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Condition of partial hyperbolicity modulo a linear subvariety for operators with constant coefficients

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1. Introduction

In [12] J. Leray proposed to solve the Cauchy problem for the initial hyperplane $x_1=0$ with the Cauchy data which are holomorphic with respect to the variables parallel to some analytic subvariety S of the initial hyperplane. He solved this problem in the Gevrey class assuming the following two conditions; 1) the characteristic roots are real when the variables dual to S are fixed to 0; 2) the characteristic roots are distinct. Later Hamada-Leray-Wagschal [2] extended the result to the case of operators of constant multiplicity. Limiting the problem to the case of operators with constant coefficients, one of the authors discussed in [4] that condition 1) is not sufficient whereas condition 2) is more than necessary, and proposed a new sharper sufficient condition when S is a hyperplane. We generalized this condition to the case of general linear subvariety and showed that it is sufficient for the solvability of Cauchy problem for the hyperfunction Cauchy data which contain variables parallel to S as holomorphic parameters [4, 11]. In this article, we show that it is also necessary for the solvability of Cauchy problem and establish theorems corresponding to distribution solutions. Finally, we give the micro-local variants of these.

After the first version of our manuscript was submitted, Prof. T. Ôaku kindly remarked us that our operator is an example of micro-hyperbolic system in the sense of Kashiwara-Schapira [8] when it is considered together with $\bar{\partial}_{z'''}$. Thus we should say that the sufficiency part of our work is already known even in the micro-local level. We expect, however, that our simple direct proof has some meaning though the stress of our work is rather on the necessity part.

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2. Hyperfunction Cauchy data

Let $P(D)$ be an m-th order linear partial differential operator with constant coefficients. Let $p_m(D)$ be its principal part. Assume that $x_1=0$ is non-characteristic with respect to P. We employ the following notation for the separation of the independent variables; $x=(x_1, x')=(x_1, x'', x''')$ with $x''=(x_2, \cdots, x_{k+1})$, $x'''=(x_{k+2}, \cdots, x_n)$ and similar one for the complexification $z=x+\sqrt{-1}y$ or for the dual variables $\zeta=\xi+\sqrt{-1}\eta$. We put

$$\Omega_A=\{x''\in \boldsymbol{R}^k;\ |x''|<A\},\ U_A=\{z'''\in \boldsymbol{C}^{n-k-1};\ |z'''|<A\},$$
$$T_A^+=\{x_1\in \boldsymbol{R},\ 0<x_1<A\},\ T_A^-=\{x_1\in \boldsymbol{R};\ -A<x_1<0\},$$
$$T_A=T_A^+\cup T_A^-\cup\{0\}.$$

We let $\mathcal{BO}(\Omega_A\times U_A)$ denote the space of hyperfunctions on $\Omega_A\times U_A$ containing $z'''\in U_A$ as holomorphic parameters. $\mathcal{BO}(T_A\times\Omega_A\times U_A)$ etc. denote similar spaces in the variables (x_1, x'', z'''). For the notions in the theory of hyperfunctions which are not explained here, we refer to [7].

In this section we give the following somewhat semiglobal result:

THEOREM 1. *The following are equivalent*:

a) *Given $A>0$, there exist positive constants $a, B<A$ such that for any data $u_j(x'', z''')\in\mathcal{BO}(\Omega_A\times U_A)$ the unilateral boundary value problem*

$$\text{(2.1)}\qquad \begin{cases} P(D)u=0 & \text{on}\quad T_a^+\times\Omega_B\times U_B,\\ \left(\dfrac{\partial}{\partial x_1}\right)^j u\Big|_{x_1\to+0}=u_j(x'', z'''), & \text{on}\quad \Omega_B\times U_B,\ \ j=0, \cdots, m-1\end{cases}$$

always admits a hyperfunction solution $u(x_1, x'', z''')\in\mathcal{BO}(T_a^+\times\Omega_B\times U_B)$.

b) *There exist positive constants b, c such that for any $\varepsilon>0$ we can find $C_\varepsilon>0$ such that*

$$\operatorname{Im}\zeta_1\geqq-\varepsilon|\zeta'|-b|\operatorname{Im}\zeta''|-c|\zeta'''|-C_\varepsilon$$

if $\zeta\in\boldsymbol{C}^n$ and $P(\zeta)=0$.

PROOF. The proof of sufficiency of the condition b) can be shown in a way similar to the case $k=1$ given in [4] (where the natation was such that $x''=x_n$); we can employ Bony-Schapira's method of sweeping out and construct the defining functions of the solution in the complex domain. The detailed calculation on the behavior of the characteristic hyperplanes needed for this will be given in [11]. For another proof, we can construct the solution via the partial Fourier-Laplace transform. See the proof of Theorem 2 below for the case of distribution solutions. The same construction is rediscussed in a more general situation in Theorem 4. Therefore we here focus our attention to the proof of necessity of the condition b).

Assume that u_j have compact support in x''. Then a solution u has compact support in x'' by Holmgren's uniqueness theorem. Also, it follows from Sato' Fundamental theorem that u contains x_1 as a real analytic parameter. Hence the restriction $u|_{x_1=a'}$, $0<a'<a$ is well defined as a hyperfunction in x'. Therefore we can define the mapping

$$\begin{array}{ccc} (\mathscr{B}[K]\hat{\otimes}\mathcal{O}(U_A))^m & \longrightarrow & (\mathscr{B}[L]\hat{\otimes}\mathcal{O}(U_B))^m \\ \cup & & \cup \\ \{u_j\}_{j=0}^{m-1} & \longmapsto & \left\{\left(\dfrac{\partial}{\partial x_1}\right)^j u\Big|_{x_1=a'}\right\}_{j=0}^{m-1}, \quad a'<a, \end{array}$$

where K, L are a suitable pair of compact sets in $\boldsymbol{R}^k$ such that $K\Subset L$, and we are denoting by $\mathscr{B}[K]\hat{\otimes}\mathcal{O}(U_A)$ the space of hyperfunctions with support in $K\times U_A$ and containing $z'''\in U_A$ as holomorphic parameters. As is well-known $\mathscr{B}[K]\hat{\otimes}\mathcal{O}(U_A)$ is a Fréchet space and in view of the Paley-Wiener theorem its seminorms are given by

$$\|v\|_{K,A',\varepsilon}=\sup_{\substack{\zeta''\in\boldsymbol{C}^k\\ |z'''|<A'}}|\hat{v}(\zeta'',z''')e^{-\varepsilon|\zeta''|-H_K(\mathrm{Im}\,\zeta'')}|,\quad A'<A,\ \varepsilon>0,$$

where $\hat{v}(\zeta'', z''')$ denotes the partial Fourier-Laplace transform of $v(x'', z''')\in\mathscr{B}[K]\hat{\otimes}\mathcal{O}(U_A)$ with respect to x''. We denote by $\|\cdot\|_{L,B',\varepsilon}$ corresponding seminorms for $\mathscr{B}[L]\hat{\otimes}\mathcal{O}(U_B)$.

The above mapping is continuous in view of the closed graph theorem. In fact, we can easily find out that our Cauchy problem always admits an $\mathcal{O}(\tilde{L})'\hat{\otimes}\mathcal{O}(U_B)$-valued solution, depending continuously on the data, for some polydisc $\tilde{L}\subset\boldsymbol{C}^k$. Because we have a continuous inclusion $\mathscr{B}[L]\hat{\otimes}\mathcal{O}(U_B)\hookrightarrow\mathcal{O}(\tilde{L})'\hat{\otimes}\mathcal{O}(U_B)$ for $\tilde{L}\supset L$, our mapping obviously has a closed graph. Therefore for any $\varepsilon>0$, $B'<B$, there exist positive constants $A'<A$, ε' and C_ε such that

$$\sum_{j=0}^{m-1}\left\|\left(\frac{\partial}{\partial x_1}\right)^j u\Big|_{x_1=a'}\right\|_{L,B',\varepsilon}\leqq C_\varepsilon\sum_{j=0}^{m-1}\|u_j\|_{K,A',\varepsilon'}. \tag{2.2}$$

Now set the initial value $(u_0, \cdots, u_{m-1})$ to $(0, \cdots, 0, \delta(x'')e^{iz'''\zeta'''})$. By the uniqueness of the solution of the Cauchy problem for the equation after the partial Fourier-Laplace transform with respect to x'', we may assume that the solution for these data has the form $u(x)=v(x_1, x'', \zeta''')e^{iz'''\zeta'''}$, where $v(x_1, x'', \zeta''')$ is the solution of

$$\begin{cases} P(D_1, D'', \zeta''')v(x_1, x'', \zeta''')=0, \\ \left(\dfrac{\partial}{\partial x_1}\right)^j v\Big|_{x_1\to+0}=0,\ j=0, \cdots, m-2, \left(\dfrac{\partial}{\partial x_1}\right)^{m-1} v\Big|_{x_1\to+0}=\delta(x''). \end{cases}$$

Denote by $Y(x_1)v$ the canonical extension of v at the boundary $x_1=0$ for the sake of simplicity. By the Leibniz rule and by the definition of the boundary values, we have

$$\begin{aligned} \text{(2.3)}\qquad & P(D_1, D'', \zeta''')(Y(a'-x_1)Y(x_1)v(x_1, x'', \zeta''')) \\ &=(-i)^m p_m(N)\delta(x_1)\delta(x'') \\ &\quad +i\sum_{k=0}^{m-1}\sum_{j=0}^{m-k-1} P_j(D'', \zeta''')(D_1^{m-k-1-j}\delta(a'-x_1)D_1^k v|_{x_1=a'}) \\ &=(-i)^m p_m(N)\delta(x_1)\delta(x'')+f_{a'}(x_1, x'', \zeta'''), \end{aligned}$$

where $P_j(D')$ denotes the coefficient of D_1^{m-j} in $P(D)$. In particular, $P_0(D')=p_m(N)$ with $N=(1, 0, \cdots, 0)$. If (ζ_1, ζ'') satisfies $P(\zeta_1, \zeta'', \zeta''')=0$, then we obtain by taking Fourier-Laplace transform in (2.3)

$$\text{(2.4)}\qquad (-i)^m p_m(N)+\hat{f}_{a'}(\zeta_1, \zeta'', \zeta''')=0$$

where

$$\begin{aligned} &\hat{f}_{a'}(\zeta_1, \zeta'', \zeta''') \\ &=ie^{-ia'\zeta_1}\sum_{k=0}^{m-1}\sum_{j=0}^{m-k-1} P_j(\zeta')\zeta_1^{m-k-1-j}\widehat{D_1^k v}(a', \zeta'', \zeta'''). \end{aligned}$$

It follows that

$$|e^{ia'\zeta_1}\hat{f}_{a'}(\zeta_1, \zeta'', \zeta''')|\leqq C(1+|\zeta_1|+|\zeta'|)^{m-1}\left|\sum_{j=0}^{m-1}\widehat{D_1^j v}(a', \zeta'', \zeta''')\right|.$$

Hence

$$\begin{aligned} \text{(2.5)}\qquad & \sup_{\substack{\zeta_1\in C \\ \zeta''\in C^k \\ |z'''|<B'}} \frac{1}{(1+|\zeta_1|+|\zeta'|)^{m-1}}|e^{ia'\zeta_1}\hat{f}_{a'}(\zeta_1, \zeta'', \zeta''')e^{iz'''\zeta'''}e^{-\varepsilon|\zeta''|-H_L(\mathrm{Im}\zeta'')}| \\ &\leqq C\sum_{j=0}^{m-1}\|(\partial/\partial x_1)^j u|_{x_1=a'}\|_{L,B',\varepsilon}. \end{aligned}$$

By (2.2) the right-hand side is estimated by

$$\begin{aligned}&\leqq C\cdot C_\varepsilon \sum_{j=0}^{m-1} \|u_j\|_{K,A',\varepsilon'}\\&=C\cdot C_\varepsilon \|\delta(x'')e^{iz'''\zeta'''}\|_{K,A',\varepsilon'}\\&=C\cdot C_\varepsilon \sup_{\substack{\zeta''\in C^k\\ |z'''|<A'}} |e^{-\varepsilon'|\zeta''|}e^{iz'''\zeta'''}|\\&\leqq C'_\varepsilon e^{A'|\zeta'''|}.\end{aligned}$$

Therefore, assuming that $L\subset\{|x''|\leqq r\}$ we have

$$\frac{1}{(1+|\zeta_1|+|\zeta'|)^{m-1}}|\hat{f}_a(\zeta_1, \zeta'', \zeta''')|e^{-\varepsilon|\zeta''|-r|\mathrm{Im}\,\zeta''|-B'|\zeta'''|-a'\mathrm{Im}\,\zeta_1}\leqq C'_\varepsilon e^{A'|\zeta'''|}.$$

Assume here that $\zeta=(\zeta_1, \zeta'', \zeta''')$ satisfies $P(\zeta)=0$. We have $|\zeta_1|\leqq C(1+|\zeta''|+|\zeta'''|)$ for some $C>0$ by the assumption of non-characteristic property. Therefore it follows from (2.4) that there exist positive constants b, c, C such that for any $\varepsilon>0$,

$$-\mathrm{Im}\,\zeta_1\leqq\varepsilon|\zeta''|+b|\mathrm{Im}\,\zeta''|+c|\zeta'''|+C\log(1+|\zeta'|)+\log C_\varepsilon.$$

This obviously implies the condition b). Q.E.D.

Similarly, for the boundary value problem to $x_1<0$, we can deduce the necessary condition

$$\mathrm{Im}\,\zeta_1\leqq\varepsilon|\zeta'|+b|\mathrm{Im}\,\zeta''|+c|\zeta'''|+C_\varepsilon \qquad \text{if } P(\zeta)=0$$

for some constants $b, c>0$.

These two conditions are obviously equivalent to the following ones (with different constants b, c) for the principal part:

$$\mp\mathrm{Im}\,\zeta_1\leqq b|\mathrm{Im}\,\zeta''|+c|\zeta'''| \qquad \text{if } p_m(\zeta)=0. \tag{2.6}$$

Hence by the homogeneity, these are also equivalent to

$$|\mathrm{Im}\,\zeta_1|\leqq b|\mathrm{Im}\,\zeta''|+c|\zeta'''| \qquad \text{if } p_m(\zeta)=0, \tag{2.7}$$

and hence to

$$|\mathrm{Im}\,\zeta_1|\leqq\varepsilon|\zeta'|+b|\mathrm{Im}\,\zeta''|+c|\zeta'''|+C_\varepsilon \qquad \text{if } P(\zeta)=0.$$

Thus we obtain

COROLLARY. *Each of the conditions* (2.6), (2.7) *is necessary and sufficient for the Cauchy problem*

$$(2.8)\qquad \begin{cases} P(D)u=0 \\ \left(\dfrac{\partial}{\partial x_1}\right)^j u\Big|_{x_1=0}=u_j(x'', z'''), \qquad j=0, \cdots, m-1 \end{cases}$$

to admit a solution $u\in\mathcal{BO}(T_a\times\Omega_B\times U_B)$ *for any* $u_j\in\mathcal{BO}(\Omega_A\times U_A)$, *where* a, B *are constants determined from* A *and* $P(D)$.

3. Distribution Cauchy data

We shall now prove the theorem corresponding to distribution solutions.

THEOREM 2. *The following are equivalent*:

a) *Given* $A>0$, *there exist positive constants* $a, B<A$ *such that the Cauchy problem*

$$(3.1)\qquad \begin{cases} P(D)u=0 \\ \left(\dfrac{\partial}{\partial x_1}\right)^j u\Big|_{x_1\to+0}=u_j(x'', z''') \end{cases}$$

where each $u_j(x'', z''')$, $j=0, \cdots, m-1$, *is a distribution defined on* $\Omega_A\times U_A$ *and holomorphic in* $z'''\in U_A$, *always admits a distribution solution* $u(x_1, x'', z''')$ *which is defined on* $T_a\times\Omega_B\times U_B$ *and holomorphic in* $z'''\in U_B$.

b) *There exist positive constants* b, c, C *such that*

$$-\operatorname{Im}\zeta_1\leqq b|\operatorname{Im}\zeta''|+c|\zeta'''|+C\log(1+|\zeta'|)+C$$

if $\zeta\in C^n$ *and* $P(\zeta)=0$.

PROOF. 1) sufficient condition.

First fix a compact set $K\subset\Omega_A$ and consider the data of the form $u_j(x'', z''')=v_j(x'')e^{iz'''\zeta'''}$ with $v_j\in\mathcal{E}'(K)$ and fixed constants ζ'''. Then we obtain the Cauchy problem

$$(3.2)\qquad \begin{cases} P(D_1, D'', \zeta''')v(x_1, x'', \zeta''')=0 \\ \left(\dfrac{\partial}{\partial x_1}\right)^j v\Big|_{x_1\to+0}=v_j(x''), \qquad j=0, \cdots, m-1. \end{cases}$$

By Holmgren's uniqueness theorem, the solution of (3.2) should have compact support when x_1 is bounded. Therefore we can study the Cauchy problem by taking the partial Fourier-Laplace transform $\hat{v}(x_1, \zeta'', \zeta''')$ of v with respect to x''. Then we obtain a Cauchy problem for ordinary differential equations

$$
(3.3)\qquad \begin{cases} P(D_1, \zeta'', \zeta''')\hat{v}(x_1, \zeta'', \zeta''')=0 \\ \left(\dfrac{\partial}{\partial x_1}\right)^j \hat{v}\Big|_{x_1\to+0}=\hat{v}_j(\zeta''), \qquad j=0, \cdots, m-1. \end{cases}
$$

Let $\zeta''\in C^k$, $\zeta'''\in C^{n-k-1}$ be fixed constants. Then for each $l=0, \cdots, m-1$ there exists a solution of the Cauchy problem

$$
(3.4)\qquad \begin{cases} P(D_1, \zeta'', \zeta''')f_l=0 \\ \left(\dfrac{\partial}{\partial x_1}\right)^j f_l\Big|_{x_1\to+0}=\delta_{jl}, \qquad j=0, \cdots, m-1. \end{cases}
$$

Furthermore, if $P(\zeta_1, \zeta'', \zeta''')=0$,

$$
\begin{aligned}
&|f_l(x_1, \zeta'', \zeta''')| \\
&\leqq 2^m(C(1+|\zeta''|+|\zeta'''|)+1)^{m-l}\exp\{(b|\mathrm{Im}\,\zeta''|+c|\zeta'''|+C\log(1+|\zeta'|)+C)x_1\}.
\end{aligned}
$$

(Cf. [3, Lemma 12.7.7].) Then the solution of (3.3) is given by

$$
(3.5)\qquad \hat{v}(x_1, \zeta'', \zeta''')=\sum_{j=0}^{m-1} \hat{v}_j(\zeta'')f_j(x_1, \zeta'', \zeta''').
$$

We use the Paley-Wiener theorem to estimate $\hat{v}_j(\zeta'')$. It follows that the function $\hat{v}$ defined by (3.5) has the estimate for $x_1\leqq b'/b$,

$$
\begin{aligned}
&|\hat{v}(x_1, \zeta'', \zeta''')| \\
&\leqq C'(1+|\zeta''|+|\zeta'''|)^{N+m+C}e^{c|\zeta'''||x_1|}e^{b'|\mathrm{Im}\,\zeta''|+H_K(\mathrm{Im}\,\zeta'')} \\
&\leqq C'(1+|\zeta''|+|\zeta'''|)^{N+m+C}e^{c|\zeta'''||x_1|}e^{H_{K_{b'}}(\mathrm{Im}\,\zeta'')}
\end{aligned}
$$

where $K_{b'}$ is the b'-neighborhood of K. Hence it follows that there exists a solution $v\in \mathcal{D}'$ with compact support $K_{b'}$ in x''.

Thus we obtained the solution $v(x_1, x'', \zeta''')e^{iz'''\zeta'''}$ for our special Cauchy data. Since the correspondence $\{v_j(x'')e^{iz'''\zeta'''}\}\mapsto v(x_1, x'', \zeta''')e^{iz'''\zeta'''}$ given above is continuous in the obvious meaning, the solution for general data $\{u_j(x'', z''')\}$ with support in K in x'' can be obtained if we notice that distributions of the form $\sum v_k(x'')e^{iz'''\zeta'''_k}$, $v_k\in \mathcal{E}'(K)$, are dense in the space $\mathcal{O}(U_A, \mathcal{E}'(K))$. To prove the existence of solution for the data with no restriction on their supports, it suffices to choose K such that $\mathrm{Int}(K)\supset \Omega_a$. (If required, we can patch up as usual these local solutions with compact support in x'' along the initial surface to construct a more global solution on a neighborhood of $\{0\}\times\Omega_A\times U_A$.)

2) necessary condition.

As in the proof of Theorem 1 we obtain a well-defined mapping

$$\begin{array}{ccc} (\mathcal{E}'(K)\hat{\otimes}\mathcal{O}(\overline{U_A}))^m & \longrightarrow & (\mathcal{E}'(L)\hat{\otimes}\mathcal{O}(\overline{U_B}))^m \\ \Psi & & \Psi \\ \{u_j\}_{j=0}^{m-1} & \longmapsto & \left\{\left(\dfrac{\partial}{\partial x_1}\right)^j u\Big|_{x_1=a'}\right\}_{j=0}^{m-1}, \quad a'<a. \end{array}$$

These are DFS spaces. Hence by the closed graph theorem the mapping is continuous. Therefore the image of the bounded set $\{(0,\cdots,0,\delta(x'')e^{iz'''\zeta'''});\ |\zeta'''|\leqq C\}$ is mapped to a bounded set. By Paley-Wiener's theorem, a bounded set of $\mathcal{E}'(K)\hat{\otimes}\mathcal{O}(\overline{U_A})$ is characterized by the boundedness of a seminorm

$$\|v\|_{K,A',(N)}=\sup_{\substack{\zeta''\in C^k\\ |z'''|<A'}}|\hat{v}(\zeta'',z''')(1+|\zeta''|)^{-N}e^{-H_K(\mathrm{Im}\,\zeta'')}|,\qquad A'>A.$$

Therefore, by the similarity transform it follows that there exists a constant $b>0$ such that for some $B'>B$, $C>0$ depending on A', N we have

$$\frac{1}{(1+|\zeta_1|+|\zeta''|)^{m-1}(1+|\zeta''|)^N}e^{-b|\mathrm{Im}\,\zeta''|-B'|\zeta'''|-a'\mathrm{Im}\,\zeta_1}\leqq Ce^{A'|\zeta'''|},\qquad \text{if } P(\zeta)=0.$$

Hence

$$-\mathrm{Im}\,\zeta_1\leqq b|\mathrm{Im}\,\zeta''|+c|\zeta'''|+C\log(1+|\zeta'|)+C$$

for some constants $b, c, C>0$.

The proof is complete.

As usual (cf. [3], proof of Theorem 12.3.1) we can omit the term $C\log(1+|\zeta'|)$ from the above condition employing Seidenberg-Tarski's theorem. Further, in view of an argument similar to the case of hyperbolic operator employing the relation $\sum_{j=1}^{m}\tau_j=$constant times the coefficient of ζ_1^{m-1}, it is also equivalent to each of

$$\mp\mathrm{Im}\,\zeta_1\leqq b|\mathrm{Im}\,\zeta''|+c|\zeta'''|+C\qquad \text{if } P(\zeta)=0.$$

Hence the condition implies the solvability of the Cauchy problem (i.e. to both sides) just as in the case of hyperfunction solutions.

4. Localization and micro-localization

The result of Section 2 can be localized to the level of germs as follows:

THEOREM 3. *The following are equivalent*:

a) *The Cauchy problem with hyperfunction data containing holomorphic parameters* z''';

$$(4.1)\qquad \begin{cases} P(D)u=0 \\ \left(\dfrac{\partial}{\partial x_1}\right)^j u\Big|_{x_1=0}=u_j(x'', z'''), \qquad j=0,\cdots,m-1 \end{cases}$$

always admits a local hyperfunction solution $u(x_1, x'', z''')$.

b) *There exist positive constants* b, c *such that*

$$|\operatorname{Im}\zeta_1|\leqq b|\operatorname{Im}\zeta''|+c|\zeta'''| \qquad \textit{if } p_m(\zeta)=0.$$

In view of the discussion in 2, what remains to prove is that the assumption a) of local existence implies a fixed size of domain for the solutions depending only on the domain of the data. This can be shown in a usual way, employing the Baire category argument: Assume that for every data $\{u_j\}\in(\mathscr{B}[K]\hat{\otimes}\mathcal{O}(U_A))^m$, there exists a solution $u\in C^m(T_a^+, \mathscr{B}[L]\hat{\otimes}\mathcal{O}(U_B))$ as above, where a, B may depend on $\{u_j\}$. (However, L can be chosen independently of $\{u_j\}$ in view of Holmgren's uniqueness theorem.) Thus we obtain a well-defined continuous mapping

$$(4.2)\qquad (\mathscr{B}[K]\hat{\otimes}\mathcal{O}(U_A))^m \longrightarrow \varinjlim_{a,B\downarrow 0} C^m(T_a^+, \mathscr{B}[L]\hat{\otimes}\mathcal{O}(U_B)).$$

By Grothendieck's theorem (which is a variant of Baire's category theorem; cf. [1], p. 16, Theorem A), we conclude that the image of (4.2) is contained in the Fréchet space $C^m(T_a, \mathscr{B}[L]\hat{\otimes}\mathcal{O}(U_B))$ for some a, B. Note that we have a canonical imbedding

$$(4.3)\qquad C^m(T_a^+, \mathscr{B}[L]\hat{\otimes}\mathcal{O}(U_B)) \hookrightarrow \mathscr{B}\mathcal{O}(T_a^+\times \boldsymbol{R}^k\times U_B).$$

In fact, for $0<\varepsilon<a'<a$, we have a canonical mapping

$$\chi_{[\varepsilon,a']}(x_1)C^m(T_a^+, \mathscr{B}[L]\hat{\otimes}\mathcal{O}(U_B)) \hookrightarrow \mathscr{B}[[\varepsilon, a']\times L]\hat{\otimes}\mathcal{O}(U_B)$$

defined e.g. via the duality concerning the part (x_1, x'') of the variables.

This is obviously patched to the mapping (4.3). From this it is also obvious that the imbedded image has support in $T_a^+\times L\times U_B$. Thus we have reached the situation as treated in Theorem 1.

In the above we assumed that the solution u depends on x_1 in a differentiable way as a $\mathscr{B}[L]\hat{\otimes}\mathcal{O}(U_B)$-valued function. This is always true for a solution in $\mathscr{B}\mathcal{O}(T_a^+\times R^k\times U_B)$ supported by $T_a^+\times L\times U_B$. See Appendix for an elementary proof of this fact.

Similar localization obviously holds for the unilateral boundary value problem to $\pm x_1>0$. The distribution analogue is also valid.

We can further localize the result with respect to the variables ξ:

THEOREM 4. *Let $\Delta\subset S^{k-1}$ be an open subset. The following are equivalent*:

a) *The hyperfunction boundary value problem*

$$(4.4)\qquad \begin{cases} P(D)u=0 \\ \left(\dfrac{\partial}{\partial x_1}\right)^j u\Big|_{x_1\to+0}=u_j(x'',z'''), \quad j=0,\cdots,m-1 \end{cases}$$

is locally solvable for the data $u_j(x'',z''')$ which contain z''' as holomorphic parameters and which satisfy

$$\mathrm{S.S.}u_j\Subset R^{n-1}\times\Delta dx''.$$

b) *For any compact subset L of Δ there exists b, $c>0$ such that*

$$(4.5)\qquad \mathrm{Im}\,\zeta_1\geqq -b|\mathrm{Im}\,\zeta''|-c|\zeta'''| \quad \text{if } p_m(\zeta)=0 \text{ and } \mathrm{Re}\,\zeta''/|\mathrm{Re}\,\zeta''|\in L.$$

The same condition corresponds to the solvability of the boundary value problem to the side $x_1<0$ for the data satisfying $\mathrm{S.S.}u_j\Subset R^{n-1}\times(-\Delta)dx''$.

PROOF. 1) sufficient condition: This is proved in [11]. We give here a second proof employing the Fourier transform. Choose the open convex cones Γ_l, $l=1,\cdots,N$ such that $\bigcup_{l=1}^N \Gamma_l^\circ\cap S^{k-1}\Subset\Delta$ and $\Omega_A\times U_A\times\bigcup_{l=1}^N\Gamma_l^\circ\cap S^{k-1}\Supset\mathrm{S.S.}u_j$ on $\Omega_A\times U_A$. Then there exist $F_{lj}(z')\in\mathcal{O}((\Omega_A\times i\Gamma_l)\cap\{|\mathrm{Im}\,z''|<\varepsilon\}\times U_A)$ such that $u_j(x'',z''')=\sum_{l=1}^N F_{lj}(x''+i\Gamma_lQ,z''')$, $j=0,\cdots,m-1$. Therefore, for fixed l and $\Gamma\Subset\Gamma_l$, it suffices to prove the condition a). First consider the existence of a solution of the Cauchy problem

$$\begin{cases} P(D)F(z)=0, \\ \left(\dfrac{\partial}{\partial z_1}\right)^j F\Big|_{z_1=0}=F_j(z'), \qquad j=0,\cdots,m-1, \end{cases}$$

where $F_j(z')\in\mathcal{O}((\Omega_A+i\Gamma)\cap\{|\mathrm{Im}\, z''|<\varepsilon\})\hat{\otimes}\mathcal{O}(U_A)$. By the flabbiness of hyperfunctions with holomorphic parameters there exists an extension $[f_j(x'',z''')]\in\mathcal{B}[\bar{\Omega}_A]\hat{\otimes}\mathcal{O}(U_A)$ of $f_j(x'',z''')=F_j(x''+i\Gamma 0,z''')$. Then decompose it as

$$[f_j(x'',z''')]=[f_j]\underset{x''}{*}W(x'',\Gamma^\circ)+[f_j]\underset{x''}{*}W(x'',S^{k-1}\diagdown\Gamma^\circ)$$

where

$$W(x'',\Gamma^\circ)=\int_{\Gamma^\circ\cap S^{k-1}}W(x'',\omega'')d\omega'',$$

$$(4.6)\qquad W(x'',\omega'') \\ =\frac{(k-1)!\,\Psi(x'',\omega'')e^{-x''^2}}{(-2\pi\sqrt{-1})^k[x''\omega''+\sqrt{-1}(x''^2-(x''\omega'')^2)/\sqrt{1+x''^2}+\sqrt{-1}0]^k}$$

is a variant of twisted Radon decomposition of $\delta(x'')$ introduced in [6]. Then we can decompose as

$$F_j(z')=G_j(z')+H_j(z'),$$

where

$$G_j(z')\in\tilde{\mathcal{O}}^{-\delta}((D^k+i\Gamma)\cap\{|\mathrm{Im}\, z''|<\varepsilon'\})\hat{\otimes}\mathcal{O}(U_A),$$
$$H_j(z')\in\mathcal{O}(\Omega_{A'}+i\{|\mathrm{Im}\, z''|<\varepsilon'\})\hat{\otimes}\mathcal{O}(U_A),$$

with some constants $A'<A$, $\varepsilon'<\varepsilon$, $\delta>0$, and $\tilde{\mathcal{O}}^{-\delta}$ denotes the sheaf $e^{-\sqrt{1+z''^2}}\tilde{\mathcal{O}}$ and $\tilde{\mathcal{O}}$ is the sheaf of slowly increasing holomorphic functions introduced in [10]. Since this Cauchy problem with Cauchy data $H_j(z')$ is locally solvable by the Cauchy-Kowalevsky theorem, we can suppose from the beginning that the Cauchy data $F_j(z')$ belongs to $\tilde{\mathcal{O}}^{-\delta}((D^k+i\Gamma)\cap\{|\mathrm{Im}\, z''|<\varepsilon'\})\hat{\otimes}\mathcal{O}(U_A)$. First assume that these are of the form $F_j(z'')e^{iz'''\zeta'''}$ with $F_j(z'')\in\tilde{\mathcal{O}}^{-\delta}((D^k+i\Gamma)\cap\{|\mathrm{Im}\, z''|<\varepsilon'\})$ and a fixed vector $\zeta'''\in C^{n-k-1}$. Consider the Cauchy problem

$$(4.7)\qquad \begin{cases} P(D_1,\zeta'',\zeta''')\hat{F}(x_1,\zeta'',\zeta''')=0, \\ \left(\dfrac{\partial}{\partial x_1}\right)^j \hat{F}\Big|_{x_1\to+0}=\hat{F}_j(\zeta''), \qquad j=0,\cdots,m-1. \end{cases}$$

Here $\hat{F}_j \in \widetilde{\mathcal{O}}^{-\delta}(D^k + i\{|\mathrm{Im}\,\zeta''| < \delta\})$ satisfies the following conditions: i) for any closed convex cone Γ'° containing Γ° as a proper subcone we have with some $\delta' > 0$

$$|\hat{F}_j(\zeta'')| \leqq Ce^{-\delta'|\mathrm{Re}\,\zeta''|}, \qquad \text{if } \mathrm{Re}\,\zeta'' \in \boldsymbol{R}^k \setminus \Gamma'^{\circ}, \tag{4.8}$$

and ii) in general for every $\varepsilon > 0$,

$$|\hat{F}_j(\zeta'')| \leqq C_{\varepsilon} e^{\varepsilon|\zeta''|}. \tag{4.9}$$

Let us fix ζ''. Then there exists a solution of the ordinary differential equation.

$$\begin{cases} P(D_1, \zeta'', \zeta''')v_l(x_1, \zeta'', \zeta''') = 0 \\ \left(\dfrac{\partial}{\partial x_1}\right)^j v_l \Big|_{x_1 \to +0} = \delta_{jl}, \qquad j = 0, \cdots, m-1. \end{cases}$$

Also, if $P(\zeta_1, \zeta'', \zeta''') = 0$ and $\Gamma'^{\circ} \cap \boldsymbol{S}^{k-1} \Subset \Delta$, we have, for any $\varepsilon > 0$,

$$|v_l(x_1, \zeta'', \zeta''')| \leqq C(1 + |\zeta''| + |\zeta'''|)^m \exp(\varepsilon|\zeta''| + b|\mathrm{Im}\,\zeta''| + c|\zeta'''| + C_{\varepsilon})x_1, \\ \text{for } \mathrm{Re}\,\zeta'' \in \Gamma'^{\circ},$$

and

$$|v_l(x_1, \zeta'', \zeta''')| \leqq C(1 + |\zeta''| + |\zeta'''|)^m \exp c(|\zeta''| + |\zeta'''| + 1)x_1, \\ \text{for } \mathrm{Re}\,\zeta'' \in \boldsymbol{R}^k \setminus \Gamma'^{\circ}.$$

Then the solution of (4.4) is given by

$$\hat{F}(x_1, \zeta'', \zeta''') = \sum_{j=0}^{m-1} \hat{F}_j(\zeta'')v_j(x_1, \zeta'', \zeta'''). \tag{4.10}$$

If $\mathrm{Re}\,\zeta'' \in \Gamma'^{\circ}$ and $|\mathrm{Im}\,\zeta''| < \varepsilon''$ for some $\varepsilon'' > 0$, then it follows from (4.9) that for any $\varepsilon' > 0$, $0 < x_1 < T_0$, the function $\hat{F}$ defined by (4.10) has the estimates

$$\begin{aligned} &|\hat{F}(x_1, \zeta'', \zeta''')| \\ &\leqq C'(1 + |\zeta''| + |\zeta'''|)^m \exp[\varepsilon|\zeta''| + (\varepsilon|\zeta''| + b|\mathrm{Im}\,\zeta''| + c|\zeta'''| + C_{\varepsilon})x_1] \\ &\leqq C'C_{\varepsilon'} \exp(\varepsilon'|\zeta''| + cT_0|\zeta'''|). \end{aligned}$$

Also, if $\mathrm{Re}\,\zeta'' \in \boldsymbol{R}^k \setminus \Gamma'^{\circ}$ and $|\mathrm{Im}\,\zeta''| < \varepsilon''$, then by (4.8) there exists positive constants ε', T_1 such that for $0 < x_1 < T_1$,

$$\begin{aligned} &|\hat{F}(x_1, \zeta'', \zeta''')| \\ &\leqq C''(1 + |\zeta''| + |\zeta'''|)^m \exp(-\delta'|\mathrm{Re}\,\zeta''| + c|\mathrm{Re}\,\zeta''|x_1 + c(|\mathrm{Im}\,\zeta''| + |\zeta'''|)x_1) \\ &\leqq C''C_{\varepsilon'} \exp(-\varepsilon'|\mathrm{Re}\,\zeta''| + cT_1|\zeta'''|). \end{aligned}$$

Therefore, for $x_1<T=\min\{T_0, T_1\}$, $F(x_1, \zeta'', \zeta''')$ not only increases slowly but also decreases exponentially outside Γ'°. Hence it follows that for $x_1<T$, there exists the solution $f(x_1, x'', \zeta''')$ with defining function $F(x_1, z'', \zeta''')\in\tilde{\mathcal{O}}^{-\delta}(D^k+i\Gamma')$ in z''. Thus we obtained the solution $f(x_1, x'', \zeta''')e^{iz'''\zeta'''}$ for our special Cauchy data. Note that $\tilde{\mathcal{O}}^{-\delta}(D^k+i\Gamma\cap\{|\mathrm{Im}\, z''|<\varepsilon'\})$ is a Fréchet space in the obvious manner, and the correspondence $\{F_j(z'')e^{iz'''\zeta'''}\}_{j=0}^{m-1}\longmapsto F(z_1, z'', \zeta''')e^{iz'''\zeta'''}$ given above is continuous. Therefore the solution for general data $F_j(z')$ can be obtained.

2) necessary condition: First assume that $\mathrm{S.S.}u_j\Subset R^{n-1}\times\Gamma^{\circ}dx''$ with a convex cone Γ and let u be a solution of (4.4) defined locally on $x_1>0$. Then by the watermelon theorem, there exists a convex open cone $\Gamma'\Subset\Gamma$ and a neighborhood U of $0\in R^{n-1}$ such that

$$\mathrm{S.S.}[u]\subset(U\cap\{x_1\geqq 0\})\times\{(\pm(1-\theta)dx_1+\theta\omega'dx')\infty,\ \omega'\in\Gamma'^{\circ},\ 0\leqq\theta\leqq 1\}.$$

Therefore, if $|x_1|$ is sufficiently small, then dx''-direction of $\mathrm{S.S.}u$ is contained in Γ'°. Further, by Sato's Fundamental theorem, u contains x_1 as a real analytic parameter.

Hence u can be represented by only one term $F((x_1, x'')+i\tilde{\Gamma}'0, z''')$ where $\tilde{\Gamma}'\cap\{y_1=0\}=\Gamma'$.

Now assume that u_j is further of the form $u_j(x'', z''')=v_j(x'')e^{iz'''\zeta'''}$ with a fixed constant vector ζ''' and $v_j\in\underset{\approx}{\mathcal{O}}(D^k\setminus K)$ where K is a compact subset of R^k, and $\underset{\approx}{\mathcal{O}}$ is the sheaf of rapidly decreasing holomorphic functions of modified type introduced in [10] (its sections are exponentially decreasing on some conical complex neighborhoods of points at infinity). Then by the uniqueness of the Cauchy problem, there exists a compact subset L such that $K\Subset L$ and $\left(\dfrac{\partial}{\partial x_1}\right)^j u\Big|_{x_1=a'}\in\underset{\approx}{\mathcal{O}}(D^k\setminus L)\hat{\otimes}\mathcal{O}(U_B)$. Therefore we can define the mapping

$$\begin{array}{ccc}(\mathcal{O}(R^k+i\Gamma\cap\{|\mathrm{Im}\, z''|<\varepsilon\})\cap\underset{\approx}{\mathcal{O}}(D^k\setminus K))\hat{\otimes}\mathcal{O}(U_A) & & \\ & \longrightarrow & (\mathcal{O}(R^k+i\Gamma'\cap\{|\mathrm{Im}\, z''|<\varepsilon'\})\cap\underset{\approx}{\mathcal{O}}(D^k\setminus L))\hat{\otimes}\mathcal{O}(U_B) \\ \cup & & \cup \\ \{F_j(z')\}_{j=0}^{m-1} & \longmapsto & \left\{\left(\dfrac{\partial}{\partial z_1}\right)^j F\Big|_{z_1=a'}\right\}_{j=0}^{m-1},\qquad 0<a'<a.\end{array}$$

If $\mu\in\underset{\approx}{\mathcal{O}}(D^k\setminus K)$, then by Kawai's theorem [6, Lemma 5.1.2], for any $\delta>0$, there exists a constant c_δ such that $\hat{\mu}(\zeta'')$ is holomorphic in $D_\delta=\{\zeta''\in C^k|\ |\mathrm{Im}\,\zeta''|<c_\delta(|\mathrm{Re}\,\zeta''|+1)\}$ and $|\hat{\mu}(\zeta'')|\leqq A_{\delta,\eta}\exp(\eta|\zeta''|+\delta|\mathrm{Im}\,\zeta''|+H_K(\mathrm{Im}\,\zeta''))$ in D_δ, where $\eta>0$ runs independently of δ.

Now let us fix $\delta>0$. Then $(\mathcal{O}(\boldsymbol{R}^k+i\Gamma\cap\{|\mathrm{Im}\,z''|<\varepsilon\})\cap\underset{\approx}{\mathcal{Q}}(\boldsymbol{D}^k\setminus K))\hat{\otimes}\mathcal{O}(U_A)$ contains a Fréchet space corresponding to a fixed conical complex neighborhood of $\boldsymbol{D}^k\setminus K$ and a fixed exponential decreasing type for elements of $\underset{\approx}{\mathcal{Q}}(\boldsymbol{D}^k\setminus K)$. In view of Bochner type theorem and Kawai's theorem mentioned above, its seminorms are represented by

$$\|v\|_{\Gamma',A',\eta}=\sup_{\substack{\zeta''\in D_\delta\\ |z'''|<A'}}|\hat{v}(\zeta'',z''')e^{-\eta|\zeta''|-\delta|\mathrm{Im}\,\zeta''|-H_K(\mathrm{Im}\,\zeta'')}|+\sup_{\substack{\mathrm{Re}\,\zeta''\notin\Gamma'^\circ\\ |\mathrm{Im}\,\zeta''|<\delta\\ |z'''|<A'}}|\hat{v}(\zeta'',z''')e^{(\delta-\eta)|\zeta''|}|,$$

where $\Gamma'\Subset\Gamma$, $A'<A$, $\eta>0$. The above mapping induces a continuous mapping between two spaces of this type with larger L, Γ and smaller K, Γ'. By the closed graph theorem it is continuous.

Now set the initial value to $(u_0,\cdots,u_{m-1})$ to $(0,\cdots,0,W(x'',\Gamma^\circ)e^{iz'''\zeta'''})$. As in the proof of Theorem 1, we obtain

$$\begin{aligned}(4.11)\qquad &\sup_{\substack{\zeta_1\in C\\ \zeta''\in D_\delta\\ |z'''|<B'}}\left|\frac{1}{(1+|\zeta_1|+|\zeta'|)^{m-1}}e^{ia'\zeta_1}\hat{f}_{a'}(\zeta_1,\zeta'',\zeta''')e^{iz'''\zeta'''}e^{-\eta|\zeta''|-\delta|\mathrm{Im}\,\zeta''|-H_L(\mathrm{Im}\,\zeta'')}\right|\\ &\leqq CC_\eta\sup_{\substack{\zeta''\in D_\delta\\ |z''|<A'}}|\hat{W}(\zeta'',\Gamma^\circ)e^{iz'''\zeta'''}e^{-\eta'|\zeta''|-\delta|\mathrm{Im}\,\zeta''|}|\\ &\quad+CC_\eta\sup_{\substack{\mathrm{Re}\,\zeta''\notin\Gamma'^\circ\\ |\mathrm{Im}\,\zeta''|<\delta\\ |z'''|<A'}}|W(\zeta'',\Gamma^\circ)e^{iz'''\zeta'''}e^{(\delta-\eta)|\zeta''|}|\\ &\leqq C'_\eta e^{A'|\zeta'''|}.\end{aligned}$$

Also, in view of Lemma 2.3 in [6], there exists positive constant C_δ such that

$$|\hat{W}(\zeta'',S^{k-1}\setminus\Gamma^\circ)|\leqq C_\delta\exp\left(\delta|\mathrm{Im}\,\zeta''|-\frac{1}{C_\delta}|\mathrm{Re}\,\zeta''|\right),$$

if $\mathrm{Re}\,\zeta''\in\Gamma''^\circ(\Gamma''^\circ\Subset\Gamma^\circ)$ and $|\mathrm{Im}\,\zeta''|\leqq\frac{1}{C_\delta}|\mathrm{Re}\,\zeta''|$. Therefore we obtain

$$(4.12)\qquad |\hat{W}(\zeta'',\Gamma^\circ)|=|1-\hat{W}(\zeta'',S^{k-1}\setminus\Gamma^\circ)|\geqq 1-\varepsilon,$$

if $\mathrm{Re}\,\zeta''\in\Gamma''^\circ$, $|\mathrm{Im}\,\zeta''|\leqq\frac{1}{2C_\delta}|\mathrm{Re}\,\zeta''|$ and $|\mathrm{Re}\,\zeta''|\gg 1$.

Hence we can use $\hat{W}(\zeta'',\Gamma^\circ)$ as a substitute for 1 in this region, and as in the proof of Theorem 1 it follows from (4.11) that there exist positive constants b, c, C such that for any $\eta>0$,

$$-\mathrm{Im}\,\zeta_1\leqq\eta|\zeta''|+b|\mathrm{Im}\,\zeta''|+c|\zeta'''|+C\log(1+|\zeta'|)+\log C_\eta,$$
$$\text{if } P(\zeta)=0,\ \zeta''\in D_\delta \text{ and } \mathrm{Re}\,\zeta''\in\Gamma''^\circ.$$

This obviously implies the condition b).

The distribution analogue of Theorem 4 is true. It suffices to replace S.S.u by $WF_A u$ and the inequality (4.5) by the one in Theorem 2 (but now assumed only for $\mathrm{Re}\,\zeta''/|\mathrm{Re}\,\zeta''| \in L \Subset \Delta$). The proof is obvious. (Note that $W(x'', \Gamma^\circ) \in \mathcal{D}'$ with $WF_A W(x'', \Gamma^\circ) \subset \{0\} \times \Gamma^\circ$. Therefore we can prove it by the same method as for the hyperfunction data.)

Following suggestion of the referee, we shall give a true microlocal version of the above theorem employing the notion of microfunctions with holomorphic parameter. A germ of microfunction (represented by a hyperfunction) $f(x'', z''')$ at $(0, 0; \xi'' dx'') \in \boldsymbol{R}^k \times \boldsymbol{C}^{n-k-1} \times \boldsymbol{S}^{k+2(n-k-1)-1}$ is said to contain z''' as holomorphic parameters if $\bar{\partial}_{z'''} f = 0$ in the sense of microfunction, i.e. if $\bar{\partial}_{z'''} f$ is micro-analytic at this point. The following lemma establishes the equivalence of this intuitive definition with the cohomological one $\mathcal{C}_{x''}\mathcal{O}_{z'''} = \mathcal{H}^k_{S^*R^k \times C^{n-k-1}}(\pi^{-1}\mathcal{O}_{z'', z'''})$. (See e.g. [14] for the latter sheaf.)

LEMMA. *We can always choose a hyperfunction representative for* f, *i.e.*, $g(x'', z''') \in \mathcal{B}\mathcal{O}_{(0,0)}$ *such that* $f - g$ *is micro-analytic at* $(0, 0; \xi'' dx'')$.

PROOF. The solution of $\bar{\partial}_{z'''} f = 0$ has S.S. in $\zeta''' + i\eta''' = 0$. Hence without loss of generality we can take a representative g such that $\bar{\partial}_{z'''} g$ is micro-analytic on, say, $\Omega \times U \times \Delta'' dx'' \times \boldsymbol{S}^{2(n-k-1)-1}$, where $\Omega \times U$ is a neighborhood of $(0, 0) \in \boldsymbol{R}^k \times \boldsymbol{C}^{n-k-1}$ and Δ'' is a neighborhood of ξ'' in $\boldsymbol{S}^{k-1}$. In view of the fact that $\bar{\partial}_{z_j}$ are operators with constant coefficients, we can cut off the directional components of the S.S. of g to a closed neighborhood $\overline{\Gamma''} \subset \Delta''$ of $\xi'' dx''$, by first cutting the support and then convoluting with the components of Kashiwara's twisted Radon decomposition of δ. Since the replaced g also satisfies $\bar{\partial}_{z'''} g = 0$, we find that without loss of generality we can assume that S.S.$g \subset \Omega \times U \times \overline{\Gamma''} \times \{0\}$. Hence it further satisfies

$$\bar{\partial}_{z_j} g = g_j(x'', x''', y'''), \qquad j = k+2, \cdots, n,$$

where $g_j(x'', x''', y''')$ are real analytic functions on $\Omega \times U$. They obviously satisfy the compatibility condition

$$\bar{\partial}_{z_l} g_j = \bar{\partial}_{z_j} g_l, \qquad j, l = k+2, \cdots, n.$$

Hence the system of equations

$$\bar{\partial}_{z_j} h = g_j, \qquad j = k+2, \cdots, n,$$

admits a real analytic solution h on a smaller neighborhood $\Omega' \times U'$. Now $g-h$ defines the same microfunction as f and is holomorphic in z''' in the sense of hyperfunction. Q.E.D.

COROLLARY TO THEOREM 4. *The following are equivalent*:

a) *The microfunction boundary value problem* (4.4) *is solvable for every germ at* $(0, \xi'' dx'')$ *of microfunctions* u_j *containing* z''' *as holomorphic parameters.*

b) *There exists a neighborhood* L *of* ξ'' *such that* (4.5) *holds.*

In fact, b)$\Longrightarrow$a) follows from the corresponding part of Theorem 4, because in view of the above Lemma the germs u_j can be represented by hyperfunctions containing z''' as holomorphic parameters and satisfying $\mathrm{S.S.} u_j \subset \Omega \times U \times L dx''$.

a)$\Longrightarrow$b) can be deduced as follows: Assume that the data u_j are hyperfunctions as in Theorem 4. Then (4.4) in the sense of microfunctions means that we have

$$\begin{cases} P(D)u = f(x_1, x'', z'''), \\ \left(\dfrac{\partial}{\partial x_1}\right)^j u \Big|_{x_1 \to +0} = u_j(x'', z''') + v_j(x'', z'''), \end{cases}$$

where u, f are mild hyperfunctions and $\rho\text{-S.S.} f \cap \Delta dx'' = \emptyset$. (Here ρ-S.S. denotes the reduced singular spectrum of a mild hyperfunction. See Kataoka [9] for the notion of mild hyperfunctions and their boundary values.) v_j are hyperfunctions whose S.S. do not contain components in $\Delta dx''$. Decomposing u by the S.S. we can assume without loss of generality that $\rho\text{-S.S.} u$ is contained in a small neighborhood of $L dx''$. In that case the residue terms become real analytic. Hence we can omit them by the Cauchy-Kowalevsky theorem. Thus also this part was reduced to the corresponding assertion of Theorem 4.

Appendix.

We give an elementary proof of the differentiable dependence on x_1 of the solution $u(x_1, x'', z''') \in \mathcal{BO}(T_a^+ \times \boldsymbol{R}^k \times U_B)$ with support contained in $T_a^+ \times L \times U_B$, as a $\mathcal{B}[L] \hat{\otimes} \mathcal{O}(U_B)$-valued function. Let $W(z'', \omega'')$ denote the component of a twisted Radon decomposition. (We can use (4.6) though a simpler one suffices here.) Let Γ_σ denote the σ-th orthant of

R^k and let $W(z'', \Gamma_\sigma)=\int_{\Gamma_\sigma \cap S^{k-1}} W(z'', \omega'')d\omega''$. Put $F_\sigma(x_1, z'', z''')=\int_{R^k} W(z''-x'', \Gamma_\sigma)u(x_1, x'', z''')dx''$, where z'' denotes a set of complex variables independent of x''. By Sato's fundamental theorem u contains x_1 as real analytic parameter. Hence we can easily see that $F_\sigma(z)$ is holomorphic on a neighborhood of $T_a^+ \times U_\sigma^L \times U_B$, where $U_\sigma^L \subset C^k$ is a domain containing an infinitesimal wedge of the form $R^k+i\Gamma_\sigma 0$ and the real set $R^k \backslash L$, determined by the domain of definition of $W(z'', \omega'')$. It obviously satisfies

$$P(D_{x_1}, D_{z''}, D_{z'''})F_\sigma(x_1, z'', z''')=0.$$

Thus by a precise version of Cauchy-Kowalevsky theorem (cf. [13], Lemma 9.1), $F_\sigma(z)$ can be continued to a domain of the form

$$\{z \in C^n; |y_1|+(x_1)_-+(x_1-a)_+<C[\mathrm{dis}(z'', C^k \setminus U_\sigma^L)+\mathrm{dis}(z''', C^{n-k-1} \setminus U_B)],$$

where $t_+=\max\{t, 0\}$, $t_-=\max\{-t, 0\}$. Thus $F_\sigma(x_1, z'', z''')$ becomes an $\mathcal{O}(U_\sigma^L \times U_B)$-valued continuous function of x_1 for $0 \leqq x_1 \leqq a$. Recall that the locally uniform convergence of the defining functions $F_\sigma(x_1, z'', z''')$ in $U_\sigma^L \times U_B$ implies the convergence of $u(x_1, x'', z''')$ in $\mathcal{B}[L]\hat{\otimes}\mathcal{O}(U_B)$. (See e.g. [7], Theorem 4.3.4.) Thus we have shown the continuity of u in x_1 as a $\mathcal{B}[L]\hat{\otimes}\mathcal{O}(U_B)$-valued function. The differentiability is verified just similarly.

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