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Meromorphic transformation to Birkhoff standard form in dimension three

Dedicated to Professor Tosihusa Kimura on occasion of his sixtieth birthday

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0. Introduction

Consider an arbitrary *meromorphic differential equation* of dimension $n\times n$ and Poincaré rank $r\geqq 0$:

$$zx'=A(z)x,\ A(z)=z^{r}\sum_{k=0}^{\infty}A_{k}z^{-k},\ A_{0}\neq 0 \text{ or } r=0; \tag{0.1}$$

the power series being convergent for $|z|>a$, say, with some $a\geqq 0$.

We say that (0.1) is *meromorphically equivalent* to another equation

$$zy'=B(z)y, \tag{0.2}$$

iff there exists a *meromorphic transformation*, i.e. an $n\times n$ matrix $T(z)$, analytic and invertible for $|z|$ sufficiently large and so that $T^{\pm 1}(z)$ has at most a pole at $z=\infty$, for which

$$B(z)=T^{-1}(z)[A(z)T(z)-zT'(z)]. \tag{0.3}$$

It follows from (0.3) that (for some natural number $\hat{r}$)

$$B(z)=z^{\hat{r}}\sum_{k=0}^{\infty}B_{k}z^{-k},\quad B_{0}\neq 0 \text{ or } \hat{r}=0,$$

and the series converges, say, for $|z|>\hat{a},\ \hat{a}\geqq 0$.
Generally, both $\hat{a}$ and $\hat{r}$ will be different from a, resp. r. However, in case of *analytic equivalence* (i.e. if we can find a $T(z)$, analytic at $z=\infty$ and $T(\infty)$ invertible, so that (0.2) holds), then clearly $\hat{r}=r$ follows.

In 1913, G. D. Birkhoff [5] claimed that *every* equation (0.1) is analytically equivalent to a polynomial equation, i.e. an equation (0.2) with $B(z)$ being a matrix of polynomials in z. The proof he gave was only valid for such equations whose monodromy matrix can be diagonalized, and by counterexamples (in dimension 2×2) he was proven false in the general case (see [1] for references).

If we allow meromorphic transformations (instead of analytic ones), then it immediately follows from Birkhoff's results that indeed every equation (0.1) is meromorphically equivalent to a polynomial equation, however, the Poincaré rank of this equation might be larger than the original one, which means that the polynomial equation may contain too many parameters in order to be a "good" representative for the equations within a certain equivalence class. Hence the problem arises to meromorphically transform (0.1) into *a polynomial equation* (0.2) *having a minimal Poincaré rank* r_0 (i.e. r_0 is the minimal Poincaré rank of *all* equations which are meromorphically equivalent to (0.1). Every such equation shall be called a *Birkhoff standard form* for (0.1). Since we may very well assume that (0.1) already has minimal Poincaré rank, this problem is the same as saying that we wish to meromorphically transform (0.1) into a polynomial equation (0.2) without increasing the Poincaré rank.

In 1963, H. L. Turrittin [9] succeeded in showing that such a transformation exists provided A_0 (in (0.1)) has *n distinct eigenvalues*, and in [6], W. B. Jurkat announced that he could show the same under somewhat more general assumptions. The general case, however, is still an open problem. Only for dimension $n=2$, W. B. Jurkat, D. A. Lutz, and A. Peyerimhoff [7] were able to show that every equation (0.1) is meromorphically equivalent to a Birkhoff standard form.

In this article, the author shows that meromorphic equivalence to Birkhoff standard form also holds true for $n=3$. So far, the techniques used in the proof do not generalize to larger values of n.

1. A factorization of formal meromorphic transformations

A formal series of the form

$$T(z)=\sum_{k=N}^{\infty} T_k z^{-k}, \qquad T_N\neq 0,\ T(z)\, n\times n, \tag{1.1}$$

will be called *a formal meromorphic transformation*, if $\det T(z)$ is not the zero series (so that $T^{-1}(z)$ again is a formal meromorphic transformation). Obviously, $T(z)$ is a meromorphic transformation (as defined in the introduction) iff the series converges for sufficiently large $|z|$, and we use the word *proper* meromorphic transformation to distinguish the convergent case from the formal one. If $T(z)$ is a (formal) power series in z^{-1} beginning with an invertible matrix for its constant term, we call $T(z)$ a (formal) *analytic transformation*, and if the constant term equals the identity matrix, we speak of a (formal) *Birkhoff transformation.*

The following Lemma is an analogue to the factorization of a constant invertible matrix into a product of a lower and an upper triangular matrix. For a convergent transformation, its proof was communicated to the author by Y. Sibuya in a seminar held at USC, Los Angeles. The same proof holds for a formal transformation and is included here for the sake of completeness:

LEMMA 1. *Let* $T(z)$ *be an arbitrary formal analytic transformation* (*of type* $n\times n$), *and let* k_j *be integers* ($j=1, \cdots, n$) *so that*

$$k_1\leqq k_2\leqq\cdots\leqq k_n. \tag{1.2}$$

Then there exists a permutation matrix R *so that*

$$T(z)R=L(z)U(z), \tag{1.3}$$

with $U(z)=[u_{\nu\mu}(z)]$ *being a formal analytic transformation so that, as* $z\to\infty$ (*formally*):

$$u_{\nu\nu}(z)=u_\nu+0(z^{-1}), \qquad u_\nu\neq 0,\ 1\leqq\nu\leqq n, \tag{1.4}$$

$$u_{\nu\mu}(z)=0(z^{k_\nu-k_\mu-1}), \qquad 1\leqq\mu<\nu\leqq n, \tag{1.5}$$

and $L(z)=[l_{\nu\mu}(z)]$ *being a lower triangular matrix with ones along the diagonal, and for* $1\leqq\mu<\nu\leqq n$, *the entry* $l_{\nu\mu}(z)$ *being a polynomial in* z^{-1} *of degree not larger than* $k_\nu-k_\mu$.

PROOF. (Induction with respect to n). For $n=1$, the Lemma is trivially correct. For $n\geqq 2$, block

$$T(z)R=\left(\begin{array}{c|c}\hat{T}(z) & t_1(z)\\ \hline t_2^T(z) & t(z)\end{array}\right)$$

(with R to be determined later),

$$L(z)=\left(\begin{array}{c|c}\hat{L}(z) & 0\\ \hline l_2^T(z) & 1\end{array}\right),$$

$$U(z)=\left(\begin{array}{c|c}\hat{U}(z) & u_1(z)\\ \hline u_2^T(z) & u(z)\end{array}\right)$$

(where the first diagonal block always is of type $(n-1)\times(n-1)$).

Then (1.3) holds iff

$$\hat{T}(z)=\hat{L}(z)\hat{U}(z), \tag{1.6}$$

$$t_1(z)=\hat{L}(z)u_1(z), \tag{1.7}$$

$$t_2^T(z)=l_2^T(z)\hat{U}(z)+u_2^T(z), \tag{1.8}$$

$$t(z)=l_2^T(z)u_1(z)+u(z). \tag{1.9}$$

Permuting columns of $T(z)$ (i.e. selecting R), one can ensure that $\hat{T}(z)$ is a formal analytic transformation and (using the induction hypothesis) admits a factorization (1.6), with $\hat{L}(z)$, $\hat{U}(z)$ having the required forms. Defining $u_1(z)$ by (1.7), we certainly have that the components of $u_1(z)$ are formal power series in z^{-1}, and this is all that is required for those positions of $U(z)$. Next, let

$$t_2^T(z)\hat{U}^{-1}(z)=l_{2]}^T(z)+\hat{u}_2^T(z),$$

where $l_2^T(z)=[l_{n1}(z),\cdots,l_{n,n-1}(z)]$ is a row-vector of polynomials in z^{-1} which are the truncation of the corresponding positions of $t_2^T(z)\hat{U}^{-1}(z)$ (so that their degree is as required), and $\hat{u}_2^T(z)=[\hat{u}_{n1}(z),\cdots,\hat{u}_{n,n-1}(z)]$ is a row-vector of formal power series in z^{-1}, so that

$$\hat{u}_{n\mu}(z)=0(z^{k_\mu-k_n-1}),\qquad 1\leqq\mu\leqq n-1.$$

If we define $u_2^T(z)=\hat{u}_2^T(z)\hat{U}(z)$, then (1.8) holds, and it can be verified that (1.5) holds, with $\nu=n$. Finally, define $u(z)$ by (1.9), then $u(z)$ certainly is a formal power series in z^{-1}, and (1.3) holds (formally). If the constant term of $u(z)$ would vanish, then (1.3) would imply that $\det T(\infty)=0$, which is a contradiction, hence (1.4) holds for $\nu=n$. This completes the proof.

We use Lemma 1 to generalize a result, stated without proof by T. Kimura [8] for a proper meromorphic transformation, to the case of a formal one:

LEMMA 2. *Let an arbitrary formal meromorphic transformation*

$T(z)$ be given. Then there exists a diagonal matrix

$$K = \operatorname{diag}[k_1, \cdots, k_n] \tag{1.10}$$

of integer diagonal entries k_j, so that

$$T(z) = P(z) T_a(z) z^K, \tag{1.11}$$

where $P(z)$ is a matrix of polynomials in z with constant (non-zero) determinant, and $T_a(z)$ is a formal analytic transformation.

PROOF. Every formal meromorphic transformation $T(z)$ can be factored in various ways as $T(z) = F(z)\tilde{T}_a(z)$ with a proper meromorphic transformation $F(z)$ and a formal analytic transformation $\tilde{T}_a(z)$ (for example, apply a Proposition in [6], p. 52 to the transpose of $T(z)$). From G. D. Birkhoff [4] (applied to the transpose of $F(z)$) we obtain the existence of $K = \operatorname{diag}[k_1, \cdots, k_n]$ with integers k_j, so that

$$F(z) = \hat{P}(z) z^K \hat{T}_a(z)$$

with a proper analytic transformation $\hat{T}_a(z)$ and a matrix $\hat{P}(z)$ of entire functions (in z) with $\det \hat{P}(z) \neq 0$ everywhere. Since $F(z)$ only has a pole at $z = \infty$, we find that the entries of $\hat{P}(z)$ must be polynomials, hence $\det \hat{P}(z)$ is a polynomial without roots, and consequently a constant. Therefore

$$T(z) = \hat{P}(z) z^K T_a(z)$$

with a formal analytic transformation $T_a(z)$ $(= \hat{T}_a(z)\tilde{T}_a(z))$. Without loss in generality, we may assume (1.10) (otherwise replace $\hat{P}(z)$, K, $T_a(z)$ by $\hat{P}(z)\hat{R}$, $\hat{R}^{-1}K\hat{R}$, $\hat{R}^{-1}T_a(z)$, resp., with a permutation matrix $\hat{R}$). Applying Lemma 1 to $T_a(z)$ and observing that $z^K L(z) z^{-K}$ is a matrix of polynomials in z with constant non-zero determinant, while $z^K U(z) z^{-K}$ is a formal analytic transformation, we obtain (1.11) with $\hat{P}(z) z^K L(z) z^{-K} = P(z)$, $z^K U(z) z^{-K} R^{-1}$ in place of $T_a(z)$, and RKR^{-1} in place of K.

REMARK 1.1. Obviously, in (1.11) one can always *normalize* the factors $P(z)$, $T_a(z)$ by requiring

$$\begin{array}{ll} \text{either} & P(0) = \mathrm{I}, \\ \text{or} & T_a(\infty) = \mathrm{I}. \end{array}$$

In each case, it is easily seen that (given K) the factors $P(z)$, $T_a(z)$ are unique (subject to this normalization).

2. Polynomial equations with normalized formal solutions

As pointed out in the introduction, every equation (0.1) is meromorphically equivalent to a polynomial equation (whose rank may be larger than that of the original one). We are going to show that in addition one may arrange a formal fundamental solution of the polynomial equation to have a particular form:

THEOREM 1. *Every meromorphic differential equation* (0.1) *is meromorphically equivalent to an equation* (0.2) *with*

$$B(z)=z^{\hat{r}}\sum_{k=0}^{\hat{r}} B_k z^{-k}, \qquad B_0\neq 0 \text{ or } \hat{r}=0, \tag{2.1}$$

so that $B_{\hat{r}}$ *has eigenvalues within a fixed, but arbitrary system of representatives modulo one, and so that* (0.2) *has a formal fundamental solution*

$$H(z)=F_b(z)z^K G(z), \tag{2.2}$$

with a formal Birkhoff transformation $F_b(z)$, *a diagonal matrix* K *of integer diagonal entries, and* $G(z)$ (*the formal meromorphic invariant of* (0.1)) *as described in* [6], [2].

PROOF. It is known that one always may find (0.2) with $B(z)$ as in (2.1), and eigenvalues of $B_{\hat{r}}$ as stated, so that (0.2) is meromorphically equivalent to (0.1) (compare, e.g., [6], p. 171). Every such equation (0.2) has a formal fundamental solution

$$H(z)=F(z)G(z),$$

with $G(z)$ as above, and a formal meromorphic transformation $F(z)$ (see [6], p. 32, or [2]). According to Lemma 2, resp. Remark 2.1,

$$F(z)=P(z)F_b(z)z^K,$$

and the transformation $y=P(z)\tilde{y}$ takes (0.2) into

$$z\tilde{y}'=\tilde{B}(z)\tilde{y}, \quad \tilde{B}(z)=P^{-1}(z)[B(z)P(z)-zP'(z)].$$

This completes the proof.

For later use, we wish to show that, generally, there are several equations (2.1), with eigenvalues of $B_{\hat{r}}$ in the same system of representatives, having formal fundamental solutions (2.2) with the same $G(z)$, but with different K. For this reason, let a formal Birkhoff transformation $F_b(z)$ be given and try to find a diagonal matrix $D=\operatorname{diag}[d_1, \cdots, d_n]$ of integer diagonal entries, so that

$$F_b(z)=P(z)\tilde{F}_b(z)z^{-D}, \tag{2.3}$$

with $P(z)$ a matrix of polynomials in z, having non-zero constant determinant, and $\tilde{F}_b(z)$ another formal Birkhoff transformation. If we succeed in finding $P(z)$, $\tilde{F}_b(z)$ as stated, then $P(z)$ may be used as a transformation of (0.2) into an equation of the same kind, but with $\hat{K}=K-D$ in place of K. The question of factoring $F_b(z)$ in the form (2.3) is closely related to BHP-factorizations studied in [1], and we give some results which we wish to apply later:

PROPOSITION 1. *Let* $F_b(z)=[f_1(z), \cdots, f_n(z)]$, $f_j(z)=\sum_{k=0}^{\infty} f_j^{(k)} z^{-k}$, *be an arbitrary formal Birkhoff transformation (i.e.* $f_j^{(0)}=e_j$, *the* j^{th} *unit vector). For* $k\leqq -1$, $1\leqq j\leqq n$, *let* $f_j^{(k)}=0$. *For* $D=\operatorname{diag}[d_1, \cdots, d_n]$, *we can factor* $F_b(z)$ *as in* (2.3) *iff*

$$\operatorname{tr} D=\sum_{j=1}^{n} d_j=0, \tag{2.4}$$

and

$$\det{}_D(F_b)=\det\begin{pmatrix} f_1^{(0)} & \cdots f_1^{(d_1+d)} & \cdots f_n^{(0)} & \cdots f_n^{(d_n+d)} \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(-d)} & \cdots f_1^{(d_1)} & \cdots f_n^{(-d)} & \cdots f_n^{(d_n)} \end{pmatrix}\neq 0 \tag{2.5}$$

with

$$d=-\min\{d_1, \cdots, d_n\}. \tag{2.6}$$

Note that in (2.5), the entries $f_j^{(k)}$ are n-vectors, hence the determinant is well-defined.

PROOF. Make a change of variable $z=w^{-1}$ and apply [1], Theorem 1, to $S(w)=F_b(z)$, observing that the convergence of the expansion of $S(w)$ is not made use of in the proof.

PROPOSITION 2. *Let $F_b(z)=[f_{jk}(z)]$ be an arbitrary Birkhoff transformation. Let I_1, I_2 be such that*

$$I_1 \cup I_2=\{1, \cdots, n\}, \quad I_1 \cap I_2=\varnothing,$$

and assume that for every μ, $1\leqq\mu\leqq\tau$ (with arbitrarily fixed natural τ) and arbitrary $j\in I_1$, $k\in I_2$, we have

$$\det_D(F_b)=0, \quad D=\operatorname{diag}[d_1, \cdots, d_n],$$

for

$$d_\nu=\begin{cases} \mu & \text{if } \nu=k, \\ -\mu & \text{if } \nu=j, \\ 0 & \text{if } \nu\neq j,\ k;\ 1\leqq\nu\leqq n. \end{cases} \tag{2.7}$$

Then for every $j\in I_1$, $k\in I_2$

$$f_{jk}(z)=0(z^{-1-\tau}) \qquad (z\to\infty, \textit{formally}).$$

PROOF. Make a change of variable $z=w^{-1}$ and apply [1], Proposition 1, to $S(w)=F_b(z)$ (again observing that convergence of $S(w)$ is not really required).

3. Three-dimensional equations

In what follows, we restrict to equations (0.1) with $n=3$. In order to prove our main result (Theorem 2), we wish to distinguish five different cases, depending upon the structure of the formal meromorphic invariant $G(z)$ of (0.1):

It can be seen from [6], or [2], that

$$G(z)=z^J U e^{Q(z)}, \tag{3.1}$$

with constant matrices J and U, and a diagonal matrix $Q(z)$. If we order the diagonal elements of $Q(z)$ in an appropriate way (which one can always do according to the general theory developped in [6], [2]), then J, U, and $Q(z)$ are of one of the following forms:

Case I.
$$Q(z)=\operatorname{diag}[q_1(z),\ q_2(z),\ q_3(z)],$$
$$J=\operatorname{diag}[\lambda_1',\ \lambda_2',\ \lambda_3'],$$
$$U=I,$$

with polynomials $q_j(z)$ *in the variable* z (which may or may not be distinct) with $q_j(0)=0$, and complex constants λ_j' satisfying

$$0 \leqq \operatorname{Re} \lambda_j' < 1, \quad 1 \leqq j \leqq 3.$$

Case II. $Q(z)$ as above, with $q_2(z) \equiv q_3(z)$,

$$J=\begin{pmatrix} \lambda' & 0 & 0 \\ 0 & \lambda_2' & 0 \\ 0 & 1 & \lambda_2' \end{pmatrix},$$

$$U=I,$$

with complex constants λ_j' satisfying

$$0 \leqq \operatorname{Re} \lambda_j' < 1, \quad 1 \leqq j \leqq 2.$$

Case III. $Q(z)$ as above, with $q_1(z) \equiv q_2(z) \equiv q_3(z)$,

$$J=\begin{pmatrix} \lambda' & 0 & 0 \\ 1 & \lambda' & 0 \\ 0 & 1 & \lambda' \end{pmatrix},$$

$$U=I,$$

with a complex constant λ' satisfying

$$0 \leqq \operatorname{Re} \lambda' < 1.$$

Case IV. $Q(z)=\operatorname{diag}[q_1(z), q_2(z), q_2(ze^{2\pi i})]$,

$$J=\operatorname{diag}[\lambda_1', \lambda_2', \lambda_2'+1/2]$$

$$U=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix},$$

with $q_1(z)$ a polynomial in z, $q_2(z)$ a polynomial in $z^{1/2}$, but not in z, both satisfying $q_j(0)=0$, and λ_1', λ_2' complex constants satisfying

$$0 \leqq \operatorname{Re} \lambda_1' < 1,$$
$$0 \leqq \operatorname{Re} \lambda_2' < 1/2.$$

Case V. $Q(z)=\operatorname{diag}[q(z), q(ze^{2\pi i}), q(ze^{4\pi i})]$,

$$J=\operatorname{diag}[\lambda', \lambda'+1/3, \lambda'+2/3],$$

$$U=\begin{pmatrix} 1 & 1 & 1 \\ 1 & \varepsilon & \varepsilon^2 \\ 1 & \varepsilon^2 & \varepsilon^4 \end{pmatrix},$$

with $q(z)$ a polynomial in $z^{1/3}$, but not in z, satisfying $q(0)=0$, λ' a complex constant satisfying

$$0 \leqq \operatorname{Re} \lambda' < 1/3,$$

and

$$\varepsilon = e^{2\pi i/3}.$$

Although we will, in the proof of Theorem 2, use different arguments for all five cases, the main difference in the arguments stems from the fact that in Cases I–III, the matrix $Q(z)$ consists of polynomials in z, while in the remaining cases it contains polynomials in a root of z. It is for this reason that we sometimes will refer to Cases I–III as *the cases without roots*, and to Cases IV, V as *the cases including roots.*

We are now ready to state

THEOREM 2. *Every meromorphic differential equation* (0.1), *with* $n=3$, *is meromorphically equivalent to an equation* (0.2), *with* $B(z)$ (*given by* (0.3)) *a polynomial in* z *of degree* r_0, *where* r_0 *is the minimum of the Poincaré rank of all equations being meromorphically equivalent to* (0.1). *In other words, every equation* (0.1) *with* $n=3$ *is meromorphically equivalent to one in Birkhoff standard form.*

REMARK 3.1. One can easily see that the minimal Poincaré rank r_0 is equal to the Poincaré rank of *the formal meromorphic normal form* of (0.1), i.e. of the equation

$$z\tilde{x}' = \tilde{B}(z)\tilde{x},$$

with

$$\tilde{B}(z) = zG'(z)G^{-1}(z),$$

and one can see that r_0 is, in fact, the smallest natural number larger than or equal to the (rational) degree of the elements in $Q(z)$.

4. Proof of the main theorem for cases without roots

Consider a fixed, but arbitrary, equation (0.1) such that its formal meromorphic invariant $G(z)$ is as in Case I, II or III. According to Theorem 1, (0.1) is meromorphically equivalent to an equation (0.2), $B(z)$ as in (2.1), having a formal fundamental solution as in (2.2), and the eigenvalues of $B_{\hat{r}}$ having real parts in the half-open interval $[0, 1)$. Defining

$$\begin{aligned}\hat{B}(z)&=z[z^{K}G(z)]'[z^{K}G(z)]^{-1}\\&=K+zQ'(z)Q^{-1}(z)+z^{K}Jz^{-K}\end{aligned}\tag{4.1}$$

(note that in every case considered here, $Q(z)$ and J commute), we see that (0.2) is formally Birkhoff equivalent to

$$z\hat{y}'=\hat{B}(z)\hat{y},$$

hence $\hat{r}$ equals the Poincaré rank of $\hat{B}(z)$. In Case I, it is immediately clear from (4.1) that $\hat{r}=r_0$, hence the proof is completed. In Case II, we have $\hat{r}=r_0$ iff

$$k_3-k_2\leqq r_0.\tag{4.2}$$

Suppose that (3.4) is violated. Applying Proposition 2, with $I_1=\{1, 2\}$, $I_2=\{3\}$, we may find a natural number μ and $j\in I_1$, $k\in I_2$, so that $\det_D(F_b)\neq 0$, with D as in (2.7), except for cases where $F_b(z)=[f_{jk}(z)]$ is such that

$$f_{13}(z)=f_{23}(z)=0.\tag{4.3}$$

If we can find such a D, then we may transform (0.2), by means of a polynomial transformation with constant determinant, to a new equation having the same properties as (0.2), but with $K-D$ in place of K. Obviously, the difference k_3-k_2 for the new equation will be smaller, either by the amount of μ (if $j=1$) or even 2μ (if $j=2$). Hence a series of such steps either leads to an equation (0.2) for which (4.2) holds, or to one with (4.3). In the latter case, we apply Proposition 2 with $I_1=\{2\}$, $I_2=\{1, 3\}$, and in quite the same manner as before, we see that finitely many steps either lead to an equation (0.2) with (4.2), or to one with (4.3) and

$$f_{21}(z)\,(=f_{23}(z))\equiv 0.\tag{4.4}$$

Suppose now that (4.3) and (4.4) hold. Applying the constant transformation $y=R\hat{y}$,

$$R=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and replacing $H(z)=F_b(z)z^K G(z)$ by

$$\begin{aligned}
&\hat{H}(z)=R^{-1}H(z)R=\hat{F}_b(z)z^{\hat{K}}\hat{G}(z),\\
&\hat{F}_b(z)=R^{-1}F_b(z)R,\\
&\hat{K}=R^{-1}KR,\\
&\hat{G}(z)=R^{-1}G(z)R,
\end{aligned}$$

we find that $\hat{F}_b(z)$ and $\hat{G}(z)$ (hence $\hat{H}(z)$) are lower triangular matrices. Consequently, $\hat{B}(z)=R^{-1}B(z)R=[\hat{b}_{jk}(z)]$ is also lower triangular, and the j^{th} diagonal element $\hat{h}_{jj}(z)=\hat{f}_{jj}(z)z^{\hat{k}_j}\hat{g}_{jj}(z)$ of $\hat{H}(z)$ is a (fundamental) solution of

$$z\hat{x}_j'=\hat{b}_{jj}(z)\hat{x}_j \qquad (j=1, 2, 3).$$

Since $\hat{b}_{jj}(z)$ is a polynomial in z, it is easily seen that this implies $\hat{f}_{jj}(z)\equiv 1$, and $\hat{b}_{jj}(o)=\hat{k}_j+\hat{\lambda}_j'$ $(j=1, 2, 3)$, with

$$\operatorname{diag}[\hat{\lambda}_1', \hat{\lambda}_2', \hat{\lambda}_3']=R^{-1}\operatorname{diag}[\lambda_1', \lambda_2', \lambda_3']R.$$

Since $\hat{B}(o)=\hat{B}_r$ is lower triangular, $\hat{b}_{jj}(o)$ are its eigenvalues, and since both $\hat{b}_{jj}(o)$ and $\hat{\lambda}_j'$ have real parts in $[0, 1)$, we conclude $\hat{k}_j=0$ $(j=1, 2, 3)$. This in turn implies (4.2), hence completes the proof in Case II.

In Case III, we have (compare [6], [2]) that $F_b(z)$ converges. Hence $T(z)=F_b(z)z^K$ is a proper meromorphic transformation that takes (0.2) into an equation having $G(z)$ as a fundamental solution, and this equation clearly is in Birkhoff standard form. Hence the proof of Theorem 2 in Cases I, II, III is completed.

REMARK 4.1. It may be seen from the proof that, in Cases I, II, III, one can always find an equation (0.2) in Birkhoff standard form (equivalent to (0.1)), so that the real parts of the eigenvalues of the matrix $B_r=B(o)$ ly in the half-open interval $[0, 1)$ (or in any other system of representatives modulo 1).

5. Proof of the main theorem for cases including roots

Consider a fixed, but arbitrary, equation (0.1) such that its formal meromorphic invariant is as in Case IV or V. According to [1], Theorem 2, every equation (0.1) can be analytically transformed into Birkhoff standard form, except for equations being analytically equivalent to a block-triangular equation. Hence we may assume without loss in generality

$$A(z)=\begin{pmatrix} A_1(z) & 0 \\ A_{21}(z) & A_{22}(z) \end{pmatrix}, \tag{5.1}$$

with diagonal blocks of dimensions one, resp. two, or vice versa. From [6], Section 13, or [3], we obtain that then $Q(z)$ has to split into two parts (of dimensions 1 resp. 2) which must each be "closed under analytic continuation". This implies that such an equation cannot belong to Case V (i.e. every equation in Case V is analytically equivalent to an equation in Birkhoff standard form). Consequently, we are in Case IV, and we may assume that the diagonal blocks $A_j(z)$ are polynomials in z of degree not larger than r_0, $j=1, 2$; this is obvious for the scalar block and follows for the two-dimensional one using that, according to [7], every two-dimensional equation is meromorphically equivalent to one in Birkhoff standard form. Moreover, we may arrange that the real parts of the eigenvalues of $A_2(o)$ are large in comparison with the real parts of eigenvalues of $A_1(o)$; if this does not hold, apply the transformation

$$x=\operatorname{diag}[z^N I_s, I_{3-s}], \tag{5.2}$$

with s being the size of $A_1(z)$ (i.e. $s=1$ or $s=2$), and N being a sufficiently large natural number. From [3] we conclude the existence of a formal fundamental solution

$$\tilde{H}(z)=\tilde{F}(z)z^{\tilde{L}}e^{\tilde{Q}(z)},$$

where either $\tilde{Q}(z)=\operatorname{diag}[q_1(z), q_2(z), q_2(ze^{2\pi i})]$ or $\tilde{Q}(z)=\operatorname{diag}[q_2(z), q_2(ze^{2\pi i}), q_1(z)]$ (with $q_1(z)$, $q_2(z)$ as described in Section 3, Case IV), a constant matrix $\tilde{L}$ and a formal meromorphic transformation $\tilde{F}(z)$ which both are lower triangularly blocked (in the block structure of $A(z)$), and so that

$$\tilde{H}(ze^{2\pi i})=\tilde{H}(z)e^{2\pi i\tilde{L}}. \tag{5.3}$$

Since $\tilde{Q}(ze^{2\pi i})=R^{-1}\tilde{Q}(z)R$ with

$$R=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{or } R=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we conclude from (5.3) that $e^{2\pi i\tilde{L}}R^{-1}$ and $\tilde{Q}(z)$ commute, i.e. $e^{2\pi i\tilde{L}}R^{-1}$ is a diagonal matrix. This implies that $\tilde{L}$ is diagonally blocked (in the block structure of $A(z)$). In the same way as in [3], Section 5, one can now construct an equation (0.2) in Birkhoff standard form, being meromorphically equivalent to (0.1) (observe that in [3], formula (5.1), one need not require $E_{\nu,jj}(0)$ to have non-zero determinant, but can still ensure convergence of the integrals at $\zeta=0$ by choosing N in (5.2) sufficiently large). This completes the proof in Cases IV and V.

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