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Extension groups for modular Hecke algebras

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Introduction.

Hecke algebras were introduced as a fundamental tool in the character theory of Chevalley groups: they are used to deal with characters involved in induced representations from a Borel subgroup up to the full group. Their standard presentation by generators and relations has a formal analogue where, roughly speaking, the cardinality $q=p^a$ of the root subgroups is considered as an indeterminate. Specializations where $q=0$ ("0-Hecke algebras") were studied by P. N. Norton ([4]): they have a commutative semi-simple quotient and one may find an explicit decomposition into principal indecomposable ideals. At the same time, Sawada introduced ([5]) the algebra $\mathcal{H}_k := \mathrm{End}_{kG}\,\mathrm{Ind}_U^G k$, an analogue where the Borel subgroup is replaced by a Sylow p-subgroup U and k is of characteristic p. A decomposition of $\mathcal{H}_k$ gives rise to a remarkable bijection between simple modules for $\mathcal{H}_k$ and for kG, leading to a good description of the indecomposable summands of the induced module $\mathrm{Ind}_U^G k$.

In the present paper we study the extension groups $\mathrm{Ext}^1_{\mathcal{H}_k}(S, S')$ for all simple $\mathcal{H}_k$-modules S, S'. Since simple $\mathcal{H}_k$-modules are one-dimensional, this amounts to a study of representations of dimension 2 (see 4.). We use a presentation (2.) of $\mathcal{H}_k$ which is very similar to the one discovered by Yokonuma ([8]) in the case of Chevalley groups. Another ingredient is a decomposition (3.) of the radical $J(\mathcal{H}_k)$ that leads to a clear dichotomy for Ext^1 groups (Theorem 16). Some internal consequences for modular and 0-Hecke algebras are given (5. 6.). But the main applications of this work are for kG-modules, as it was the case in Sawada's work. This will be published later[1].

1) "A criterion for complete reducibility and some applications", Actes du colloque sur les représentations des groupes finis (Luminy 1988), to appear in *Astérisque* (1989).

Notation.

Through the whole paper we fix p a prime, k an algebraically closed field of characteristic p, G a finite group with a split BN-pair of characteristic p.

The subgroup B of G satisfies $B=U\rtimes T=N_G(U)$ where U is a Sylow p-subgroup of G, T is an abelian p'-group. We denote by $W=N/T$ the Weyl group of G, R its set of generators, Φ the (non crystallographic) root system of basis Δ and same Weyl group: we denote by $\alpha\mapsto r_\alpha\in R$ the associated indexation. Refering to the Coxeter graph of W, we say r and s in R are connected if, and only if, they do not commute. We denote this by $r-s$. If I and J are subsets of R, we denote their boolean sum $(I\setminus J)\cup(J\setminus I)$ by $I+J$.

If $\alpha\in\Phi$, we denote by X_α the associated root subgroup. If $\Sigma\subset\Delta$ corresponds to $S\subset R$, one gets the parabolic subgroup $P_S:=BW_SB=U_S\rtimes L_S$, where $L_S=T.\ \langle X_\alpha;\ \alpha\in\Phi_\Sigma\rangle$ is the "Levi subgroup". One denotes $G_S=\langle X_\alpha;\ \alpha\in\Phi_\Sigma\rangle$. Then the split BN-pair of G endows G_S and L_S with split BN-pairs of rank $|S|$ by intersection.

We use the standard notations for k-algebras and modules. If A is a k-algebra, A' a subalgebra, M an A-module, one denotes by $\mathrm{Res}_{A'}M$ the restriction of M to A'. The (Jacobson) radical of A is denoted by $J(A)$. If S is a subset of A, one denotes by $\langle S\rangle$ the two-sided ideal generated by S. If $a, b\in A$, one denotes $[a, b]:=ab-ba$. One defines $[A, A]:=\langle[a, b];\ a, b\in A\rangle$.

1. Modular Hecke algebras.

We denote by $\mathcal{H}_k(G)$ (or simply $\mathcal{H}_k$) the endomorphism algebra of the induced kG-module $\mathrm{Ind}_U^G k=kG\otimes_{kU}k$, where k stands for the one dimensional trivial kU-module. A k-basis is indexed by $N:(a_n)_{n\in N}$ defined by $a_n(1\otimes 1)=\sum_{g\in C} g\cdot(n\otimes 1)$ where C is a representative system of $U/U\cap nUn^{-1}$ (see [3] 7).

If $r\in R$, one may choose $(r)\in r\cap G_r$, we assume it is the case through the whole paper. We denote $T_r=T\cap G_r$. We also denote $T'_r:=\langle t(r)t^{-1}(r)^{-1};\ t\in T\rangle\subset T_r$ (see [6] § 1). Then the law of $\mathcal{H}_k$ obeys the following rules (see [3] Theorems 5 and 7(b)):

$$(\mathcal{L})\qquad a_n a_{n'}=a_{n'n}\quad\text{when}\quad l(n'n)=l(n')+l(n),$$

$$a_n a_{(r)} = -|T_r|^{-1} \sum_{t \in T_r} a_{tn} \quad \text{when} \quad l((r)n) < l(n),$$

where $l(n)$ is defined as the length in W of the corresponding class mod. T.

This implies at once that $\langle a_t;\, t \in T \rangle$ is a semi-simple subalgebra isomorphic to kT, we denote it by kT. One also gets

F1. *$\mathcal{H}_k$ is generated as a k-algebra by kT and the $a_{(r)}$'s for $r \in R$.*

2. The standard presentation.

In order to construct certain representations of $\mathcal{H}_k$ (see 4), we need a presentation. We use the relations first introduced by Yokonuma ([8]), slightly adapted to avoid reference to Chevalley groups.

Let us recall we already chose for every $r \in R$ a representative in G_r.

DEFINITION 1. If $r, s \in R$ and rs is of order m, let $t_{rs} \in G$ defined by $\cdots (s)(r) = \cdots (r)(s) t_{rs}$ with m terms on the left and $m+1$ on the right.

LEMMA 2. $t_{rs} \in T_r \cap T_s$.

PROOF. Standard, see for instance [6] p. 141.

The presentation of $\mathcal{H}_k$ is the following (compare [8] Théorème 3):

PROPOSITION 3. *$\mathcal{H}_k$ is generated by the generators $\{a_t;\, t \in T\} \cup \{a_{(r)};\, r \in R\}$ subject to the relations*:

(Y0) $a_t a_{t'} = a_{t't}$

(Y1) $a_t a_{(r)} = a_{(r)} a_{(r)t(r)^{-1}}$

(Y2) $a_{(r)}^2 = -|T_r|^{-1} \sum_{t \in T_r} a_t a_{(r)}$

($\mathrm{Y3}_{rs}$) $a_{(r)} a_{(s)} \cdots = a_{t_{rs}} a_{(s)} a_{(r)} \cdots$

PROOF. The proof is standard, we restate the main ideas for the convenience of the reader.

Denote by $\mathcal{A}$ the algebra defined above. The law ($\mathcal{L}$) implies $\mathcal{H}_k$ is a quotient. So let us check $\dim_k \mathcal{A} \leq |N|$. Consider the monoid of subsets of $\mathcal{A}$ with multiplication $AA' = \{aa';\, a \in A,\ a' \in A'\}$. If $w \in W$ has reduced expression $w = r_1 \cdots r_l$ let $A_w := \{a_{(r_l)} \cdots a_{(r_1)} a_t;\ t \in T\}$. By Y1, this is $A_{r_l} \cdots A_{r_1}$. It only depends on w: use Y3 and Iwahori-Matsumoto "word lemma" ([2] § 1 Proposition 5). Then Y2 implies the k-span of $\bigcup_{w \in W} A_w$ is an ideal, hence equals $\mathcal{A}$. Each A_w has cardinality $\leq |T|$, so $\dim_k \mathcal{A} \leq |N|$.

REMARK. It is indeed possible to choose the representatives (r) so that all the t_{rs}'s are 1. This is clear for (untwisted) Chevalley groups (see [8]). This is also proved in general by Tits to be a consequence of a formal analogue of Chevalley's commutator formula ([7] Proposition 2.9). We thank Professor Tits for having pointed this to us. The commutator formula (P1 in [7]) might be included in the axioms of split BN-pairs since the notion is devised mainly to study Chevalley groups and their twisted analogues (who clearly satisfy it), moreover those are the only split BN-pairs thanks to Fong-Seitz classification.

However, Lemma 2 will be enough for our purpose.

3. The radical.

The irreducible representations of $\mathcal{H}_k$ were classified by Sawada (see [8]). They are given by the following (see also 6.):

DEFINITION. If $\chi \in \mathrm{Mor}(T, k^*)$ one denotes $R(\chi) := \{r \in R;\ \chi(T_r)=1\}$. An "admissible pair" is any pair (χ, J) with $J \subset R(\chi)$.

PROPOSITION 4. (Sawada) *The irreducible representations of $\mathcal{H}_k$ are one-dimensional. They are under the following form. If (χ, J) is an admissible pair, there is a unique morphism*

$$\phi(\chi, J) : \mathcal{H}_k \longrightarrow k^*,$$

such that $\forall t \in T$, $\phi(\chi, J)(a_t)=\chi(t)$ and, $\phi(\chi, J)(a_{(r)})$ is -1 if $r \in J$, 0 if $r \in R \backslash J$.

If $I \subset R$, we consider the subalgebras corresponding to Levi subgroups and G_I. By the law in $\mathcal{H}_k$, it is clear that $\bigoplus_{n \in N \cap G_I} ka_n \subset \bigoplus_{n \in N \cap L_I} ka_n$ are subalgebras isomorphic to $\mathcal{H}_k(G_I)$ and $\mathcal{H}_k(L_I)$ respectively. Since the irreducible representations of $\mathcal{H}_k(G)$ are one-dimensional, they remain irreducible when restricted to any subalgebra, hence

F2. *If $I \subset J \subset R$ then $J(\mathcal{H}_k(G_I)) \subset J(\mathcal{H}_k(G_J)) \subset J(\mathcal{H}_k(L_J)) \subset J(\mathcal{H}_k(G))$.*

The one-dimensionality of the irreducible representations of $\mathcal{H}_k$ also implies that $\mathcal{H}_k / J(\mathcal{H}_k)$ is isomorphic to some power of k, hence commutative. So

F3. $[\mathcal{H}_k, \mathcal{H}_k] \subset J(\mathcal{H}_k)$.

We now give a generation property of $J(\mathcal{H}_k)$.

THEOREM 5. *If G is a finite group with a split BN-pair of characteristic p, the radical of the modular Hecke algebra is given by the following*: $J(\mathcal{H}_k)=[\mathcal{H}_k, \mathcal{H}_k]+\langle J(\mathcal{H}_k(L_r));\ r\in R\rangle=\langle J(\mathcal{H}_k(G_{rs}));\ r, s\in R$ *such that* $r-s$ *or* $r=s\rangle$.

The following corollary is an immediate consequence:

COROLLARY 6. *If M is a $\mathcal{H}_k$-module, then M is semi-simple if, and only if,* $\forall r, s\in R$ *such that* $r-s$ *or* $r=s$ *the restriction* $\mathrm{Res}_{\mathcal{H}_k(G_{rs})}M$ *is semi-simple.*

PROOF OF THE THEOREM. In view of **F2** and **F3**, we just have to check $J(\mathcal{H}_k)\subset[\mathcal{H}_k, \mathcal{H}_k]+\langle J(\mathcal{H}_k(G_r));\ r\in R\rangle$ and $J(\mathcal{H}_k)\subset\langle J(\mathcal{H}_k(G_{rs}));\ r, s\in R$ such that $r-s$ or $r=s\rangle$.

Let $\pi:\mathcal{H}_k\to\mathcal{H}_k/([\mathcal{H}_k, \mathcal{H}_k]+\langle J(\mathcal{H}_k(L_r));\ r\in R\rangle)$ be the canonical surjection. Then, by **F1**, $\pi(\mathcal{H}_k)$ is generated by the semisimple subalgebras $\pi(kT)$ and $\pi(\mathcal{H}_k(G_r))$ for $r\in R$. On the other hand, $\pi(\mathcal{H}_k)$ is commutative. So it is semi-simple (see for instance [1] §7), hence $J(\mathcal{H}_k)=[\mathcal{H}_k, \mathcal{H}_k]+\langle J(\mathcal{H}_k(L_r));\ r\in R\rangle$.

In view of the above, there just remains to check $[\mathcal{H}_k, \mathcal{H}_k]\subset\langle J(\mathcal{H}_k(G_{rs})); r, s\in R\ r-s$ or $r=s\rangle$, or equivalently that $A:=\mathcal{H}_k/\langle J(\mathcal{H}_k(G_{rs}));\ r, s\in R\ r-s$ or $r=s\rangle$ is commutative. By **F1**, A is generated by the images of the a_t's and the $a_{(r)}$'s. The a_t's commute. By **F3**, if $r, s\in R$, then $[a_{(r)}, a_{(s)}]\in J(\mathcal{H}_k(G_{rs}))$, so the images commute when $r-s$. If $r-/-s$ then $[a_{(r)}, a_{(s)}]=(a_{t_{rs}}-1)a_{(s)}a_{(r)}$ by $Y3_{rs}$, but $t_{rs}\in T_s$. It is clear by Proposition 4 that any irreducible representation $\psi(\chi, J)$ of $\mathcal{H}_k(G_r)$ kills $(a_{t_{rs}}-1)a_{(s)}$, thus the images of $a_{(r)}$ and $a_{(s)}$ commute. Let us take $t\in T$ and $r\in R$, then, by Y1, $[a_{(r)}, a_t]=a_t a_{(r)}(a_{t'}-1)$ where $t'=[t, (r)]\in T_r$. Again $a_{(r)}(a_{t'}-1)$ is in the radical of $\mathcal{H}_k(G_r)$, so the images of a_t and $a_{(r)}$ commute. This finishes the proof of the theorem.

The following is stated for future reference. It is a refinement in a particular case.

PROPOSITION 7. *If M is an $\mathcal{H}_k$-module with all composition factors isomorphic, then M is semi-simple if, and only if,* $\forall r\in R$, $\mathrm{Res}_{\mathcal{H}_k(G_r)}M$ *is semi-simple.*

PROOF. The "only if" part is clear by **F2**. We assume $\forall r\in R$, $\mathrm{Res}_{\mathcal{H}_k(G_r)}M$

is semi-simple. Then each of these restrictions is a sum of isomorphic lines since a Jordan-Hölder sequence for $\mathcal{H}_k$ remains Jordan-Hölder when restricted. So $\mathcal{H}_k(G_r)$ acts by scalar matrices on M. On the other hand kT is commutative semi-simple. Hence, by **F1**, there is a decomposition of M as a sum of $\mathcal{H}_k$-stable lines. This proves the proposition.

4. Extension groups.

In order to compute all groups $\mathrm{Ext}^1_{\mathcal{H}_k}(S, S')$ for S, S' simple (1-dimensional) $\mathcal{H}_k$-modules, we study 2-dimensional $\mathcal{H}_k$-modules. Thus we have to find almost all subsets of $\mathrm{Mat}_2(k)$ satisfying the relations of Proposition 3. A preliminary lemma rules the case when G is of rank 1.

LEMMA 8. *Let (χ, J), (χ', J') be two admissible pairs for G. Let $r \in R$ and denote $\varepsilon = \phi(\chi, J)(a_{(r)})$, $\varepsilon' = \phi(\chi', J')(a_{(r)})$. Let $\lambda \in k$. Then*

$$a_t \longmapsto \begin{pmatrix} \chi(t) & 0 \\ 0 & \chi'(t) \end{pmatrix}, \qquad a_{(r)} \longmapsto \begin{pmatrix} \varepsilon & \lambda \\ 0 & \varepsilon' \end{pmatrix}$$

defines a morphism from $\mathcal{H}_k(L_r)$ to $\mathrm{Mat}_2(k)$ if, and only if, either $\lambda = 0$, or $\lambda \neq 0$ and one of the following is satisfied:
i) $(\varepsilon, \varepsilon') = (0, 0)$, $\chi' = \chi^r$ *and* $r \notin R(\chi)$,
ii) $(\varepsilon, \varepsilon') = (0, -1)$ *or* $(-1, 0)$, *and* $\chi' = \chi$.
In particular $(\varepsilon, \varepsilon') \neq (-1, -1)$ *if* $\lambda \neq 0$.

PROOF. The map defined above is a morphism if, and only if, the matrices satisfy relations Y0-1-2 of Proposition 3. Y0 is clear. The pairs being admissible, the diagonal elements satisfy Y1-2. So Y1-2 amount to the equalities $\chi' = \chi^r$ and $\varepsilon + \varepsilon' = -(\mathrm{Res}_{T_r} \chi, 1)$ (usual scalar product). Case $(\varepsilon, \varepsilon') = (0, 0)$ is then clear. Case $(\varepsilon, \varepsilon') = (-1, -1)$ is impossible since the scalar product is 0 or 1. In case $(\varepsilon, \varepsilon') = (0, -1)$, we have $r \in R(\chi')$, so $\chi'^r = \chi'$ since $T'_r \subset T_r$ (see 1.). Then the condition is equivalent to $\chi' = \chi$. This completes the proof.

LEMMA 9. *If*

$$\phi : \mathcal{H}_k \longrightarrow \mathrm{Mat}_2(k),$$
$$h \longmapsto \begin{pmatrix} \phi(\varphi, I)(h) & \phi_{12}(h) \\ 0 & \phi(\chi, J)(h) \end{pmatrix}$$

is a morphism and $u, v \in I \setminus J$ or $u, v \in J \setminus I$, then $\phi(a_{(u)}) = \phi(a_{(v)})$.

PROOF. If $\varphi\neq\chi$, then Lemma 8 implies $\phi_{12}(a_{(u)})=\phi_{12}(a_{(v)})=0$, so $\phi(a_{(u)})=\phi(a_{(v)})$. Let us assume $\varphi=\chi$. If $u, v\in J\setminus I\subset R(\chi)$, we have $\chi(t_{uv})=1$, $\phi(a_{(u)})=\begin{pmatrix}0 & \phi_{12}(a_{(u)})\\ 0 & -1\end{pmatrix}$ and $\phi(a_{(v)})=\begin{pmatrix}0 & \phi_{12}(a_{(v)})\\ 0 & -1\end{pmatrix}$. Then $\mathrm{Y3}_{uv}$ gives $(-1)^{|uv|-1}\phi(a_{(u)})=(-1)^{|uv|-1}\phi(a_{(v)})$. The same if $u, v\in I\setminus J$.

In the next three sections we take two admissible pairs (φ, I) and (χ, J), and study the possible extensions corresponding to the associated simple modules. The decomposition $J(\mathcal{H}_k)=[\mathcal{H}_k, \mathcal{H}_k]+\langle J(\mathcal{H}_k(L_r));\ r\in R\rangle$ will give rise to two cases of non split extensions.

4.1. A first case.

Case 1 for the pair $((\varphi, I), (\chi, J))$ is when

(1) $\varphi=\chi$, $I\setminus J\neq\varnothing$, $J\setminus I\neq\varnothing$ and $\forall r\in I\setminus J$, $\forall s\in J\setminus I$, $r-s$.

We consider a morphism

$$\begin{aligned}\phi : \mathcal{H}_k &\longrightarrow \mathrm{Mat}_2(k),\\ h &\longmapsto \begin{pmatrix}\phi(\chi, I)(h) & \phi_{12}(h)\\ 0 & \phi(\chi, J)(h)\end{pmatrix}.\end{aligned}$$

Then:

PROPOSITION 10. *Choose $r\in I\setminus J$, $s\in J\setminus I$ and $\lambda, \mu\in k$. Then there is exactly one ϕ as above satisfying $\phi_{12}(a_{(r)})=\lambda$ and $\phi_{12}(a_{(s)})=\mu$.*

PROOF. Let us check first existence. We define ϕ on generators and check the relations Y0-3. If $t\in T$, let $\phi(a_t)=\chi(t)I_2$. If $u\in R\setminus(I+J)$, let $\phi_{12}(a_{(u)})=0$, so that $\phi(a_{(u)})=0$ or $-I_2$. If $u\in I\setminus J$, let $\phi_{12}(a_{(u)})=\lambda$, so that $\phi(a_{(u)})=\phi(a_{(r)})=\begin{pmatrix}-1 & \lambda\\ 0 & 0\end{pmatrix}$. If $u\in J\setminus I$, let $\phi(a_{(u)})=\phi(a_{(s)})=\begin{pmatrix}0 & \mu\\ 0 & -1\end{pmatrix}$. Then each restriction to $\mathcal{H}_k(L_v)$ is a morphism by Lemma 8, so Y0-2 are satisfied. Let us check $\mathrm{Y3}_{uv}$ for $u, v\in R$. If $\phi(a_{(u)})$ or $\phi(a_{(v)})$ is scalar, or if $\phi(a_{(u)})=\phi(a_{(v)})$, then $\mathrm{Res}_{\mathcal{H}_k(L_{uv})}\phi$ is diagonalizable similar to $\begin{pmatrix}\phi(\chi, I) & 0\\ 0 & \phi(\chi, J)\end{pmatrix}$, so Y3 is satisfied. There remains the case when $u\in I\setminus J$, $v\in J\setminus I$. Then $u-v$ by (1). But $\phi(a_{(u)})\phi(a_{(v)})=0$, so we have 0 on both sides of Y3.

We show uniqueness. Let ϕ satisfy the hypotheses of the proposition. Then $\phi_{12}(kT)=0$ since the corresponding 2-dimensional representation of kT is semi-simple with isomorphic composition factors $(\chi=\varphi)$. If $v\in R\setminus(I+J)$, let us check $\phi_{12}(a_{(v)})=0$. We cannot have both $v-r$ and $v-s$, since this would provide a cycle $v-r-s-v$. Suppose $v-/-r$ and apply $Y3_{vr}$. We have $t_{vr} \in T_r \subset \ker \chi$ since $r \in R(\chi)$ (Lemma 2), so $[\psi(a_{(v)}), \psi(a_{(r)})]=0$. On the other hand $\psi(\chi, I)(a_{(v)})=\psi(\chi, J)(a_{(v)})$, hence $\psi_{12}(a_{(v)})=0$. There remains to check that if $v \in I \setminus J$, $\psi(a_{(v)})=\psi(a_{(r)})$. If $v \in I \setminus J$ (resp. $v \in J \setminus I$) then $\psi(a_{(v)})=\psi(a_{(r)})$ (resp. $\psi(a_{(v)})=\psi(a_{(s)})$) by Lemma 9.

4.2. A second case.

Case 2 for the pair $((\varphi, I), (\chi, J))$ is when

(2) there is $r \in R \setminus R(\varphi)$ such that $\varphi=\chi^r$, $\{s \in R;\ s-r\} \supset I+J$ and $(|rs|=3 \Longrightarrow s \notin I \cap J)$.

We consider a morphism

$$\psi : \mathcal{H}_k \longrightarrow \mathrm{Mat}_2(k),$$
$$h \longmapsto \begin{pmatrix} \psi(\varphi, I)(h) & \psi_{12}(h) \\ 0 & \psi(\chi, J)(h) \end{pmatrix}.$$

Then:

PROPOSITION 11. *Suppose $I=J$. Let R' be the set of elements $s \in R \setminus R(\chi)$ such that $\varphi=\chi^s$ and there is no $v \in I$ such that $|sv|=3$. Then if $(\lambda_s)_{s \in R'}$ is a family of elements of k, there exists exactly one ψ such that $\forall t \in T$, $\psi_{12}(a_t)=0$ and $\forall s \in R'$, $\psi_{12}(a_{(s)})=\lambda_s$.*

PROPOSITION 12. *Suppose $|I+J|=1$ and $\varphi \neq \chi$. Let $R'' :=\{s \in R \setminus R(\chi);$ $\varphi=\chi^s$ and $s-(I+J)\}$. Then, if $(\lambda_s)_{s \in R''}$ is a family of elements of k, there exists exactly one ψ such that $\forall s \in R''$, $\psi_{12}(a_{(s)})=\lambda_s$.*

PROPOSITION 13. *Suppose $|I+J|=1$ and $\varphi=\chi$. Keep R'' as above. Then, if $(\lambda_s)_{s \in R'' \cup (I+J)}$ is a family of elements of k, there exists exactly one ψ such that $\forall t \in T$, $\psi_{12}(a_t)=0$ and $\forall s \in R'' \cup (I+J)$, $\psi_{12}(a_{(s)})=\lambda_s$.*

PROPOSITION 14. *Suppose $|I+J|>1$ and $\varphi \neq \chi$. There is a unique r satisfying* (2). *If $\lambda \in k$, there is exactly one ψ such that $\psi_{12}(a_{(r)})=\lambda$ and $\forall t \in T$, $\psi_{12}(a_t)=0$.*

PROPOSITION 15. *Suppose $|I+J|>1$ and $\varphi=\chi$. There is a unique r satisfying* (2). *Choose $s \in I+J$. If $\lambda, \mu \in k$, there is exactly one ψ such that $\psi_{12}(a_{(r)})=\lambda$ and $\psi_{12}(a_{(s)})=\mu$.*

PROOF OF PROPOSITION 11. Existence: if $v \in R \setminus R'$, let $\psi_{12}(a_{(v)})=0$. Then Y0-2 are easy to check by Lemma 8. Let us check $Y3_{uv}$ for all $u, v \in R$. If $\psi(a_{(v)})$ or $\psi(a_{(u)})=0$, or if $\psi(a_{(u)})=\psi(a_{(v)})=-I_2$, it is clear. In the remaining cases one may assume $v\in R'$, then $\phi(a_{(v)})=\begin{pmatrix}0 & \lambda_v\\ 0 & 0\end{pmatrix}$, and $\phi(a_{(u)})=-I_2$ or $\begin{pmatrix}0 & \lambda_u\\ 0 & 0\end{pmatrix}$. The second case is trivial: both sides of Y3 are 0. In the first case $u\in I\subset R(\chi)\cap R(\varphi)$, so $\phi(t_{uv})=I_2$. If $u-/-v$, $Y3_{uv}$ is a commutation, it is satisfied. If $u-v$ then $|uv|>3$ and there is zero on both sides of $Y3_{uv}$.

Uniqueness. Let ϕ be a morphism as in Proposition 11, and let us check that $\phi_{12}(a_{(v)})=0$ when $v\in R\setminus R'$. If $\phi_{12}(a_{(v)})\neq 0$, then Lemma 8 implies $v\in R\setminus R(\chi)$ and $\varphi=\chi^v$ (case ii) does not occur since $I=J$). So, if $v\notin R'$ there is some $s\in I$ with $|vs|=3$. Then in $Y3_{sv}$ there is $\phi(a_{(v)})=\begin{pmatrix}0 & \phi_{12}(a_{(v)})\\ 0 & 0\end{pmatrix}$ on the left and 0 on the right. This completes the proof of Proposition 11.

OTHER PROOFS. Existence. Take everywhere $\phi_{12}(kT)=0$. For Propositions 12–13, take $\phi_{12}(a_{(v)})=0$ where non specified by the statement. For Proposition 14, take $\phi_{12}(a_{(v)})=0$ if $v\neq r$. For Proposition 15, when $v\in I+J$ take $\phi_{12}(a_{(v)})=\mu$ if v and s are both in $I\setminus J$ or $J\setminus I$, $-\mu$ otherwise; when $v\notin I+J\cup\{r\}$, take $\phi_{12}(a_{(v)})=0$.

Uniqueness is proved much as for Propositions 10 and 11, using Lemmas 8-9 and Y3.

The uniqueness of r in Propositions 14–15 comes from acyclicity: if r, r' satisfy (2) and s, s' are in $I+J$, (2) implies $r-s-r'-s'-r$.

4.3. Extension groups.

We now can prove our main result

THEOREM 16. *Let* $\mathcal{H}_k:=\mathrm{End}_{kG}\mathrm{Ind}_U^G k$ *be the modular Hecke algebra in characteristic p of a finite group with a split BN-pair of the same characteristic, let* $\phi(\varphi, I)$ *and* $\phi(\chi, J)$ *be two irreducible representations of* $\mathcal{H}_k$ *(see Proposition 4). The group*

$$\mathrm{Ext}^1_{\mathcal{H}_k}(\phi(\chi, J), \phi(\varphi, I))$$

is non zero if, and only if, one of the following is satisfied:

(1) $\varphi=\chi$, $I\setminus J\neq\emptyset$, $J\setminus I\neq\emptyset$, *and* $\forall r\in I\setminus J$, $\forall s\in J\setminus I$, $r-s$,

(2) *there is* $r\in R\setminus R(\chi)$ *such that* $\varphi=\chi^r$, $\{s\in R;\ s-r\}\supset I+J$ *(boolean sum) and* $(|rs|=3\Longrightarrow s\notin I\cap J)$,

the first case corresponding to non-split extensions that are split when restricted to each $\mathcal{H}_k(L_r)$ *for* $r\in R$. *The dimension of* $\mathrm{Ext}^1_{\mathcal{H}_k}(\phi(\chi, J), \phi(\varphi, I))$ *is given by the following: it is* 1 *when* $|I+J|>1$ *(this includes case* (1)).

In case (2) *with* $I=J$ *it is* $|\{s\in R\setminus R(\chi);\ \varphi=\chi^{s}$ *and* $\forall v\in I\ |sv|\neq 3\}|$. *In case* (2) *with* $|I+J|=1$, *it is* $|\{s\in R\setminus R(\chi);\ \varphi=\chi^{s}$ *and* $s-(I+J)\}|$.

PROOF. The dimensions are easy to derive from Propositions 10-15. They are clearly non zero.

Let us take $\phi(\chi, J)$ and $\phi(\varphi, I)$ two irreducible representations of $\mathcal{H}_k$, and assume

$$\phi:\mathcal{H}_k\longrightarrow \mathrm{Mat}_2(k),$$
$$h\longmapsto\begin{pmatrix}\phi(\varphi, I)(h) & \phi_{12}(h)\\ 0 & \phi(\chi, J)(h)\end{pmatrix}$$

is a morphism such that $\phi(J(\mathcal{H}_k))\neq 0$. Using the decomposition $J(\mathcal{H}_k)=\langle J(\mathcal{H}_k(L_r));\ r\in R\rangle+[\mathcal{H}_k, \mathcal{H}_k]$ of Theorem 5, one of the following must happen:

(1′) $\forall r\in R$, $\phi(J(\mathcal{H}_k(L_r)))=0$, and $\phi([\mathcal{H}_k, \mathcal{H}_k])\neq 0$.

(2′) There exists $r\in R$ such that $\phi(J(\mathcal{H}_k(L_r)))\neq 0$.

To complete the proof of the theorem it suffices to show that 1′ implies 1 and 2′ implies 2.

Case 1′. Since $\forall r\in R$, $\phi([\mathcal{H}_k(L_r), \mathcal{H}_k(L_r)])=0$, we have $\forall r\in R$, $t\in T$, $\phi([a_{(r)}, a_t])=0$. So there are $r, s\in R$ such that $\phi([a_{(r)}, a_{(s)}])\neq 0$. Then $\phi(a_{(r)}), \phi(a_{(s)})$ are semi-simple non scalar, so case i) of Lemma 8 does not occur: $\{r, s\}\subset I+J$. Moreover, case ii) must occur for r or s: otherwise $\phi_{12}(a_{(r)})=\phi_{12}(a_{(s)})=0$ contradicts non commutation. Thus $\varphi=\chi$. For the same reason, Lemma 9 implies r and s are not both in $I\setminus J$ or $J\setminus I$. Assume $r\in I\setminus J$ and $s\in J\setminus I$, then $I\setminus J$ and $J\setminus I$ are non empty. There remains to check $\forall u\in I\setminus J$, $u-s$. Lemma 9 implies $\phi(a_{(u)})=\phi(a_{(r)})$. Since $I\setminus J\subset R(\chi)$, $\chi(t_{us})=1$. If we had $u-/-s$, $\mathrm{Y3}_{us}$ would say $[\phi(a_{(r)}), \phi(a_{(s)})]=[\phi(a_{(u)}), \phi(a_{(s)})]=0$, a contradiction.

Case 2′. We have $\phi_{12}(a_{(r)})\neq 0$ and case i) of Lemma 8 is satisfied since case ii) is semi-simple. So $r\in R\setminus R(\chi)$ and $\varphi=\chi^{r}$. If $s\in I+J$, let us check $s-r$. We have $\phi(a_{(r)})=\begin{pmatrix}0 & \phi_{12}(a_{(r)})\\ 0 & 0\end{pmatrix}$ and $\phi(a_{(s)})=\begin{pmatrix}0 & \phi_{12}(a_{(s)})\\ 0 & -1\end{pmatrix}$ or $\begin{pmatrix}-1 & \phi_{12}(a_{(s)})\\ 0 & 0\end{pmatrix}$. So $\phi(a_{(r)})\phi(a_{(s)})$ or $\phi(a_{(s)})\phi(a_{(r)})$ is zero while the other is $-\phi(a_{(r)})\neq 0$, so $\mathrm{Y3}_{rs}$ implies $r-s$.

REMARK. In case (1), $I\setminus J$ or $J\setminus I$ has a single element: if $r, r'\in I\setminus J$ and $s, s'\in J\setminus I$ then $r-s-r'-s'-r$, a cycle.

The reader may have noticed that our convention for $\psi(\chi, J)$ is opposite to the classical one: in [5], our $\psi(\chi, J)$ would be denoted by $\psi(R(\chi)\setminus J, \chi)$. An argument for our convention is that conditions (1) and (2) would then become more complicated.

5. Blocks.

We now derive from Theorem 16 some lemmas leading to the determination of the blocks of $\mathcal{H}_k$.

Let us recall that W acts on $\mathrm{Mor}(T, k^*)$ by $\chi(ntn^{-1})=\chi^w(t)=(w^{-1}\cdot\chi)(t)$ when $n\in w\in N/T$. Let (χ, J) be an admissible pair for G.

LEMMA 17. *If $\varphi\in W\cdot\chi$, then $\psi(\varphi, \emptyset)$ is in the block of $\psi(\chi, \emptyset)$.*

LEMMA 18. *If we have $r\in R\setminus R(\chi)$ and $s\in J$ with $r-s$, then $(\varphi, I):=(\chi^r, \{s\in J;\ s-/-r\})$ is admissible and $\psi(\varphi, I)$ is in the block of $\psi(\chi, J)$.*

LEMMA 19. *If we have $r\in R(\chi)\setminus J$ and $s\in J$ such that $r-s$, then $\psi(\chi, J+\{r, s\})$ is in the block of $\psi(\chi, J)$.*

PROOFS. In 17, one may assume $\varphi=\chi^r$ for $r\in R$. Then, if $r\in R(\chi)$, $\chi^r=\chi=\varphi$ and it is trivial. If $r\in R\setminus R(\chi)$, then Theorem 16 (2) applies.

In 18, $I\subset R(\varphi)$: if $s-/-r$, r normalizes T_s by the action of W on roots, so $\chi^r(T_s)=\chi(T_s)$ which is 1 if $s\in J\subset R(\chi)$. Then Theorem 16 (2) applies.

In 19, Theorem 16 (1) applies.

Concerning the action of W on $\mathrm{Mor}(T, k^*)$, we know that if $r\in R(\chi)$ then $\chi^r=\chi$. So, if $R(\chi)=R$, χ is W-fixed; it is even extendible to a unique $\hat{\chi}\in\mathrm{Mor}(N, k^*)$ such that $\forall r\in R\hat{\chi}((r))=1$: a presentation of N is given by $T\cup\{(r);\ r\in R\}$ subject to relations defining T, the action of (r) on T by conjugation, the square of (r) (in T_r), and $\cdots(r)(s)t_{rs}=\cdots(s)(r)$ as in Definition 1 with $t_{rs}\in T_r\cap T_s\subset\ker\chi$.

DEFINITION 20. $M_0:=\{\chi;\ R(\chi)=R\}\subset\mathrm{Mor}(T, k^*)$, $M':=\mathrm{Mor}(T, k^*)\setminus M_0$. If $\chi\in\mathrm{Mor}(T, k^*)$, let $e_\chi:=|T|^{-1}\sum_{t\in T}\chi(t^{-1})a_t$. If $\chi\in M_0$, let $b_\chi^\emptyset:=|T|^{-1}\sum_{n\in N}\hat{\chi}(n^{-1})a_n$, $b_\chi^R:=(-1)^{l(w_0)}|T|^{-1}\sum_{n\in w_0}\hat{\chi}(n^{-1})a_n$, $b'_\chi:=e_\chi-b_\chi^\emptyset-b_\chi^R$, where $w_0\in W$ is the element of maximal length. If $X\in M'/W$ (:=the set of W-orbits in M'), let $b_X:=\sum_{\chi\in X}e_\chi$.

THEOREM 21. *Let* $\mathcal{H}_k := \mathrm{End}_{kG}\,\mathrm{Ind}_U^G k$ *be the modular Hecke algebra in characteristic p of a finite group G with irreducible split BN-pair of the same characteristic and rank* ≥ 2.
i) *There are* $2(T : T\cap G_R) + |W|^{-1}\sum_{w\in W}(T : [T, w])$ *blocks in* $\mathcal{H}_k$, *given by the following primitive idempotents in the center* $Z(\mathcal{H}_k)$:

$$\{b_X;\ X\in M'/W\}\cup\{b_\chi^\varnothing;\ \chi\in M_0\}\cup\{b_\chi^R;\ \chi\in M_0\}\cup\{b'_\chi;\ \chi\in M_0\}.$$

ii) *If* (χ, J) *is an admissible pair, the corresponding representation* $\phi(\chi, J)$ *is in the block corresponding to*: b_X *if* $R(\chi)\neq R$ *and* $\chi\in X$, $b_\chi^\varnothing$ *if* $R(\chi)=R$ *and* $I=\varnothing$, b_χ^R *if* $R(\chi)=R$ *and* $I=R$, b'_χ *if* $R(\chi)=R$ *and* $R\neq I\neq\varnothing$.
iii) *If* $|R|=1$, *the number of blocks is* $\frac{1}{2}(|T|+(T : T'_r))+(T : T_r)$, *the last set in the union above being empty. The blocks corresponding to the* $b_\chi^\varnothing$*'s and the* b_χ^R *for* $\chi\in M_0$ *are simple. The blocks corresponding to the* b_X*'s are uniserial of length two.* $\mathcal{H}_k$ *is then uniserial with indecomposable modules of dimensions* 1 *and* 2.
iv) *If* $|R|\geq 2$, *the simple blocks correspond to the* $b_\chi^\varnothing$*'s and the* b_χ^R*'s. The other uniserial blocks correspond to the* b'_χ*'s for* $\chi\in M_0$ *when* $|R|=2$, *their length is* $\frac{1}{2}|W|-1$.

PROOF. The e_χ's are clearly idempotents of kT and, if $n\in N$, $e_\chi a_n = a_n e_{n\cdot\chi}$ by $(\mathcal{L})$. So the b_X's are central idempotents. On the other hand, if $R(\chi)=R$, $(\mathcal{L})$ implies $e_\chi b_\chi^\varnothing = b_\chi^\varnothing e_\chi = b_\chi^\varnothing$, $e_\chi b_\chi^R = b_\chi^R e_\chi = b_\chi^R$, $e_\chi b'_\chi = b'_\chi e_\chi = b'_\chi$, $a_r b_\chi^\varnothing = b_\chi^\varnothing a_r = 0$, $a_r b_\chi^R = b_\chi^R a_r = -b_\chi^R$ for all $r\in R$. So $b_\chi^\varnothing, b_\chi^R, b'_\chi$ are central idempotents. The images by an irreducible representation $\phi(\varphi, I)$ are as follows:

$\phi(\varphi, I)(b_X)=1$ if, and only if, $\varphi\in X\subset M'$,
$\phi(\varphi, I)(b_\chi^\varnothing)=1$ if, and only if, $\varphi=\chi$ and $I=\varnothing$,
$\phi(\varphi, I)(b_\chi^R)=1$ if, and only if, $\varphi=\chi$ and $I=R$,
$\phi(\varphi, I)(b'_\chi)=1$ if, and only if, $\varphi=\chi$ and $\varnothing\neq I\neq R$.

The cases being disjoint, we have orthogonality. Their number is clearly $|\mathrm{Mor}(T, k^*)/W|+2|M_0| = |W|^{-1}\sum_{w\in W}(T : [T, w])+2(T : T_R)$ where $T_R := \langle T_r;\ r\in R\rangle$, this subgroup equals $T\cap G_R$ by [3] 11.4.

There remains to check primitivity, or equivalently that two arbitrary irreducible representations are in the same block when they send the same idempotent to 1 above.

First case is for a representation $\psi(\chi, J)$ with $R(\chi)\neq R$, let B be its block. We must show that all representations $\psi(\varphi, J)$ with $\varphi\in W\cdot\chi$ are in B. By Lemma 17, it is enough to check it is the case for some $\psi(\varphi, \emptyset)$ with $\varphi\in W\cdot\chi$. Let m be the smallest cardinality of sets I such that some $\psi(\varphi, I)$ is in B, then $m\leq|J|\leq R(\chi)<|R|$. Assume m is not zero. Among the $\psi(\varphi, I)$'s in B with $|I|=m$, choose one such that the distance $d\geq 1$ of I to $R\setminus R(\varphi)$ in the Coxeter diagram is minimal. If $d=1$ there is $r\in R\setminus R(\varphi)$ and $s\in I$ such that $s-r$, then Lemma 18 says $\psi(\varphi^{r}, \{s\in I; s-r\})$ is in B, a contradiction with the minimality of m. So $d\geq 2$, and there are $r_1-r_2-\cdots-r_d$ with $r_1\in R\setminus R(\varphi)$ and $r_d\in I$. Then $r_{d-1}\in R(\varphi)\setminus I$ and Lemma 19 says $\psi(\varphi, I+\{r_{d-1}, r_d\})$ is in B. But $|I+\{r_{d-1}, r_d\}|=m$ and the distance to $R\setminus R(\varphi)$ is now $d-1$, a contradiction.

The second case is when $\psi(\chi, J)$ satisfies $R(\chi)=R$ and $|J|=1$. If B denotes its block, we must show that any representation $\psi(\chi, I)$ with $R\neq I\neq\emptyset$ is in B. If I has just one element, repeated application of Lemma 19 allows to move it to the element of J along the Coxeter graph. If $|R|>|I|>1$, Lemma 19 allows to assume there is a "gap" in $I: r-a-s$ with $r\neq s$ in I and $a\in R\setminus I$. Then $\mathrm{Ext}^1_{\mathcal{H}_k}(\psi(\chi, I), \psi(\chi, I+\{a, r, s\}))\neq 0$ by Theorem 16 (1), so $\psi(\chi, I+\{a, r, s\})$ is in B. We have decreased the cardinality of I. Repeating this, we would get down to 1. The proof of i) and ii) is now complete. Adaptations in the case $|R|=1$ are clear.

Concerning uniserial and simple blocks, one knows that $\psi(\chi, J)$ is in a uniserial (resp. simple) block if, and only if, $\sum_{\varphi, I}\dim \mathrm{Ext}^1_{\mathcal{H}_k}(\psi(\chi, J), \psi(\varphi, I))\leq 1$ (resp. $=0$). So they are easily derived from Theorem 16. Lengths equal dimensions, so they follow from $\dim e_\chi\mathcal{H}_k=|W|$ when $\chi\in M_0$. This finishes the proof of the theorem.

REMARK. The computation of blocks can be made by giving explicitly the principal indecomposable modules for $\mathcal{H}_k$, as done in [4] for the 0-Hecke algebra $\mathcal{H}_k(G, B)=e_1\mathcal{H}_k$. If (χ, J) is admissible and $\hat{\chi}$ is the extension of χ to $N\cap L_{R(\chi)}$ such that $\hat{\chi}(R(\chi))=1$, let us denote $a_{\chi,J}:=\sum_{n\in N_{R(\chi)\setminus J}}\hat{\chi}(n^{-1})a_n$. A slight adaptation of [4] would show that the p.i.m. corresponding to $\psi(\chi, J)$ is

$$U_{\chi,J}=\bigoplus_{w\in X_{R(\chi)\setminus J}\setminus X_J} k\cdot a_{(w)}a_{\chi,J}$$

where $X_S:=\{w\in W;\ \forall s\in S\ l(sw)>l(w)\}$. As in [4], a filtration is given by taking elements of increasing lengths. The Cartan invariant $c(\psi(\chi, J), \psi(\varphi, I))$ is then proved to be $|\{w\in(X_{R(\chi)\setminus J}\setminus X_J)\cap(X_{R(\varphi)\setminus I}\setminus X_I)^{-1};\ \chi^w=\varphi\}|$. This may also lead to the determination of blocks, but it is not clear whether one can find the Ext^1 groups in the same fashion: one needs (two first terms of) all filtrations of $U_{\chi,J}$.

REMARK. The 0-Hecke algebra $\mathcal{H}_k(G,B):=\mathrm{End}_{kG}\,\mathrm{Ind}_U^G k$ equals $e_1\mathcal{H}_k$ with e_1 as above (see [3] 9.4). Since e_1 is central, irreducible representations and Ext^1's for $\mathcal{H}_k(G,B)$ are easy to derive from what we know about $\mathcal{H}_k$. Irreducible representations are the restrictions of $\psi(1,I)$'s for $I\subset R$, and $\mathrm{Ext}^1_{\mathcal{H}_k(G,B)}(\psi(1,I),\psi(1,J))$ is non zero if, and only if, $I\setminus J\neq\emptyset$, $J\setminus I\neq\emptyset$ and $\forall r\in I\setminus J$, $\forall s\in J\setminus I, r-s$. Then its dimension is 1. The blocks would be easy to derive, thus providing another proof of [4] 5.2 (see also next section).

6. Concluding remark: generalized 0-Hecke algebras.

The arguments used in Sections 3. 4. 5. seem a bit more general and might cover other examples of rings associated to a Coxeter system acting on an algebra (replacing the group algebra kT). An instance in arbitrary characteristic would be the "0-Hecke algebra" of [4], or the specialized version where $u=0$ of the generic algebra of [8].

Let us take now K an algebraically closed field, A an algebra over K and (W,R) a Coxeter system. We assume W acts on A by algebra automorphisms: $(a\mapsto w(a))\in\mathrm{Aut}\,A$. We assume there is a map

$$\begin{aligned} z: R &\longrightarrow A\\ r &\longmapsto z(r). \end{aligned}$$

Let us define the algebra over K, denoted by $\mathcal{H}_A(W)$, generated by $A\cup\{[r], r\in R\}$ subject to:

(Y0) $a\cdot b=ab$ when $a,b\in A$,

(Y1) $a\cdot[r]=[r]\cdot r(a)$,

(Y2) $[r]^2=z(r)\cdot[r]$,

($\mathrm{Y3}_{rs}$) $[r]\cdot[s]\cdots=[s]\cdot[r]\cdots$ ($|rs|$ terms on each side) for all $r,s\in R$ such that rs is of finite order.

The last relation implies that if $w=r_1\cdots r_{l(w)}$, then $[w]:=[r_1]\cdots[r_{l(w)}]$ only depends on w. Let B be a basis of A over K. Using the same idea as in [2] 23 p.55, one may prove that $(b[w])_{b\in B, w\in W}$ is a basis of $\mathcal{H}_A(W)$ if, and only if, the following are satisfied:

($\mathcal{R}$) $\forall r\in R\quad \forall a\in A\quad az(r)=z(r)r(a)$,

($\mathcal{W}$) $\forall r, s\in R$ and $w\in W$ such that $rw=ws$ and $l(rw)>l(w)$, $z(r)=w(z(s))$.

Assume $\mathcal{R}$ and $\mathcal{W}$ are satisfied, then we have the following (compare with [5] 3.5):

THEOREM 22. *If W is finite and A is finite dimensional, the simple $\mathcal{H}_A(W)$-modules remain simple when restricted to A. If moreover $A/J(A)$ is commutative, the simple $\mathcal{H}_A(W)$-modules are in bijection with pairs (χ, I) where $\chi\in \mathrm{Mor}(A, K)$ and $I\subset R_A(\chi):=\{r\in R;\ \chi(z(r))\neq 0\}$; one has $r\in R_A(\chi)\Rightarrow \chi=r\cdot\chi$.*

PROOF. (see also [5]) Let V be a simple $\mathcal{H}_A(W)$-module. Let $0\neq S\subset \mathrm{Res}_A V$ be a submodule of minimal dimension. Let $w\in W$ of maximal length such that $[w]\cdot S\neq 0$. Then $[w]\cdot S$ is A-stable by Y1, so it is simple as A-module by minimality of dimension. It suffices to check $[w]\cdot S$ is stable under $\mathcal{H}_A(W)$. Let $r\in R$, if $l(rw)>l(w)$ $[r]\cdot[w]\cdot S=[rw]\cdot S=0$ by maximality of $l(w)$, if $l(rw)<l(w)$ $[r]\cdot[w]\cdot S=z(r)\cdot[w]\cdot S\subset[w]\cdot S$. This completes the proof of the first statement.

If $A/J(A)$ is commutative, each simple A-module is one-dimensional, identified with some $\chi\in\mathrm{Mor}(A, K)$. Moreover, if χ is the restriction of $\phi\in\mathrm{Mor}(\mathcal{H}_A(W), K)$ then, by Y2, $(\phi([r]))^2=\phi([r])\chi(z(r))$. So $\phi([r])$ is 0 or $\chi(z(r))$, and necessarily 0 if $r\in R\setminus R_A(\chi)$. Moreover, if $r\in R_A(\chi)$, condition ($\mathcal{R}$) implies $\chi=r\cdot\chi$.

Conversely, if $\chi\in\mathrm{Mor}(A, K)$ and $I\subset R_A(\chi)$, there is one $\phi\in \mathrm{Mor}(\mathcal{H}_A(W), K)$ such that $\mathrm{Res}_A\phi=\chi$, $\phi([r])=\chi(z(r))$ if $r\in I$ and $\phi([r])=0$ if $r\in R\setminus I$: it suffices to check Y0-3 are satisfied. Y0-2 are clear. $Y3_{rs}$ is clear when $|rs|$ is even or $\{r, s\}\not\subset I$. If $r, s\in I\subset R_A(\chi)$ and $|rs|$ is odd, then r and s are conjugate by some $w\in W_{\{r,s\}}$, so w fixes χ and ($\mathcal{W}$) implies $\chi(z(s))=\chi(z(r))$, hence $Y3_{rs}$ is satisfied in that case.

From this, it would be easy to derive analogues of Theorems 5, 16, 21 for this kind of general "0-Hecke algebra" $\mathcal{H}_A(W)$, when $A/J(A)$ is commutative. In Theorem 5, one has to add $J(A)$ which is not necessarily 0. Non semi-simplicity of A causes also a slight change in Theorem 16. The value of $\dim \mathrm{Ext}^1_{\mathcal{H}_A(W)}(\phi(\chi, J), \phi(\varphi, I))$ is the same as in Theorem 16 except in one case:

(3) $I=J$ and $\mathrm{Ext}^1_A(\chi, \varphi)\neq 0$.

In case (3) one proves easily, using the defining relations Y1-3, that $\dim \mathrm{Ext}^1_{\mathcal{H}_A(W)}(\phi(\chi, J), \phi(\varphi, J))=\dim \mathrm{Ext}^1_A(\chi, \varphi)+|\{r\in R\setminus R_A(\chi);\ \varphi=r\cdot\chi,\ \forall s\in J\ |sr|\neq 3\}|$. The change on blocks is as follows. The group W permutes the blocks of A. Using $(\mathscr{R})$, one proves easily that if $R_A(\chi)=R$ and $\mathrm{Ext}_A^1(\chi,\varphi)\neq 0$, then $R_A(\varphi)=R$. Then, the blocks of $\mathscr{H}_A(W)$ are obtained as in Theorem 21, replacing $\mathrm{Mor}(T,k^*)$ with the set of blocks of A and M_0 with the set of blocks of irreducible representations χ of A such that $R_A(\chi)=R$.

Some adaptations are also possible to study the two-dimensional representations of $\mathscr{H}_A(W)$ when W is no longer finite. One must take care of squares and triangles in the Coxeter diagram, since we used repeatedly acyclicity in the discussions of 5. Cycles of larger size do not seem to cause any change.

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