (A proof can be found in Mizohata [7], for example.) Hence u vanishes identically even for $p=(n+2+2l)/(n-2)$.

Appendix. Proof of Theorem A

In this Appendix, we will give a proof of Theorem A by Joseph-Lundgren's method. However, we shall study the existence of the solution orbit Q and its asymptotic behavior around the critical point in detail.

As for radial solutions $u=u(r)$, where $r=|x|$, the problem (P_λ) can be reduced to the ordinary differential equation

$$(RP_\lambda) \qquad \begin{aligned} &(r^{n-1}u')'+\lambda r^{l+n-1}e^u=0 \quad \text{for } r\in(0,1), \\ &u'(0)=0 \quad \text{and} \quad u(1)=0. \end{aligned}$$

We first deal with the case $n>2$. The case $n=2$ will be treated later.

Similarly in § 2, we consider the initial value problem

$$(IVP_\lambda) \qquad \begin{aligned} &(r^{n-1}u')'+\lambda r^{l+n-1}e^u=0, \\ &u(0)=A \quad \text{and} \quad u'(0)=0, \end{aligned}$$

for $A>0$.

Introducing the change of variables:

$$u(r)=v(s)+A \quad \text{and} \quad r=Bs, \tag{A.1}$$

where

$$B=\{(n-2)(2+l)/\lambda e^A\}^{1/(l+2)}, \tag{A.2}$$

we transforms (IVP_λ) into the problem

$$(s^{n-1}v')'+(2+l)(n-2)s^{l+n-1}e^v=0, \tag{A.3}$$

$$v(0)=v'(0)=0. \tag{A.4}$$

The boundary value condition $u(1)=0$ corresponds to the condition

$$v(B^{-1})=-A. \tag{A.5}$$

Further we make another change of variables:

$$v(s)=w(t)-(2+l)t \quad \text{and} \quad s=e^t, \tag{A.6}$$

which transforms the problem (A.3), (A.4) into

$$w''+(n-2)w'+(2+l)(n-2)(e^w-1)=0, \tag{A.7}$$

$$\lim_{t\to-\infty}\{w(t)-(2+l)t\}=\lim_{t\to-\infty}e^{-t}\{w'(t)-(2+l)\}=0. \tag{A.8}$$

LEMMA A.1. *The problem* (A.7), (A.8) *has a unique classical solution* $w=w(t)$.

PROOF. Substituting $y(t)=w(t)-(2+l)t$ into (A.7) and (A.8), we have

$$y''+(n-2)y'+(2+l)(n-2)e^{(2+l)t}e^y=0, \tag{A.7$'$}$$

$$\lim_{t\to-\infty}y(t)=\lim_{t\to-\infty}e^{-t}y'(t)=0. \tag{A.8$'$}$$

The problem (A.7)$'$, (A.8)$'$ can be transformed into an equivalent integral equation

$$y(t)=-(2+l)(n-2)\int_{-\infty}^{t}e^{-(n-2)\eta}d\eta\int_{-\infty}^{\eta}e^{(n+l)\xi}e^{y(\xi)}d\xi. \tag{A.9}$$

The unique existence of a solution of the equation (A.9), hence of the problem (A.7), (A.8), can be proved with the same procedure as in the proof of Lemma 2.1, which will not be repeated here.

For the solution $w=w(t)$, we put $z(t)=w'(t)$ and define the orbit $Q=\{(w(t),z(t))\,|\,t>-\infty\}$. Consequently, Q is the orbit of the autonomous system

$$\frac{d}{dt}\begin{pmatrix} w \\ z \end{pmatrix}=\begin{pmatrix} z \\ -(2+l)(n-2)(e^w-1)-(n-2)z \end{pmatrix}, \tag{A.10}$$

whose singular point is only O(0, 0). Now we are going to investigate the properties of the orbit Q, some of which have been shown essentially in [2] and [6].

PROPOSITION A.2. *There exists only one orbit of* (A.10) *that is asymptotic to the line* $m: z=2+l$ *from below as* $w\to-\infty$, *Moreover, this orbit is identical with* Q.

PROOF. Suppose that there exist two distinct orbits having such a property. These orbits near $w=-\infty$ can be expressed with $z=\zeta_i(w)$ $(i=1,2)$ because $\dfrac{dw}{dt}=z>0$. Then we have from (A.7), (A.11) $$\frac{d\zeta_i}{dw}=-(n-2)-(2+l)(n-2)\frac{e^w-1}{\zeta_i} \qquad (i=1,2).$$

For $\chi(w)\equiv\zeta_2(w)-\zeta_1(w)$, we may assume $\chi(w_0)>0$ for some negative w_0 without loss of generality and have from (A.11),

(A.12) $$\frac{d\chi}{dw}=-(2+l)(n-2)(1-e^w)\frac{\chi}{\zeta_1\zeta_2}<0 \qquad \text{at } w=w_0.$$

Consequently we conclude from (A.12) that $\chi=\chi(w)$ is monotone decreasing in $(-\infty, w_0)$ and

(A.13) $$0<\chi(w_0)<\chi(w) \qquad \text{for } w\in(-\infty, w_0),$$

which contradicts the fact that both $\zeta_i(w)$ $(i=1,2)$ converge to $2+l$ as w tends to $-\infty$. Thus we have obtained the uniqueness.

On the other hand, the condition (A.8) show that the orbit Q is asymptotic to the line m as w tends to $-\infty$. Further, from (A.9) we derive

(A.14) $$z(t)-(2+l)=-(2+l)(n-2)e^{-(n-2)t}\int_{-\infty}^{t} e^{(n-2)\xi}e^{w(\xi)}d\xi<0,$$

which shows that Q stays below m.

PROPOSITION A.3. *The orbit Q starts asymptotic to the line m and remains in a domain*

$$\mathcal{E}:(2+l)(1-e^w)<z<2+l,$$

until it crosses the z-axis or approaches O(0, 0).

PROOF. Assume Q meets the curve $C:z=(2+l)(1-e^w)$ at (w_0, z_0) and put

(A.15) $$\theta(w)=\zeta(w)-(2+l)(1-e^w),$$

for $Q:z=\zeta(w)$, then we have

(A.16) $$\theta'(w_0)=(2+l)e^{w_0}>0.$$

The relation (A.16) means

(A.17) $$\theta(w)<0 \quad \text{and} \quad \frac{d\zeta}{dw}(w)=-(n-2)\frac{\theta(w)}{\zeta(w)}>0$$

for $w\in(-\infty, w_0)$. From (A.17) we have

$$\zeta(w)<\zeta(w_0)=(2+l)(1-e^{w_0})<2+l$$

for $w\in(-\infty, w_0)$, which contradicts the asymptotic approach to the line m.

Assume that Q stays below the curve C for w near $-\infty$, then we are led to the same contradiction.

Next the behavior of Q near the critical point O(0, 0) will be analyzed. The linearized system of (A.10) at O is

$$\text{(A.18)} \qquad \frac{d}{dt}\begin{pmatrix} h \\ k \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ -(2+l)(n-2) & -(n-2) \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix},$$

where the eigenvalues

$$\text{(A.19)} \qquad \mu_\pm=\frac{1}{2}\{2-n\pm\sqrt{(n-2)(n-10-4l)}\}$$

of the matrix in (A.18) satisfy the relations

$$\text{(A.20)} \qquad \mu_+ + \mu_- <0 \quad \text{and} \quad \mu_+\cdot\mu_->0.$$

Consequently the real parts of $\mu_\pm$ are negative. Further, $\mu_\pm$ become imaginary if and only if $n<10+4l$. Hence the following propositions follow directly from the general theory of autonomous systems.

PROPOSITION A.4. *Every orbit of* (A.10) *near* O *necessarily approaches* O *as* $t\to+\infty$. *Moreover, it approaches* O *spirally if* $2<n<10+4l$.

PROPOSITION A.5. *Every orbit of* (A.10) *never meets itself and has no limit cycle.*

With these preparations, we are going to trace the orbit Q.

LEMMA A.6. *Let* $n\geq 10+4l$. *The orbit* Q *starts asymptotically to the line* $m: z=2+l$ *and goes right and downwards in* $\mathcal{E}$. *Eventually it approaches* O *in one direction as* $t\to+\infty$. (cf. Figure 5.)

PROOF. Owing to Propositions A.2 and A.3, we have to prove only the last part.

Assume Q meets the line $n_+: z=\mu_+ w$ at (w_0, z_0) in $\mathcal{E}$, then we have

$$\frac{d}{dw}\{\mu_+ w-\zeta(w)\}|_{w=w_0}>\mu_+ +(n-2)+\frac{(n-2)(2+l)}{\mu_+}=0, \tag{A.21}$$

where $\zeta(w)$ is that in (A.15). Because of (A.21), Q never cuts the line n_+ from the left to the right. Consequently, Q stays in the domain: $(2+l)(1-e^w)<z<\min.(2+l, \mu_+ w)$ and approaches O in a certain limiting direction there.

We remark here that Q approaches O along the line n_+. However, this fact will not be needed later.

LEMMA A.7. *Let $2<n<10+4l$. The orbit Q starts asymptotically to the line $m: z=2+l$, goes right and downwards in $\mathcal{E}$ and meets the positive part of the z-axis. Further, crossing the w- and z-axis alternately, it approaches* O *spirally as $t\to+\infty$.* (cf. Figure 6.)

PROOF. Going right and downwards in $\mathcal{E}$, the orbit Q never approaches O directly from Proposition A.4. Hence, it eventually meets the positive part of the z-axis and enters into the domain $\mathcal{F}: w>0$ and $z>0$. Going right and downwards in $\mathcal{F}$, it crosses the w-axis vertically at a point $(w^*, 0)$, where w^* is a certain positive number.

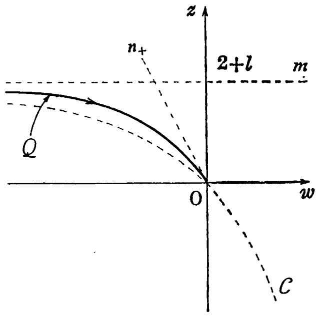

Figure 5.

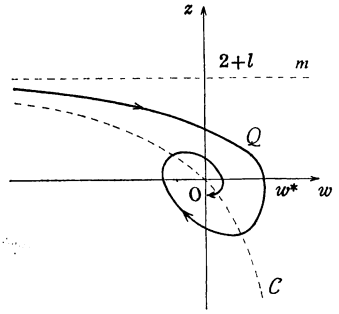

Figure 6.

A further trace of Q, which can be accomplished quite the same as in Lemmas 2.5 and 2.9, will prove the assertion. However, we shall skip it here.

Now we deal with the case $n=2$ briefly.

We make the changes of variables (A.1) and (A.6) again, where B is taken to be $(\lambda e^A)^{-1/(l+2)}$ instead of $\{(n-2)(2+l)/\lambda e^A\}^{1/(l+2)}$. Accordingly, the problem (IVP_λ) is transformed into the problem

$$w''+e^w=0, \tag{A.22}$$

$$\lim_{t\to-\infty}\{w(t)-(2+l)t\}=\lim_{t\to-\infty}e^{-t}\{w'(t)-(2+l)\}=0. \tag{A.23}$$

With a slight modification of the proof of Lemma A.1, we get a unique solution of the problem (A.22), (A.23). Hence we can define the orbit Q as before.

LEMMA A.8. *When $n=2$, the orbit Q is expressed by the equation*

$$z^2+2e^w=(2+l)^2. \tag{A.24}$$

PROOF. From (A.22), we have

$$\frac{d}{dt}\{z^2+2e^w\}=0. \tag{A.22$'$}$$

Since (w, z) on Q approaches $(-\infty, 2+l)$ as t tends to $-\infty$, (A.24) follows immediately from (A.22)$'$.

At this point, what is left to be shown is how Lemmas A.6, A.7 and A.8 lead to

PROOF OF THEOREM A. First, we repeat the remark that there is a one-to-one correspondence between the points on Q and real numbers t through the condition (A.8). Hence, for a point (w_0, z_0) on Q, we can determine a corresponding number t_0. Further A and λ are chosen by the relations

$$-A=w_0-(2+l)t_0 \quad\text{and}\quad e^{t_0}=B^{-1},$$

that is,

$$A=-w_0+(2+l)t_0, \tag{A.25}$$

$$\lambda=(n-2)(2+l)e^{w_0} \qquad \text{in the case } n>2, \tag{A.26.1}$$

and

$$\lambda=e^{w_0} \qquad \text{in the case } n=2. \tag{A.26.2}$$

The solution $u=u(r)$ of (IVP_λ) with A and λ above satisfies the condition (A.5). Consequently it turns out to be a solution of (RP_λ).

Conversely, for given λ the number of solutions of (RP_λ) is equal to the number of the common points of Q and the line: $w=w_0$, where w_0 is determined by (A.26).

Finally, part (**1**) of the assertion follows from Lemma A.6 because $w_0<0$ means $0<\lambda<(n-2)(2+l)$ in (A.26.1). As a consequence of Lemma A.7, we have part (**2**) with $\lambda^*=(n-2)(2+l)e^{w^*}$, where w^* is that in the proof of Lemma A.7. Part (**3**) with $\lambda^*=\frac{1}{2}(2+l)^2$ follows from Lemma A.8.

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Department of Mathematics
Chiba Institute of Technology
Narashino, Chiba
275 Japan