J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
36 (1989), 19-41.

Propagation of micro-analyticities of solutions to some class of linear differential equations with non-involutive double characteristics

By Kenji HASEGAWA

(Communicated by H. Komatsu)

0. Introduction

The purpose of this paper is to study on propagation of micro-analyticities of solutions to some class of linear differential equations which have doubly characteristic points on non-involutive submanifolds.

In studying our problem, we apply the theory of second microlocalization. Kashiwara and Laurent [4] formulated second microfunctions and second microdifferential operators along involutive submanifolds, and Laurent [5] investigated the algebraic properties of second microdifferential operators. According to [4], estimation of supports of second microfunctions is useful for the study on propagation of micro-analyticities of hyperfunctions along isotropic leaves of involutive submanifolds. Tose [11]-[14] proved that some class of microdifferential operators which have doubly characteristic points on involutive submanifolds is equivalent to the de Rham operator if they are considered as second microdifferential operators. Assuming that the microdifferential operator P belongs to this class, he extracted some results on propagation of micro-analyticities of solutions to the equation $Pu=0$ from the structure of second microfunctions induced from u.

On the other hand, Lebeau [7] presented the theory of second microlocalization along isotropic submanifolds in the fashion of Sjöstrand [9]. He gave the definition of second microsupports along isotropic submanifolds and proved the watermelon cut theorem. We shall estimate second microsupports defined in [7] of solutions to our equations to investigate propagation of their micro-analyticities.

For k, l satisfying $k+l<n$, we set

$$\boldsymbol{R}_x^n=\boldsymbol{R}_{x^*}^k\times\boldsymbol{R}_{x'}^l\times\boldsymbol{R}_{x''}^{n-k-l},$$

with $x=(x_1,\cdots,x_n)$, $x^*=(x_1,\cdots,x_k)$, $x'=(x_{k+1},\cdots,x_{k+l})$, $x''=(x_{k+l+1},\cdots,x_n)$.

We denote by $\xi=(\xi^*, \xi', \xi'')$ the dual coordinates of $x=(x^*, x', x'')$. We put

$$\Lambda=\{(x, \xi)\in T^*\boldsymbol{R}^n \mid \xi^*=x'=\xi'=0\}. \tag{0.1}$$

We remark that Λ is not an involutive submanifolds of $T^*\boldsymbol{R}^n$ if $l\neq 0$. Let Ω be an open subset of $\boldsymbol{R}^n$ which meets the set $\{(x^*, x', x'') \mid x'=0\}$ and let $P(x, D_x)$ be a linear differential operator of order m with real analytic coefficients defined on Ω, where $D_x=-i\partial/\partial x$. We denote by p_m the principal symbol of P. We fix a point $\rho_0=(x_0^*, 0, x_0''; 0, 0, \xi_0'')$ satisfying $(x_0^*, 0, x_0'')\in\Omega$ and $\xi_0''\neq 0$. Then we put

$$\Gamma=\{(x, \xi)\in T^*\boldsymbol{R}^n \mid x'=0,\ x''=x_0'',\ \xi^*=0,\ \xi'=0,\ \xi''=\xi_0''\}. \tag{0.2}$$

We assume that the differential operator P satisfies the following hypotheses:

(H.1) *$p_m=dp_m=0$ on Λ.*
(H.2) *$\partial^2 p_m/\partial\xi^*\partial x'=\partial^2 p_m/\partial\xi^*\partial\xi'=0$ on Γ.*
(H.3) *The matrix $(\partial^2 p_m/\partial\xi^{*2})$ has one positive eigenvalue and $k-1$ negative eigenvalues on Γ.*
(H.4) *The matrix*

$$\begin{pmatrix} \partial^2 p_m/\partial x'^2 & \partial^2 p_m/\partial x'\partial\xi' \\ \partial^2 p_m/\partial\xi'\partial x' & \partial^2 p_m/\partial\xi'^2 \end{pmatrix}$$

is positive definite on Γ.

For example, if $\rho_0=(0; 0, \cdots, 0, 1)$ and p_m has the form

$$p_m=a(x, \xi)\left(\xi_1^2-\sum_{j=2}^{k}\xi_j^2+\xi'^2+x'^2\xi_n^2\right),$$

with $a\neq 0$, then p_m satisfies (H.1)-(H.4).

We regard x^* as the coordinates of Γ and denote by $(x^*, \tilde{\xi}^*)$ those of $T^*\Gamma$. To state our theorem, we introduce the function $q(x^*, \tilde{\xi}^*)$ on $T^*\Gamma$ as follows:

$$q(x^*, \tilde{\xi}^*)=\sum_{1\leq i, j\leq k}\tilde{\xi}_i\tilde{\xi}_j\partial_{\xi_i}\partial_{\xi_j}p_m(x^*, 0, x_0''; 0, 0, \xi_0'')/2. \tag{0.3}$$

For a C^1-function f on $T^*\Gamma$, we define the so-called Hamilton vector field H_f as follows:

$$H_f=\sum_{j=i}^{k}\frac{\partial f}{\partial\tilde{\xi}_j}\frac{\partial}{\partial x_j}-\frac{\partial f}{\partial x_j}\frac{\partial}{\partial\tilde{\xi}_j}.$$

It follows from (H.3) that H_q is not vanishing everywhere on $q^{-1}(0)$. For

$(x^*, \xi^*) \in T^*\Gamma$, let $\gamma(x^*, \xi^*)(s)$ be the integral curve of H_q with a parameter $s \in \boldsymbol{R}$ satisfying

$$\frac{d\gamma(x^*, \xi^*)}{ds} = H_q(\gamma(x^*, \xi^*)), \quad \gamma(x^*, \xi^*)(0) = (x^*, \xi^*).$$

Then we set $C^+(x^*, \xi^*) = \{\gamma(x^*, \xi^*)(s)\}_{s \geq 0}$.

Now we fix a vector $\vartheta \in T^*_{x_0^*}\Gamma$ satisfying $q(x_0^*, \vartheta) > 0$, we denote by V the connected component of the set $\{\xi^* \in \boldsymbol{R}^k |\, q(x_0^*, \xi^*) > 0\}$ which contains ϑ and denote by A the boundary of V. It follows from (H.3) that $A \setminus \{0\}$ is a smooth hypersurface. Moreover we denote by S the projection to Γ of the union $\bigcup_{\xi^* \in A} C^+(x_0^*, \xi^*)$. We remark that making a suitable coordinate transformation of Γ near x_0^*, we have $\vartheta = (0, \cdots, 0, 1)$, $x_0^* = 0$, and

$$q = \xi_1^2 - \sum_{2 \leq i, j \leq k} \xi_i \xi_j a_{ij}(x^*), \tag{0.4}$$

on some neighborhood of x_0^*, where the matrix (a_{ij}) is positive definite. This implies that $S \setminus \{x_0^*\}$ is a smooth hypersurface in a neighborhood of x_0^*. We denote by $\tilde{\omega}$ this neighborhood. Moreover there exists a connected component of $\tilde{\omega} \setminus S$ which is contained in the semi-space $\{x^* |\, (x^* - x_0^*, \vartheta) > 0\}$. We denote by W this component.

We denote by π the projection of $T^*\Gamma$ to Γ. Our main result is the following theorem:

THEOREM 0.1. *Let ω be a neighborhood of x_0^* in Γ such that $\omega \subset \tilde{\omega}$. Assume that a distribution $u(x) \in \mathcal{D}'(\Omega)$ satisfies*

(H.5) $(WF_a(Pu) \cap \Gamma) \cap (\bigcup_{\xi^* \in A} \pi(C^+(x_0^*, \xi^*)) \cup \omega) = \emptyset$.

(H.6) $(WF_a(u) \cap \Gamma) \cap W \cap \omega = \emptyset$.

(H.7) *For each $\xi^* \in A$, there exists $s \geqq 0$ such that $\pi(\gamma(x_0^*, \xi^*)(s)) \notin WF_a(u)$.*

Then we have $\rho_0 \notin WF_a(u)$.

Here $WF_a(u)$ means the analytic wave front set of u (see [9]).

We remark that in the case that $l = 0$, we can replace (H.6) by the following: *the closure of the complement of $WF_a(u)$ contains ρ_0.* Here we apply the argument of Tose [11]-[14]. This hypothesis is weaker than (H.6).

The outline of the proof of Theorem 0.1 is as follows: We shall investigate the second microsupports of $u(x)$. Using an argument similar to that in the microhyperbolic case due to Sjöstrand [9], we can verify the propagation of second micro-analyticities of $u(x)$ along the integral curves of H_q. Then from (H.5) and (H.7), we can obtain a sharp estimate of second microsupports of $u(x)$ near ρ_0, which yields the conclusion under (H.6) in view of the microlocal Holmgren theorem.

REMARK 1. Because P is not microhyperbolic with respect to conormals of S, we cannot apply the results of Kashiwara-Kawai [3] or Sjöstrand [9] under the hypothesis (H.6).

REMARK 2. If the principal symbol of a differential operator P satisfies (H.1)-(H.4) after a suitable contact transformation, then Theorem 0.1 holds for P. But the author does not know how we interpret (H.2) in terms of the symplectic geometry.

The result in this paper in the case that P has a special form is announced in [2].

The author would like to express his hearty gratitude to Prof. A. Kaneko for his uncreasing encouragement and to the referee for his useful advices.

1. Second microsupports of distributions

We review the theory of second microlocalization along isotropic submanifolds due to Lebeau [7].

We recall the Fourier-Bros-Iagolitzer (hereafter abbreviated as F.B.I.) transformation, which is denoted by T. For $u\in\mathcal{E}'(\boldsymbol{R}^n)$, we define Tu as

$$Tu(z, \lambda)=\int e^{-\lambda((z-x)^2/2)-i\lambda x\xi_0}u(x)dx\,, \tag{1.1}$$

for $(z, \lambda)\in\boldsymbol{C}^n\times\boldsymbol{R}^+$, where ξ_0 is the ξ coordinate of ρ_0 which is fixed in previous section. Then $Tu\in H^{loc}_{|\mathrm{Im}\, z|^2/2}$, where H^{loc}_φ with a continuous function $\varphi(z)$ follows [9]. According to [9] we can see that $(x, \xi)\in WF_a(u)$ if and only if $Tu\sim 0$ at $(x-i(\xi-\xi_0), \xi-\xi_0)$ in the sense of $H^{loc}_{|\mathrm{Im}\, z|^2/2}$.

Next we recall the second F.B.I. transformation for $H^{loc}_{|\mathrm{Im}\, z|^2/2}$. We put $z^*=(z_1, \cdots, z_k)$, $z'=(z_{k+1}, \cdots, z_{k+l})$, $z''=(z_{k+l+1}, \cdots, z_n)$. Let ω be an open subset of $\boldsymbol{C}^n$. For $z_0^*\in\boldsymbol{C}^k$, we define the operator $\delta_{z_0^*}$ for $H^{loc}_{|\mathrm{Im}\, z|^2/2}(\omega)$ as follows:

$$\delta_{z_0^*}(z, \lambda, \mu)=\int_{\Sigma_{z_0^*,\mu}} e^{-\lambda\rho(z^*-w^*)^2/2}U(w^*, \mu z', \mu z'', \lambda)dw^*\,, \tag{1.2}$$

for $U\in H^{loc}_{|\mathrm{Im}\, z|^2/2}(\omega)$, where $\rho=\mu^2/1-\mu^2$ and the contour $\Sigma_{z_0^*,\mu}$ is defined by

$$w^*=\mathrm{Re}\, z_0^*+t^*+i\mu^2\,\mathrm{Im}\, z_0^*\,, \tag{1.3}$$

with $t^* \in \boldsymbol{R}^k$, $|t^*| \leqq r$. If we choose r small enough so that $\Sigma_{z_0^*, \mu} \subset \omega$, then (1.3) is well-defined.

For $z^* \in \boldsymbol{C}^k$, $C>0$, $\mu>0$, we set

$$D(z^*, C, \mu) = \{w \in \boldsymbol{C}^n \mid |\mathrm{Re}\, z^* - \mathrm{Re}\, w^*| \leqq C, |\mathrm{Im}\, z^* - \mathrm{Im}\, w^*| \leqq C/\mu,$$

$$|z'| \leqq C/\mu, |z''| \leqq C/\mu\}.$$

If we take C small enough, then we can verify by the Stokes' formula that there exists $\varepsilon>0$ such that

$$(1.4) \qquad |\delta_{z^*}(U)(z, \lambda, \mu) - \delta_{w^*}(U)(z, \lambda, \mu)| \leqq \frac{1}{\varepsilon} e^{\lambda \mu^2(-\varepsilon + |\mathrm{Im}\, z|^2/2)} \sup_{w \in \omega} |U(w, \lambda) e^{-\lambda |\mathrm{Im}\, w|^2/2}|,$$

$$(1.5) \qquad |\delta_{z^*}(U)(z, \lambda, \mu)| \leqq \frac{1}{\varepsilon} e^{\lambda \mu^2 |\mathrm{Im}\, z|^2/2} \sup_{w \in \omega} |U(w, \lambda) e^{-\lambda |\mathrm{Im}\, w|^2/2}|,$$

if (z^*, z', z''), $(w^*, z', z'') \in D(z_0^*, C, \mu)$ for $z_0^* \in \boldsymbol{C}^k$ satisfying $\mathrm{Re}\, z_0^* \in \omega$. We call δ_{z^*} the second F.B.I. transformation.

We shall introduce function spaces which contain the image of δ_{z^*}. Let $\hat{\omega}$ be an open subset of $\boldsymbol{C}^n$ and $\psi(z)$ be a continuous functions on $\hat{\omega}$. We denote by $\mathrm{H}^2_\psi(\hat{\omega})$ the space consisting of all functions $u(z, \lambda, \mu)$ that are defined on $\hat{\omega} \times \boldsymbol{R}^+ \times (0, \mu_0)$ for some $\mu_0>0$, holomorphic with respect to z, and satisfy the following: for any $\varepsilon>0$ and any compact subset K of $\hat{\omega}$, there exists a function $\lambda(\mu)$ such that

$$(1.6) \qquad |u(z, \lambda, \mu)| \leqq e^{\lambda \mu^2(\psi(z)+\varepsilon)},$$

for $z \in K$, $\lambda \geqq \lambda(\mu)$. We give an equivalence relation $\sim$ to $\mathrm{H}^2_\psi(\hat{\omega})$ as follows: for $u, v \in \mathrm{H}^2_\psi(\hat{\omega})$, we write $u \sim v$ if for any compact subset $K \subset \hat{\omega}$, there exist $\varepsilon>0$ and $\lambda(\mu)$ such that

$$(1.7) \qquad |u(z, \lambda, \mu) - v(z, \lambda, \mu)| \leqq e^{\lambda \mu^2(\psi(z)-\varepsilon)} \qquad \text{for } z \in K, \ \lambda \geqq \lambda(\mu).$$

We fix $z_0^* \in C^k$, $\mu_0>0$, and $C>0$ such that $D(z_0^*, C, \mu_0) \subset \omega$. It follows from (1.4), (1.5) that $\delta_{z^*}(U) \in \mathrm{H}^2_{|\mathrm{Im}\, z|^2/2}(D(z_0^*, C, \mu_0))$ and $\delta_{z^*}(U) \sim \delta_{w^*}(U)$ for $(z^*, 0), (w^*, 0) \in D(z_0^*, C, \mu_0)$. Using these spaces, we shall define the *second microsupports* for distributions, which are denoted by SS^2_Γ.

DEFINITION 1.1. Let $u(x)$ be a distribution on some open subset Ω of $\boldsymbol{R}^n$ which meets the set $\{x \in \boldsymbol{R}^n \mid x'=0, x''=x_0''\}$. For $(y, \tilde{\eta}^*) \in T^*\Gamma$, we write $(y^*, \tilde{\eta}^*) \in \mathrm{SS}^2_\Gamma(u)$ if $\delta_{(y^*-i\tilde{\eta}^*)}(T(\chi u)) \sim 0$ in a neighborhood of $(y^*-i\tilde{\eta}^*, 0)$ in $\boldsymbol{C}^n$, where $\chi(x)$ is an element of $C_0^\infty(\Omega)$ such that $\chi(x) \equiv 1$ near $(y^*, 0, x_0'')$.

$\mathrm{SS}^2_\Gamma(u)$ is a closed set of $T^*\Gamma$ and conic in the sense that it is invariant under multiplication to $\tilde{\xi}^*$ by positive numbers. We remark that (1.5) implies that if $(y^*, 0, x_0''; 0, 0, \xi_0'') \notin WF_a(u)$, then $(y^*, \tilde{\eta}^*) \notin SS_\Gamma^2(u)$ for any $\tilde{\eta}^* \in \boldsymbol{R}^k$.

The following proposition is a collorary of Theorémè 4.4 in [7]. It gives a relation between WF_a and SS_Γ^2. We call it the *microlocal Holmgren theorem.*

PROPOSITION 1.2. *Let y^* be a point of Γ and let Ω be an open subset of $\boldsymbol{R}^n$ containing $(y^*, 0, x_0''; 0, 0, \xi_0'')$. We take a real valued analytic function $\phi(x^*)$ on Γ such that $\phi(y^*)=0$ and that $d\phi(y^*) \neq 0$. If $u \in \mathcal{D}'(\Omega)$ satisfies*

$$(WF_a(u) \cap \Gamma) \cap \{x^* \in \Gamma \mid \phi(x^*) > 0\} = \emptyset ,$$

and

$$(y^*, 0, x_0''; 0, 0, \xi_0'') \in WF_a(u) ,$$

then we have $(y^*, \pm d\phi(y^*)) \in SS_\Gamma^2(u)$.

Therefore the estimation of the second microsupports of distributions gives an information for propagation of their micro-analyticities along Γ.

2. Symbolic calculus

In this section, we shall give a method of symbolic calculus to investigate the second microsupports of solutions to linear differential equations. We recall the definition of classical analytic symbol in [9]. Let Ω be an open subset of $\boldsymbol{C}^n$ and let $\varphi(z)$ be an element of $C^{1,1}(\Omega)$, where $C^{1,1}(\Omega)$ is the set consisting of all $\varphi(z) \in C^1(\Omega)$ such that $\partial_z \varphi(z)$ is Lipschtz continuous. For a neighborhood V of $\Lambda_\varphi = \{(z, \zeta) \in \boldsymbol{C}^n \times \boldsymbol{C}^n \mid \zeta = -2i\partial_z \varphi(z)\}$ in $\boldsymbol{C}^n \times \boldsymbol{C}^n$, we take a sequence $\{a_k(z, \zeta)\}_{k=0}^\infty$ such that a_k is a holomorphic function on V. Suppose that for any compact subset K of V, there exists $C>0$ such that

$$|a_k(z, \zeta)| \leqq C^{k+1} k! \qquad \text{for } (z, \zeta) \in K . \tag{2.1}$$

We define a symbol $a(z, \zeta, \lambda)$ on K, which is an asymptotic sum of $\{a_k(z, \zeta)\}_{k=0}^\infty$, as follows:

$$a(z, \zeta, \lambda) = \sum_{0 \leq k \leq \lambda/eC} \lambda^{-k} a_k(z, \zeta) . \tag{2.2}$$

For convenience, we write (2.2) formally as

$$a(z, \zeta, \lambda) \sim \sum_{k=0}^\infty \lambda^{-k} a_k(z, \zeta).$$

We call $a(z,\zeta,\lambda)$ the classical analytic symbol. For $a(z,\zeta,\lambda)$, we define the pseudo-differential operator $\Phi_\varphi(a)$ on $\mathrm{H}^{\mathrm{loc}}_\varphi(\Omega)$. Let $\tilde{\Omega}$ be an open subset of $\boldsymbol{C}^n$ such that $\tilde{\Omega}\Subset\Omega$. For $z\in\tilde{\Omega}$, we put

$$\Gamma_\varphi(z)=\{(w,\zeta)\in\boldsymbol{C}^n\times\boldsymbol{C}^n \mid \zeta=-2i\partial_z\varphi(z)+iT(\overline{z-w}),\ |z-w|\leqq r\},\tag{2.3}$$

where T is large enough so that there exists $C>0$ such that

$$e^{-\lambda(\varphi(z)-\varphi(w))}|e^{i\lambda(z-w)\eta}|\leqq e^{-\lambda|z-w|^2/C},\tag{2.4}$$

for $(w,\zeta)\in\Gamma_\varphi(z)$ and r is small enough so that $\Gamma_\varphi(z)\subset\Omega$ for $z\in\tilde{\Omega}$. $\Phi_\varphi(a)$ is defined as (2.16) in [10] with $P=a$. We put

$$\boldsymbol{C}^n_z=\boldsymbol{C}^k_{z^*}\times\boldsymbol{C}^l_{z'}\times\boldsymbol{C}^{n-k-l}_{z''}.$$

For $z_0\in\boldsymbol{C}^n$, $C>0$, $\mu>0$, we set

$$W(z_0,C,\mu)=\{z\in\boldsymbol{C}^n \mid |\mathrm{Re}\,z^*-\mathrm{Re}\,z_0^*|\leqq C,\ \mathrm{Im}\,z^*=\mu^2\,\mathrm{Im}\,z_0^*,$$
$$z'=\mu z_0',\ z''=\mu z_0''\}.$$

If $W(z_0^*,C,\mu)\subset\Omega$, then we can apply the second F.B.I. transformation $\delta_{z_0^*}$ to $U\in\mathrm{H}^{\mathrm{loc}}_{|\mathrm{Im}\,z|^2/2}(\Omega)$. We put $\delta(U)(z,\lambda,\mu)=\delta_{z^*}(z,\lambda,\mu)$. Let ω, $\tilde{\omega}$ be open subsets of $\boldsymbol{C}^n$ satisfying the following: $\tilde{\omega}\Subset\omega$ and there exist $C>0$ and $\mu_0>0$ such that $\bigcup_{z\in\omega}W(z,C,\mu)\Subset\tilde{\Omega}$ for $0\leqq\mu\leqq\mu_0$. Then there exists $\varepsilon>0$ such that the point

$$(z^*+2(1-\mu^2)\partial_{z^*}\varphi(z),\mu z',\mu z''\,;\,-2i\mu^2\partial_{z^*}\varphi(z),\,-2i\mu\partial_{z'}\varphi(z),\,-2i\mu\partial_{z''}\varphi(z)),$$

is contained in the domain of the symbol a if $z\in\omega$, $0\leqq\mu\leqq\mu_0$, and $\|\varphi-(|\mathrm{Im}\,z|^2/2)\|_{C^{1,1}(\omega)}<\varepsilon$, where we put

$$\|\phi\|_{C^{1,1}(\omega)}=\|\phi\|_{C^1(\omega)}+\sup_{z,w\in\omega}|\partial_z\phi(z)-\partial_w\phi(w)|/|z-w|,$$

for $\phi\in C^{1,1}(\omega)$. Assuming that

$$\|\varphi-(|\mathrm{Im}\,z|^2/2)\|_{C^{1,1}(\omega)}\leqq\min(\varepsilon,1/100n),\tag{2.5}$$

we define $\Psi_\varphi(a)(U)$ for $U\in\mathrm{H}^2_{|\mathrm{Im}\,z|^2/2}(\omega)$ and a symbol $a(z,\zeta,\lambda)$ as follows:

$$\Psi_\varphi(a)(U)(z,\lambda,\mu)=\Big(\frac{\lambda\mu^2}{2\pi}\Big)^{n+k}\Big(\frac{-i\rho}{\mu^4}\Big)^k\int_{\Delta_\mu(z)}\exp\Big\{-\frac{\lambda\rho}{2}(z^*-u^*)^2$$
$$+i\lambda(u^*-v^*)\zeta^*+\frac{\lambda\rho}{2}(v^*-w^*)^2+i\lambda\mu^2(z'-w')\zeta'+i\lambda\mu^2(z''-w'')\zeta''\Big\}$$
$$\times a(u^*,\mu z',\mu z''\,;\,\zeta^*,\mu\zeta',\mu\zeta''\,;\lambda)U(w,\lambda,\mu)du^*\wedge dv^*\wedge d\zeta\wedge dw,\tag{2.6}$$

for $(z, \lambda, \mu) \in \tilde{\omega} \times \boldsymbol{R}^+ \times (0, \mu_0)$, where $\rho = \mu^2/1-\mu^2$ and the contour $\varDelta_\mu(z)$ is defined by

$$(2.7) \qquad \begin{cases} u^* = z^* + 2(1-\mu^2)\partial_{z^*}\varphi(z) + \dfrac{1+\mu^2}{4}\alpha^* + \dfrac{3-\mu^2}{4}\overline{\alpha^*}, \\ v^* = u^* + x^*, \quad \zeta^* = -2i\mu^2\partial_{z^*}\varphi(z) - i\mu^2 T\overline{x^*}, \\ w^* = z^* + x^* + \alpha^*, \\ w' = z' + x', \quad \zeta' = -2i\partial_{z'}\varphi(z) - iT\overline{x'}, \\ w'' = z'' + x'', \quad \zeta'' = -2i\partial_{z''}\varphi(z) - iT\overline{x''}, \end{cases}$$

with $x \in \boldsymbol{C}^n$, $\alpha^* \in \boldsymbol{C}^k$ and $|\alpha^*|$, $|x^*|$, $|x'|$, $|x''| \leqq r$, where T is large enough so that

$$(2.8) \qquad \begin{aligned} & -\lambda\mu^2\varphi(z) + \mathrm{Re}\left\{ -\frac{\lambda\rho}{2}(z^*-u^*)^2 + i\lambda(u^*-v^*)\zeta^* + \frac{\lambda\rho}{2}(v^*-w^*)^2 \right. \\ & \qquad \left. + i\lambda\mu^2(z'-w')\zeta' + i\lambda\mu^2(z''-w'')\zeta'' \right\} + \lambda\mu^2\varphi(w) \\ & \leqq -\lambda\mu^2(|x|^2 + |\alpha^*|^2)/10, \end{aligned}$$

if $(u^*, v^*, w, \zeta) \in \Delta_\mu(z)$, and r is small enough so that if $z \in \tilde{\omega}$ and $(u^*, v^*, w, \zeta) \in \varDelta_\mu(z)$, then $(u^*, \mu z', \mu z''; \zeta^*, \mu\zeta', \mu\zeta'')$ is contained in the domain of a and $w \in \omega$. It is possible to take T satisfying (2.8) under the hypothesis (2.5). After these preparations, we obtain the following:

PROPOSITION 2.1. *Let $a(z, \zeta, \lambda)$ be a classical analytic symbol defined on a neighborhood of the set $\{(z, \zeta) \in \boldsymbol{C}^n \times \boldsymbol{C}^n \mid z \in \Omega,\ \zeta = -2i\partial_z\varphi(z)\}$. Then there exist $C>0$ and $\varepsilon>0$ such that*

$$(2.9) \qquad \begin{aligned} & |\Psi_{|\mathrm{Im}\,z|^2/2}(a)(\delta(U))(z, \lambda, \mu) - \delta(\Phi_{|\mathrm{Im}\,z|^2/2}(a)(U))(z, \lambda, \mu)| \\ & \leqq C\mu^{-k}\exp\{\lambda\mu^2(|\mathrm{Im}\,z|^2-\varepsilon)/2\} \sup_{w\in\Omega} |e^{-\lambda|\mathrm{Im}\,w|^2/2}U(w, \lambda)|, \end{aligned}$$

for $U \in \mathrm{H}^{\mathrm{loc}}_{|\mathrm{Im}\,z|^2/2}(\Omega)$, $z \in \tilde{\omega}$.

PROOF. For convenience, we put

$$I(\varepsilon, z, \lambda, \mu) = \exp\{\lambda\mu^2(|\mathrm{Im}\,z|^2-\varepsilon)/2\} \sup_{w\in\Omega} |e^{-\lambda|\mathrm{Im}\,w|^2/2}U(w, \lambda)|.$$

For a contour $\Gamma(z)$, we set

$$A_{\Gamma}(z, \lambda, \mu)(U)(z, \lambda, \mu)=\left(\frac{\lambda}{2\pi}\right)^{n}\int_{\Gamma(z^*, \mu z', \mu z'')} \exp\left\{-\frac{\lambda\rho}{2}(z^*-u^*)^2 + i\lambda(u^*-v^*)\zeta^* + i\lambda(\mu z'-v')\zeta' + i\lambda(\mu z''-v'')\zeta''\right\} \times a(u^*, \mu z', \mu z'', \zeta)\, U(v, \lambda, \mu)\, du^* \wedge dv \wedge d\zeta .$$

Then recalling the contours of δ and $\Phi_{|\mathrm{Im}\, z|^2/2}$, we have $\delta \circ \Phi_{|\mathrm{Im}\, z|^2/2} = A_{\tilde{\Sigma}_\mu}$, where $\tilde{\Sigma}_\mu(z)$ is defined by

$$(2.10)\qquad \begin{cases} u^* = z^* - i(1-\mu^2)\,\mathrm{Im}\, z^* + t^* , \\ v^* = u^* + x^*, \quad \zeta^* = -\mu^2\,\mathrm{Im}\, z^* - iT\overline{x^*} , \\ v' = z' + x', \quad \zeta' = -\mathrm{Im}\, z' - iT\overline{x'} , \\ v'' = z'' + x'', \quad \zeta'' = -\mathrm{Im}\, z'' - iT\overline{x''} , \end{cases}$$

with $t^* \in \boldsymbol{R}^k$, $x \in \boldsymbol{C}^n$, $|t^*|$, $|x| \leqq r$, $T>0$, and r is small enough.

We make a deformation of the contour $\tilde{\Sigma}_\mu$. For $a>0$, let $\tilde{\Sigma}_{\mu,a}(z)$ denote the following contour:

$$(2.11)\qquad \begin{cases} v^* = u^* + ax^*, \quad \zeta^* = -\mu^2\,\mathrm{Im}\, z^* - iTa\overline{x^*} , \\ v' = z' + ax', \quad \zeta' = -\mathrm{Im}\, z' - iTa\overline{x'} , \\ v'' = z'' + ax'', \quad \zeta'' = -\mathrm{Im}\, z'' - iTa\overline{x''} , \end{cases}$$

and we leave the argument u^* as in (2.10). Then we can see that $\tilde{\Sigma}_{\mu,1} = \tilde{\Sigma}_\mu$. Taking T large enough, we obtain

$$(2.12)\qquad -\frac{\lambda\mu^2}{2}|\mathrm{Im}\, z|^2 + \mathrm{Re}\left\{-\frac{\lambda\rho}{2}(z^*-u^*)^2 + i\lambda(u^*-v^*)\zeta^* + i\lambda(\mu z'-v')\zeta' + i\lambda(\mu z''-v'')\zeta''\right\} + \frac{\lambda}{2}|\mathrm{Im}\, z|^2 \leqq -\frac{\lambda\mu^2}{2}(|t^*|^2+|x|^2) ,$$

for $(u^*, v, \zeta) \in \tilde{\Sigma}_{\mu,a}(z^*, \mu z', \mu z'')$ with $\mu \leqq a \leqq 1$. In view of the Stokes' formula, we can see that (2.12) implies that there exist $C>0$ and $\varepsilon>0$ such that

$$(2.13)\qquad |\delta(\Phi_{|\mathrm{Im}\, z|^2/2}(a)(U))(z, \lambda, \mu) - A_{\tilde{\Sigma}_{\mu,\mu}}(U)(z, \lambda, \mu)| \leqq CI(\varepsilon, z, \lambda, \mu) ,$$

for $\lambda, \mu>0$, $z \in \omega$.

We recall an approximate left inverse of δ, which is constructed in [7]. Let Ω_0, Ω_1 be open subsets of $\boldsymbol{C}^n$ satisfying $\tilde{\Omega} \Subset \Omega_1 \Subset \Omega_0 \Subset \Omega$. For Ω_0, we put

$$C(\Omega_0)=\{z\in \boldsymbol{C}^n \mid (\operatorname{Re} z^*, 0, 0)\in\Omega_0\}.$$

Then $\delta(U)\in \mathrm{H}^2_{|\operatorname{Im} z|^2/2}(C(\Omega_0))$ for $U\in \mathrm{H}^{\mathrm{loc}}_{|\operatorname{Im} z|^2/2}(\Omega)$. For this $\delta(U)$, we set

$$K(\delta(U))(z, \lambda, \mu)=\left(\frac{i\lambda\rho}{2\pi}\right)^k\int_{\Sigma'_\mu(z)} e^{(\lambda\rho/2)(z^*-w^*)^2}\delta(U)(w^*, z'/\mu, z''/\mu)dw^*, \tag{2.14}$$

where the contour $\Sigma'_\mu(z)$ is defined by

$$w^*=z^*+i(1-\mu^2/\mu^2)\operatorname{Im} z^*+is^*/\mu, \tag{2.15}$$

with $s^*\in\boldsymbol{R}^k$, $|s^*|\leqq r$. Taking r small enough, we have $K(\delta(U))\in \mathrm{H}^2_{|\operatorname{Im} z|^2/2}(\Omega_1)$. From Lemma (3.28) in [7], we have

LEMMA 2.2. *There exist $C>0$ and $\varepsilon>0$ such that*

$$\begin{aligned}&|K(\delta(U))(z, \lambda, \mu)-U(z, \lambda, \mu)|\\ &\leqq C\mu^{-k}\exp\left\{\frac{\lambda}{2}(|\operatorname{Im} z|^2-\varepsilon\mu^2)\right\}\sup_{w\in\Omega}|e^{-\lambda|\operatorname{Im} w|^2/2}U(w, \lambda)|,\end{aligned} \tag{2.16}$$

for $U\in \mathrm{H}^{\mathrm{loc}}_{|\operatorname{Im} z|^2/2}(\Omega)$, $z\in\Omega_1$.

It follows from Lemma 2.2 that it is sufficient to prove that there exist $C>0$, $\varepsilon>0$ such that

$$|A_{\tilde{\Sigma}_{\mu}\,\mu}(K(\delta(U)))(z, \lambda, \mu)-\Psi_{|\operatorname{Im} z|^2/2}(\delta(U))(z, \lambda, \mu)|\leqq CI(\varepsilon, z, \lambda, \mu). \tag{2.17}$$

Composing $\tilde{\Sigma}_{\mu,\mu}$ and Σ'_μ, we can express $A_{\tilde{\Sigma}_{\mu,\mu}}(K(\delta(U)))$ as

$$\begin{aligned}&A_{\tilde{\Sigma}_{\mu}\,\mu}(K(\delta(U)))(z, \lambda, \mu)\\ &=\left(\frac{\lambda\mu^2}{2\pi}\right)^{n+k}\left(\frac{-i\rho}{\mu^4}\right)^k\int_{\tilde{\Delta}_\mu(z)}\exp\left\{-\frac{\lambda\rho}{2}(z^*-u^*)^2+i\lambda(u^*-v^*)\zeta^*\right.\\ &\quad\left.+\frac{\lambda\rho}{2}(v^*-w^*)^2+i\lambda\mu^2(z'-w')\zeta'+i\lambda\mu^2(z''-w'')\zeta''\right\}\\ &\quad\times a(u^*, \mu z', \mu z''\,;\zeta)(\delta(U))(w, \lambda, \mu)du^*\wedge dv^*\wedge d\zeta\wedge dw,\end{aligned} \tag{2.18}$$

where $\tilde{\Delta}_\mu(z)$ is defined by

$$\begin{cases}u^*=z^*-i(1-\mu^2)\operatorname{Im} z^*+t^*,\\ v^*=u^*+x^*,\quad \zeta^*=-\mu^2\operatorname{Im} z^*-iT\mu\overline{x^*},\\ w^*=z^*+\left(\mu\operatorname{Re} x^*+i\dfrac{\operatorname{Im} x^*}{\mu}\right)+\left(t^*+i\dfrac{s^*}{\mu}\right),\\ w'=z'+x',\quad \zeta'=-\operatorname{Im} z'-iT\overline{x'},\\ w''=z''+x'',\quad \zeta''=-\operatorname{Im} z''-iT\overline{x''},\end{cases} \tag{2.19}$$

where $t^*, s^* \in \boldsymbol{R}^k$, $x \in \boldsymbol{C}^n$, $|t^*|, |s^*|, |x^*|, |x'|, |x''| \leqq r$. Since $\tilde{\Delta}_\mu(z)$ is not uniformly bounded when μ tends to 0, we shall make a deformation of $\tilde{\Delta}_\mu(z)$. For $b>0$, let $\tilde{\Delta}_{\mu,b}(z)$ denote the contour defined by

$$(2.20)\qquad \begin{cases} u^* = z^* - i(1-\mu^2)\operatorname{Im} z^* + t^* - i\dfrac{\mu(1-b^2)}{2b}s^*, \\ v^* = u^* + \dfrac{\mu}{2}x^*, \quad \zeta^* = -\mu^2 \operatorname{Im} z^* - iTb\mu x^*, \\ w^* = z^* + \left(\dfrac{\mu}{b}\operatorname{Re} x^* + i\dfrac{b}{\mu}\operatorname{Im} x^*\right) + \left(t^* + i\dfrac{bs^*}{\mu}\right), \end{cases}$$

and we leave the arguments w', w'', ζ' and ζ'' as in (2.19). Then we can see that $\tilde{\Delta}_{\mu,1}(z) = \tilde{\Delta}_\mu(z)$ and $\tilde{\Delta}_{\mu,\mu}(z) = \Delta_\mu(z)$ with $\alpha^* = t^* + is^*$. Choosing T large enough, we obtain

$$(2.21)\qquad -\frac{\lambda\mu^2}{2}|\operatorname{Im} z|^2 + \operatorname{Re}\Big\{-\frac{\lambda\rho}{2}(z^*-u^*)^2 + i\lambda(u^*-v^*)\zeta^* + \frac{\lambda\rho}{2}(v^*-w^*)^2$$

$$+ i\lambda\mu^2(z'-w')\zeta' + i\lambda\mu^2(z''-w'')\zeta''\Big\} + \frac{\lambda\mu^2}{2}|\operatorname{Im} w|^2$$

$$\leqq -\frac{\lambda\mu^2}{4}(|t^*|^2 + |s^*|^2 + |x|^2),$$

if $(u^*, v^*, \zeta, w) \in \tilde{\Delta}_{\mu,b}(z)$ and $\mu \leqq b \leqq 1$. In view of the Stokes' formula, it follows from (2.6), (2.18), and (2.21) that there exist $C>0$ and $\varepsilon>0$ such that

$$(2.22)\qquad |A_{\tilde{\Sigma}_{\mu,\mu}}(K(\delta(U)))(z,\lambda,\mu) - \Psi_{|\operatorname{Im} z|^2/2}(\delta(U))(z,\lambda,\mu)|$$

$$\leqq C\Big(\exp\frac{\lambda\mu^2}{2}(|\operatorname{Im} z|^2 - \varepsilon)\Big) \sup_{w\in D(z,2r,\mu)} |e^{-\lambda\mu^2|\operatorname{Im} z|^2/2}\delta(U)(w,\lambda,\mu)|,$$

for $z \in \hat{\omega}$, $\lambda>0$, $\mu>0$. Combinating (1.5) and (2.22), we obtain (2.17). The proof is complete. □

Next we assert that we can reduce the operator $\Psi_\varphi(a)$ to the multiplication of the principal symbol of a by using the method of stationary phase when we seek an a priori estimate for $\Psi_\varphi(a)$.

Before stating the following proposition, we introduce norms. Let $\hat{\omega}$ be an open subset of $\boldsymbol{C}^n$ and let ψ be a continuous function on $\hat{\omega}$. For a square integral function $U(z)$ on $\hat{\omega}$, we put

$$\|U\|^2_{\psi,\hat{\omega},\lambda} = \int_{\hat{\omega}} |U(z)|^2 e^{-2\lambda\psi(z)} L(dz),$$

where $L(dz)$ means the Lebesgue measure on $\boldsymbol{C}^n$. We denote by $L^2_{\varphi,\lambda}(\hat{\omega})$ the set of all measurable functions U such that $\|U\|_{\varphi,\hat{\omega},\lambda}<\infty$. Then in view of (2.8) we have

$$\|\Psi_\varphi(a)(U)\|_{\varphi,\tilde{\omega},\lambda\mu^2}\leqq C\|U\|_{\varphi,\omega,\lambda\mu^2}, \tag{2.23}$$

for $U\in L^2_{\varphi,\lambda\mu^2}(\omega)$ if φ satisfies (2.5), where C is independent of U. Using these norms, we have

PROPOSITION 2.3. *There exists $\varepsilon>0$ satisfying the following: If φ satisfying $\|\varphi-(|\mathrm{Im}\, z|^2/2)\|_{C^{1,1}(\omega)}<\varepsilon$, then there exists $C>0$ such that*

$$\|\Psi_\varphi(a)(U)-b(U)\|_{\varphi,\tilde{\omega},\lambda\mu^2}\leqq C(\lambda\mu^2)^{-1/2}\|U\|_{\varphi,\omega,\lambda\mu^2}, \tag{2.24}$$

for $U\in L^2_{\varphi,\lambda\mu^2}(\omega)$, where we put

$$b(U)=a_0(z^*+2(1-\mu^2)\partial_{z^*}\varphi(z),\ \mu z',\ \mu z''\ ;\ -2i\mu^2\partial_{z^*}\varphi(z),\ -2i\mu\partial_{z'}\varphi(z),\ -2i\mu\partial_{z''}\varphi(z))$$
$$\times U(z,\lambda,\mu),$$

where a_0 is the principal symbol of a.

PROOF. Noting that $|a(z,\zeta,\lambda)-a_0(z,\zeta)|\leqq C\lambda^{-1}$ for some $C>0$, we have

$$\|\Psi_\varphi(a)(U)-\Psi_\varphi(a_0)(U)\|_{\varphi,\tilde{\omega},\lambda\mu^2}\leqq C\mu^2(\lambda\mu^2)^{-1}\|U\|_{\varphi,\omega,\lambda\mu^2}. \tag{2.25}$$

Moreover putting

$$\tilde{b}(z,\zeta)=a_0(z^*+i(1-\mu^2)\zeta^*,\ \mu z',\ \mu z''\ ;\ \mu^2\zeta^*,\ \mu\zeta',\ \mu\zeta''),$$

we have

$$|a_0(u^*,\ \mu z',\ \mu z''\ ;\ \zeta^*,\ \mu\zeta',\ \mu\zeta'')-\tilde{b}(z,\ \zeta^*/\mu^2,\ \zeta',\ \zeta'')|\leqq C(|\alpha^*|+|x|), \tag{2.26}$$

where (u,ζ) has the form in (2.7).

If φ satisfies (2.5), then it follows from (2.8) and (2.26) that the kernel function of $\Psi_\varphi(a_0)-b(z,-2i\partial_z\varphi(z))\Psi_\varphi(1)$ is estimated by

$$C(|x|+|\alpha^*|)e^{-\lambda\mu^2(|x|^2+|\alpha^*|^2)/10}e^{\lambda\mu^2(\varphi(z)-\varphi(z+x+\alpha^*))}$$
$$\leqq C(\lambda\mu^2)^{-1/2}(\lambda\mu^2(|x|^2+|\alpha^*|^2))^{1/2}e^{-\lambda\mu^2(|x|^2+|\alpha^*|^2)/10}e^{\lambda\mu^2(\varphi(z)-\varphi(z+x+\alpha^*))}$$
$$\leqq C(\lambda\mu^2)^{-1/2}e^{\lambda\mu^2(\varphi(z)-\varphi(z+x+\alpha^*))}.$$

Then we have

(2.27) $$\|\Psi_\varphi(a_0)(U)-\tilde{b}(\cdot,\, -2i\partial_z\varphi(\cdot))\Psi_\varphi(1)(U)\|_{\varphi,\tilde{\omega},\lambda\mu^2} \leqq C(\lambda\mu^2)^{-1/2}\|U\|_{\varphi,\omega,\lambda\mu^2}\,.$$

Recalling (2.6) and (2.7), $\Psi_\varphi(1)(U)$ can be rewritten as

(2.28) $$\Psi_\varphi(1)(U)=\left(\frac{\lambda\mu^2}{2\pi}\right)^{n+k}\int_{|x|\leq r}\int_{|\alpha^*|\leq r}\exp\left\{-\frac{\lambda\rho}{2}\left(\frac{3-\mu^2}{2}|\alpha^*|^2\right)-\lambda\mu^2|x|^2\right\}$$
$$\times\exp\left\{\lambda\mu^2\left(\frac{\alpha^{*2}}{4}-2(x+(\alpha^*,\,0,\,0))\partial_z\varphi(z)\right)\right\}U(z+x+(\alpha^*,0,0))L(dx)L(d\alpha^*).$$

Applying Théorème 2.1 in [9] to (2.28), we obtain

(2.29) $$\|\Psi_\varphi(1)(U)-U\|_{\varphi,\tilde{\omega},\lambda\mu^2}\leqq Ce^{-\lambda\mu^2/C}\|U\|_{\varphi,\omega,\lambda\mu^2}\,,$$

for some $C>0$. Finally combinating (2.25), (2.27), and (2.29), we obtain (2.24). The proof is complete. □

3. Proof of Theorem 0.1

Let Ω_0 be an open subset of $\boldsymbol{R}^n$ such that $\Omega_0\Subset\Omega$. For Ω_0, we take a function $\chi(x)$ on Ω such that $\chi(x)\in C_0^\infty(\Omega)$ and $\chi(x)\equiv 1$ on Ω_0. Let $u\in\mathcal{D}'(\Omega)$. We put

(3.1) $$U(z,\,\lambda)=T(\chi u)(z,\,\lambda)\,.$$

According to Chap. 7 in [9], for the differential operator P, there exists a classical analytic symbol

$$\tilde{p}(z,\,\zeta,\,\lambda)\sim\sum_{j=0}^{\infty}\lambda^{-j}\tilde{p}_j(z,\,\zeta)\,,$$

such that $\Phi_{|\mathrm{Im}\,z|^2/2}(\tilde{p})(U)\sim T(\chi Pu)$ in the sense of $\mathrm{H}^{\mathrm{loc}}_{|\mathrm{Im}\,z|^2/2}$, and that $\tilde{p}_0(z,\,\zeta)=p_m(z+i\zeta,\,\zeta+\xi_0'')$.

For Ω_0, we put $\omega_0=\{(z^*,z',z'')\mid(\mathrm{Re}\,z^*,0,x_0'')\in\Omega_0\}$. If $w=(w^*,w',w'')\in\omega_0$ satisfies $(\mathrm{Re}\,w^*,0,x_0''\,;\,0,0,\xi_0'')\notin WF_a(Pu)$ then using Proposition 2.1 with $a=\tilde{p}$ and (1.5), we have

(3.2) $$\Psi_{|\mathrm{Im}\,z|^2/2}(\tilde{p})(\delta(U))\sim 0\,,$$

in the sense of $\mathrm{H}^2_{|\mathrm{Im}\,z|^2/2}$ on a neighborhood ω_1 of w. We replace $|\mathrm{Im}\,z|^2/2$ in (3.2) by $\varphi(z)\in C^{1,1}(\omega_1)$. If $\|\varphi-(|\mathrm{Im}\,z|^2/2)\|_{C^{1,1}}$ is small enough, then by Proposition 2.3, (3.2) implies that there exist $C>0$, $\varepsilon>0$, $\lambda(\mu)$, and a neighborhood ω_2 of w satisfying $\omega_2\Subset\omega_1$ such that

(3.3) $$\|\tilde{q}(\cdot, -2i\partial_z\varphi(\cdot))\delta(U)\|_{\varphi,\omega_2,\lambda\mu^2} \leqq C((\lambda\mu^2)^{-1/2}\|\delta(U)\|_{\varphi,\omega_1,\lambda\mu^2} + e^{-\varepsilon\lambda\mu^2}),$$

for $\lambda \geqq \lambda(\mu)$, where

$$\tilde{q}(z, \zeta) = \tilde{p}_0(z^* - i(1-\mu^2)\zeta^*, \mu z', \mu z''\,;\, \mu^2\zeta^*, \mu\zeta', \mu\zeta'').$$

We assume that P satisfies (H.1)-(H.4). Then by the Taylor expansion of $\tilde{q}$ at $\mu=0$, $\tilde{q}$ has the form

(3.4) $$\tilde{q} = \mu^2\tilde{a}(z, \zeta) + \mu^3\tilde{b}(z, \zeta) + \mu^4(q(z^* + i\zeta^*, \zeta^*) + \tilde{c}(z, \zeta)) + \mu^5 r(z, \zeta, \mu),$$

where q is defined in (0.3) and $\tilde{a}, \tilde{b}, \tilde{c}$ and r satisfy

(3.5) $$|\tilde{a}(z, \zeta)| + |\tilde{b}(z, \zeta)| + |\tilde{c}(z, \zeta)| \leqq C(|z'|^2 + |\zeta'|^2),$$

(3.6) $$|r(z, \zeta, \mu)| \leqq C,$$

for some $C>0$. Moreover (H.4) gives

(3.7) $$\tilde{a}(z, -\operatorname{Im} z) \geqq c|z'|^2,$$

for some $c>0$.

Firstly we shall investigate the growth of $\delta(U)$ in (3.2) when $\lambda \to \infty$ for a fixed $\mu>0$ by an a priori estimate induced from the ellipticities of the principal symbol p_0.

PROPOSITION 3.1. *Let u be a distribution on Ω and $w=(w^*, w', w'')$ be a point of ω_0. We assume*

(3.8) $$(\operatorname{Re} w, 0, x_0''\,;\, 0, 0, \xi_0'') \notin WF_a(Pu).$$

If $w' \neq 0$ or $q(\operatorname{Re} w^*, -\operatorname{Im} w^*)>0$, then the function U defined in (3.1) *satisfies*

(3.9) $$\delta_{w^*}(U) \sim 0,$$

in the sense of $\mathrm{H}^2_{|\operatorname{Im} z|^2/2}$ *on a neighborhood of w.*

PROOF. Firstly we assume that $w' \neq 0$. In view of (3.7), we can take a neighborhood ω_1 of w such that $\tilde{a}(z, -\operatorname{Im} z) \geqq 3c$ on ω_1 for some $c>0$. Let ω_2, ω_3 be neighborhood of w such that $w \in \omega_3 \Subset \omega_2 \Subset \omega_1$, and let $\varphi \in C^{1,1}(\omega_1)$ satisfying

(3.10) $$\varphi(z) = |\operatorname{Im} z|^2/2 \qquad \text{on } \omega_3,$$

(3.11) $$\varphi(z) > |\operatorname{Im} z|^2/2 \qquad \text{on } \omega_1 \setminus \omega_2.$$

If $\|\varphi-|\mathrm{Im}\, z|^2/2\|_{C^{1,1}}$ and ω_i $(i=1,2,3)$ is small enough, then (3.8) implies that (3.3) holds and $|\tilde{a}(z, -2i\partial_z\varphi(z))|\geqq 2c$ on ω_1. Taking $\mu>0$ small enough, we can see that

$$|\tilde{q}(z, -2i\partial_z\varphi(z))|\geqq c\mu^2 \quad \text{on } \omega_2, \tag{3.12}$$

in view of (3.4)-(3.7). Then (3.3) and (3.12) give

$$\|\delta(U)\|_{\varphi,\omega_2,\lambda\mu^2}\leqq C\mu^{-2}(\lambda\mu^2)^{-1/2}\|\delta(U)\|_{\varphi,\omega_1,\lambda\mu^2}+Ce^{-\varepsilon\lambda\mu^2}. \tag{3.13}$$

We take $\lambda(\mu)$ such that $C\mu^{-2}(\lambda\mu^2)^{-1/2}<1/2$ and $\mu^{-2}e^{-\varepsilon\lambda\mu^2}<1$ when $\lambda\geqq\lambda(\mu)$. Then by transposing the first term of the right-hand side of (3.13) to the left-hand side, we obtain

$$\|\delta(U)\|_{\varphi,\omega_2,\lambda\mu^2}\leqq C(\|\delta(U)\|_{\varphi,\omega_1\setminus\omega_2,\lambda\mu^2}+e^{-\varepsilon\lambda\mu^2}) \quad \text{for } \lambda\geqq\lambda(\mu). \tag{3.14}$$

Remarking (1.5) and (3.11), we have

$$\|\delta(U)\|_{\varphi,\omega_1\setminus\omega_2,\lambda\mu^2}\leqq Ce^{-\varepsilon\lambda\mu^2}. \tag{3.15}$$

Then it follows from (3.10), (3.14), and (3.15) that

$$\|\delta_{w^*}(U)\|_{|\mathrm{Im}\, z|^2/2,\omega_3,\lambda\mu^2}\leqq Ce^{-\varepsilon\lambda\mu^2}. \tag{3.16}$$

Here we remark that $\delta_{w^*}(U)e^{-|\mathrm{Im}\, z|^2/2}$ is a plurisubharmonic function on ω_3. Then by the mean value theorem, we have (3.9).

Secondly we assume that $w'=0$ and

$$q(\mathrm{Re}\, w^*, -\mathrm{Im}\, w^*)>0. \tag{3.17}$$

Let $\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3$ be neighborhoods of (w^*, w'') in $C^k_{z^*}\times C^{n-k-l}_{z''}$ such that $\tilde{\omega}_1\supset\tilde{\omega}_2\supset\tilde{\omega}_3$, and let $\tilde{\tilde{\omega}}_1, \tilde{\tilde{\omega}}_2, \tilde{\tilde{\omega}}_3$ be neighborhoods of 0 in $C^l_{z'}$ such that $\tilde{\tilde{\omega}}_1\supset\tilde{\tilde{\omega}}_2\supset\tilde{\tilde{\omega}}_3$. We take $\tilde{\varphi}\in C^{1,1}(\tilde{\omega}_1)$ satisfying

$$\tilde{\varphi}(z^*, z'')=(|\mathrm{Im}\, z^*|^2/2+|\mathrm{Im}\, z''|^2/2) \quad \text{on } \tilde{\omega}_3, \tag{3.18}$$

$$\tilde{\varphi}(z^*, z'')>(|\mathrm{Im}\, z^*|^2/2+|\mathrm{Im}\, z''|^2/2) \quad \text{on } \tilde{\omega}_1\setminus\tilde{\omega}_2. \tag{3.19}$$

If $\|\tilde{\varphi}-(|\mathrm{Im}\, z^*|^2/2+|\mathrm{Im}\, z''|^2/2)\|_{C^{1,1}}$ and $\tilde{\omega}_i$ is small enough, then it follows from (3.4)-(3.7) and (3.17) that

$$\begin{aligned}|\tilde{q}(z, -2i\partial_z\varphi(z))|&\geqq \mathrm{Re}\, q(z, -2i\partial_z\varphi(z))\\ &\geqq c(\mu^2|z'|^2+\mu^4)\geqq c\mu^4 \quad \text{on } \tilde{\omega}_2\times\tilde{\tilde{\omega}}_2,\end{aligned} \tag{3.20}$$

for some $c>0$, where we set

$$\varphi(z)=\tilde{\varphi}(z^*, z'')+|\mathrm{Im}\, z'|^2/2.$$

We put $\omega_i=\tilde{\omega}_i\times\tilde{\tilde{\omega}}_i$ $(i=1,2,3)$. As in the first case, it follows from (3.3) and (3.20) that (3.14) holds for some $\lambda(\mu)$. Note that $\omega_1\setminus\omega_2=\tilde{\omega}_1\times(\tilde{\tilde{\omega}}_1\setminus\tilde{\tilde{\omega}}_2)\cup(\tilde{\omega}_1\setminus\tilde{\omega}_2)\times\tilde{\tilde{\omega}}_1$. Since there exists $\gamma>0$ such that $z'\notin\tilde{\tilde{\omega}}_2$ implies that $|z'|\geqq\gamma$, from the result (3.9) in the first case, we can see that

$$\|\delta(U)\|_{\varphi,\tilde{\omega}_1\times(\tilde{\tilde{\omega}}_1\setminus\tilde{\tilde{\omega}}_2)}\leqq Ce^{-\varepsilon\lambda\mu^2}, \tag{3.21}$$

for some $\varepsilon>0$, $C>0$. Moreover from (1.5) and (3.19), we have

$$\|\delta(U)\|_{\varphi,(\tilde{\omega}_1\setminus\tilde{\omega}_2)\times\tilde{\tilde{\omega}}_1}\leqq Ce^{-\varepsilon\lambda\mu^2}. \tag{3.22}$$

Combination of (3.21) and (3.22) gives (3.15). Thus we have (3.9) in this case. The proof is complete. □

Recalling the definition of the second microsupports, we have

COROLLARY 3.2. *Let u be a distribution on Ω and let $(y^*,\tilde{\eta}^*)\in T^*\Gamma$ satisfying $(y^*,0,x_0^*;0,0,\xi_0'')\notin WF_a(Pu)$ and $q(y^*,\tilde{\eta}^*)>0$. Then $(y^*,\tilde{\eta}^*)\notin SS_\Gamma^2(u)$.*

Next we shall prove propagation of second micro-analyticities along the integral curves of H_q on the hypersurface $q^{-1}(0)$.

PROPOSITION 3.3. *Let u be a distribution on Ω satisfying* (H.5) *and $(y^*,\tilde{\eta}^*)$ be a point of $T^*\Gamma$ with $q(y^*,\tilde{\eta}^*)=0$ and $y^*\in\bigcup_{\xi^*\in A}\pi(C^+(x_0^*,\tilde{\xi}^*))$. If $(y^*,\tilde{\eta}^*)\notin SS_\Gamma^2(u)$, then the integral curve $\gamma(y^*,\tilde{\eta}^*)$ of H_q through $(y^*,\tilde{\eta}^*)$ satisfies*

$$\gamma(y^*,\tilde{\eta}^*)\cap SS_\Gamma^2(u)=\emptyset. \tag{3.23}$$

PROOF. We take a point $(y_0^*,\tilde{\eta}_0^*)\in\gamma(y^*,\tilde{\eta}^*)$ which belongs to the closure of the complement of $SS_\Gamma^2(u)$. It is sufficient to prove that

$$(y_0^*,\tilde{\eta}_0^*)\notin SS_\Gamma^2(u), \tag{3.24}$$

because we can apply the inductive method with respect to the parameter of $\gamma(y^*,\tilde{\eta}^*)$. From (H.3), there exists a real analytic contact transformation χ on $T^*\Gamma$ such that

$$q(\chi(x^*,\tilde{\xi}^*))=\tilde{\xi}_1^*,$$

and

$$\chi^{-1}(y_0^*,\tilde{\eta}_0^*)=(0;0,\cdots,0,1).$$

Then $\gamma(y^*,\tilde{\eta}^*)$ is transformed by χ to the curve $\{(u,0,\cdots,0;0,\cdots,0,1)\}_{u\in I}$ with some interval I containing 0. From the assumption for $(y_0^*,\tilde{\eta}_0^*)$, we can take a sequence $\{(u_\nu, 0, \cdots, 0\,;\, 0, \cdots, 0, 1)\}_{\nu=1}^{\infty}$ such that

$$(u_\nu, 0, \cdots, 0\,;\, 0, \cdots, 0, 1) \notin \chi^*(\mathrm{SS}_\Gamma^2(u)), \tag{3.25}$$

for any ν. Without loss of generality, we may assume that $u_\nu > 0$ (3.25) implies that there exists a sequence of positive numbers $\{\varepsilon_\nu\}_{\nu=1}$ such that

$$\{(u, 0, \cdots, 0\,;\, 0, \cdots, 0, 1) \mid |u - u_\nu| \leqq 2\varepsilon_\nu\} \cap \chi^*(\mathrm{SS}_\Gamma^2(u)) = \emptyset, \tag{3.26}$$

for any ν. For ν, we introduce the function ϕ_ν on $T^*\Gamma$ as

$$\phi_\nu(x^*, \tilde{\xi}^*) = -x_1 + \frac{4u_\nu}{\varepsilon_\nu^2}\Big(\sum_{i=2}^{k} x_i^2 + \sum_{i=1}^{k-1} \tilde{\xi}_i^2 + (\tilde{\xi}_k - 1)^2\Big). \tag{3.27}$$

Using ϕ_ν, we put

$$\tilde{\omega}_{1,\nu} = \{z^* \in \boldsymbol{C}^k \mid |\chi^{-1} \circ H(z^*) - (u_\nu, 0, \cdots, 0\,;\, 0, \cdots, 0, 1)| < \varepsilon_\nu\}$$
$$\cup \{z^* \in \boldsymbol{C}^k \mid \phi_\nu(\chi^{-1} \circ H(z^*)) < 2u_\nu,\ x_1(\chi^{-1} \circ H(z^*)) < u_\nu\},$$
$$\tilde{\omega}_{2,\nu} = \Big\{z^* \in \boldsymbol{C}^k \mid |\chi^{-1} \circ H(z^*) - (u_\nu, 0, \cdots, 0\,;\, 0, \cdots, 0, 1)| < \frac{3}{4}\varepsilon_\nu\Big\}$$
$$\cup \{z^* \in \boldsymbol{C}^k \mid \phi_\nu(\chi^{-1} \circ H(z^*)) < u_\nu,\ x_1(\chi^{-1} \circ H(z^*)) < u_\nu\},$$

where $H(z^*) = (\mathrm{Re}\, z^*, -\mathrm{Im}\, z^*)$. Then we can see that

$$H^{-1}((y_0^*, \tilde{\eta}_0^*)) \in \tilde{\omega}_{2,\nu} \Subset \tilde{\omega}_{1,\nu}, \tag{3.28}$$

and that

$$(\chi^{-1} \circ H)(\tilde{\omega}_{1,\nu} \setminus \tilde{\omega}_{2,\nu}) \subset \{(x^*, \tilde{\xi}^*) \in T^*\Gamma \mid |(x^*, \tilde{\xi}^*) - (u_\nu, 0, \cdots, 0\,;\, 0, \cdots, 0, 1)| < \varepsilon_\nu\} \tag{3.29}$$
$$\cup \{(x^*, \tilde{\xi}^*) \in T^*\Gamma \mid \phi_\nu(x^*, \tilde{\xi}^*) \geqq u_\nu\}.$$

We set

$$\omega_{i,\nu} = \tilde{\omega}_{i,\nu} \times \{(z', z'') \in \boldsymbol{C}^{n-k} \mid |z'| < \delta_{i,\nu},\ |z''| < \delta_{i,\nu}\},$$

for $i = 1, 2$, where $0 < \delta_{2,\nu} < \delta_{1,\nu}$ and $\delta_{1,\nu}$ is small enough so that

$$\delta(U) \sim 0 \quad \text{on } \{z^* \in \boldsymbol{C}^k \mid |\chi^{-1} \circ H(z^*) - (u_\nu, 0, \cdots, 0\,;\, 0, \cdots, 0, 1)| < \varepsilon_\nu\} \tag{3.30}$$
$$\times \{(z', z'') \in \boldsymbol{C}^{n-k} \mid |z'| < \delta_{1,\nu},\ |z''| < \delta_{1,\nu}\}.$$

It follows from (3.26) that there exists such $\delta_{1,\nu} > 0$.

We put

$$\tilde{\phi}_\nu(z^*, \zeta^*) = \phi_\nu(\chi^{-1}(z^* + i\zeta^*, \zeta^*)),$$

where we extend the domain of χ to a complex neighborhood of $T^*\Gamma$.

Then $\tilde{\phi}_\nu$ has real values on $\Lambda_{|\mathrm{Im}\, z|^2/2}$. We shall consider the following Cauchy problem:

$$\begin{cases} -2\partial_t \Phi_\nu(t, z^*) + \tilde{\phi}_\nu(z^*, -2i\partial_{z^*}\Phi_\nu(t, z^*)) = 0\,, \\ \Phi_\nu(0, z^*) = |\mathrm{Im}\, z^*|^2/2\,. \end{cases}$$

By the Taylor expansion at $t=0$, we have

$$\Phi_\nu(t, z^*) = |\mathrm{Im}\, z^*|^2/2 + t\tilde{\phi}_\nu(z^*, -\mathrm{Im}\, z^*) + O(t^2)\,, \tag{3.31}$$

for small $t>0$. For $t>0$, $a>0$, we set

$$\Phi_{t,a,\nu}(z) = \Phi_\nu(t, z^*) + (|\mathrm{Im}\, z'|^2 + |\mathrm{Im}\, z''|^2 + a|z''|^2)/2\,. \tag{3.32}$$

For $(z, \zeta) \in \boldsymbol{C}^n \times \boldsymbol{C}^n$, we define the complex curve $s \mapsto \tilde{\gamma}_s^\nu(z, \zeta)$ as follows: $\tilde{\gamma}_s^\nu(z, \zeta)$ satisfies

$$\begin{cases} \dfrac{d}{ds}\tilde{\gamma}_s^\nu(z, \zeta) = H_{\tilde{\phi}_\nu}(\tilde{\gamma}_s^\nu(z, \zeta))\,, \\ \tilde{\gamma}_0^\nu(z, \zeta) = (z, \zeta)\,. \end{cases}$$

By the Hamilton-Jacobi theory, we can see that

$$\Lambda_{\Phi_{t,a,\nu}} = \tilde{\gamma}_{it}^\nu(\Lambda_{\Phi_{0,a,\nu}})\,,$$

that is, if $(\mathring{z}, \mathring{\zeta}) \in \Lambda_{\Phi_{t,a,\nu}}$, then there exists $(z, \zeta) \in \Lambda_{\Phi_{0,a,\nu}}$ can be rewritten as

$$\begin{cases} \mathring{z}^* = z^* + it\partial_{\zeta^*}\tilde{\phi}_\nu(z^*, \zeta^*) + t^2 r_1(z^*, t)\,, \\ \mathring{\zeta}^* = \zeta^* - it\partial_{z^*}\tilde{\phi}_\nu(z^*, \zeta^*) + t^2 r_2(z^*, t)\,, \\ \mathring{z}' = z'\,, \qquad \mathring{\zeta}' = \zeta'\,, \\ \mathring{z}'' = z''\,, \qquad \mathring{\zeta}'' = \zeta''\,, \end{cases}$$

with $(z, \zeta) \in \Lambda_{\Phi_{0,a,\nu}}$. Here it follows from (3.27) that

$$|\partial_{z^*}\tilde{\phi}_\nu| + |\partial_{\zeta^*}\tilde{\phi}_\nu| \leqq \frac{Cu_\nu}{\varepsilon_\nu} \quad \text{on } \tilde{\omega}_2\,, \tag{3.33}$$

and

$$|r_i| \leqq \frac{Cu_\nu^2}{\varepsilon_\nu^3} \quad \text{on } \tilde{\omega}_2\,, \tag{3.34}$$

for $i=1, 2$ if $t>0$ and $a>0$ are small enough, where C is independent of t, a and ν.

We set

$$s(z, \zeta, \mu)=\mu^2\tilde{a}(z, \zeta)+\mu^3\tilde{b}(z, \zeta)+\mu^4(q(z^*+i\zeta^*, \zeta^*)+\tilde{c}(z, \zeta)) .$$

Substituting $(\mathring{z}, \mathring{\zeta})$ to $s(z, \zeta, \mu)$, in view of (3.5) and the fact that $\tilde{a}$, $\tilde{b}$ and q have real values on $\Lambda_{|\mathrm{Im}\, z|2/2}$, we have

$$\mathrm{Re}\, s(z, \zeta, \mu)=\mu^2(\tilde{a}(z, -\mathrm{Im}\, z)+r_3(z, t, a, \mu))+\mu^4(q(H(z^*))+t^2r_4(z, t)) ,$$

$$\mathrm{Im}\, s(z, \zeta, \mu)=\mu^2r_5(z, t, a, \mu)-\mu^4t(H_q((\chi^{-1})^*\phi_\nu)(H(z^*))+tr_6(z, t)) ,$$

where r_i $(i=3, 5)$ satisfies

$$|r_i|\leqq C|z'|^2\Big(\Big(\frac{tu_\nu}{\varepsilon_\nu}\Big)+\mu+a\Big) \quad \text{on } \omega_{2,\nu} , \tag{3.35}$$

and r_i $(i=4, 6)$ satisfies

$$|r_i|\leqq\frac{Cu_\nu^2}{\varepsilon_\nu^3} \quad \text{on } \omega_{2,\nu} . \tag{3.36}$$

Recalling the definition of $\tilde{\omega}_{2,\nu}$, we have

$$\sup_{z^*\in\tilde{\omega}_{2,\nu}}|q(H(z^*))|<\varepsilon_\nu . \tag{3.37}$$

Since χ is a contact transformation, we have

$$H_q((\chi^{-1})^*\phi_\nu)=H_{\xi_1}\phi_\nu=-1 . \tag{3.38}$$

It follows from (3.7) and (3.35)-(3.38) that $\mathrm{Re}\, s$ and $\mathrm{Im}\, s$ satisfy

$$\mathrm{Re}\, s\geqq\mu^2|z'|^2\Big(c-C\Big(\Big(\frac{tu_\nu}{\varepsilon_\nu}\Big)+a+\mu\Big)\Big)-\mu^4\Big(\varepsilon_\nu+\frac{t^2u_\nu^2}{\varepsilon_\nu^3}\Big),$$

$$\mathrm{Im}\, s\geqq\mu^4t\Big(1-\frac{Ctu_\nu^2}{\varepsilon_\nu^3}\Big)-\mu^2C|z'|^2\Big(\frac{tu_\nu}{\varepsilon_\nu}+a+\mu\Big),$$

for small $t>0$ with some $c>0$. For ν, we put

$$\delta(\nu)=\min\Big(\frac{c\varepsilon_\nu}{9Cu_\nu}, \frac{\varepsilon_\nu^2}{u_\nu}, \frac{\varepsilon_\nu^3}{3Cu_\nu^2}\Big).$$

If $0<t<\delta(\nu)$, $0<a<tu_\nu/\varepsilon_\nu$, $0<\mu<tu_\nu/\varepsilon_\nu$, then

$$\mathrm{Re}\, s\geqq\frac{\mu^4\varepsilon_\nu(c-27Cu_\nu)}{15Cu_\nu} \quad \text{when } \mu^2\leqq 9Cu_\nu|z'|^2,$$

$$\mathrm{Im}\, s\geqq\frac{\mu^4t}{3} \quad \text{when } \mu^2\geqq 9Cu_\nu|z'|^2.$$

We take ν large enough so that $54Cu_\nu<c$. Then there exists $b>0$ such that $|s(z, \zeta, \mu)| \geqq 2b\mu^4$ for small $t>0$, $a>0$. Moreover remarking (3.6), we obtain

$$|q(z, \zeta, \mu)| \geqq b\mu^4, \tag{3.39}$$

on $\Lambda_{\Phi_{t,a,\nu}}$. By (3.3) with $\varphi=\Phi_{t,a,\nu}$ and $\omega_i=\omega_{i,\nu}$, (3.39) implies as in the proof of Proposition 3.1 that there exist $\varepsilon>0$ and $\lambda(\mu)$ such that

$$\|\delta(U)\|_{\Phi_{t,a,\nu},\omega_{2,\nu},\lambda\mu^2} \leqq C((\lambda\mu^2)^{-1/2}\|\delta(U)\|_{\Phi_{t,a,\nu},\omega_{1,\nu}\setminus\omega_{2,\nu},\lambda\mu^2}+e^{-\varepsilon\lambda\mu^2}) \tag{3.40}$$

for $\lambda \geqq \lambda(\mu)$. We put

$$\Omega_{1,\nu}=H^{-1}\circ\chi(\{(x^*, \tilde{\xi}^*)\in T^*\Gamma \mid |(x^*, \tilde{\xi}^*)-(u_\nu, 0, \cdots, 0 ; 0, \cdots, 0, 1)|<\varepsilon_\nu\})$$

$$\times\{(z', z'')\in \boldsymbol{C}^{n-k} \mid |z'|<\delta_{1,\nu}, |z''|<\delta_{1,\nu}\},$$

$$\Omega_{2,\nu}=H^{-1}\circ\chi(\{(x^*, \tilde{\xi}^*)\in T^*\Gamma \mid \phi_\nu(x^*, \tilde{\xi}^*)\geqq u_\nu\}\cap\tilde{\omega}_{1,\nu})$$

$$\times\{(z', z'')\in \boldsymbol{C}^{n-k} \mid |z'|<\delta_{1,\nu}, |z''|<\delta_{1,\nu}\},$$

$$\Omega_{3,\nu}=\tilde{\omega}_{1,\nu}\times\{(z', z'')\in \boldsymbol{C}^{n-k} \mid \delta_{2,\nu}\leqq|z'|<\delta_{1,\nu}, |z''|<\delta_{1,\nu}\},$$

$$\Omega_{4,\nu}=\tilde{\omega}_{1,\nu}\times\{(z', z'')\in \boldsymbol{C}^{n-k} \mid |z'|<\delta_{1,\nu}, \delta_{2,\nu}\leqq|z''|<\delta_{1,\nu}\}.$$

From (3.29), we have

$$\omega_{1,\nu}\setminus\omega_{2,\nu}\subset\Omega_{1,\nu}\cup\Omega_{2,\nu}\cup\Omega_{3,\nu}\cup\Omega_{4,\nu}.$$

By Proposition 3.1 and (3.30), we can see that $\delta(U)\sim 0$ on $\Omega_{1,\nu}\cup\Omega_{3,\nu}$ in the sense of $\mathrm{H}^2_{|\mathrm{Im}\, z|^2/2}$. It follows from (3.31) and (3.32) that there exists $\varepsilon>0$ such that

$$\Phi_{t,a,\nu}(z)\geqq|\mathrm{Im}\, z|^2/2+\varepsilon \quad \text{on } \Omega_{2,\nu}\cup\Omega_{4,\nu}.$$

Hence remarking that $\delta(U)\in\mathrm{H}^2_{|\mathrm{Im}\, z|^2/2}$, we have

$$\|\delta(U)\|_{\Phi_{t,a,\nu},\omega_{1,\nu}\setminus\omega_{2,\nu},\lambda\mu^2}\leqq Ce^{-\varepsilon\lambda\mu^2/2}, \tag{3.41}$$

for some $\varepsilon>0$. (3.40) and (3.41) imply that

$$\|\delta(U)\|_{\Phi_{t,a,\nu},\omega_{2,\nu},\lambda\mu^2}\leqq e^{-\varepsilon\lambda\mu^2},$$

when $\lambda\geqq\lambda(\mu)$ for some $\varepsilon>0$, $C>0$, $\lambda(\mu)$. Remarking (3.28) and that $\Phi_{t,a,\nu}<|\mathrm{Im}\, z|^2/2$ near $(y_0^*-i\tilde{\eta}_0^*, 0)$ by (3.32), we have $\delta(U)\sim 0$, there in the sense of $\mathrm{H}^2_{|\mathrm{Im}\, z|^2/2}$ when ν is large enough. Recalling the definition of $\mathrm{SS}^2_\Gamma(u)$, we obtain (3.24). This completes the proof. □

PROOF OF THEOREM 0.1. From Corollary 3.2, we have

$$\{(x_0^*, \tilde{\xi}^*) \in T_{x_0^*}^* \Gamma \mid q(x_0^*, \tilde{\xi}^*) > 0\} \cap SS_\Gamma^2(u) = \emptyset . \tag{3.42}$$

Let $(x_0^*, \tilde{\xi}^*)$ be a point of A. From (H.7) there exists $s \geqq 0$ such that $(x_0^*, \tilde{\xi}^*)(s) \notin SS_\Gamma^2(u)$. Moreover we note that (H.5) holds. Then using Proposition 3.3, we conclude that $(x_0^*, \tilde{\xi}^*) \notin SS_\Gamma^2(u)$. Thus we have

$$A \cap SS_\Gamma^2(u) = \emptyset . \tag{3.43}$$

It follows from (0.4) that there exists an affine coordinate transformation χ on Γ such that

$$\chi^{-1}(x_0^*) = 0 , \quad q((\chi^{-1})^*(0, \tilde{\xi}^*)) = \tilde{\xi}_1^2 - \sum_{j=2}^k \tilde{\xi}_j^2 .$$

Then (3.42) and (3.43) imply that there exists $\varepsilon > 0$ such that

$$\{(x^*, \tilde{\xi}^*) \in T^*\Gamma \mid |x^*| < \varepsilon,\ (1+2\varepsilon)^2 \tilde{\xi}_1^2 > \sum_{j=2}^k \tilde{\xi}_j^2,\ \tilde{\xi}_1 > 0\} \cap \chi^*(SS_\Gamma^2(u)) = \emptyset . \tag{3.44}$$

Moreover (H.6) implies that

$$\{x^* \in \Gamma \mid |x^*| < \varepsilon,\ x_1^2 > (1+\varepsilon)^2 \sum_{j=2}^k x_j^2,\ x_1 > 0\} \cap \chi^{-1}(\Gamma \cap WF_a(u)) = \emptyset , \tag{3.45}$$

where we replace ε by a smaller one if necessary. Hence any conormals of the hypersurface $x_1^2 = (1+\varepsilon)^2 \sum_{j=2}^k x_j^2$ do not contain $SS_\Gamma^2(u)$ in $|x^*| < \varepsilon$, $x_1 > 0$. In view of Proposition 1.2, we have

$$\left(\left\{ x^* \in \Gamma \,\middle|\, \frac{\varepsilon(1+\varepsilon)}{4} < x_1 < \frac{\varepsilon(1+\varepsilon)}{2},\ \left(\frac{x_1}{1+\varepsilon} + \delta \right)^2 > \sum_{j=2}^k x_j^2 \right\} \right. \tag{3.46}$$

$$\left. \cup \left\{ x^* \in \Gamma \,\middle|\, 0 < x_1 \leqq \frac{\varepsilon(1+\varepsilon)}{4},\ \frac{x_1^2}{(1+\varepsilon)^2} > \sum_{j=2}^k x_j^2 \right\} \right) \cap \chi^{-1}(WF_a(u) \cap \Gamma) = \emptyset ,$$

for some $\delta > 0$. We put $c_0 = (2 + 2\varepsilon + \varepsilon^2)^{1/2}$ and

$$\psi(x_2, \cdots, x_k) = \begin{cases} \dfrac{\delta}{1+\varepsilon}(c_0+1) - \left(\left(\dfrac{\delta}{1+\varepsilon} \right)^2 (c_0+1)^2 - \displaystyle\sum_{j=2}^k x_j^2 \right)^{1/2} & \text{if } \displaystyle\sum_{j=2}^k x_j^2 \leqq \delta^2 \left(1 + \frac{1}{c_0} \right)^2 , \\ (1+\varepsilon) \left(\left(\displaystyle\sum_{j=2}^k x_j^2 \right)^{1/2} - \delta \right) & \text{if } \displaystyle\sum_{j=2}^k x_j^2 \geqq \delta^2 \left(1 + \frac{1}{c_0} \right)^2 . \end{cases}$$

For $a \geqq 0$, we define the open set Ω_a by

$$\Omega_a=\left\{x^*\in\Gamma \;\middle|\; (1+\varepsilon)\left(\frac{\varepsilon}{2}-\delta\right)>x_1>\psi(x_2,\cdots,x_k)+a\right\}$$

Then (3.46) implies that

$$\Omega_{\frac{\varepsilon(1+\varepsilon)}{4}}\cap\chi^{-1}(WF_a(u)\cap\Gamma)=\emptyset,$$

and

$$\Omega_a\cap\left\{x^*\in\Gamma \;\middle|\; \left(\frac{\varepsilon}{4}+\delta\right)^2<\sum_{j=2}^{k}x_j^2\right\}\cap\chi^{-1}(WF_a(u)\cap\Gamma)=\emptyset,$$

for $\varepsilon(1+\varepsilon)/4\geqq a>0$. Since from (3.44) any conormals of the hypersurface $x_1=\psi(x_2,\cdots,x_k)$ do not contain $SS^2_\Gamma(u)$ in $|x^*|<\varepsilon$, $x_1>0$, in view of Proposition 1.2, we can use the method of sweeping out. Then we have

$$\Omega_0\cap\chi^{-1}(WF_a(u)\cap\Gamma)=\emptyset.$$

Finally applying Proposition 1.2 at x_0^*, we conclude that $x_0^*\notin WF_a(u)\cap\Gamma$. The proof is complete. □

References

[1] Bony, J. M. and P. Schapira, Propagation des singularités analytiques pour les solutions des équations aux dérivées partielles, Ann. Inst. Fourier (Grenoble) **26**-1 (1976), 81-140.

[2] Hasegawa, K., On the propagation of analyticity for some class of differential equations with non-involutive double characteristics, Proc. Japan Acad. Ser. A **63** (1987), 47-49.

[3] Kashiwara, M. and T. Kawai, Microhyperbolic pseudo-differential operators I, J. Math. Soc. Japan **27** (1975), 359-404.

[4] Kashiwara, M. and Y. Laurent, Théorèmes d'annulation et deuxième microlocalisation, Prépublication d'Orsay, Univ. Paris Sud., Orsay, 1983.

[5] Laurent, Y., Théorie de la deuxième microlocalisation dans le domaine complexe, Progress in Math. vol. 53, Birkhäuser, Boston, 1985.

[6] Lebeau, G., Deuxième microlocalisation à croissance, Séminaire Goulaouic-Meyer-Schwartz 1982-1983, exposé **15**

[7] Lebeau, G., Deuxième microlocalisation sur les sous-variétés isotropes, Ann. Inst. Fourier (Grenoble) **35**-2 (1985), 145-216.

[8] Sato, M., Kawai, T. and M. Kashiwara, Hyperfunctions and pseudo-differential equations, Lecture Notes in Math. vol. 287, Springer, Berlin-Heidelberg-New York, 1973, 265-529.

[9] Sjöstrand, J., Singularités analytiques microlocales, Astérisque **95** (1982), 1-166.

[10] Sjöstrand, J., Analytic wavefront sets and operators with multiple characteristics, Hokkaido Math. J. **12** (1983), 392-433.

[11] Tose, N., 2nd microlocalization and conical refraction, Ann. Inst. Fourier (Grenoble) **37**-2 (1987), 239-259.

[12] Tose, N., 2nd microlocalization and conical refraction II, to appear in "Prospects of Algebraic Analysis" (a volume dedicated to Prof. Sato in his 60th birthday), Academic Press.

[13] Tose, N., On a class of 2-microhyperbolic systems, J. Math. Pures Appl. **67**-1 (1988), 23-37.

[14] Tose, N., Propagation theorem for sheaves and applications to microdifferential systems, to appear in J. Math. Pures Appl.

(Received March 20, 1987)
(Revised June 21, 1988)

Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan