J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
36 (1989), 247-262.

Some remarks on the asymptotic existence theorem for meromorphic differential equations

Dedicated to Professor Tosihusa Kimura on the occasion of his sixtieth birthday

By Donald G. BABBITT and V. S. VARADARAJAN*)

0. In the local theory of linear meromorphic differential equations with an irregular singularity a fundamental role is played by the theorem that asserts that formal solutions of such equations are asymptotic, over sufficiently small sectors, to analytic solutions. It was Poincaré who first proved a germinal version of this result [Po], thereby discovering the analytic significance of the formal solutions to these equations that had been obtained earlier by Fabry [Fa]. Poincaré's work was eventually generalized and refined by Trijitzinsky [Tr], Malmquist [Malm], Hukuhara [Hu], and Turrittin [Tu] (see Majima [Maj] for a very nice historical account). This theorem is also closely related to the theorem that asserts that any formal reduction of a system of linear meromorphic differential equations can be lifted to an analytic reduction on sufficiently small sectors (cf. [Si], [W]).

A main ingredient of the proofs of these results is a remarkable and far-reaching asymptotic existence theorem concerning certain *nonlinear* ordinary differential equations, which in turn is a consequence of a result that asserts the existence of *flat* solutions to a class of such equations. The requirement of flatness introduces essential complications in the proof along conventional lines of the existence of solutions; these are overcome with the help of some unusual and beautiful variants of the standard arguments.

There are several places in the literature where this latter problem is treated in some detail ([Iw], [W] [Ra-Si]). Nevertheless, in view of the beauty and importance of this result, it appears to us a worthwhile exercise to present an exposition of the proof which, although it is constructed along lines similar to those in [W] and [Ra-Si], differs from them in certain places. We also treat the parametric case with a view to

*) Research supported in part by NSF Grants ♯ DMS 85-01742 and DMS 88-03289.

making applications to the asymptotics of *isoformal families* of differential equations. We are very grateful to Professor Sibuya for pointing out an error in our original exposition and for several extremely helpful discussions.

1. We begin with some preliminary notation and remarks. A *sector* in the complex plane C is a subset of $C^{\times}=C\backslash(0)$ of the form

$$\{z=re^{i\theta}:\alpha<\theta<\beta\} \qquad \varphi\leq\alpha<\beta\leq 2\pi+\varphi.$$

Sectors are *proper* subsets of $C^{\times}$; the *angle* of the sector is then $\beta-\alpha$. For any subset A of C and any $\delta>0$ we write A_δ for the subset of A of all points in it that are at a distance $<\delta$ from the origin. If A and B are subsets of $C^{\times}$ we write $A\Subset B$ to mean $\mathrm{Cl}(A)\subset B$. If Γ is a sector, an open subset Ω of Γ is said to be *asymptotic to* Γ ($\Omega\sim\Gamma$) if for any sector $\Gamma''\Subset\Gamma'$, there exists a $\delta>0$ such that $\Gamma'_\delta\subset\Omega$ (see Figure 1 below):

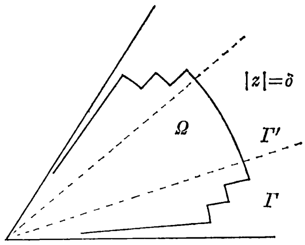

Figure 1.

For instance, if $\Gamma(n)$ is a sequence of subsectors of Γ whose union is Γ and $\delta(n)$ is a sequence of positive numbers that converge to 0, the set

$$\Omega=\bigcup_n \Gamma(n)_{\delta(n)}$$

is an open subset of Γ that is asymptotic to Γ. Let Γ be a sector in $C^{\times}$. We consider analytic functions defined on open sets $\Omega\subset\Gamma$ that are asymptotic to Γ, two such functions being regarded equivalent if they coincide on an open subset of Γ that is asymptotic to Γ. The equivalence classes are the *germs*, but we shall allow ourselves as usual to abuse the notation and work with the functions rather than the germs. Let a be such an analytic function. Let us write

$$\mathcal{F}=C[[z]][z^{-1}].$$

If $\alpha=\sum_r c_r z^r \in \mathcal{F}$, we say that a is *asymptotic to α in Γ* (or Ω) $a \sim \alpha(\Gamma)$, if for any sector $\Gamma' \Subset \Gamma$ and any integer $N \geq 0$,

$$a(z)=\sum_{r \leq N} c_r z^r + O(|z|^{N+1}) \qquad (z \in \Gamma', z \to 0).$$

The element α is then uniquely determined by a and is denoted by $a^\wedge$. In this case, we have, for any integer $r \geq 0$,

$$(d/dz)^r a \sim (d/dz)^r a^\wedge(\Gamma).$$

The set of all germs of such a is thus a differential C-algebra, which we denote by $A(\Gamma)$. The map $a \to a^\wedge$ is a homomorphism from $A(\Gamma)$ to $\mathcal{F}$. If $\Gamma = C^\times$, then $A(\Gamma)=\mathcal{F}_{cgt}$, $a^\wedge = a$; if $\Gamma \neq C^\times$, then the classical theorem of Borel-Ritt asserts that the homomorphism $a \to a^\wedge$ maps $A(\Gamma)$ *onto* $\mathcal{F}$. The kernel of this map is the ideal that consists of the so-called germs of *flat* functions in Γ, namely, germs of functions a such that

$$a \sim 0(\Gamma).$$

We also need to consider asymptotic expansions when parameters are present. Fix an integer $d \geq 1$ and let Δ denote, with or without suffixes, a polydisc in C^d centered at the origin of C^d; these polydiscs will be the domains of variations of our parameters. We write $\mathcal{O}_d$ for the ring of germs of analytic functions defined around the origin of C^d, $\mathcal{O}_d(\Delta)$ for the subring of analytic functions that are defined on Δ, and define $\mathcal{O}_{d,1}$ as the subring of $\mathcal{O}_d[[z]][z^{-1}]$ consisting of those Laurent series in z whose coefficients are defined on some common polydisc:

$$\mathcal{O}_{d,1}=\bigcup_{\Delta} \mathcal{O}_d(\Delta)[[z]][z^{-1}].$$

Let Γ be a sector in $C^\times$. An open subset Ω of $C^d \times \Gamma$ is said to be *asymptotic to Γ* if for any sector $\Gamma' \Subset \Gamma$ there is $\Delta=\Delta(\Gamma')$ and $\delta=\delta(\Gamma')>0$ such that $\Delta(\Gamma') \times \Gamma'_\delta \subset \Omega$; we write $\Omega \sim \Gamma$. For instance, if $\Gamma'_n=\Gamma(n)_{\delta(n)}$ where $\Gamma(n)$ and $\delta(n)$ are as in the previous example and Δ_n are polydiscs that shrink to (0), then,

$$\Omega=\bigcup_n (\Delta_n \times \Gamma'_n)$$

is an open set asymptotic to Γ. We now consider germs of analytic functions defined on open sets Ω asymptotic to Γ, germs being equivalence classes for the obvious notion of equivalence: f' defined on Ω' is equivalent to f defined on Ω if there is an Ω'' asymptotic to Γ on which $f=f'$. If f is defined on Ω, f is said to *have an asymptotic expansion* if there is a formal power series $f^{\wedge}\in\mathcal{O}_{d,1}$ with the following property: for any sector $\Gamma'\Subset\Gamma$ there are $\Delta(\Gamma')\subset\Delta$ and $\delta=\delta(\Gamma')>0$ for which $\Delta(\Gamma')\times\Gamma'_{\delta}\subset\Omega$ such that for any integer $N\geq 0$ we have

$$f(\lambda:z)=\sum_{r\leq N}a_r(\lambda)z^r+O(|z|^{(N+1)})\qquad(z\in\Gamma',\ z\rightarrow 0)$$

the O being uniform in $\lambda\in\Delta(\Gamma')$. We denote this by

$$f\sim f^{\wedge}(\Omega),$$

the element $f^{\wedge}$ being uniquely determined by f. We write $A_d(\Gamma)$ for the differential algebra of (the germs of) such f. When there are no parameters, i.e., when $d=0$, $A_d(\Gamma)$ reduces to the algebra $A(\Gamma)$. The map $f\rightarrow f^{\wedge}$ is a homomorphism, and for any differential operator

$$D=(\partial/\partial\lambda_1)^{m(1)}\cdots(\partial/\partial\lambda_d)^{m(d)}(d/dz)^r,$$

we have,

$$Df\sim Df^{\wedge}(\Gamma).$$

The Borel-Ritt theorem remains true in the parametric setting, and we formulate it in the following sharp form: if Γ is a sector $\neq \boldsymbol{C}^{\times}$, and $\varphi=\sum_{r\in\boldsymbol{Z}}a_rz^r\in\mathcal{O}_{d,1}$, then, for any $\Delta'\Subset\Delta$ and $\alpha>0$, we can find f defined and analytic on $\Delta'\times\Gamma_{\alpha}$ such that $f\sim\varphi(\Delta'\times\Gamma_{\alpha})$. Indeed, let us define the numbers t_m as follows: t_m is 0 when $a_m=0$ and $t_m=(\sup_{\lambda\in\Delta'}|a_m(\lambda)|)^{-1}$ otherwise. If $\alpha>0$, and $0<\beta<1$ is so small that $\cos(\beta\arg z)\geq 1/2$ for all $z\in\Gamma$, then the function

$$f(\lambda:z)=\sum_m a_m(\lambda)(1-\exp(-t_m\alpha^{-m}z^{-\beta}))z^m$$

is analytic on $\Delta'\times\Gamma_{\alpha}$ and $f\sim\varphi(\Delta'\times\Gamma_{\alpha})$ (see [W], pp. 41–42; the O-estimates for the differences between f and the initial segments of φ are actually uniform on all of $\Delta'\times\Gamma_{\alpha}$).

The extension of the notion of order (in z) to the rings $A_d(\Gamma)$ is immediate; if $f\in A_d(\Gamma)$ and $f\sim f^{\wedge}(\Gamma)$, then the order of f is the order of $f^{\wedge}$, namely, the smallest of the numbers r such that $a_r\neq 0$.

2. We fix a sector Γ in the z-plane, an integer $m\geq 1$, and consider a system of n ordinary differential equations in $u=(u_1,\cdots,u_n)$ of the form

(1) $$z^{m+1}du_i/dz=\delta_i u_i+f_i(z:u_1,\cdots,u_n)\qquad(1\leq i\leq n),$$

where the following conditions are satisfied:

(a) the δ_i are units of $\mathcal{O}_d$
(b) the f_i are polynomials in $u_1,\cdots,u_n$ with coefficients in $A_d(\Gamma)$
(c) the coefficients of the f_i have order ≥ 0; and those of the terms of degree (in the u_i) ≤ 1 are of order >0.

We say that $v=(v_1,\cdots,v_n)$, $v_i\in\mathcal{O}_{d,1}$ is a *formal solution* to (1) if it satisfies

(1_f) $$z^{m+1}dv_i/dz=\delta_i v_i+f_i^{\wedge}(z:v_1,\cdots,v_n)\qquad(1\leq i\leq n),$$

where $f_i^{\wedge}$ is the polynomial in $v_1,\cdots,v_n$ whose coefficients are the elements of $\mathcal{O}_{d,1}$ that are the asymptotic expansions of the corresponding coefficients of the f_i.

THEOREM 2.1. *Suppose the angle of Γ is $\leq\pi/m$, and the system* (1) *has a formal solution $v=(v_1,\cdots,v_n)$, with* $\mathrm{ord}(v_i)>0$, $1\leq i\leq n$. *Then, we can find $u_i\in A_d(\Gamma)$ such that*
(a) *$u=(u_1,\cdots,u_n)$ satisfies* (1)
(b) *$u_i\sim v_i(\Gamma)$.*

PPOOF. Let $v=(v_1,\cdots,v_n)$, $\mathrm{ord}(v_i)>0$, be a formal solution to (1). Choose Δ and functions w_i analytic on $\Delta\times\Gamma_\alpha$ (for some $\alpha, 0<\alpha<1$) so that $v_i\in\mathcal{O}_d(\Delta)[[z]][z^{-1}]$ and

$$w_i\sim v_i(\Delta\times\Gamma)$$

for all i. Let Ω ($\sim\Gamma$) be the intersection with $\Delta\times\Gamma_\alpha$ of the domain where the coefficients of the f_i are all defined. Then for all i,

$$z^{m+1}dw_i/dz-\delta_i(\lambda)w_i-f_i^{\wedge}(\lambda:z:w_1,\cdots,w_n)\sim 0(\Gamma).$$

We then seek a solution to (1) in the form $u_i=w_i+a_i$, a_i being defined on a domain $\Omega'\subset\Omega$ asymptotic to Γ and $\sim 0(\Gamma)$. The equations for the a_i become

(2) $$z^{m+1}da_i/dz=\delta_i(\lambda)a_i+g_i(\lambda:z:a_1,\cdots,a_n)\qquad(1\leq i\leq n),$$

where the g_i are polynomials in $a_1,\cdots,a_n$ with coefficients $g_{i,0}$ (constant term), $g_{i,j}$ (coefficient of a_j), and $g_{i,\mu}$ (coefficient of monomials in the a's with multi-index $\mu=(\mu_1,\cdots,\mu_n)$ where $|\mu|=\mu_1+\cdots+\mu_n\geq 2$). The coefficients are analytic on Ω and satisfy

(3) $\quad g_{i,0}\sim 0(\Gamma), \quad \mathrm{ord}(g_{i,j})>0 \ (1\leq i, j\leq n), \quad \mathrm{ord}(g_{i,\mu})\geq 0 \ (|\mu|\geq 2).$

We write

$$c_i(\lambda)=-(1/m)\delta_i(\lambda)$$

and rewrite (2) as

(4) $\quad d/dz(a_i\exp(-c_i(\lambda)z^{-m}))=\exp(-c_i(\lambda)z^{-m})z^{-m-1}g_i(\lambda:z:a_1,\cdots,a_n)$

so that we can go over to the equivalent integral equation

(5a)
$$a_i(\lambda:z)=k_i(\lambda)\exp(c_i(\lambda)(z^{-m})) + \int_{C_i(z)}\exp(c_i(\lambda)(z^{-m}-\zeta^{-m}))\zeta^{-m-1}g_i(\lambda:\zeta:a_1,\cdots,a_n)d\zeta$$

for suitable "initial values" $k_i(\lambda)$ and "boundary conditions"

(5b)
$$a_i\sim 0(\Gamma),$$

the $C_i(z)$ being paths in Γ that end at the point z.

The technique of proving this theorem is essentially the usual one of formulating the solution as the fixed point of a suitable (non linear) integral operator in a Banach space of analytic functions. However, the flatness condition (5b) creates substantial complications because it cannot be formulated in a single Banach space. We shall now proceed to explain how these difficulties are overcome.

For any open set θ contained in the unit disc in $\boldsymbol{C}$ not containing 0 but with $0\in\mathrm{Cl}(\theta)$, any polydisc $\Delta\subset\boldsymbol{C}^d$, and any real number $k\geq 0$, let $B_k(\Delta\times\theta)$ be the Banach space of functions g that are defined and holomorphic on $\Delta\times\theta$ and satisfy

$$\|g\|_k=\sup_{z\in\theta,\lambda\in\Delta}|z|^{-k}|g(\lambda:z)|<\infty.$$

The B_k form a *filtration of Banach algebras*:

$$B_r\subset B_k \quad (r\geq k\geq 0),$$
$$g\in B_r,\ h\in B_k \Longrightarrow gh\in B_{k+r},\ \|gh\|_{k+r}\leq\|g\|_r\|h\|_k.$$

The first step of the proof of the existence of flat solutions a_i to (4) is to show that for each $\Gamma'\Subset\Gamma$ and $k>0$, the right side of (5a) defines a *contraction operator* in a sufficiently small ball in the Banach space

$B_k(\Delta'\times\Gamma'(\delta))$ where $0<\delta\leq\alpha$ and $\Gamma'(\delta)$ is an open subset of Γ' that is asymptotic to Γ', provided the $k_i(\lambda)$ are sufficiently small. The fixed points of this operator will then give solutions to (4). The second step, which is the key to the entire proof from now on, is Lemma 2.4 which asserts that *any solution in* $B_k(\Delta'\times\Gamma'(\delta))$, *for a fixed* $k>0$, *is automatically flat* (cf. [Ra-Si], Theorem 2.3.1). This not only implies the flatness of the special fixed point solutions constructed above but shows further that *any solution in* $B_k(\Delta'\times\Gamma'(\delta))$, *when* δ *is sufficiently small, satisfies the special conditions guaranteeing the existence of fixed points.* This is a strong uniqueness statement for the solutions of (4) (cf. [Ra-Si], Theorem 2.1.1, (ii)) that allows one to extend this solution by successively enlarging the sector Γ' so that the fixed points of the contraction operators associated to Γ' *vary coherently.* This extension process finally leads to a solution defined on a region that is asymptotic to Γ.

The paths $C_i(z)$ and the domains $\Gamma'(\delta)$ are chosen as follows. The idea is to choose the paths so that the kernel $\exp(c_i(\lambda)(z^{-m}-\zeta^{-m}))\zeta^{-m-1}$ in (5a) is *small* on them. We shall suppose from now on that all subsectors of Γ considered by us are *symmetric*, namely, symmetric with respect to the bounding rays of Γ. Let $T(C^\times\to C^\times)$ be the map $z\to t=z^{-m}$; we write $t=Tz$. T is then one-one on sectors in the z-plane of angles $<2\pi/m$, in particular on Γ whose angle is $\leq\pi/m$; $T\Gamma$ is then a sector of angle $\leq\pi$ in the t-plane. Let $0<\delta\leq 1$ and let $z(\delta)$ be the mid-point of the arc of $|z|=\delta$ within Γ and $t(\delta)=Tz(\delta)$. We form the domain $T\Gamma+t(\delta)$ (which has no point in common with the domain $|t|\leq\delta^{-m}$) and define $\Gamma(\delta)$ by

$$\Gamma(\delta)=T^{-1}(T\Gamma+t(\delta))$$

(see Figure 2 below):

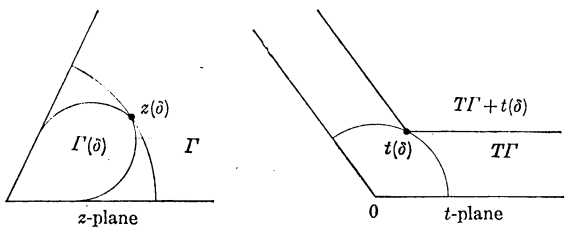

Figure 2.

The bounding curves of $\Gamma(\delta)$ make zero angle with the bounding rays of Γ at the origin and so $\Gamma(\delta)$ is asymptotic to Γ. These definitions apply to any sector $\Gamma' \subset \Gamma$ in place of Γ (and symmetric by agreement, so that $t(\delta)$ and $z(\delta)$ remain the same).

We shall now describe the paths. If $1 \leq i \leq n$, we shall say that i is of *type* I if

$$\text{(I)} \qquad \exists \gamma_i \in \Gamma, \ |\gamma_i| = 1, \text{ such that } \operatorname{Re}(c_i(0)\gamma_i^{-m}) > 0.$$

We choose, once and for all, γ_i satisfying (I) for all i of type I and write $t_i = T\gamma_i$. If i is not of type I, we shall say it is *of type* II. It is easy to see that

$$\text{(II)} \qquad i \text{ is of type II} \Longleftrightarrow \operatorname{Re}(c_i(0)\gamma^{-m}) < 0 \quad \text{for all } \gamma \in \Gamma.$$

Indeed, if $c_i(0) = \rho e^{i\varphi}$ and $\gamma = re^{i\theta}$, $\operatorname{Re}(c_i(0)\gamma^{-m}) = 0 \Longleftrightarrow \cos(m\theta - \varphi) = 0$, and this function takes values of both signs close to its zeros. So, in this case, if $\Gamma' \Subset \Gamma$, there is a $\xi' = \xi'(\Gamma')$ with $0 < \xi' < 1$ such that $\operatorname{Re}(c_i(0)\gamma^{-m}) \leq -\xi'$ for all $\gamma \in \Gamma'$ with $|\gamma| = 1$. If $\Gamma_1 \subset \Gamma$ is an open sector, we write $\Gamma_1 \ll \Gamma$ to indicate that $\Gamma_1 \Subset \Gamma$, is symmetric with respect to Γ, and is large enough so that it contains all the γ_i for i of type I; we shall from now on consider only such sectors. Let now z be a variable point of $\Gamma(\delta)$. If i is of type I, the path $C_i(z)$ is the one that corresponds, under T, to the infinite line segment that comes from ∞ to $t = Tz$ in the direction $-T\gamma_i$; if $\Gamma' \ll \Gamma$ and $z \in \Gamma'$, this path lies in $\Gamma'(\delta)$. If i is of type II, $C_i(z)$ is the path from $z(\delta)$ to z that corresponds under T to the line segment $[t(\delta), t]$; again these paths lie in $\Gamma'(\delta)$ if $z \in \Gamma'$ and $\Gamma' \ll \Gamma$. Let $TC_i(z) = B_i(t)$ (see Figure 3 below):

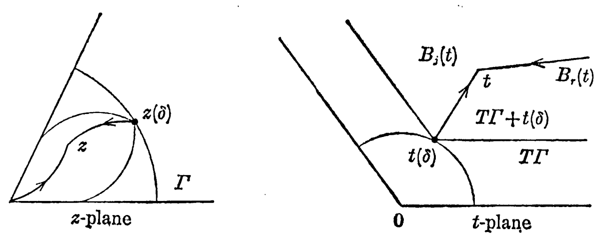

Figure 3.

We choose a Δ sufficiently small so that we have the following:

(a) if i is of type I

$$\text{(I}'\text{)} \qquad \operatorname{Re}(c_i(\lambda)\gamma_i^{-m}) \geq u_0 > 0 \qquad (\lambda \in \Delta)$$

(b) for each sector $\Gamma' \ll \Gamma$ we choose $\Delta' = \Delta'(\Gamma') \Subset \Delta$, $\delta' = \delta'(\Gamma')$, with $0 < \delta' < 1$, and $\xi' = \xi'(\Gamma')$ with $0 < \xi' < 1$, such that $\mathrm{Cl}(\Delta' \times \Gamma'_{\delta'}) \subset \Omega$, and for any i of type II,

$$\text{(II}'\text{)} \qquad \operatorname{Re}(c_i(\lambda)\gamma^{-m}) \leq -\xi' |\gamma|^{-m} \qquad (\gamma \in \Gamma',\ \lambda \in \Delta').$$

(c) $|c_i(\lambda)| \geq u_0 > 0$ for all $\lambda \in \Delta$ and for all i.

Regarding these paths we have the following easily established estimates:

(i) If i is of type I, $B_i(t)$ is the path $\tau : s \longrightarrow t + st_i$ $(\infty > s \geq 0)$, and

$$\text{(I}''\text{)} \qquad |\exp(-c_i(\lambda)(t - \tau(s)))| \leq \exp(-su_0), \quad t \in T\Gamma'(\delta),\ \lambda \in \Delta,\ s \geq 0;$$

moreover, the cosine of the angle between t and t_r is bounded away from -1, so that

$$\text{(I}'''\text{)} \qquad |t + st_i| \geq \beta |t| \text{ for some constant } \beta > 0, \text{ and all } t \in T\Gamma \text{ and } s \geq 0.$$

(ii) Suppose i is of type II. Let ρ be the midpoint of the line segment $[t(\delta), t]$ from $t(\delta)$ to t. Then for $\tau \in [\rho, t]$, $t \in T\Gamma' + t(\delta)$, $\lambda \in \Delta'$

$$\text{(II}''\text{)} \qquad |\tau| \geq (1/2)|t|, \qquad \operatorname{Re} c_i(\lambda)(t - \tau) \leq -\xi' |t - \tau|.$$

Moreover, for $\tau \in [t(\delta), \rho]$, $\lambda \in \Delta'$, we have,

$$\begin{aligned} &\operatorname{Re} c_i(\lambda)(t-\tau) = \operatorname{Re} c_i(\lambda)(t-\rho) + \operatorname{Re} c_i(\lambda)(\rho - \tau) \\ \text{(II}'''\text{)} \quad &|\tau| \geq |t(\delta)| \geq (1/2)|t| \quad \text{in the regime} \quad |t - t(\delta)| < (1/2)|t| \\ &|\tau| \geq |t(\delta)| \geq 1,\ |t - \rho| \geq (1/4)|t| \quad \text{in the regime} \quad |t - t(\delta)| \geq (1/2)|t|. \end{aligned}$$

LEMMA 2.2. *Fix i, $1 \leq i \leq n$, and define*

$$(J_{i,\delta} g)(\lambda : z) = \int_{C_i(z)} \exp(c_i(\lambda)(z^{-m} - \zeta^{-m}))\zeta^{-m-1} g(\lambda : \zeta) d\zeta.$$

Fix any sector $\Gamma' \ll \Gamma$, let Δ' and δ' be associated to Γ' as above, and let $k \geq 0$ be any real number. Then for any sector $\Gamma_1 \ll \Gamma$ and $\subset \Gamma'$, and any δ with $0 < \delta \leq \delta'$, $J_{i,\delta}$ defines a bounded operator on the Banach space $B_k(\Delta' \times (\Gamma'(\delta) \cap \Gamma_1))$, and the norm of this operator is uniformly bounded when δ and Γ_1 vary but Γ' and k are fixed.

PROOF. The point is that the paths $C_i(z)$ stay within $\Gamma'(\delta) \cap \Gamma_1$ when $z \in \Gamma'(\delta) \cap \Gamma_1$, as may be seen by going over to the t-plane. Write $\tau = T\zeta$,

$t_i=T\gamma_i$ (i of type I), $h(\lambda:\tau)=g(\lambda:\zeta)$. We must show that

$$\left|\int_{B_i(t)}\exp(c_i(\lambda)(t-\tau))h(\lambda:\tau)d\tau\right|\leq K\cdot|t|^{-k/m}\sup(|h(\lambda:\tau)||\tau|^{k/m})$$

where the sup is over $\lambda\in\Delta'$, $\tau\in T(\Gamma'(\delta)\cap\Gamma_1)$, and K denotes a constant independent of δ, Γ_1 and h. This comes to proving that

$$(*)\qquad I=\int_{B_i(t)}\exp(\mathrm{Re}\, c_i(\lambda)(t-\tau))|\tau|^{-k/m}d\tau\leq K\cdot|t|^{-k/m}.$$

If i is of type I we use (I″) and (I‴) to obtain the estimate

$$\int_{B_i(t)}\exp(\mathrm{Re}\, c_i(\lambda)(t-\tau))|\tau|^{-k/m}d\tau\leq(\beta^{-k/m}u_0^{-1})\cdot|t|^{-k/m}.$$

These estimates show also that the integral defining $J_{i,\delta}g$ is uniformly convergent, so that the holomorphy of $J_{i,\delta}g$ is clear. If i is of type II, we are integrating over a finite segment and the holomorphy is obvious, but the proof of (*) is more delicate. We now split I as

$$\int_{[t(\delta),t]}=\int_{[t(\delta),\rho]}+\int_{[\rho,t]}=I_1+I_2.$$

$\boxed{I_2}$: Here (II″) is applicable, and so,

$$I_2\leq 2^{k/m}|t|^{-k/m}\cdot\int_0^{|t-\rho|}\exp(-\xi'x)dx\leq 2^{k/m}\xi'^{-1}|t|^{-k/m}.$$

$\boxed{I_1 \text{ with } |t-t(\delta)|<(1/2)|t|}$ Here, we use (II‴) to get

$$I_1\leq 2^{k/m}|t|^{-k/m}\cdot\int_{|t-\rho|}^{|t-t(\delta)|}\exp(-\xi'x)dx\leq 2^{k/m}\xi'^{-1}|t|^{-k/m}.$$

$\boxed{I_1 \text{ with } |t-t(\delta)|\geq(1/2)|t|}$ Here again we use (II‴) to get

$$I_1\leq\exp(-\xi'|t|/4)\cdot\int_0^{|\rho-t(\delta)|}\exp(-\xi'x)dx\leq(1/\xi')\sup_{y>0}(y^{k/m}e^{-\xi'y/4})\cdot|t|^{-k/m}. \qquad \blacklozenge$$

Let us now introduce the Banach spaces

$$\boldsymbol{B}_k(\Gamma',\Gamma_1,\delta)=\boldsymbol{B}_k(\Delta'\times(\Gamma'(\delta)\cap\Gamma_1))^n \qquad (k>0)$$

where the n-fold product has the sup norm. Let $A=(A_i)_{1\leq i\leq n}$ where the A_i are bounded analytic functions on Δ' with $A_i=0$ whenever i is of type I. We put $\|A\|=\sup_i \sup_{\lambda\in\Delta'}|A_i(\lambda)|$. We now define a non linear operator

$$\boldsymbol{P}'=\boldsymbol{P}'_{k,\delta,A}$$

on $\boldsymbol{B}_k(\Gamma',\Gamma_1,\delta)$ $(0<\delta\leq\delta')$ given by

$$\boldsymbol{P}'\boldsymbol{b}=\boldsymbol{\beta}=(\beta_1,\cdots,\beta_n),\qquad \text{for}\quad \boldsymbol{b}=(b_1,\cdots,b_n), \tag{6}$$

where (remembering that the $B_k(\Delta'\times(\Gamma'(\delta)\cap\Gamma_1))$ are algebras)

$$\beta_i(\lambda:z)=(J_{i,\delta}b_i^*)(\lambda:z)+A_i(\lambda)\exp(c_i(\lambda)(z^{-m}-z(\delta)^{-m}) \tag{7a}$$

$$b_i^*(\lambda:\zeta)=g_i(\lambda:\zeta:b_1(\lambda:\zeta),\cdots,b_n(\lambda:\zeta)). \tag{7b}$$

Observe that if i is of type II, then for any $r>0$, $|A_i(\lambda)\exp(c_i(\lambda)(z^{-m}-z(\delta)^{-m}))|$ is majorized on $\Delta'\times\Gamma'(\delta)$ by

$$\begin{aligned}&|z|^r\cdot\|A\|\cdot\sup_{t\in T\Gamma'(\delta)}\{|t|^{r/m}\exp(\mathrm{Re}(c_i(\lambda)(t-t(\delta))))\}\\ &\leq|z|^r\cdot\|A\|\cdot\sup_{\tau\in T\Gamma'}\{|\tau+t(\delta)|^{r/m}e^{-\xi'|\tau|}\}\\ &\leq|z|^r\cdot\|A\|\cdot\delta^{-r}M'_r;\end{aligned}$$

here we use the estimates

$$|t(\delta)|=\delta^{-m}\geq1,\qquad |\tau+t(\delta)|\leq|t(\delta)|(1+|\tau|),$$

and M'_r is defined by

$$M'_r=\sup_{x>0}\{(1+x)^{r/m}e^{-\xi'x}\}.$$

So, if we write

$$\varepsilon_{i,\delta,A}(\lambda:z)=A_i(\lambda)\exp(c_i(\lambda)(z^{-m}-z(\delta)^{-m}), \tag{8a}$$

then $\varepsilon_{i,\delta,A}$ lies in every $B_r(\Delta'\times\Gamma'(\delta))$ and satisfies

$$\|\varepsilon_{i,\delta,A}\|_r\leq M'_r\|A\|\delta^{-r}\qquad (r>0). \tag{8b}$$

This estimate and Lemma 2.2 make it clear that $\boldsymbol{P}'_{k,\delta,A}$ maps $\boldsymbol{B}_k(\Gamma',\Gamma_1,\delta)$ into itself for any δ $(0<\delta\leq\delta')$ and Γ_1 $(\subset\Gamma'$ and $\ll\Gamma)$.

If $\boldsymbol{b}=(b_i)_{1\leq i\leq n}\in\boldsymbol{B}_k(\Gamma',\Gamma_1,\delta)$, then $\lambda\to b_i(\lambda:z(\delta))$ are bounded analytic functions of λ on Δ'. We now note that the fixed point equation

$$\boldsymbol{P}'_{k,\delta,A}\boldsymbol{b}=\boldsymbol{b} \tag{9a}$$

is equivalent to the differential equation

(9b) $$d/dz(b_i \exp(-c_i(\lambda)z^{-m})) = \exp(-c_i(\lambda)z^{-m})z^{-m-1}g_i(\lambda : z : b_1, \cdots, b_n)$$

together with the "initial conditions"

(9c) $$b_i(\lambda : z(\delta)) = A_i(\lambda) \qquad (i \text{ of type II}).$$

Indeed, (9b) and (9c) follow from (9a) by differentiation. Conversely, if (9b) and (9c) are satisfied by $\boldsymbol{b}$, we can integrate both sides of (9a) over the paths $C_i(z)$, which is permissible by Lemma 2.2. If i is of type II, this gives, in view of (9b),

(10a) $$b_i = (J_{i,\delta}b_i^*) + \varepsilon_{i,\delta,A}.$$

If i is of type I, we get

(10b) $$b_i = (J_{i,\delta}b_i^*),$$

since, for these indices,

(11) $$\lim_{\zeta \to 0} b_i(\lambda : \zeta) \exp(c_i(\lambda)(z^{-m} - \zeta^{-m})) = 0,$$

the point ζ going to 0 along the path $C_i(z)$. The relations (10) are the same as the equation (8).

LEMMA 2.3. *Fix $\Gamma' \ll \Gamma$, $k \geq 1$. Then we can find δ_k', θ_k', σ_k' that satisfy $0 < \delta_k' \leq \delta'$, $0 < \theta_k' \leq 1$ and $0 < \sigma_k' \leq 1$, and have the following property: for any δ with $0 < \delta \leq \delta_k'$, any A with $\|A\| < \sigma_k'\delta^k$, and any $\Gamma_1 \ll \Gamma$ and $\subset \Gamma'$, $\boldsymbol{P}' = \boldsymbol{P}_{k,\delta,A}'$ defines a contraction operator on the closed ball of radius θ_k' centered at the origin of the Banach space $\boldsymbol{B}_k(\Gamma', \Gamma_1, \delta)$.*

PROOF. Let $L_k' \geq 1$ be an upper bound for the norms of the operators $J_{i,\delta}$, $1 \leq i \leq n$, $0 < \delta \leq \delta'$ on the Banach spaces $\boldsymbol{B}_k(\Gamma', \Gamma_1, \delta)$, $\Gamma_1 \ll \Gamma$ and $\subset \Gamma'$. Then, with $\boldsymbol{b}^* = (b_1^*, \cdots, b_n^*)$, we have, in $\boldsymbol{B}_k(\Gamma', \Gamma_1, \delta)$,

$$\|\boldsymbol{P}'\boldsymbol{b}\|_k \leq L_k'\|\boldsymbol{b}^*\|_k + M_k'\|A\| \cdot \delta^{-k}$$
$$\|\boldsymbol{P}'\boldsymbol{b} - \boldsymbol{P}'\boldsymbol{b}'\|_k \leq L_k'\|\boldsymbol{b}^* - \boldsymbol{b}'^*\|_k.$$

Let $\alpha_k(\delta)$ (resp. $\beta(\delta)$) be the maximum of the norms of $g_{i,0}$ (resp. $g_{i,j}$) in $B_k(\Delta' \times \Gamma'(\delta))$ (resp. $B_0(\Delta' \times \Gamma'(\delta))$), and γ the maximum of $\sup|g_{i,\mu}|$ over $\Delta' \times \Gamma_{\delta'}'$. Then,

(12) $$\alpha_k(\delta) \longrightarrow 0, \quad \beta(\delta) \longrightarrow 0 \quad (\delta \longrightarrow 0).$$

It is then easy to see that there is a constant $N>1$ with the property: if $\|b\|_k\leq 1$, $\|b'\|_k\leq 1$, then

$$\|b^*\|_k\leq\alpha_k(\delta)+n\beta(\delta)\|b\|_k+\gamma N\|b\|_k^2,$$
$$\|b^*-b'^*\|_k\leq\{n\beta(\delta)+\gamma N\max(\|b\|_k,\ \|b'\|_k)\}\|b-b'\|_k.$$

So, if $0<\theta\leq 1$, $0<\sigma\leq 1$, $\|b\|_k\leq\theta$, $\|b'\|_k\leq\theta$, and $\|A\|<\sigma\delta^k$, then

$$\|P'b\|_k\leq L_k'\{\alpha_k(\delta)+n\beta(\delta)\theta+\gamma N\theta^2\}+\sigma M_k',$$
$$\|P'b-P'b'\|_k\leq L_k'\{n\beta(\delta)+\gamma N\theta\}\|b-b'\|_k.$$

Take $\theta_k'=(4N(1+\gamma)L_k')^{-1}$ and let σ_k' and δ_k' be such that

$$L_k'\{\alpha_k(\delta)+n\beta(\delta)\}<\theta_k'/4 \qquad \text{for} \quad \delta\leq\delta_k',$$
$$\sigma M_k'\leq\theta_k'/4 \qquad \text{for} \quad \sigma\leq\sigma_k'.$$

The assertions of the lemma are now clear with these choices. ◆

LEMMA 2.4. *Let $k>0$, δ $(0<\delta\leq\delta')$ be arbitrary and let us assume that $b\in B_k(\Gamma',\Gamma_1,\delta)$ satisfy the system of differential equation* (4). *Then b is flat, viz.,*

$$b\in B_r(\Gamma',\Gamma_1,\delta) \qquad \textit{for all} \quad r>0.$$

PROOF. By our earlier remarks b satisfies the relations (9a)-(9c). We write $q=\min(1,k)$ and prove by induction on $r\geq 1$ that $b\in B_{rq}(\Gamma',\Gamma_1,\delta)$ for all $r\geq 1$. Suppose that $b_i\in B_{rq}=B_{rq}(\Delta'\times(\Gamma'(\delta)\cap\Gamma_1))$ for all i and some $r\geq 1$. We have

$$b_i^*(\lambda:\zeta)=g_{i,0}(\lambda:\zeta)+\sum_{1\leq j\leq n}g_{i,j}(\lambda:\zeta)b_j(\lambda:\zeta)$$
$$+\sum_{|\mu|\geq 2}g_{i,\mu}(\lambda:\zeta)b_1(\lambda:\zeta)^{\mu(1)}\cdots b_n(\lambda:\zeta)^{\mu(n)}$$

μ being $(\mu(1),\cdots,\mu(n))$. Since $g_{i,0}\sim 0(\Gamma)$, the first term on the right is certainly in $B_{(r+1)q}$. Further, as $\mathrm{ord}(g_{i,j})>0$, $g_{i,j}\in B_1$ for all j so that the second term is also in $B_{(r+1)q}$. Finally, each term in the last summation is in B_{sq} where $s=r\cdot|\mu|\geq 2r\geq r+1$, hence in $B_{(r+1)q}$. Hence $b_i^*\in B_{(r+1)q}$ for all i. Lemma 2.3.2 now shows that $J_{i,\delta}b_i^*\in B_{(r+1)q}$ for all i. On the other hand, for any i of type II, the estimates (8) derived earlier show that $\varepsilon_{i,\delta,A}$ lies in $B_{(r+1)q}$. This shows that $b_i\in B_{(r+1)q}$ for all i. ◆

REMARK. This lemma is true even when $k=0$ [Ra-Si], although the above method does not seem to work. We do not however need this extended form of the lemma.

Since we can always take $A=0$, Lemmas 2.3 and 2.4 show that there are flat solutions to (4), namely solutions that lie in every $\boldsymbol{B}_r$ $(r>0)$. We shall now prove that *there are no solutions to* (4) *other than those obtained as the fixed points of the* $\boldsymbol{P}'_{k,\delta,A}$ *satisfying the conditions of Lemma* 2.3. More precisely, we have the following.

LEMMA 2.5. *Fix* Γ', Γ_1, *and* $k>0$ *as above. Let* $\boldsymbol{b}^\sim$ *be a solution to* (4) *that lies in some* $\boldsymbol{B}_k(\Gamma', \Gamma_1, \delta^\sim)$ $(0<\delta^\sim\leq\delta')$. *Then we can find an* $\varepsilon^\sim<\min(\delta^\sim, \delta'_k)$ *with the following property. For any* $\delta<\varepsilon^\sim$, $\exists A=A(\delta)$ *such that* $\|A\|<\sigma'_k\delta^k$, *the restriction of* $\boldsymbol{b}^\sim$ *to* $\Delta'\times(\Gamma'(\delta)\cap\Gamma_1)$ *lies in the ball of radius* θ'_k *in the space* $\boldsymbol{B}_k(\Gamma', \Gamma_1, \delta)$, *and is therefore the unique fixed point of* $\boldsymbol{P}'_{k,\delta,A(\delta)}$ *there.*

PROOF. By the preceding lemma we know that $\boldsymbol{b}^\sim\in \boldsymbol{B}_{k+1}(\Gamma', \Gamma_1, \delta^\sim)$. This means the supremum of $\|\boldsymbol{b}^\sim(\lambda:z)\|\cdot|z|^{-k}$ over $\Delta'\times(\Gamma'(\delta)\cap\Gamma_1)$ *tends to zero* as $\delta\to 0$, so that the restriction $\boldsymbol{b}_\delta$ of $\boldsymbol{b}^\sim$ to $\Delta'\times(\Gamma'(\delta)\cap\Gamma_1)$ lies in the ball of radius θ'_k in the space $\boldsymbol{B}_k(\Gamma', \Gamma_1, \delta)$, if δ is small enough. Moreover $\|\boldsymbol{b}_\delta(\lambda:z(\delta))\|=\|\boldsymbol{b}^\sim(\lambda:z(\delta))\|\leq\text{const.}\,\delta^{k+1}<\sigma'_k\delta^k$ if δ is small enough. For any such δ we now put $A_i(\lambda)=b_i^\sim(\lambda:z(\delta))$ if i is of type II and 0 if i is of type I. This defines a vector $A=A(\delta)$ with $\|A\|<\sigma'_k\delta^k$. It is now clear that $\boldsymbol{b}_\delta$ is the unique fixed point of $\boldsymbol{P}'_{k,\delta,A(\delta)}$ in the ball of radius θ'_k in the space $\boldsymbol{B}_k(\Gamma', \Gamma_1, \delta)$. ◆

LEMMA 2.6. *Suppose* Γ'' *is an open sector with* $\Gamma'\subset\Gamma''\Subset\Gamma$ *and* $\boldsymbol{b}'^\sim$ *is a solution to* (4) *defined and analytic on some* $\Delta'\times\Gamma'(\delta'^\sim)$. *Then we can find a solution* $\boldsymbol{b}''^\sim$ *to* (4) *defined and analytic on* $\Delta''\times\Gamma''(\delta''^\sim)$ *for some* $\Delta''\subset\Delta'$ *and* $\delta''^\sim<\delta'^\sim$ *such that* $\boldsymbol{b}''^\sim$ *coincides with the restriction of* $\boldsymbol{b}'^\sim$ *to* $\Delta''\times\Gamma''(\delta''^\sim)$.

PROOF. By diminishing Γ' (but still satisfying $\Gamma'\ll\Gamma$) we may assume that $\boldsymbol{b}'^\sim$ is defined on $\Gamma'\cap\Delta(\delta'^\sim)$ where $\Delta(\delta'^\sim)$ is the disc of radius $\delta'^\sim$ in the z-plane. We place ourselves in the setting of the previous lemma, but with Γ'' and Γ' in place of Γ' and Γ_1 respectively. Then if δ is sufficiently small, the restriction of $\boldsymbol{b}'^\sim$ to $\Gamma''(\delta)\cap\Gamma'$ is the fixed point of a contraction operator $\boldsymbol{P}''_{k,\delta,A}$ associated to $\Gamma''(\delta)\cap\Gamma'$. Since this fixed point is the restriction of the fixed point defined on $\Delta''\times\Gamma''(\delta)$ for the same contraction operator, we are through. ◆

Theorem 2.1 is now an immediate consequence of the preceding lemmas. ◆

REMARKS 1. One of the main applications of Theorem 2.1 is to the theory of asymptotics of *isoformal families* of differential equations

$$(*) \qquad du/dz = A(\lambda : z)u.$$

Here Theorem 2.1, in conjunction with the formal reduction theory of isoformal families developed in [BV 2], allows us to prove that any formal solution to $(*)$ can be lifted, on sufficiently small sectors, to an analytic solution. This will then imply that the map, that takes λ to the Stokes class of $(*)$ determined by the given formal reduction, is analytic in λ.

2. It will be clear to the reader from the proof of Theorem 2.1 that one can relax considerably the assumption that the f_i in (2.1) are polynomials in the u_i. We leave it to the reader to formulate and prove such a generalization (see [W], [Ra-Si]).

3. The reader should also consult [Ra-Si] for the formulations and proofs of the existence and uniqueness theorems in the context of the Gevrey theory.

Bibliography

[BV 1] Babbitt, D. G. and V. S. Varadarajan, Local moduli for meromorphic differential equations near an irregular singularity. To appear in Asterisque, 1989.

[BV 2] Babbitt, D. G. and V. S. Varadarajan, Deformations of nilpotent matrices over rings and reduction of analytic families of meromorphic differential equations, Mem. Amer. Math. Soc. **55** (1985).

[Fa] Fabry, E., Sur les intégrales des équations différentielles linéaires à coefficients rationnels, Thèse, Paris, 1885.

[Hu 1] Hukuhara, M., Sur les points singuliers des équations différentielles linéaires, II, J. Fac. Sci. Hokkaido Univ. **5** (1937), 123-166.

[Hu 2] Hukuhara, M., Sur les points singuliers des équations différentielles linéaires, III, Mem. Fac. Sci. Kyushu Univ. **2** (1942), 125-137.

[Iw] Iwano, M., Bounded solutions and stable domains of nonlinear ordinary differential equations, Lecture Notes in Math. vol. 183, Springer-Verlag, Berlin-Heidelberg-New York, 1971, 59-127.

[Maj] Majima, H., Asymptotic analysis for integrable connections with irregular singular points, Lecture Notes in Math. vol. 1075, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1984.

[Malm] Malmquist, J., Sur l'études analytiques des solutions d'un système des équations différentielles dans le voisinage d'un point singulier d'indétermination, I, II, III, Acta Math. **73** (1940), 8-129, **74** (1941), 1-64, 109-128.

[Po] Poincaré, H., Sur les intégrales des équations linéaires, Acta Math. **8** (1886), 295-344.

[Ra-Si] Ramis, J.-P. and Y. Sibuya, Hukuhara's domains and fundamental existence and uniqueness theorems for asymptotic solutions of Gevrey type. To appear in Asymptotic Analysis **2** (1989).

[Si] Sibuya, Y., Simplification of a system of linear ordinary differential equations about a singular point, Funkcial. Ekvac. **4** (1962), 29-56.

[Tri] Trijitzinsky, W. J., Analytic theory of linear differential equations, Acta Math. **62** (1934), 167-226.

[Tu] Turrittin, H., Convergent solutions of ordinary differential equations in the neighborhood of an irregular singular point, Acta Math. **93** (1955), 27-66.

[W] Wasow, W., Asymptotic Expansions for Ordinary Differential Equations, Interscience, New York, 1965, Dover 1987.

(Received January 27, 1989)

University of California at Los Angeles
Los Angeles, CA 90024
U. S. A.