J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
36 (1989), 373-386.

Jacobi forms and a Maass relation for Eisenstein series (II)

By Tadashi YAMAZAKI

(Communicated by Y. Ihara)

Introduction

This is a continuation or supplement to our previous work [Y]. There we proved basic properties of Jacobi forms for the Siegel modular group and defined an action of the Hecke algebra for the Siegel modular group on the space of Jacobi forms. In particular we obtained explicit formulas for the action of some of the generators of the Hecke algebra on the Eisenstein series. This gave a generalization of the Maass relation ([M]) for the Siegel-Eisenstein series. In this paper we shall calculate the action of the remaining generators on the Eisenstein series. Thus we get all the relations among the Fourier coefficients of the Siegel-Eisenstein series which can be formulated in terms of Jacobi forms and the Hecke algebra (§5 Theorem 5.2). In the course of the calculation we find that the action of the Hecke algebra on the Eisenstein series has a close connection with the Siegel operator (§4 Theorem 4.1). This explains the strange nature of the Euler factor which appeared in [Y] (see the proof of Theorem 5.2).

Notation. Let $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ denote the ring of integers, the field of rational numbers, the field of real numbers and the field of complex numbers, respectively. For any commutative ring A, $M_{m,n}(A)$ denotes the set of m by n matrices with entries in A and $M_m(A)$ denotes $M_{m,m}(A)$. The identy and zero elements of $M_m(A)$ are denoted by 1_m and 0_m, respectively. For any matrix X we denote by ${}^tX$ the transpose of X. If $X_1, \cdots, X_r$ are square matrices, $\mathrm{diag}[X_1, \cdots, X_r]$ denotes the matrix with $X_1, \cdots, X_r$ in the diagonal blocks and 0 in all the other blocks.

The symbol $\boldsymbol{e}(x)$ denotes $e^{2\pi i x}$ and $\boldsymbol{e}^m(x)$ denotes $\boldsymbol{e}(mx)$.

For a finite set S, we denote by $\sharp(S)$ the cardinality of S.

1. Jacobi forms

The basic facts and definitions from the theory of Jacobi forms can be found in [Y]. We also refer to [M], [Zi] for more generalities. To fix our notation we briefly summarize those items that we shall need in the following.

The Siegel upper half plane H_n of degree n is the set of n by n symmetric complex matrices with positive definite imaginary parts. The letters τ and z will always be reserved for variables in H_n and C^n respectively. Let G_n, Γ_n, S_n be the real symplectic group, the Siegel modular group, the group of similitudes, respectively; namely we define

$$G_n=Sp(n, \boldsymbol{R})=\{M\in M_{2n}(\boldsymbol{R});\ {}^tMJ_nM=J_n\},$$
$$\Gamma_n=Sp(n, \boldsymbol{Z})=G_n\cap M_{2n}(\boldsymbol{Z}),$$
$$S_n=\{M\in M_{2n}(\boldsymbol{R});\ {}^tMJ_nM=\nu J_n \text{ for some } \nu>0\},$$

in which $J_n=\begin{bmatrix} 0_n & 1_n \\ -1_n & 0_n \end{bmatrix}$. Let M be in S_n. If ${}^tMJ_nM=\nu J_n$, we write $\nu=\nu(M)$. We make the direct product $S_n^J=S_n\times \boldsymbol{R}^{2n}\times \boldsymbol{R}$ into a group in the following way. For $g_i=[M_i, X_i, \kappa_i]$ in S_n^J $(i=1, 2)$ we define

$$g_1g_2=[M_1M_2,\ \nu_2^{-1}X_1M_2+X_2,\ \nu_2^{-1}\kappa_1+\kappa_2+\nu_2^{-1}X_1J_n{}^tX_2], \tag{1}$$

where $\nu_2=\nu(M_2)$. Similarly we define subgroups $G_n^J=G_n\times \boldsymbol{R}^{2n}\times \boldsymbol{R}$ and $\Gamma_n^J=\Gamma_n\times \boldsymbol{Z}^{2n}\times \boldsymbol{Z}$ of S_n^J.

Let k and m be non-negative integers. Take an element $g=[M, X, \kappa]$ in S_n^J and decompose M and X into $n\times n$ blocks $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and n-vectors (λ, μ), respectively. For any (τ, z) in $H_n\times C^n$ we put $g(\tau, z)=(M\tau, \nu(z+\lambda\tau+\mu)(c\tau+d)^{-1})$, in which $M\tau=(a\tau+b)(c\tau+d)^{-1}$ and $\nu=\nu(M)$. Also for any function $\varphi(\tau, z)$ on $H_n\times C^n$ we define

$$\begin{aligned}(\varphi|_{k,m}g)(\tau, z) \\ &=\boldsymbol{e}^{m\nu}(\kappa+\lambda\tau{}^t\lambda+2\lambda{}^tz+\lambda{}^t\mu-(z+\lambda\tau+\mu)(c\tau+d)^{-1}c{}^t(z+\lambda\tau+\mu)) \\ &\quad\times\det(c\tau+d)^{-k}\varphi(g(\tau, z)).\end{aligned} \tag{2}$$

A holomorphic function φ on $H_n\times C^n$ is called a Jacobi form of weight k and index m if it satisfies the functional equation $\varphi|_{k,m}g=\varphi$ for all g in Γ_n^J. When $n=1$, we impose a regularity condition at infinity. A basic example of Jacobi form is the Eisenstein series:

(3) $$E_{k,m}=\sum_{g\in\Gamma^J_{n,0}\backslash\Gamma^J_n} 1|_{k,m}g,$$

in which 1 denotes the constant one function and

$$\begin{aligned}\Gamma^J_{n,0}&=\{g\in\Gamma^J_n;\ 1|_{k,m}g=1\}\\&=\{[M,(\lambda,\mu),\kappa]\in\Gamma^J_n;\ M\in\Gamma_{n,0},\ \lambda=0\},\\ \Gamma_{n,0}&=\left\{\begin{bmatrix}a&b\\c&d\end{bmatrix}\in\Gamma_n;\ c=0_n\right\}.\end{aligned}$$

For $k>n+2$ the series in the right hand side of (3) converges absolutely and uniformly on any compact subset of $H_n\times C^n$ and defines a non-zero Jacobi form of weight k and index m.

Let $\Delta_n=S_n\cap M_{2n}(Z)$. We shall define an action of the Hecke algebra $\mathcal{H}(\Gamma_n,\Delta_n)$ of the Siegel modular group on the space of Jacobi forms. Take an element M in Δ_n and decompose the double coset $\Gamma_nM\Gamma_n$ into left cosets;

$$\Gamma_nM\Gamma_n=\bigcup_i\Gamma_nM_i.$$

For a Jacobi form φ of weight k and index m, we define

(4) $$\varphi|_{k,m}(\Gamma_nM\Gamma_n)=\sum_i\varphi|_{k,m}[M_i,0,0].$$

Then $\varphi|_{k,m}(\Gamma_nM\Gamma_n)$ is a Jacobi form of weight k and index $m\nu(M)$. We remark that here we do not use any normalization factors. It makes easy to relate the Hecke algebra and the Siegel Φ-operators (see §4).

2. Double coset decompositions

In this section we determine double coset decompositions of the generators of the Hecke algebra $\mathcal{H}(\Gamma_n,\Delta_n)$ of the Siegel modular group. Let p be a prime number. We know that the p-part of the Hecke algebra is generated by the following elements,

(1) $$T_n(p)=\Gamma_n\operatorname{diag}[1_n,p1_n]\Gamma_n$$

and

(2) $$T_{i,n-i}(p^2)=\Gamma_n\operatorname{diag}[1_i,p1_{n-i},p^21_i,p1_{n-i}]\Gamma_n\qquad(0\le i\le n).$$

We put

$$T_n(p^2)=\bigcup_{0\le i\le n} T_{i,n-i}(p^2)$$

and for $0\le i\le j\le n$

(3) $$\delta_{ij}=\mathrm{diag}[1_i,\ p1_{j-i},\ p^2 1_{n-j}].$$

PROPOSITION 2.1. *We have the following decomposition of* $T_n(p^2)$;

(4) $$T_n(p^2)=\bigcup_{0\le i\le j\le n}\ \bigcup_{[x]} \Gamma_{n,0}\delta_{ij}(x)\Gamma_n,$$

in which

$$\delta_{ij}(x)=\begin{bmatrix} p^2\delta_{ij}^{-1} & x \\ 0_n & \delta_{ij}\end{bmatrix}$$

and

$$x=\mathrm{diag}[0_i,\ x_{22},\ 0_{n-j}],\quad x_{22}={}^t x_{22}\in M_{j-i}(Z).$$

For any x, y of the above form we say that they are equivalent and write $[x]=[y]$ if and only if there exists a matrix u in $GL(n, Z)\cap(\delta_{ij}GL(n, Z)\delta_{ij}^{-1})$ such that $u_{22}x_{22}{}^t u_{22}\equiv y_{22} \bmod p$, where $u_{22}\in M_{j-i}(Z)$ is the $(2, 2)$-entry of u when we decompose it into blocks of type $(i, j-i, n-j)$. And in the right hand side of (4) *we take a union over the set of equivalence classes $[x]$.*

REMARK. If $(i, j)\neq(0, n)$, then the condition $[x]=[y]$ is equivalent to say that there exists u_{22} in $M_{j-i}(Z)$ such that $u_{22}x_{22}{}^t u_{22}\equiv y_{22} \bmod p$.

PROOF. It is obvious that we can take matrices of the form $\delta_{ij}(x)$, with $x=\mathrm{diag}[0_i, x_{22}, 0_{n-j}]$ and $x_{22}={}^t x_{22}\in M_{j-i}(Z)$ as representatives of $\Gamma_{n,0}\backslash T_n(p^2)/\Gamma_n$. Moreover if $\delta_{ij}(x)$ and $\delta_{kl}(y)$ determine the same double coset, then $i=k$ and $j=l$. Let $x=\mathrm{diag}[0_i, x_{22}, 0_{n-j}]$ and $y=\mathrm{diag}[0_i, y_{22}, 0_{n-j}]$. We assume that $\delta_{ij}(x)\in\Gamma_{n,0}\delta_{ij}(y)\Gamma_n$. Then there exist w, u in $GL(n, Z)$ and symmetric integral matrices s, b such that

$$\delta_{ij}(x)=\mathrm{diag}[w,\ {}^t w^{-1}]\begin{bmatrix} 1_n & s \\ 0_n & 1_n\end{bmatrix}\delta_{ij}(y)\begin{bmatrix} 1_n & b \\ 0_n & 1_n\end{bmatrix}\mathrm{diag}[u^{-1},\ {}^t u].$$

Therefore we have $w=\delta_{ij}^{-1}u\delta_{ij}$ and $x=w(y+s\delta_{ij}+p^2\delta_{ij}^{-1}b){}^t u$. Since $\delta_{ij}x=px$ and $\delta_{ij}y=py$, it follows that u is in $GL(n, Z)\cap(\delta_{ij}GL(n, Z)\delta_{ij}^{-1})$ and $x=u(y+p^{-1}\delta_{ij}s\delta_{ij}+pb){}^t u$. It is easy to see that the last two conditions are equivalent to say that there exists u in $GL(n, Z)\cap(\delta_{ij}GL(n, Z)\delta_{ij}^{-1})$ such that $u_{22}x_{22}{}^t u_{22} \equiv y_{22} \bmod p$. Conversely if there exists such a matrix u then, following the above argument in the reverse order, we get $\delta_{ij}(x) \in \Gamma_{n,0}\delta_{ij}(y)\Gamma_n$. Q.E.D.

For an integral matrix A, we define $\mathrm{rank}_p(A)$ to be the rank of $A \bmod p$ over the finite prime field $\boldsymbol{F}_p$. Let A be in $T_n(p^2)$. Then A is in $T_{\alpha,n-\alpha}(p^2)$ if and only if $\mathrm{rank}_p(A)=\alpha$. Therefore we get the following

COROLLARY 2.2. *For any* $0 \leq \alpha \leq n$, *we have*

$$T_{\alpha,n-\alpha}(p^2)=\bigcup_{\substack{0\leq i\leq j\leq n\\ \alpha=i+n-j+\mathrm{rank}_p(x)}}\ \bigcup_{[x]} \Gamma_{n,0}\delta_{ij}(x)\Gamma_n,$$

in which the union is taken over the set of equivalence classes satisfying $\alpha=i+n-j+\mathrm{rank}_p(x)$.

3. Actions of the Hecke operators

In this section, using the double coset decompositions in the previous section, we calculate the action of the Hecke algebra on the Eisenstein series.

We start with an elementary lemma.

LEMMA 3.1. *Let p be a prime number. For any λ in $\boldsymbol{Z}^n$ and $0 \leq j \leq n$, we define*

$$G_j^n(\lambda)=\sum_{\substack{x={}^t x\in M_n(\boldsymbol{Z}) \bmod p\\ \mathrm{rank}_p(x)=j}} e(p^{-1}\lambda x {}^t\lambda). \tag{1}$$

Then we have $G_0^n(\lambda)=1$ *and for* $1 \leq j \leq n$

$$G_j^n(\lambda)=\begin{cases}(-1)^j p^{[j/2]([j/2]+1)} g_p(n-1, 2[j/2]) \prod_{\alpha<j,\alpha:\mathrm{odd}} (p^\alpha-1) & \lambda \not\equiv 0 \bmod p\\ p^{[j/2]([j/2]+1)} g_p(n, j) \prod_{\alpha\leq j,\alpha:\mathrm{odd}} (p^\alpha-1) & \lambda\equiv 0 \bmod p,\end{cases}$$

where $g_p(n,j)=\prod_{1\leq\alpha\leq j}(p^{n-j+\alpha}-1)(p^\alpha-1)^{-1}$, *and for a real number x $[x]$ denotes the largest integer satisfying the inequality* $[x] \leq x$.

PROOF. The case $j=0$ is trivial, so let us assume that $1 \leq j \leq n$. First consider the case where p is odd. Then any symmetric matrix $x \bmod p$ of rank j is equivalent under $GL(n, \boldsymbol{F}_p)$ to

$$x_0=\mathrm{diag}\,[1_j, 0_{n-j}]$$

or

$$x_1=\mathrm{diag}\,[1_{j-1}, \gamma, 0_{n-j}], \qquad \text{with } \left(\frac{\gamma}{p}\right)=-1,$$

where $\left(\frac{*}{p}\right)$ denotes the quadratic residue symbol modulo p. For $i=0, 1$ we write $x_i=\mathrm{diag}\,[y_i, 0_{n-j}]$. Then the orthogonal group $O(x_i)$ for x_i is given by

$$O(x_i)=\left\{\begin{bmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{bmatrix} \in GL(n, \boldsymbol{F}_p);\ \begin{matrix} g_{11}\in O(y_i),\ g_{22}\in GL(n-j, \boldsymbol{F}_p) \\ g_{12}\in M_{j,n-j}(\boldsymbol{F}_p) \end{matrix}\right\}.$$

We note that the order of $O(y_i)$ is given by ([S])

$$(2) \qquad \#(O(y_i))=\begin{cases} 2p^{(j-1)^2/4}\prod_{\alpha=1}^{[j/2]}(p^{2\alpha}-1) & \text{if } j=\text{odd}, \\ 2p^{j(j-2)/4}\left(p^{j/2}-\left(\frac{-1}{p}\right)^{j/2}(-1)^i\right)\prod_{\alpha=1}^{j/2-1}(p^{2\alpha}-1) & \text{if } j=\text{even}. \end{cases}$$

By using the orthogonal group, we find that

$$\begin{aligned} G_j^n(\lambda) &= \sum_{i=0,1}\ \sum_{\xi\in GL(n,\boldsymbol{F}_p)/O(x_i)} e(\lambda\xi x_i{}^t\xi^t\lambda/p) \\ &= \sum_{i=0,1}(\#(O(x_i)))^{-1}\sum_{\xi\in GL(n,\boldsymbol{F}_p)} e(\lambda\xi x_i{}^t\xi^t\lambda/p). \end{aligned}$$

If $\lambda\equiv 0 \bmod p$, then

$$G_j^n(\lambda)=\sum_{i=0,1}\#(GL(n, \boldsymbol{F}_p))(\#(O(x_i)))^{-1}.$$

Since $\#(O(x_i))=p^{j(n-j)}(\#(O(y_i)))\cdot\#(GL(n-j, \boldsymbol{F}_p))$, we get the desired formula. When $\lambda\not\equiv 0 \bmod p$, we have

$$G_j^n(\lambda)=\sum_{i=0,1}p^{n-1}\#(GL(n-1, \boldsymbol{F}_p))(\#(O(x_i)))^{-1}\sum_{\substack{\mu\in \boldsymbol{Z}^n \bmod p \\ \mu\not\equiv 0 \bmod p}} e(\mu x_i{}^t\mu/p).$$

Now we introduce the usual Gaussian sum $g(\varepsilon)=\sum_{a=0}^{p-1}e(\varepsilon a^2/p)=\left(\frac{\varepsilon}{p}\right)g(1)$. Then the sum in the right hand side becomes as follows

$$\begin{aligned} \sum_{\substack{\mu\in \boldsymbol{Z}^n \bmod p \\ \mu\not\equiv 0 \bmod p}} e(\mu x_i{}^t\mu/p) &= -1+\sum_{\mu\in \boldsymbol{Z}^n \bmod p} e(\mu x_i{}^t\mu/p) \\ &= -1+(-1)^i p^{n-j} g(1)^j. \end{aligned}$$

We note that $g(1)^2=\left(\frac{-1}{p}\right)p$. This completes the proof for the case when $p>2$. Now let us assume that $p=2$. We remark that in this case any non-singular symmetric matrix is equivalent to 1_n or $\begin{bmatrix} 0_m & 1_m \\ 1_m & 0_m \end{bmatrix}$. The orders of the orthogonal and symplectic groups over the finite prime field are given by ([H-K], [C])

$$\sharp(O_n(F_p))=p^{n(n-1)/2-[(n-1)/2]([(n-1)/2]+1)}\prod_{\alpha=1}^{[(n-1)/2]}(p^{2\alpha}-1) \tag{3}$$

and

$$\sharp(Sp(n, F_p))=p^{n^2}\prod_{\alpha=1}^{n}(p^{2\alpha}-1), \tag{4}$$

respectively. We can argue in a similar way as in the case $p>2$. So we omit the detail. Q.E.D.

Since $G_j^n(\lambda)$ takes the same value for all $\lambda\neq 0$, by abuse of notation, we denote it by $G_j^n(1)$.

Now we determine the action of the Hecke algebra on the Eisenstein series. To simplify notation, we put for $M\in S_n$ and $\lambda\in \boldsymbol{Q}^n$

$$j(k, m; M, \lambda)=1|_{k,m}[1_{2n}, (\lambda, 0), 0][M, 0, 0]. \tag{5}$$

For $0\leq i\leq j\leq n$ and $x=\mathrm{diag}[0_i, x_{22}, 0_{n-j}]$ with $x_{22}={}^t x_{22}\in M_{j-i}(Z)$, we set

$$\Gamma(\delta_{ij}(x))=\Gamma_n\cap(\delta_{ij}(x)^{-1}\Gamma_{n,0}\delta_{ij}(x)).$$

Also for $0\leq i\leq j\leq n$ and $0\leq \alpha\leq n$, we put

$$K_{ij}^{\alpha}=\sum_{\substack{[x] \\ \mathrm{rank}_p(x)=\alpha}}\ \sum_{M\in\Gamma(\delta_{ij}(x))\backslash\Gamma_n}\ \sum_{\lambda\in Z^n} j(k, 1; \delta_{ij}(x)M, \lambda),$$

where the summation is taken over a set of representatives of the equivalence classes defined in Proposition 2.1.

PROPOSITION 3.2. *As a function on $H_n\times C^n$, K_{ij}^{α} is a linear combination of the Eisenstein series $E_{k,1}(\tau, pz)$ and $E_{k,p^2}(\tau, z)$.*

PROOF. We put

$$U=\left\{\begin{bmatrix} 1_n & s \\ 0_n & 1_n \end{bmatrix};\ s={}^t s\in M_n(\boldsymbol{Z})\right\}$$

and

$$\Gamma'(\delta_{ij}(x))_u=\Gamma(\delta_{ij}(x))U.$$

Then it is easy to show that the set

$$\left\{\begin{bmatrix}1_n & s\\ 0_n & 1_n\end{bmatrix};\ s=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & s_{23}\\ 0 & s_{32} & s_{33}\end{bmatrix},\ \begin{matrix}s_{23}={}^ts_{32}\in M_{j-i,n-j}(\boldsymbol{Z}) \mod p\\ s_{33}={}^ts_{33}\in M_{n-j}(\boldsymbol{Z}) \mod p^2\end{matrix}\right\}$$

is a complete system of representatives for $\Gamma(\delta_{ij}(x))\backslash\Gamma(\delta_{ij}(x))_u$. If we replace M by $\begin{bmatrix}1_n & s\\ 0_n & 1_n\end{bmatrix}M$, with s is of the above form, then each term in the summation is multiplied by the factor

$$e(p^2\lambda\delta_{ij}^{-1}s\delta_{ij}^{-1t}\lambda)=e(2p^{-1}\lambda_2s_{23}{}^t\lambda_3+p^{-2}\lambda_3s_{33}{}^t\lambda_3),$$

where we write $\lambda=(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1\in\boldsymbol{Z}^i$, $\lambda_2\in\boldsymbol{Z}^{j-i}$ and $\lambda_3\in\boldsymbol{Z}^{n-j}$. Therefore if we sum over the set of representatives, we get

$$\sum e(p^2\lambda\delta_{ij}^{-1}s\delta_{ij}{}^t\lambda)=\begin{cases}p^{(n-j)(n-i+1)} & \text{if } \lambda_3\in p\boldsymbol{Z}^{n-j}\\ 0 & \text{otherwise.}\end{cases}$$

Since $p(\lambda_1, \lambda_2, p\lambda_3)\delta_{ij}^{-1}=(p\lambda_1, \lambda_2, \lambda_3)$, we have

$$K_{ij}^{\alpha}(\tau, z)=p^{-k(2n-i-j)+(n-j)(n-i+1)}\sum_{\substack{[x]\\ \operatorname{rank}_p(x)=\alpha}}\ \sum_{M\in\Gamma(\delta_{ij}(x))_u\backslash\Gamma_n}$$
$$\times\sum_{\lambda\in L_i}j\left(k, 1; \begin{bmatrix}1_n & p^{-1}x\\ 0 & 1_n\end{bmatrix}M, \lambda\right)(\tau, pz),$$

where we put $L_i=(p\boldsymbol{Z})^i\times\boldsymbol{Z}^{j-i}\times\boldsymbol{Z}^{n-j}$. Now we define a subgroup $\Gamma(\delta_{ij})$ of $\Gamma_{n,0}$ by

$$\Gamma(\delta_{ij})=\left\{\begin{bmatrix}a & b\\ 0 & d\end{bmatrix}\in\Gamma_{n,0};\ a\in\delta_{ij}GL(n, \boldsymbol{Z})\delta_{ij}^{-1}\right\}.$$

For any u in $GL(n, \boldsymbol{Z})$, $\operatorname{diag}[u, {}^tu^{-1}]$ is in $\Gamma(\delta_{ij}(x))_u$ if and only if $u_{22}x_{22}{}^tu_{22}\equiv x_{22} \mod p$ and $u\in GL(n, \boldsymbol{Z})\cap\delta_{ij}GL(n, \boldsymbol{Z})\delta_{ij}^{-1}$. And if u is in $\delta_{ij}GL(n, \boldsymbol{Z})\delta_{ij}^{-1}$, then it stabilizes the lattice L_i. Therefore the summation over the set of equivalence classes $[x]$ and the set of representatives of $\Gamma(\delta_{ij}(x))_u\backslash\Gamma(\delta_{ij})$ turns into a summation over x modulo p. The value of this sum is given by Lemma 3.1 (See the remark after the lemma.). Hence we obtain

$$\begin{aligned}(6)\qquad & K_{ij}^{\alpha}(\tau, z)\\ &= p^{-k(2n-i-j)+(n-j)(n-i+1)} \sum_{\substack{x \bmod p\\ \mathrm{rank}\,(x)=\alpha}} \sum_{M\in\Gamma(\delta_{ij})\backslash\Gamma_n}\\ &\quad\times \sum_{\lambda\in L_i} j\left(k, m; \begin{bmatrix}1 & p^{-1}x\\ 0 & 1\end{bmatrix}M, \lambda\right)(\tau, pz)\\ &= p^{-k(2n-i-j)+(n-j)(n-i+1)}\{(G_{\alpha}^{j-i}(0)-G_{\alpha}^{j-i}(1))\sum_{M\in\Gamma(\delta_{ij})\backslash\Gamma_n}\\ &\quad\times\sum_{\lambda\in L_j} j(k, 1; M, \lambda)(\tau, pz) + G_{\alpha}^{j-i}(1)\sum_{M\in\Gamma(\delta_{ij})\backslash\Gamma_n}\sum_{\lambda\in L_i} j(k, 1; M, \lambda)(\tau, pz)\}.\end{aligned}$$

We put $\delta_i=\mathrm{diag}[1_i, p1_{n-i}]$ and define a subgroup $\Gamma(\delta_i)$ of $\Gamma_{n,0}$ by

$$\Gamma(\delta_i)=\left\{\begin{bmatrix}a & b\\ 0 & d\end{bmatrix}\in\Gamma_{n,0};\ a\in\delta_i GL(n, Z)\delta_i^{-1}\right\}.$$

Then the two sums in the right hand side of (6) are equal to

$$[\Gamma(\delta_j):\Gamma(\delta_{ij})]\sum_{M\in\Gamma(\delta_j)\backslash\Gamma_n}\sum_{\lambda\in L_j} j(k, 1; M, \lambda)(\tau, pz)$$

and

$$[\Gamma(\delta_i):\Gamma(\delta_{ij})]\sum_{M\in\Gamma(\delta_i)\backslash\Gamma_n}\sum_{\lambda\in L_i} j(k, 1; M, \lambda)(\tau, pz),$$

respectively. The sum of this type was treated in [Y]. We have

$$\begin{aligned}&\sum_{M\in\Gamma(\delta_j)\backslash\Gamma_n}\sum_{\lambda\in L_j} j(k, 1; M, \lambda)(\tau, pz)\\ &= \sum_{\substack{L\subset Z^n\\ Z^n/L\cong(Z/p)^j}}\sum_{M\in\Gamma_{n,0}\backslash\Gamma_n}\sum_{\lambda\in L} j(k, 1; M, \lambda)(\tau, pz)\\ &= g_p(n-1, j)\sum_{M\in\Gamma_{n,0}\backslash\Gamma_n}\sum_{\lambda\in Z^n} j(k, 1; M, \lambda)(\tau, pz)\\ &\quad + p^{n-j}g_p(n-1, j-1)\sum_{M\in\Gamma_{n,0}\backslash\Gamma_n}\sum_{\lambda\in Z^n} j(k, p^2; M, \lambda)(\tau, z)\\ &= g_p(n-1, j)E_{k,1}(\tau, pz)+p^{n-j}g_p(n-1, j-1)E_{k,p^2}(\tau, z).\end{aligned}$$

Hence K_{ij}^{α} is a linear combination of $E_{k,1}(\tau, pz)$ and $E_{k,p^2}(\tau, z)$. Q.E.D.

THEOREM 3.3. *Let p be a prime number. Then for any $0\leq\alpha\leq n$, $E_{k,1}|_{k,1}T_{\alpha,n-\alpha}(p^2)(\tau, z)$ is a linear combination of $E_{k,1}(\tau, pz)$ and $E_{k,p^2}(\tau, z)$.*

PROOF. It follows from the definition that

$$\begin{aligned}&E_{k,1}|_{k,1}T_{\alpha,n-\alpha}(p^2)\\ &= \sum_{M\in\Gamma_n\backslash T_{\alpha,n-\alpha}(p^2)}\sum_{M'\in\Gamma_{n,0}\backslash\Gamma_n}\sum_{\lambda\in Z^n} 1|_{k,1}[M'M, p^{-2}(\lambda, 0)M'M, 0]\\ &= \sum_{M\in\Gamma_n\backslash T_{\alpha,n-\alpha}(p^2)}\sum_{\lambda\in Z^n} j(k, 1; M, \lambda).\end{aligned}$$

In view of the double coset decomposition of $T_{\alpha,n-\alpha}(p^2)$ given in Corollary 2.2, we get

$$\begin{aligned} &E_{k,1}|_{k,1}T_{\alpha,n-\alpha}(p^2) \\ &= \sum_{0\le i\le j\le n} \sum_{\substack{[x] \\ \alpha=i+n-j+\mathrm{rank}_p(x)}} \sum_{M\in\Gamma(\delta_{ij}(x))\backslash\Gamma_n} \sum_{\lambda\in Z^n} j(k, 1; \delta_{ij}(x)M, \lambda) \\ &= \sum_{\substack{0\le i\le j\le n \\ \alpha=i+n-j+\beta}} \sum_{0\le\beta\le n} K_{ij}^{\beta}. \end{aligned}$$

Thus the claim follows from Proposition 3.2. Q.E.D.

For a general α, the explicit formula for $E_{k,1}|_{k,1}T_{\alpha,n-\alpha}(p^2)$ is rather complicated, but in the next section we will show that it has an intimate connection with the Siegel Φ-operator. Here we just state the cases which we need later.

COROLLARY 3.4. *We have*

(i) $$(E_{k,1}|_{k,1}T_{0,n}(p^2))(\tau, z)=p^{-nk}E_{k,1}(\tau, pz),$$

(ii) $$\begin{aligned} &(E_{k,1}|_{k,1}T_{1,n-1}(p^2))(\tau, z) \\ &=p^{-(n+1)k}\left(\frac{p^{n-1}-1}{p-1}p^{2k}-p^k+\frac{p^n-1}{p-1}p^{n+1}\right)E_{k,1}(\tau, pz) \\ &\qquad +p^{-nk+n}(p^{k-1}+1)E_{k,p^2}(\tau, z). \end{aligned}$$

These formulas can be easily extracted from the proofs of Proposition 3.2 and Theorem 3.3.

4. Commutativity with the Siegel operators

Just as the case of the Siegel modular forms, we can define the Siegel Φ-operator for the Jacobi forms. Let φ be a Jacobi form of degree n. Let $\tau=\mathrm{diag}[\tau', it]\in H_n$ with $\tau'\in H_{n-1}$ and let $z=(z', z_n)\in C^n$, with $z'\in C^{n-1}$. Then the limit

(1) $$\lim_{t\to+\infty} \varphi(\tau, z)$$

exists and is independent of z_n. Hence we get a function $\varphi|\Phi$ on $H_{n-1}\times C^{n-1}$. Moreover $\varphi|\Phi$ is a Jacobi form of degree $n-1$ with the same weight and index as φ. This can be shown exactly in the same way as in the case of the Siegel modular form so we omit the proof (See [F] Chap. 1.).

Also it is easy to show that the Eisenstein series behaves nicely under the Φ-operator, namely $E_{k,m}^{(n)}|\Phi=E_{k,m}^{(n-1)}$ (for the proof, see [F] or [Zi]).

It is known that in the case of the Siegel modular form there exists an algebra homomorphism, depending on the weight k,

$$\begin{array}{c}\mathcal{H}(\Gamma_n, \Delta_n) \longrightarrow \mathcal{H}(\Gamma_{n-1}, \Delta_{n-1}) \\ T \longmapsto T^*\end{array} \tag{2}$$

which commutes with the Siegel Φ-operator [Zh]. The homomorphism is given as follows ([F] Chap. 4): Let $\Gamma_n M \Gamma_n$ be an element in $\mathcal{H}(\Gamma_n, \Delta_n)$ and decompose it into left Γ_n-cosets

$$\Gamma_n M \Gamma_n = \bigcup_i \Gamma_n M_i.$$

We may assume that M_i is the form $M_i=\begin{bmatrix} A_i & B_i \\ 0 & D_i \end{bmatrix}$ where D_i is an upper triangular matrix. We set

$$A_i=\begin{bmatrix} A_i^* & 0 \\ * & a_i \end{bmatrix}, \quad B_i=\begin{bmatrix} B_i^* & * \\ * & * \end{bmatrix}, \quad D_i=\begin{bmatrix} D_i^* & * \\ 0 & d_i \end{bmatrix}, \quad M_i^*=\begin{bmatrix} A_i^* & B_i^* \\ 0 & D_i^* \end{bmatrix},$$

where A_i^*, B_i^*, D_i^* are in $M_{n-1}(\mathbf{Z})$. Finally we put

$$(\Gamma_n M \Gamma_n)^* = \sum_i d_i^{-k} \Gamma_{n-1} M_i^*.$$

Then the right hand side is a well-defined element in $\mathcal{H}(\Gamma_{n-1}, \Delta_{n-1})$. The explicit formulas are given in [K]. From our very definition (§1 (4)) of the action of the Hecke algebra on Jacobi forms, it follows that

$$(\varphi|\Phi)|_{k,m} T^* = (\varphi|_{k,m} T)|\Phi \tag{3}$$

for a Jacobi form φ of weight k and index m and for any element T in $\mathcal{H}(\Gamma_n, \Delta_n)$.

Let p be a prime number. Let $\Delta_{n,p}=\{M \in \Delta_n; \nu(M)=p^a$ for some integer $a\}$, and let $\mathcal{H}(\Gamma_n, \Delta_{n,p})$ be the p-part of the Hecke algebra. By an iteration we get a homomorphism

$$\begin{array}{c}\mathcal{H}(\Gamma_n, \Delta_{n,p}) \longrightarrow \mathcal{H}(\Gamma_1, \Delta_{1,p}), \\ T \longmapsto T^{*(n-1)}.\end{array}$$

We consider the images of the generators $T_{0,n}(p^2), \cdots, T_{n,0}(p^2)$ for $\mathcal{H}(\Gamma_n, \Delta_{n,p})$ and put

(4) $$(T_{0,n}(p^2),\ T_{1,n-1}(p^2),\ \cdots,\ T_{n,0}(p^2))^{*(n-1)}=(T_{0,1}(p^2),\ T_{1,0}(p^2))A_n^k,$$

where A_n^k is a $2\times(n+1)$ matrix depending on p and k.

THEOREM 4.1. *Let p be a prime number. Then the action of the Hecke algebra $\mathcal{H}(\Gamma_n, \Delta_{n,p})$ on the Eisenstein series is given by the following formula*

(5) $$\begin{aligned}&E_{k,1}|_{k,1}(T_{0,n}(p^2),\ T_{1,n-1}(p^2),\ \cdots,\ T_{n,0}(p^2))(\tau, z)\\ &=(E_{k,1}(\tau, pz),\ E_{k,p^2}(\tau, z))\begin{bmatrix}p^{-k} & p^{-2k+2}-p^{-k}\\ 0 & p^{-k+1}+1\end{bmatrix}A_n^k,\end{aligned}$$

where A_n^k is given in (4).

PROOF. By Theorem 3.3, we know that $E_{k,1}|_{k,1}T_{\alpha,n-\alpha}(p^2)(\tau, z)$ is a linear combination of $E_{k,1}(\tau, pz)$ and $E_{k,p^2}(\tau, z)$. Hence there is a $2\times(n+1)$ matrix B_n^k such that

$$\begin{aligned}&E_{k,1}|_{k,1}(T_{0,n}(p^2),\ T_{1,n-1}(p^2),\ \cdots,\ T_{n,0}(p^2))(\tau, z)\\ &=(E_{k,1}(\tau, pz),\ E_{k,p^2}(\tau, z))B_n^k.\end{aligned}$$

Applying the Siegel Φ-operator $n-1$ times, we get

$$(E_{k,1}|(T_{0,1}(p^2),\ T_{1,0}(p^2))A_n^k)(\tau_{11}, z_1)=(E_{k,1}(\tau_{11}, pz_1),\ E_{k,p^2}(\tau_{11}, z_1))B_n^k,$$

where τ_{11} is the (1.1) component of τ and z_1 is the first component of z. By Corollary 3.4, we have

$$(E_{k,1}|_{k,1}T_{1,0}(p^2))(\tau_{11}, z_1)=p^{-k}E_{k,1}(\tau_{11}, pz_1)$$

and

$$(E_{k,1}|_{k,1}T_{0,1}(p^2))(\tau_{11}, z_1)=(p^{-2k+2}-p^{-k})E_{k,1}(\tau_{11}, pz_1)+(p^{-k+1}+1)E_{k,p^2}(\tau_{11}, z_1).$$

Therefore we obtain

$$\begin{aligned}&(E_{k,1}(\tau_{11}, pz_1),\ E_{k,p^2}(\tau_{11}, z_1))B_n^k\\ &=(E_{k,1}(\tau_{11}, pz_1),\ E_{k,p^2}(\tau_{11}, z_1))\begin{bmatrix}p^{-k} & p^{-2k+2}-p^{-k}\\ 0 & p^{-k+1}+1\end{bmatrix}A_n^k.\end{aligned}$$

Comparing the Fourier expansions, it is easy to see that $E_{k,1}(\tau_{11}, pz_1)$ and $E_{k,p^2}(\tau_{11}, z_1)$ are linearly independent over C ([E-Z]), hence we have

$$B_n^k=\begin{bmatrix}p^{-k} & p^{-2k+2}-p^{-k}\\ 0 & p^{-k+1}+1\end{bmatrix}A_n^k.$$

Q.E.D.

5. Maass relation

We expand the Siegel-Eisenstein series $E_k^{(n+1)}(\tau')$ of degree $n+1$ into the Fourier-Jacobi series

$$E_k^{(n+1)}(\tau')=\sum_{0\leq m} e_{k,m}(\tau, z)e(mt),$$

where $\tau'=\begin{bmatrix} \tau & {}^t z \\ z & t \end{bmatrix}$ is a general element of H_{n+1}. The Fourier-Jacobi coefficients $e_{k,m}$ and the Eisenstein series $E_{k,m}$ are related by the following

THEOREM 5.1 ([B]). *For any $m>0$, we have*

$$e_{k,m}(\tau, z)=\sum_{d^2|m, d>0} \sigma_{k-1}(md^{-2}) \sum_{a|d, a>0} \mu(a)E_{k,ma^2/d^2}(\tau, da^{-1}z),$$

in which μ is the Möbius function and $\sigma_{k-1}(a)=\sum_{d|a,d>0} d^{k-1}$.

Now we can state our main theorem.

THEOREM 5.2 (Maass relation). *The Fourier-Jacobi coefficients $e_{k,m}(\tau, z)$ of the Siegel-Eisenstein series satisfy the following relations.*
(M1) *For an element T in the Hecke algebra $\mathcal{H}(\Gamma_n, \Delta_n)$ for Γ_n such that $T^{*(n-1)}=0$, we have*

$$e_{k,m}|_{k,m}T=0.$$

(M2) *Let T be an element in $\mathcal{H}(\Gamma_n, \Delta_n)$ such that $T^{*(n-1)}=T_1(m)$, where $T_1(m)\in\mathcal{H}(\Gamma_1, \Delta_1)$ is given by*

$$T_1(m)=\sum_{ad=m, a|d} \Gamma_1 \operatorname{diag}[a, d]\Gamma_1.$$

Then we have

$$e_{k,m}=e_{k,1}|_{k,1}T.$$

PROOF. We know that $e_{k,m}$ is an image of $e_{k,1}$ by a certain element in $\mathcal{H}(\Gamma_n, \Delta_n)$ (Theorem 5.7 [Y]). Hence (M1) follows immediately from Theorem 4.1 and the commutativity of the Hecke algebra. For (M2) we remark that if we apply the homomorphism $*(n-1)$ to

$$1-\prod_{1<i\leq n}(1+p^{k-i})^{-1}T_n(p)p^{-s}+T_{0,n}(p^2)p^{(1-n)k+n(n+1)-1-2s},$$

we obtain

$$1-T_1(p)p^{-s}+T_{0,1}(p^2)p^{1-2s}$$

up to the normalization factors (see the remark at the end of section one). Now (M2) is obvious form (M1) and the well known structure theorem for the Hecke algebra of Γ_1. Q.E.D.

REMARK. When $n=1$ the relation (M1) is vacant and (M2) is the usual Maass relation for the Siegel-Eisenstein series of degree two ([M]).

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(Received November 25, 1988)

SFB 170 Mathematisches Institut
der Universität
3400 Göttingen
BRD
and
Department of Mathematics
Kyushu University
Fukuoka
812 Japan