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The Poisson equation with semilinear boundary conditions in domains with many tiny holes

Dedicated to Professor Hiroshi Fujita on his 60th birthday

By Satoshi KAIZU

§ 0. Introduction

Let Ω be a bounded domain in $\boldsymbol{R}^N$, $N\geqq 3$, with smooth boundary. Let Y be the cube $[-1/2, 1/2]^N$ in $\boldsymbol{R}^N$, and T a closed subdomain of Y with smooth boundary. We assume that the complement $\boldsymbol{R}^N\setminus T$ of T in $\boldsymbol{R}^N$ is connected and the interior T^0 of T is not empty (the set T need not be connected). Throughout this paper we assume that $\varepsilon\geqq r_\varepsilon>0$ and $\varepsilon\to 0$. Let $Y_\varepsilon^i=p_\varepsilon^i+\varepsilon Y$ and $T_\varepsilon^i=p_\varepsilon^i+r_\varepsilon T$, where we denote by p_ε^i, $i\in\boldsymbol{N}$, all the lattice points of edge length ε, by measurement in a parallel direction to each coordinate axis, i.e., $\varepsilon\boldsymbol{Z}^N=\{p_\varepsilon^i\,;\, i\in\boldsymbol{N}\}$. Let Ω' be a non-empty subdomain of Ω and let $\Omega''=\Omega\setminus\overline{\Omega'}$. We assume that the Lebesgue measure of $\partial\Omega'$ is zero. We set $T_\varepsilon=\cup\{T_\varepsilon^i\,;\, Y_\varepsilon^i\subset\overline{\Omega'}\}$ and $\Omega_\varepsilon=\Omega\setminus T_\varepsilon$ (Fig. 0). Then the number n_ε of holes of Ω_ε behaves like $|\Omega'|/\varepsilon^N$ when $\varepsilon\to 0$.

We consider the boundary value problem for $f\in L^2(\Omega)$.

$$-\Delta u_\varepsilon=f \qquad \text{a.e. in} \quad \Omega_\varepsilon. \tag{0.1}$$

$$\partial u_\varepsilon/\partial\nu+\alpha_\varepsilon g_\varepsilon(x, u_\varepsilon(x))=0 \qquad \text{on} \quad \partial T_\varepsilon. \tag{0.2}$$

$$u_\varepsilon=0 \qquad \text{on} \quad \partial\Omega. \tag{0.3}$$

Here ν denotes the outer unit normal vector of the boundary $\partial\Omega_\varepsilon$, α_ε is a positive constant and the function $g_\varepsilon(x, v), x\in\boldsymbol{R}^N, v\in\boldsymbol{R}$, satisfies the condition (m. 1).

(m. 1) g_ε is a function continuously differentiable with respect to v, x_i, $1\leqq i\leqq N$, and is monotonously increasing with respect to v for each x, and satisfies the condition $g_\varepsilon(x, 0)=0$.

We consider the asymptotic behavior of the weak solution u_ε of the problem (0.1)-(0.3) (for the existence and uniqueness of u_ε see, for example, Théorème 1.7 and Remarque I. 19 of H. Brezis [4]), as $\varepsilon\to 0$, where we assume

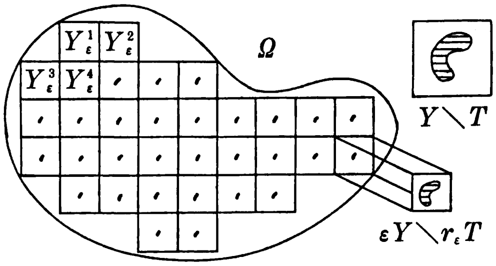

Figure 0

$$\alpha_\varepsilon\to 0,\ \text{a positive constant } C \text{ or } \infty.$$

We need several parameters as follows. Let Λ be a constant such that $0<\Lambda\leqq N$ and let $\theta_{\Lambda,\varepsilon}=r_\varepsilon^\Lambda/\varepsilon^N$. We consider only the sequence $\{\Omega_\varepsilon\}_\varepsilon$ having a limit

$$\theta_\Lambda=\lim_\varepsilon \theta_{\Lambda,\varepsilon},\ 0\leqq\theta_\Lambda\leqq\infty.$$

We have $(r_\varepsilon/\varepsilon)^N=\theta_{\Lambda,\varepsilon}r_\varepsilon^{N-\Lambda}$. We see $\varepsilon\gg r_\varepsilon$ if and only if $\theta_N=0$, while $\varepsilon\sim r_\varepsilon$ if and only if $0<\theta_N\leqq 1$. Generally, $\theta_\lambda=\infty$, if $0<\theta_\Lambda$ and $\lambda<\Lambda$. The value $\theta_{N,\varepsilon}$ behaves like usual volume density of holes as $\varepsilon\to 0$, because n_ε behaves like $|\Omega'|/\varepsilon^N$. We call θ_Λ the "*Λ dimensional volume density of holes*" analogously to the notion of the Λ dimensional Hausdorff measure; cf. Falconer [8]. For (0.2) we need two more parameters, $\bar{\alpha}$ and a. Let $\bar{\alpha}_\varepsilon=\alpha_\varepsilon r_\varepsilon$ and $a_\varepsilon=\alpha_\varepsilon r_\varepsilon^{N-1}/\varepsilon^N$. We assume that $\lim_\varepsilon\bar{\alpha}_\varepsilon$ and $\lim_\varepsilon a_\varepsilon$ exist in $[0,\infty]$. We write

$$\bar{\alpha}=\lim_\varepsilon\bar{\alpha}_\varepsilon \text{ and } a=\lim_\varepsilon a_\varepsilon \text{ with } \bar{\alpha}, a\in[0,\infty].$$

Let $\tilde{u}_\varepsilon\in H^1(\Omega)$ be the extension of u_ε such that $\tilde{u}_\varepsilon|\Omega_\varepsilon=u_\varepsilon$, $-\Delta\tilde{u}_\varepsilon=0$ on T_ε. We show that $\tilde{u}_\varepsilon\overset{w}{\to}u$ in $H_0^1(\Omega)$, where u is the solution of one of the three equations:

(a) $-\Delta u+\theta_{N-2}C_1u=f$,
(b) $-\Delta u+a|\partial T|g(x,u)=f$,
(c) $A^{\mathrm{hom}}u+a|\partial T|g(x,u)=|Y\setminus T|f$,

according to the three cases: (α) $r_\varepsilon/\varepsilon\to 0$ with a rapid way as $\theta_{N-2}<\infty$, (β) $a<\infty$ and $r_\varepsilon/\varepsilon\to 0$ with less rapidity such as $\theta_{N-2}=\infty$, (γ) $r_\varepsilon/\varepsilon\to c, 0<c<1$. Here C_1 is a value like the capacity of T, $|\partial T|$ is the surface area of ∂T and A^{hom} is an elliptic differential operator with constant coefficients. In (β) the precise velocity of $r_\varepsilon/\varepsilon$ decreasing to zero is given by the manner:

$$\theta_{\Lambda(r)}=\infty \text{ with } \Lambda(r)=N(1-2(r(N-2)+N)^{-1}),$$

if $|g(x, u)|=O(|u|^{r+1})$ as $|u|\to\infty$ (note that $\theta_{A(r)}=\infty$ implies $\theta_{N-2}=\infty$). Thus, "critical sizes" (or "special sizes") of holes are drawn as Fig. 1.

$$|g(x, u)|=O(|u|^{r+1}),\ |u|\to\infty$$

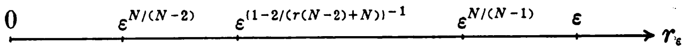

Figure 1

We review some papers around the asymptotic problem for (0.1)-(0.3). Many works are done for the study of the solutions of (0.1) satisfying a linear boundary condition, for example, Ē. Ya. Khruslov [14, 15, 16, 17], Rauch and Taylor [21], Vanninathan [22, 23], Cioranescu and Murat [6], Ozawa [20], Attouch [2] and the author [10]. Generally, in the variational formulation of (0.1)-(0.3), let I_ε be the term of the integral containing the unknown factor u_ε, over the fragmented boundaries of small holes. It is difficult to know the existence of the limit of I_ε, if it exists. Our paper [10] is one of first studies treating this point for the case: $g_\varepsilon(x, v)\equiv v$ in (0.2) and T is a fixed ball. When g_ε is nonlinear in v, we refer to [11, 12], where the forms of g_ε and T are special, and the range of the velocity of $r_\varepsilon/\varepsilon$ decreasing to zero, does not exhaust the whole region. The aim in this paper, is to conquer these points. And we have results (a)-(c), mentioned previously.

If we replace (0.2) with an inhomogeneous linear boundary condition, we also have boundary integrals over the fragmented boundaries of holes. This case is difficult too. Very recently, Conca and Donato [7], Cioranescu and Donato [5] attack this case and obtain similar results but different from ours (compare, for example, Fig. 1 and Fig. 2-(a) in our paper with Figure 1.3 in [7] and the column of "$m_{\partial T}(g)=0$" of the table in Remarque 5.6 in [5], respectively). In (a)-(c), it is noted that a value like capacity and the surface area $|\partial T|$ appear for the cases of the "small holes" and the "big holes", respectively. This also occurs in the "thick Neumann sieve" (see T. Del Vecchio [24]). We owe our main technique to Cioranescu and Murat [6]. The method is applied to the study in more complicated holes by the author [13].

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Contents

Notation. For d dimensional smooth manifold M we write $\|v\|_M=\left\{\int_M |v|^2 dM\right\}^{1/2}$, where dM is the d dimensional volume element in M. For a Banach space X with its dual space X^*, we denote by $\overset{s}{\to}$ the strong convergences in X and X^*. We denote by $\overset{w}{\to}$ and $\overset{w^*}{\to}$ the weak convergence in X and the weak* convergence in X^*, respectively.

§1. Results

We suppose other conditions (m. 2), (m. 3) and (m. 4) for $g_\varepsilon(x, v)$ satisfying (m. 1).

(m. 2) There exist a positive constant c_1 and exponents r, s, such that

$$|\partial g_\varepsilon/\partial v| \leqq c_1(1+|v|^r),$$

and

$$|\partial g_\varepsilon/\partial x_i| \leqq c_1(1+|v|^s), \qquad 1\leqq i\leqq N,$$

where the exponents r, s, satisfy the inequalities

$$\begin{cases} 0\leqq r\leqq N(N-2)^{-1}, \\ 0\leqq s< N(N-2)^{-1}+r. \end{cases}$$

(m. 3)$_g$ There exists a function $g(x, v): \boldsymbol{R}^N\times\boldsymbol{R}\to\boldsymbol{R}$ satisfying the conditions (m. 1) and (m. 2), such that

$$|g_\varepsilon(x, v)-g(x, v)| \leqq c_\varepsilon(1+|v|^t)$$

with $c_\varepsilon\to 0$, where $0\leqq t\leqq N(N-2)^{-1}+r$.

(m. 4) If $g(x, v)=0$, then $v=0$.

Let u_ε be the weak solution of (0.1)-(0.3). Let $\chi_{\Omega'}$ be the characteristic function of Ω'. For $v\in H^1(\Omega_\varepsilon)$ we use the extension $\tilde{v}\in H_0^1(\Omega)$ introduced in § 0.

THEOREM A.1. *We suppose* $\theta_N=0$,

$$0\leqq\bar{\alpha}\leqq\infty \quad \textit{for} \quad 0\leqq\theta_{N-2}<\infty$$

or

$$0<\bar{\alpha}\leqq\infty \quad \textit{for} \quad \theta_{N-2}=\infty \quad \textit{as} \quad \varepsilon\to 0.$$

We suppose the conditions (m. 1), (m. 3)$_v$ *with* $t=0$. *For* $\bar{\alpha}=\infty$, *we further suppose*

(L.1)
$$\limsup_{\varepsilon\to 0}\bar{\alpha}_\varepsilon\|g_\varepsilon(x, v)-v\|_\infty<\infty.$$

Then $\tilde{u}_\varepsilon\overset{w}{\to}u$ *in* $H_0^1(\Omega)$ *and there exists* $C_{\bar{\alpha},T}\in[0, \infty)$ *determined by* $\bar{\alpha}$ *and* T, *satisfying the properties*

$$0=C_{\bar{\alpha}_1,T}<C_{\bar{\alpha}_2,T}<C_{\bar{\alpha}_3,T}<\infty \quad \textit{for} \quad 0=\bar{\alpha}_1<\bar{\alpha}_2<\bar{\alpha}_3,$$

$$0<C_{\bar{\alpha}_3,T}-C_{\bar{\alpha}_2,T}\leqq(\bar{\alpha}_3/\bar{\alpha}_2-1)C_{\bar{\alpha}_2,T}$$

and

$$C_T=C_{\infty,T},$$

where C_T *denotes the capacity of* T *in* $\boldsymbol{R}^N$ *defined by*

$$C_T=\inf\left\{\int_{\boldsymbol{R}^N}|\nabla v|^2dx\,;\ v\in H^1(\boldsymbol{R}^N), v\geqq 1 \text{ on } T\right\}.$$

The function u *is determined by the following equations.*

(1.1)
$$-\Delta u+\theta_{N-2}C_{\bar{\alpha},T}\chi_{\Omega'}u=f \textit{ a.e. in } \Omega \textit{ for } 0\leqq\theta_{N-2}<\infty.$$

(1.2)
$$u|\Omega''\in H_0^1(\Omega''), u|\Omega'=0 \textit{ a.e. on } \Omega'$$

$$\textit{and} \qquad -\Delta u=f \textit{ a.e. on } \Omega'' \textit{ for } \theta_{N-2}=\infty \textit{ and } \Omega'\neq\Omega.$$

(1.3)
$$u=0 \textit{ a.e. in } \Omega \textit{ for } \theta_{N-2}=\infty \textit{ and } \Omega'=\Omega.$$

THEOREM A.2. *We suppose* $\bar{\alpha}=\infty, \theta_N=0$ *and* $0\leqq\theta_{N-2}\leqq\infty$ *as* $\varepsilon\to 0$, *and the conditions* (m. 1) *with the property:*

(L.2) *there exists a positive constant* c_2 *such that*

$$c_2^{-1}|v|\leqq|g_\varepsilon(x, v)|\leqq c_2|v| \quad \textit{for all } \varepsilon, x \textit{ and } v.$$

Then $\tilde{u}_\varepsilon\overset{w}{\to}u$ *in* $H_0^1(\Omega)$, *where* u *is determined by* (1.1) *with* $C_{\bar{\alpha},T}=C_T$ *for* $0\leqq\theta_{N-2}<\infty$, (1.2) *for* $\theta_{N-2}=\infty$ *and* $\Omega'\neq\Omega$, (1.3) *for* $\theta_{N-2}=\infty$ *and* $\Omega'=\Omega$.

The hypothesis $\theta_{N-2}=\infty$ with $0<\bar{\alpha}\leqq\infty$ in Theorems A.1, A.2 implies the hypothesis $\theta_{N-2}=\infty$ with $a=\infty$ in Theorem B below, because $a_\varepsilon=\bar{a}_\varepsilon\theta_{N-2,\varepsilon}$; but the conditions on g_ε in Theorems A.1, A.2 do not imply those on g_ε in Theorem B.

THEOREM B. *We suppose* $\theta_N=0$ *and* $\theta_{\Lambda(r)}=\infty$, *where*

$$\Lambda(r)=N(1-2(r(N-2)+N)^{-1}).$$

We suppose the conditions (m. 1), (m. 2), (m. 3)$_g$ *and* (m. 4) *on* g_ε. *Then* $\tilde{u}_\varepsilon\overset{w}{\to}u$ *in* $H_0^1(\Omega)$ *where* u *is determined by* (1.2) *for* $a=\infty$ *and* $\Omega'\neq\Omega$, (1.3) *for* $a=\infty$ *and* $\Omega'=\Omega$, *and*

$$-\Delta u+a|\partial T|\chi_{\Omega'}g(x,u)=f \quad a.e. \text{ in } \Omega, \text{ for } 0\leqq a<\infty. \tag{1.4}$$

THEOREM C. *We suppose* $0<\theta_N\leqq 1$ *as* $\varepsilon\to 0$, *and the conditions* (m. 1), (m. 2), (m. 3)$_g$ *and* (m. 4) *on* g_ε. *Then* $\tilde{u}_\varepsilon\overset{w}{\to}u$ *in* $H_0^1(\Omega)$, *where* u *is determined by* (1.2) *for* $a=\infty$ *and* $\Omega'\neq\Omega$, (1.3) *for* $a=\infty$ *and* $\Omega'=\Omega$, *and the following equation for* $0\leqq a<\infty$ *and* $\Omega'=\Omega$.

$$-\sum_{i,j=1}^{N}q_{ij}\partial^2u/\partial x_i\partial x_j+a|\partial T|g(x,u)=|Y\backslash\theta_N^{1/N}T|f \quad a.e. \text{ in } \Omega, \tag{1.5}$$

where q_{ij} *are constants defined by* (1.6).

In Theorem *C* we consider the case $a<\infty$ only with the case $\Omega=\Omega'$, for simplicity.

REMARK D. The technique in the proofs of Theorems B, C combining with subdifferentials allows us to remove the regularity of $g_\varepsilon(v)$. If $|g_\varepsilon(v)|=O(|v|^\rho)$, $\rho>0$, as $|v|\to\infty$, and $g_\varepsilon(\cdot)$ has a maximal monotone graph, then limit equations for a sequence of $\{\Omega_\varepsilon\}_\varepsilon$ are constructed.

REMARK E. Let $\tau_0=\theta_N^{1/N}$. The homogenized operator

$$A^{\text{hom}}=-\sum_{i,j=1}^{N}q_{ij}\frac{\partial^2}{\partial x_i\partial x_j}$$

of the minus Laplacian $-\Delta$ is defined through

$$q_{ij}=|Y\backslash\tau_0T|\delta_{ij}-\int_{Y\backslash\tau_0T}\nabla\kappa_0^i\cdot\nabla\kappa_0^j dx, \tag{1.6}$$

where κ_0^i ($\in W_0=\{v\in H^1(Y\backslash\tau_0T)$; v has an extension $\bar{v}\in H^1(Y)$ having period 1 in each variable$\}$) is defined by

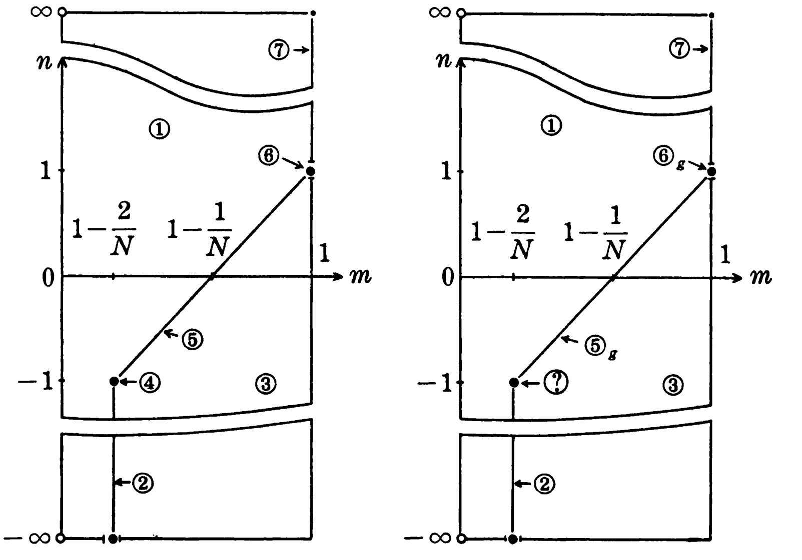
$$\int_{Y\backslash\tau_0T}\nabla\kappa_0^i\nabla v dx+\int_{\tau_0\partial T}\nu_i v dS^{\tau_0T}=0 \quad \text{for} \quad v\in W_0,$$

where $\nu=\{\nu_i\}_{i=1}^N$ is the outer unit normal to $\partial(Y\backslash\tau_0 T)$ and $dS^{\tau_0 T}$ is the standard surface element of $\tau_0\partial T$. In fact, κ_0^i is determined uniquely up to an additive constant (see [23, 24] and p. 282 of [16]).

REMARK F. Let C_1 and C_2 be positive constants. Here we apply Theorems A, B and C to the case $\Omega=\Omega'$ and

$$(1.7)\qquad \begin{cases} \varepsilon=r_\varepsilon^m/C_1, & m\in(0,1], \\ \alpha_\varepsilon=C_2 r_\varepsilon^n, & n\in(-\infty,\infty). \end{cases}$$

For the case $g(x,u)=u$, the linear case, the results are drawn in Figure 2-(a). For the nonlinear case see Figure 2-(b).



(a) (b)

In the above graphs $n=\infty$: the Neumann boundary condition, $n=-\infty$: the Dirichlet boundary condition.

Limit Equations

① $-\Delta u=f$. ② $-\Delta u+C_1^N C_T u=f$. ③ $u=0$. ④ $-\Delta u+C_1^N C(C_2,T)u=f$. ⑤ $-\Delta u+C_1^N C_2|\partial T|u=f$. ⑤$_g$ $-\Delta u+C_1^N C_2|\partial T|g(u)=f$. ⑥ $A^{\mathrm{hom}}u+C_1^N C_2|\partial T|u=|Y\backslash C_1 T|f$, $Y\supseteq C_1 T$. ⑥$_g$ $A^{\mathrm{hom}}u+C_1^N C_2|\partial T|g(u)=|Y\backslash C_1 T|f$, $Y\supseteq C_1 T$. ⑦ $A^{\mathrm{hom}}u=|Y\backslash C_1 T|f$, $Y\supseteq C_1 T$. ? An open problem.

Figure 2

By the definition of θ_A, for m in (1.7), we have $0<\theta_{mN}(=C_1^N)<\infty$. We have $n=mN-N+1$ if and only if $0<a=C_1^N C_2<\infty$. We have $0<\bar{a}=C_2<\infty$ if and only if $n=-1$. By these relations and theorems in this section we have obtained Figure 2. By Theorem A.1 and Theorem A.2 we can get limit equations for $(m, n)\in(0, 1-2/N]\times\boldsymbol{R}\cup(1-2/N, 1)\times(-\infty, -1]$ and $(0, 1)\times(-\infty, -1)$, respectively. By Theorem B we get limit equations for $(m, n)\in(1-2/N, 1)\times\boldsymbol{R}$. By Theorem C, we get limit equations for $(1, n)\in\{1\}\times\boldsymbol{R}$. Thus, the point $(1-2/N, -1)$ is covered only by Theorem A.1.

§ 2. Convergence of measures with fragmented supports I

This section is a preparation of the proof of Theorem B. Let $B^0(\rho)$ be a ball of diameter ρ with center at the origin 0 for a small number $\rho\leqq\varepsilon$ and $B(\rho)=\cup\{B^i(\rho)\,;\, B^i(\rho)=p_\varepsilon^i+B^0(\rho)\subset Y_\varepsilon^i\subset\overline{\Omega'}\}$. We often write $B_\rho=B(\rho)$ and $B_\rho^i=B^i(\rho)$, briefly. We study the measures δ_ε^B, $\delta_{r_\varepsilon}^B$ and δ_ε^T defined by

$$\langle\delta_\rho^B, \zeta\rangle=\int_{\partial B(\rho)}\zeta(x)dS_\rho^B(x),\quad \langle\delta_\varepsilon^T, \zeta\rangle=\int_{\partial T_\varepsilon}\zeta(x)dS_\varepsilon^T(x),$$

for $\zeta\in C_0^\infty(\Omega)$, where dS_ρ^B and dS_ε^T are the standard surface elements of $\partial B(\rho)$ and ∂T_ε, respectively. We often write dS for dS_ρ^B or dS_ε^T, if no confusion occurs. We regard δ_ρ^B, δ_ε^T as linear functionals on the space $C_0^\infty(\Omega)$. Thus, δ_ε^B and δ_ε^T are positive Radon measures on Ω (all Borel sets are measurable for any positive Radon measure). We see $\delta_\varepsilon^B, \delta_{r_\varepsilon}^B, \delta_\varepsilon^T\in W^{-1,p^*}(\Omega)$ by the trace theorem for each fixed ε and $p\geqq 1$, $1/p+1/p^*=1$.

THEOREM 2.1. (Cioranescu-Murat [6]).

$$\varepsilon\delta_\varepsilon^B \xrightarrow{S} 2^{1-N}|S|\chi_{\Omega'}dx \ \ in\ \ W^{-1,\infty}(\Omega)\ \ as\ \ \varepsilon\to 0,$$

where $|S|$ *denotes the surface area of the unit sphere* S.

We generalize this theorem as follows.

THEOREM 2.2. *Let* $p\in[1, N)$. *We suppose* $\theta_{N-p}=\infty$. *Then*

$$\varepsilon^N r_\varepsilon^{1-N}\delta_\varepsilon^T \xrightarrow{S} |\partial T|\chi_{\Omega'}dx \quad in \quad W^{-1,p^*}(\Omega).$$

Theorem 2.1 is obtained by Theorem 2.2 as a special case of $p=1$, $\varepsilon=r_\varepsilon$ and $T=\partial B^0(1)$, the ball of diameter one. The aim of this section is to prove Theorem 2.2. We take a constant κ such that $B^0(\kappa)\supseteqq T$ (κ may be larger than one, although $Y\supset T$).

The proof of Theorem 2.2 is done by the lemmas below.

LEMMA 2.3. *There exist positive Radon measures* ν_ε, *and a positive constant* c_1, *which does not depend on* ε, *such that*

i. $$\|\alpha_\varepsilon\delta_\varepsilon^T-\nu_\varepsilon\|_{-1,p^*}\leqq c_1\alpha_\varepsilon\kappa^{N-1}\theta_{N-p,\varepsilon}^{-1/p},$$

ii. $$0\leqq\nu_\varepsilon\leqq c_1\alpha_\varepsilon\delta_{r_\varepsilon\kappa}^B.$$

LEMMA 2.4. *There exists a constant* c_2, *which does not depend on* ε, *such that*

$$\|\alpha_\varepsilon\delta_{r_\varepsilon\kappa}^B-\kappa^{N-1}\alpha_\varepsilon\varepsilon\delta_\varepsilon^B\|_{-1,p^*}\leqq c_2\alpha_\varepsilon\kappa^{N-1}\theta_{N-p,\varepsilon}^{-1/p}.$$

LEMMA 2.5. *Under the hypotheses in Theorem* 2.2 *we have*

$$\varepsilon^N r_\varepsilon^{1-N}\delta_\varepsilon^T \xrightarrow{w^*} |\partial T|\chi_{\Omega'}dx \text{ in } W^{-1,p^*}(\Omega) \text{ as } \varepsilon\to 0.$$

LEMMA 2.6 (Cioranescu-Murat [6]). *Let* μ_ε *and* $\bar\mu_\varepsilon$ *be positive Borel measures on* Ω, *such that* $0\leqq\mu_\varepsilon\leqq\bar\mu_\varepsilon$.

i. *If* $\bar\mu_\varepsilon\in W^{-1,p^*}(\Omega)$, *then* $\|\mu_\varepsilon\|_{-1,p^*}\leqq\|\bar\mu_\varepsilon\|_{-1,p^*}$.

ii. *If* $\mu_\varepsilon\overset{w^*}{\to}\mu$ *and* $\bar\mu_\varepsilon\overset{s}{\to}\bar\mu$ *in* $W^{-1,p^*}(\Omega)$, *then*

$$\mu_\varepsilon \xrightarrow{s} \mu \quad \text{in} \quad W^{-1,p^*}(\Omega).$$

COROLLARY 2.7. *Let* μ_ε, $\bar\mu_\varepsilon$, μ *and* $\bar\mu$ *be positive Borel measures on* Ω. *We suppose* $\bar\mu_\varepsilon\overset{s}{\to}\bar\mu$ *and* $\mu_\varepsilon\overset{w^*}{\to}\mu$ *as* $\varepsilon\to 0$ *in* $W^{-1,p^*}(\Omega)$. *There exist other positive Borel measures* ν_ε, $\bar\nu_\varepsilon$ *on* Ω *such that there exists a constant c, which does not depend on* ε, *satisfying* $0\leqq\nu_\varepsilon\leqq c\bar\nu_\varepsilon$,

$$\bar\nu_\varepsilon-\bar\mu_\varepsilon \xrightarrow{s} 0 \quad \text{and} \quad \nu_\varepsilon-\mu_\varepsilon \xrightarrow{s} 0$$

as $\varepsilon\to 0$ *in* $W^{-1,p^*}(\Omega)$. *Then* $\mu_\varepsilon\overset{s}{\to}\mu$ *as* $\varepsilon\to 0$ *in* $W^{-1,p^*}(\Omega)$.

PROOF OF THEOREM 2.2. In Lemmas 2.3, 2.4 we set

$$\alpha_\varepsilon=\varepsilon^N r_\varepsilon^{1-N}.$$

Then $a_\varepsilon=1$. Then the right hand sides in the inequalities of Lemma 2.3-i and Lemma 2.4 tend to zero by the assumption $\theta_{N-p}=\infty$. By Theorem 2.1, $\{\kappa^{N-1}\varepsilon\delta_\varepsilon^B\}_\varepsilon$ converges strongly in $W^{-1,p^*}(\Omega)$. We set $\bar\mu_\varepsilon=\kappa^{N-1}\varepsilon\delta_\varepsilon^B$, $\bar\mu=(\kappa/2)^{N-1}|S|\chi_{\Omega'}dx$, $\mu_\varepsilon=\varepsilon^N r_\varepsilon^{1-N}\delta_\varepsilon^T$ and $\mu=|\partial T|\chi_{\Omega'}dx$. Further, we set $\bar\nu_\varepsilon=\varepsilon^N r_\varepsilon^{1-N}\delta_{\kappa r_\varepsilon}^B$. By Lemmas 2.3, 2.4 and 2.5, Theorem 2.2 is implied by Corollary 2.7 with the measures ν_ε in Lemma 2.3.

In the remaining part of this section we prove Lemmas 2.3–2.6. Corollary 2.7 directly follows from Lemma 2.6.

PROOF OF LEMMA 2.3. We take $\zeta\in C_0^\infty(\Omega)$ and fix it in the proof.

Since ∂T is smooth, there exist κ' such that $0<\kappa'<\kappa$, and we can decompose ∂T into J pieces of $N-1$ dimensional manifolds $(\partial T)_j$, $1\leqq j\leqq J$, (usually) having boundaries, such that $\partial T=\cup\{(\partial T)_j\,;\ 1\leqq j\leqq J\}$, $|\partial T|=\sum_{j=1}^{J}|(\partial T)_j|$ (The letter j denotes the number which distinguishes $(\partial T)_j$, while i denotes the letter showing the number of holes ∂T_ε^i, $1\leqq i\leqq n_\varepsilon$.). We assume that *each* $(\partial T)_j$ *is small in order that there exists a point* q^j *of the interior kernel of* $B^0(\kappa')$, *satisfying the conditions* (i) *and* (ii) (see Figure 3).

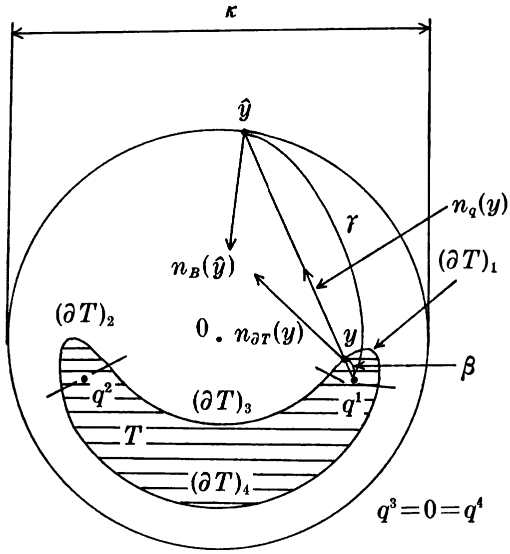


Figure 3

(i) *We take an arbitrarily* $\hat{y}\in\partial B^0(\kappa)$ *and* j, $1\leqq j\leqq J$. *For each segment* $[q^j, \hat{y}]$ *there are the alternatives: either the segment* $[q^j, \hat{y}]$ *does not intersect at any point of* $(\partial T)_j$ *or* $[q^j, \hat{y}]$ *intersects* $(\partial T)_j$ *at one and only one point* y *of* $(\partial T)_j$.

(ii) *For the latter case we have a one-to-one map:* $y\rightarrow\hat{y}$, *from* $(\partial T)_j$ *into* $\partial B^0(\kappa)$. *We set* $\beta(y, j)=|y-q^j|$, $y\in(\partial T)_j$, *and* $\gamma(y, j)=|\hat{y}-q^j|$. *We set* $n_q(y)=(y-q^j)/|y-q^j|$ *and* $n_B(\hat{y})=2\hat{y}/\kappa$. *Let* $n_{\partial T}(y)$ *be the unit normal to the surface* $(\partial T)_j$ *at the point* y. *Then there exist positive constants* κ_1, κ_2 *such that*

$$0<\kappa_1\leqq\beta(y, j), \tag{2.1}$$

$$|(n_q(y), n_{\partial T}(y))|\geqq\kappa_2 \tag{2.2}$$

for all $y\in(\partial T)_j$ *and all* j, $1\leqq j\leqq J$.

In fact, the inequality (2.1) means that (iii) *q^j is not a point of $(\partial T)_j$*. The inequality (2.2) holds, when next four conditions stand. (iv) *$(\partial T)_j$ has an orientation,* (v) *the normal unit vector $n_{\partial T}(y)$ with respect to this orientation, satisfies the Lipschitz condition,*

$$|n_{\partial T}(y_1)-n_{\partial T}(y_2)|\leqq c|y_1-y_2|,\ y_i\in(\partial T)_j,\ i=1,2,$$

with a certain constant c, (vi) *a point q^j is chosen as $||(n_q(y_0), n_{\partial T}(y_0))|-1|>1/2$ with a certain point $y_0\in(\partial T)_j$,* (vii) *the diameter of $(\partial T)_j$ is sufficiently small* (if T is a star-like shape, we can choose $J=1$ and $\partial T=(\partial T)_1$; in this case, the diameter of $(\partial T)_j$ need not be small).

From (2.2) we can derive

$$dS^T(y)\leqq\kappa_2^{-1}dS^B_\kappa(\hat{y})\qquad\text{for all}\quad y\in(\partial T)_j, \tag{2.3}$$

where dS^T and dS^B_κ are the surface elements of $(\partial T)_j$ and $\partial B^0(\kappa)$, respectively.

Now, we transform these notations to each small hole T^i_ε described in §0. That is, for $i=1,2,\cdots,n_\varepsilon$, and $j=1,2,\cdots,J$, we set

$$(\partial T^i_\varepsilon)_j=p^i_\varepsilon+r_\varepsilon(\partial T)_j,\qquad q^{i,j}_\varepsilon=p^i_\varepsilon+r_\varepsilon q^j,$$

and

$$\hat{x}_\varepsilon=q^{i,j}_\varepsilon+r_\varepsilon\gamma\Big(\frac{x-p^i_\varepsilon}{r_\varepsilon},j\Big)n_q\Big(\frac{x-p^i_\varepsilon}{r_\varepsilon}\Big)\in\partial B^i(r_\varepsilon\kappa),$$

where $x\in(\partial T^i_\varepsilon)_j$. Let $\hat{\zeta}(x)=\zeta(\hat{x}_\varepsilon)$ for $\zeta\in C^\infty_0(\Omega)$. We set

$$\beta_\varepsilon(x,i,j)=r_\varepsilon\beta\Big(\frac{x-p^i_\varepsilon}{r_\varepsilon},j\Big)\quad\text{and}\quad\gamma_\varepsilon(x,i,j)=r_\varepsilon\gamma\Big(\frac{x-p^i_\varepsilon}{r_\varepsilon},j\Big).$$

Then we have $\beta_\varepsilon(x,i,j)=|x-q^{i,j}_\varepsilon|$ for $x\in(\partial T^i_\varepsilon)_j$. By (2.1) and (2.3) we get

$$0<\kappa_1 r_\varepsilon\leqq\beta_\varepsilon(x,i,j), \tag{2.4}$$

$$dS^T_\varepsilon(x)\leqq\kappa_2^{-1}dS^B_{\kappa r_\varepsilon}(\hat{x}_\varepsilon), \tag{2.5}$$

for all $x\in(\partial T^i_\varepsilon)_j$, $1\leqq i\leqq n_\varepsilon$ and $1\leqq j\leqq J$.

Under these preparations we set

$$\langle\delta^{T,i,j}_\varepsilon,\zeta\rangle=\int_{(\partial T^i_\varepsilon)_j}\zeta(x)dS^T_\varepsilon(x)$$

and

$$\langle[\delta^{T,i,j}_\varepsilon],\zeta\rangle=\int_{(\partial T^i_\varepsilon)_j}\hat{\zeta}(x)dS^T_\varepsilon(x)$$

for $\zeta\in C_0^\infty(\Omega)$. Then, $\delta_\varepsilon^T=\sum_{i=1}^{n_\varepsilon}\sum_{j=1}^{J}\delta_\varepsilon^{T,i,j}$. Let

$$\nu_\varepsilon=\alpha_\varepsilon\sum_{i=1}^{n_\varepsilon}\sum_{j=1}^{J}[\delta_\varepsilon^{T,i,j}].$$

For small ε, using the Hölder inequality and by (2.5) we get

$$|\langle\alpha_\varepsilon\delta_\varepsilon^T-\nu_\varepsilon,\zeta\rangle|^p\leqq 2(J|\Omega|/\varepsilon^N)^{p-1}\sum_{i,j}|\alpha_\varepsilon\langle[\delta_\varepsilon^{T,i,j}]-\delta_\varepsilon^{T,i,j},\zeta\rangle|^p$$

$$=2(J|\Omega|/\varepsilon^N)^{p-1}\sum_{i,j}\left|\alpha_\varepsilon\int_{(\partial T_\varepsilon^i)_j}(\hat{\zeta}-\zeta)dS\right|^p$$

$$\leqq 2\{\kappa_2^{-1}J|S||\Omega|(\kappa r_\varepsilon/2)^{N-1}\varepsilon^{-N}\}^{p-1}\sum_{i,j}\alpha_\varepsilon^p\int_{(\partial T_\varepsilon^i)_j}|\hat{\zeta}-\zeta|^p dS_\varepsilon^T.$$

Now, by (2.4) and (2.5) we get

$$\int_{(\partial T_\varepsilon^i)_j}|\hat{\zeta}-\zeta|^p dS_\varepsilon^T$$

$$\leqq\kappa_2^{-1}\int_{\partial B^i(\kappa r_\varepsilon)}\left|\int_{\beta_\varepsilon(x,i,j)}^{\gamma_\varepsilon(x,i,j)}\frac{\partial\zeta}{\partial\rho}\rho^{(N-1)/p-(N-1)/p}d\rho\right|^p dS_{\kappa r_\varepsilon}^B$$

$$\leqq|S|C_{N,p}(\kappa r_\varepsilon/2)^{N-1}\|\nabla\zeta\|_{p,Z_\varepsilon^i}^p\{\kappa_2(\kappa_1 r_\varepsilon)^{N-p}\}^{-1},$$

where $Z_\varepsilon^i=B^i(r_\varepsilon\kappa)\cap\Omega$ and $C_{N,p}$ is a constant as follows:

$$C_{N,p}=\begin{cases}1 & \text{for } p=1,\\ \left(\dfrac{p-1}{N-p}\right)^{p-1} & \text{for } p,\ 1<p<N.\end{cases}$$

We can choose $B^0(\kappa)$ as the smallest ball containing Y. Let $G_\varepsilon=\Omega\setminus\bigcup_{i=1}^{n_\varepsilon}Y_\varepsilon^i$. For ε and r_ε such that $0<r_\varepsilon\leqq\varepsilon$, we have

$$\sum_{i,j}\|\nabla\zeta\|_{p,Z_\varepsilon^i}^p\leqq J\sum_i\|\nabla\zeta\|_{p,Z_\varepsilon^i}^p$$

$$\leqq 2NJ\sum_{i=1}^{n_\varepsilon}\|\nabla\zeta\|_{p,Y_\varepsilon^i}^p+J\|\nabla\zeta\|_{p,G_\varepsilon}^p\leqq(2N+1)J\|\nabla\zeta\|_{p,\Omega}^p.$$

Therefore, we obtain

$$|\langle\alpha_\varepsilon\delta_\varepsilon^T-\nu_\varepsilon,\zeta\rangle|^p\leqq c(a_\varepsilon\kappa^{N-1})^p\varepsilon^N r_\varepsilon^{p-N}\|\nabla\zeta\|_{L^p(\Omega)}^p,$$

with a certain constant c. We have shown i by the equality

$$(r_\varepsilon/\varepsilon)^N=r_\varepsilon^p\theta_{N-p,\varepsilon}.$$

We show ii. For $\zeta\in C_0^\infty(\Omega), \zeta\geqq 0$, by (2.5) we get

$$\langle\nu_\varepsilon, \zeta\rangle=\alpha_\varepsilon\sum_{i,j}\int_{(\partial T_\varepsilon^i)_j}\zeta(\hat{x}_\varepsilon)dS_\varepsilon^T(x)$$
$$\leqq\alpha_\varepsilon\kappa_2^{-1}\sum_{i,j}\int_{\partial B^i(\kappa r_\varepsilon)}\zeta(\hat{x}_\varepsilon)dS_{\kappa r_\varepsilon}^B(\hat{x}_\varepsilon)$$
$$=\alpha_\varepsilon\kappa_2^{-1}J\sum_i\int_{\partial B^i(\kappa r_\varepsilon)}\zeta(\hat{x}_\varepsilon)dS_{\kappa r_\varepsilon}^B(\hat{x}_\varepsilon)$$
$$=\alpha_\varepsilon\kappa_2^{-1}J\langle\delta_{\kappa r_\varepsilon}^B, \zeta\rangle.$$

This shows ii. Q. E. D.

PROOF OF LEMMA 2.4. Notice that in the notation $B^i(\varepsilon)$ the value ε denotes the diameter of the ball $B^i(\varepsilon)$, not the radius. We set $\hat{\zeta}(x)=\zeta(\hat{x}_\varepsilon)$ for $x\in\partial B^i(\kappa r_\varepsilon)$, where

$$\hat{x}_\varepsilon=p_\varepsilon^i+\frac{\varepsilon(x-p_\varepsilon^i)}{2|x-p_\varepsilon^i|}\in\partial B^i(\varepsilon).$$

We define positive Radon measures $\delta_{r_\varepsilon\kappa}^{B,i}, [\delta_{r_\varepsilon\kappa}^{B,i}], [\delta_{r_\varepsilon\kappa}^B]$ by

$$\langle\delta_{r_\varepsilon\kappa}^{B,i}, \zeta\rangle=\int_{\partial B^i(r_\varepsilon\kappa)}\zeta(x)dS_{r_\varepsilon\kappa}^B(x),$$
$$\langle[\delta_{r_\varepsilon\kappa}^{B,i}], \zeta\rangle=\int_{\partial B^i(r_\varepsilon\kappa)}\hat{\zeta}(x)dS_{r_\varepsilon\kappa}^B(x)$$

and

$$[\delta_{r_\varepsilon\kappa}^B]=\sum_{i=1}^{n_\varepsilon}[\delta_{r_\varepsilon\kappa}^{B,i}].$$

Then, we have

$$\varepsilon\delta_\varepsilon^B=\varepsilon^N(r_\varepsilon\kappa)^{1-N}[\delta_{r_\varepsilon\kappa}^B].$$

It suffices to show

$$\|\varepsilon^N(r_\varepsilon\kappa)^{1-N}([\delta_{r_\varepsilon\kappa}^B]-\delta_{r_\varepsilon\kappa}^B)\|_{-1,p^*}\leqq c\theta_{N-p,\varepsilon}^{-1/p}$$

with a constant c, which does not depend on ε. For an arbitrary $\zeta\in C_0^\infty(\Omega)$ we have

$$|\varepsilon^N(r_\varepsilon\kappa)^{1-N}\langle[\delta_{r_\varepsilon\kappa}^B]-\delta_{r_\varepsilon\kappa}^B, \zeta\rangle|^p$$
$$\leqq 2\left(\frac{(\kappa r_\varepsilon/2)^{N-1}|S||\Omega|}{\varepsilon^N}\right)^{p-1}\left|\frac{\varepsilon^N}{(r_\varepsilon\kappa)^{N-1}}\right|^p\sum_i\int_{\partial B^i(r_\varepsilon\kappa)}|\hat{\zeta}-\zeta|^p dS_{r_\varepsilon\kappa}^B.$$

Besides,

$$\int_{\partial B^i(r_\varepsilon\kappa)}|\hat{\zeta}-\zeta|^p dS^B_{r_\varepsilon\kappa}$$

$$\leqq C_{N,p}|S|(r_\varepsilon\kappa/2)^{N-1}(r_\varepsilon\kappa_1)^{p-N}\|\nabla\zeta\|^p_{L^p(Y^i_\varepsilon)}.$$

Thus, we have c, which does not depend on ε, such that

$$|\varepsilon^N(r_\varepsilon\kappa)^{1-N}\langle[\delta^B_{r_\varepsilon\kappa}]-\delta^B_{r_\varepsilon\kappa}, \zeta\rangle|^p \leqq c\varepsilon^N r_\varepsilon^{p-N}\|\nabla\zeta\|^p_{L^p(\Omega)}.$$

Multiplying both sides of the above inequality by α_ε we obtain the result.
Q. E. D.

PROOF OF LEMMA 2.5. We see that the set $\{\varepsilon^N/r_\varepsilon^{N-1}\delta^T_\varepsilon\}_\varepsilon$ is bounded in $W^{-1,p^*}(\Omega)$, because of Theorem 2.1 and Lemmas 2.3, 2.4 with $\alpha_\varepsilon=\varepsilon^N/r_\varepsilon^{N-1}$. Therefore, it suffices to show

$$\varepsilon^N r_\varepsilon^{1-N}\langle\delta^T_\varepsilon, \zeta\rangle \longrightarrow |\partial T|\int_{\Omega'}\zeta dx \qquad \text{as} \quad \varepsilon\to 0$$

for $\zeta\in C_0^\infty(\Omega)$. In fact, we have

$$\int_{\Omega'}\zeta dx=\lim_{\varepsilon\to 0}\sum_{i=1}^{n_\varepsilon}\zeta(p^i_\varepsilon)\varepsilon^N$$

and

$$\left|\sum_{i=1}^{n_\varepsilon}\zeta(p^i_\varepsilon)\varepsilon^N-|\partial T|^{-1}\varepsilon^N r_\varepsilon^{1-N}\sum_{i=1}^{n_\varepsilon}\int_{\partial T^i_\varepsilon}\zeta dS^T_\varepsilon\right|$$

$$=|\partial T|^{-1}r_\varepsilon^{1-N}\left|\sum_i\varepsilon^N\int_{\partial T^i_\varepsilon}\{\zeta-\zeta(p^i_\varepsilon)\}dS^T_\varepsilon\right|$$

$$\leqq|\partial T|^{-1}\varepsilon^N r_\varepsilon^{1-N}\sum_i\int_{\partial T^i_\varepsilon}|\zeta-\zeta(p^i_\varepsilon)|dS^T_\varepsilon \longrightarrow 0$$

because ζ is uniformly continuous and we have

$$\int_{\partial T^i_\varepsilon}dS^T_\varepsilon/(r_\varepsilon^{N-1}|\partial T|)=1 \quad \text{and} \quad \sum_{i=1}^{n_\varepsilon}\varepsilon^N\leqq|\Omega'|.$$

This proves the lemma. Q. E. D.

We omit to prove Lemma 2.6 (cf. Cioranescu and Murat [6]).

§ 3. Values like capacity

In this section the symbols ε and $\bar{\alpha}_\varepsilon$ are differently used from those in previous sections.

§ 3.1. Linear boundary condition

We set $B_\varepsilon^0 = B^0(R(\varepsilon))$ and $D_\varepsilon = B_\varepsilon^0 \setminus T$ for $R(\varepsilon) > 0, R(\varepsilon) \uparrow \infty$ as $\varepsilon \downarrow 0$. We define a function $z_\varepsilon \in H^1(D_\varepsilon)$ as follows:

$$(3.1) \qquad \begin{cases} \Delta z_\varepsilon = 0 \quad \text{in} \quad D_\varepsilon, \\ \partial z_\varepsilon / \partial \nu + \bar{\alpha}_\varepsilon z_\varepsilon = 0 \quad \text{on} \quad \partial T, \\ z_\varepsilon = 1 \quad \text{on} \quad \partial B_\varepsilon^0, \end{cases}$$

where $\bar{\alpha}_\varepsilon \in (0, \infty)$ and ν denotes the outer unit normal to ∂D_ε. Let dS^T be the surface element of ∂T. The z_ε is also defined by the minimized problem of

$$J_\varepsilon(v) = \int_{D_\varepsilon} |\nabla v|^2 dx + \bar{\alpha}_\varepsilon \int_{\partial T} |v|^2 dS^T$$

on a convex set $\boldsymbol{K}_\varepsilon = \{v \in H^1(D_\varepsilon)\,;\, v = 1 \text{ on } \partial B_\varepsilon^0 \text{ in } H^1(D_\varepsilon)\}$. $\boldsymbol{K}_\varepsilon$ is closed in $H^1(D_\varepsilon)$ and, for each $k \in [0, \infty)$, the set $J_\varepsilon^{-1}[0, k] \cap \boldsymbol{K}_\varepsilon$ is compact and convex in $L^2(D_\varepsilon)$. J_ε is lower semicontinuous on $\boldsymbol{K}_\varepsilon$, and has a minimizer on $\boldsymbol{K}_\varepsilon$ and this minimizer is determined uniquely by J_ε, because J_ε is strictly convex on $\boldsymbol{K}_\varepsilon$. By z_ε we define a constant by

$$(3.2) \qquad C_{R(\varepsilon)} = J_\varepsilon(z_\varepsilon).$$

Then, using (3.1) and an integration by parts we get

$$(3.2)^* \qquad C_{R(\varepsilon)} = \int_{\partial B_\varepsilon^0} \frac{\partial z_\varepsilon}{\partial r} dS,$$

where r denotes $|x|$, $x \in \boldsymbol{R}^N$. Since z_ε is the minimizer of J_ε on $\boldsymbol{K}_\varepsilon$ and the Sobolev space $H^1(D_\varepsilon)$ is a vector lattice (see p. 15, Kinderlehrer and Stampacchia [18]) we get

$$0 \leqq z_\varepsilon \leqq 1.$$

Actually, if a measurable set $\{x \in D_\varepsilon\,;\, z_\varepsilon(x) < 0\}$ is not a zero set, then we set $v_\varepsilon(x) = (0 \vee z_\varepsilon)(x) \in \boldsymbol{K}_\varepsilon$, $x \in D_\varepsilon$. We see $|\nabla z_\varepsilon| \geqq |\nabla v_\varepsilon|$ a. e. in D_ε and $|z_\varepsilon| \geqq |v_\varepsilon|$ on ∂T. This is a contradiction to a fact that z_ε is the unique minimizer of J_ε on $\boldsymbol{K}_\varepsilon$. If $z_\varepsilon \geqq 1$, we set $v_\varepsilon = 1 \wedge z_\varepsilon \in \boldsymbol{K}_\varepsilon$. We see $|\nabla z_\varepsilon| \geqq |\nabla v_\varepsilon|$ a. e. in D_ε and $z_\varepsilon \geqq v_\varepsilon$ on ∂T. This is a contradiction.

LEMMA 3.1. *Let $\bar{\alpha} \in [0, \infty)$. We assume $\bar{\alpha}_\varepsilon \to \bar{\alpha}$. Then we have $C \in [0, \infty)$ as a limit value of $C_{R(\varepsilon)}$ as $\varepsilon \to 0$.*

PROOF. Let $C_\varepsilon = C_{R(\varepsilon)}$. It suffices to show

$$\liminf_{\varepsilon\to0}\liminf_{\delta\to0}\{C_\varepsilon-C_\delta\}=\liminf_{\varepsilon\to0}C_\varepsilon-\limsup_{\varepsilon\to0}C_\varepsilon$$
$$\geqq 0.$$

For $\varepsilon>\delta>0, z_\varepsilon$ is regarded as an element of $\boldsymbol{K}_\delta$ by extending $z_\varepsilon=1$ on $B_\delta^0\backslash B_\varepsilon^0$. Since $C_\delta=\min\{J_\delta(v)\,;\ v\in\boldsymbol{K}_\delta\}$, we have

$$\begin{aligned}C_\delta&\leqq J_\delta(z_\varepsilon)\\&\leqq J_\varepsilon(z_\varepsilon)+(\bar\alpha_\delta-\bar\alpha_\varepsilon)\int_{\partial T}|z_\varepsilon|^2dS\\&\leqq C_\varepsilon+|\bar\alpha_\delta-\bar\alpha_\varepsilon||\partial T|.\end{aligned}$$

Q. E. D.

Let us give another definition of C_T in Theorem A.1. We regard functions of $H_0^1(B_\varepsilon^0)$ as elements of $H^1(\boldsymbol{R}^N)$ by zero extension outside B_ε^0. Since $\bigcup_{\varepsilon\downarrow0}H_0^1(B_\varepsilon^0)$ is dense in $H^1(\boldsymbol{R}^N)$, we can choose $\{u_\varepsilon\in H_0^1(B_\varepsilon^0)\}_\varepsilon$ such that $u_\varepsilon\geqq1$ on T and $\int_{\boldsymbol{R}^N\backslash T}|\nabla u_\varepsilon|^2dx\downarrow C_T$ as $\varepsilon\downarrow0$. We can suppose $u_\varepsilon\geqq0$, because we set $v_\varepsilon=u_\varepsilon\vee0\in H^1(B_\varepsilon^0)$, then $v_\varepsilon\in H^1(\boldsymbol{R}^N), v_\varepsilon\geqq1$ on ∂T and

$$\int_{\boldsymbol{R}^N\backslash T}|\nabla u_\varepsilon|^2dx\geqq\int_{\boldsymbol{R}^N\backslash T}|\nabla v_\varepsilon|^2dx.$$

We can suppose $u_\varepsilon=1$ on T, because, if we set $\Theta(t)=t$ for $t\leqq1$ and $\Theta(t)=1$ for $t\geqq1$, we see $v_\varepsilon=\Theta(u_\varepsilon)\in H^1(B_\varepsilon^0)$ and $v_\varepsilon|(\boldsymbol{R}^N\backslash B_\varepsilon^0)=0$. We get the above inequality between u_ε and v_ε. Further we can replace each u_ε by v_ε such that $\Delta v_\varepsilon=0$ in $B_\varepsilon^0\backslash T$, $v_\varepsilon=1$ on ∂T and $v_\varepsilon=0$ on ∂B_ε^0. We have $\int_{D_\varepsilon}|\nabla v_\varepsilon|^2dx=\int_{D_\varepsilon}|\nabla z_\varepsilon^D|^2dx\ (\equiv C_\varepsilon^D)$, where z_ε^D is defined by

$$(3.3)\qquad\begin{cases}\Delta z_\varepsilon^D=0 & \text{in}\quad D_\varepsilon,\\ z_\varepsilon^D=0 & \text{on}\quad \partial T,\\ z_\varepsilon^D=1 & \text{on}\quad \partial B_\varepsilon^0.\end{cases}$$

Therefore, we get

$$(3.4)\qquad C_\varepsilon^D\longrightarrow C_T\qquad\text{as}\quad\varepsilon\to0.$$

LEMMA 3.2. *We assume $\bar\alpha_\varepsilon\to\infty$. Then*

$$C_T=\lim_{\varepsilon\to0}C_\varepsilon.$$

PROOF. By an integration by parts and using (3.2)* we have

$$(3.5) \qquad -\int_{\partial T} z_\varepsilon(\partial z_\varepsilon^D/\partial\nu)dS=\int_{\partial B_\varepsilon^0}\{\partial z_\varepsilon^D/\partial r-\partial z_\varepsilon/\partial r\}dS$$

$$=C_\varepsilon^D-C_\varepsilon,$$

where we have used

$$(3.6) \qquad C_\varepsilon^D=\int_{D_\varepsilon}|\nabla z_\varepsilon^D|^2dx=\int_{\partial B_\varepsilon^0}(\partial z_\varepsilon^D/\partial r)dS.$$

By the weak minimum principle (cf. Appendix 1) and $R(\varepsilon)\uparrow\infty$ as $\varepsilon\downarrow 0$, we get $z_\varepsilon^D\geqq z_\delta^D\geqq 0$ on D_ε and $z_\varepsilon^D=0=z_\delta^D$ on ∂T for $\varepsilon>\delta>0$. Thus, by these facts with (3.5) we get

$$(3.7) \qquad \partial z_\varepsilon^D/\partial\nu\leqq\partial z_\delta^D/\partial\nu\leqq 0 \quad \text{on } \partial T \text{ for } \varepsilon>\delta>0$$

and

$$(3.8) \qquad 0\leqq C_\varepsilon\leqq C_\varepsilon^D.$$

Since ∂T is smooth, $\partial z_\varepsilon^D/\partial\nu$ is continuous on ∂T, by (3.7) we get

$$(3.9) \qquad \sup_\varepsilon\|\partial z_\varepsilon^D/\partial\nu\|_{\infty,\partial T}=k_1<\infty.$$

By the Schwarz inequality, (3.5) and (3.9) we have

$$(3.10) \qquad 0\leqq C_\varepsilon^D-C_\varepsilon\leqq k_1|\partial T|^{1/2}\|z_\varepsilon\|_{\partial T}.$$

By (3.2) and the first inequality of (3.10) we see

$$(3.11) \qquad \|z_\varepsilon\|_{\partial T}^2\leqq C_\varepsilon^D/\bar{\alpha}_\varepsilon.$$

By (3.4), (3.10) and (3.11) with $\bar{\alpha}_\varepsilon\to\infty$, the proof is completed. Q. E. D.

LEMMA 3.3. *Let $\bar{R}(\varepsilon)$ be another function such that $\bar{R}(\varepsilon)\uparrow\infty$ as $\varepsilon\downarrow 0$ and $\bar{R}(\varepsilon)\geqq R(\varepsilon)$. Then we have*

$$C_{\bar{R}(\varepsilon)}-C_{R(\varepsilon)}\longrightarrow 0 \text{ as } \varepsilon\to 0$$

provided that $\bar{\alpha}_\varepsilon\to\bar{\alpha}$, $0\leqq\bar{\alpha}\leqq\infty$.

PROOF. For $\bar{\alpha}=\infty$, by Lemma 3.2 the lemma is true. We suppose $0\leqq\bar{\alpha}<\infty$. Let $\bar{z}_\varepsilon$ be the solution of (3.1) with $\bar{R}(\varepsilon)$. Let $\bar{B}_\varepsilon^0=B^0(\bar{R}(\varepsilon))$. By an integration by parts over $\bar{B}_\varepsilon^0\backslash B_\varepsilon^0$ we get

$$-\int_{\partial\bar{B}_\varepsilon}\frac{\partial\bar{z}_\varepsilon}{\partial r}dS+\int_{\partial B_\varepsilon}\frac{\partial\bar{z}_\varepsilon}{\partial r}dS=0.$$

By this equality and an integration by parts over D_ε we get

$$\int_{\partial T}[\bar{z}_\varepsilon \partial z_\varepsilon/\partial\nu - z_\varepsilon \partial\bar{z}_\varepsilon/\partial\nu]dS = \int_{\partial\bar{B}_\varepsilon}(\partial\bar{z}_\varepsilon/\partial r)dS \tag{3.12}$$

$$-\int_{\partial B_\varepsilon}(\partial z_\varepsilon/\partial r)dS + \int_{\partial B_\varepsilon}(1-\bar{z}_\varepsilon)(\partial z_\varepsilon/\partial r)dS.$$

Note that ν is outer normal to $\partial(D\backslash T)$. We set $\bar{C}_\varepsilon = C_{\bar{R}(\varepsilon)}$. By (3.2)*, (3.12) and the same boundary condition of z_ε and $\bar{z}_\varepsilon$ on ∂T we see

$$|C_\varepsilon - \bar{C}_\varepsilon| \leqq C_\varepsilon \sup_{\partial B_\varepsilon}(1-\bar{z}_\varepsilon). \tag{3.13}$$

Choose κ such that $B^0(\kappa) \supset T$. We define $\bar{z}_\varepsilon^D$ by the problem:

$$\begin{cases} \Delta \bar{z}_\varepsilon^D = 0 & \text{on} \quad B^0(\bar{R}(\varepsilon))\backslash B^0(\kappa), \\ \bar{z}_\varepsilon^D = 0 & \text{on} \quad \partial B^0(\kappa), \\ \bar{z}_\varepsilon^D = 1 & \text{on} \quad \partial B^0(\bar{R}(\varepsilon)). \end{cases}$$

Note that κ is the diameter of $B^0(\kappa)$. By the weak minimum principle (see Appendix 1) and a concrete calculation we see

$$\begin{cases} 0 \leqq 1-\bar{z}_\varepsilon \leqq 1-\bar{z}_\varepsilon^D & \text{on} \quad B^0(\bar{R}(\varepsilon))\backslash B^0(\kappa), \\ \bar{z}^D(r) = [\kappa^{2-N} - (2r)^{2-N}]/[\kappa^{2-N} - \bar{R}(\varepsilon)^{2-N}]. \end{cases} \tag{3.14}$$

Therefore, by (3.14), for $\varepsilon \to 0$ we see

$$\sup_{\partial B_\varepsilon}(1-\bar{z}_\varepsilon) \leqq \frac{R(\varepsilon)^{2-N} - \bar{R}(\varepsilon)^{2-N}}{\kappa^{2-N} - \bar{R}(\varepsilon)^{2-N}} \tag{3.15}$$

$$\longrightarrow 0.$$

Q. E. D.

PROPOSITION 3.4. *Let $\bar{\alpha} \in [0, \infty]$. We assume $\bar{\alpha}_\varepsilon \to \bar{\alpha}$. Then there exists a constant $C_{\bar{\alpha},T} \in [0, \infty)$, which does not depend on a function $R(\varepsilon)$, but depends on $\bar{\alpha}$ and T, such that*

$$C_{\bar{\alpha},T} = \lim_{\varepsilon\to 0} C_{R(\varepsilon)}.$$

These constants $C_{\bar{\alpha},T}$ satisfy the following properties.

i. $$0 \leqq C_{\bar{\alpha},T} \leqq \bar{\alpha}|\partial T|.$$

ii. *Let $C_i = C_{\bar{\alpha},T}$ with $\bar{\alpha} = \bar{\alpha}_i$, $i=1,2$. If $0 < \bar{\alpha}_1 < \bar{\alpha}_2 < \infty$, then*

$$0 < C_1 < C_2 < \infty,$$

$$0 < C_2 - C_1 \leqq (\bar{\alpha}_2/\bar{\alpha}_1 - 1)C_1.$$

iii. $$C_T=\lim_{\bar{\alpha}\to\infty} C_{\bar{\alpha},T}\,.$$

iv. *If* $\bar{\alpha}_\varepsilon\to\infty$, *then we have*

$$\lim_{\varepsilon\to 0}\int_{D_\varepsilon}|\nabla z_\varepsilon|^2dx=C_T$$

and

$$\lim_{\varepsilon\to 0}\bar{\alpha}_\varepsilon\int_{\partial T}|z_\varepsilon|^2dS=0\,.$$

PROOF. The first assertion is proved by Lemmas 3.1–3.3. By the definition of $C_{R(\varepsilon)}$ we get the property i.

THE PROOF OF ii. Let $z_{\varepsilon,i}$, $C_{\varepsilon,i}$ and J^i_ε be the solution of (3.1), the constant $C_{R(\varepsilon)}$ and the functional J_ε with $\bar{\alpha}=\bar{\alpha}_i$, respectively, $i=1,2$. Since $z_{\varepsilon,i}$ is the minimizer of J^i_ε on $\boldsymbol{K}_\varepsilon$ and $C_{\varepsilon,i}=J^i_\varepsilon(z_{\varepsilon,i})$, we get

$$\begin{aligned}0&\leqq(\bar{\alpha}_2-\bar{\alpha}_1)\int_{\partial T}|z_{\varepsilon,2}|^2dS\\&=J^2_\varepsilon(z_{\varepsilon,2})-J^1_\varepsilon(z_{\varepsilon,2})\leqq C_{\varepsilon,2}-C_{\varepsilon,1}\leqq J^2_\varepsilon(z_{\varepsilon,1})-J^1_\varepsilon(z_{\varepsilon,1})\\&\leqq(\bar{\alpha}_2/\bar{\alpha}_1-1)\bar{\alpha}_1\int_{\partial T}|z_{\varepsilon,1}|^2dS\leqq(\bar{\alpha}_2/\bar{\alpha}_1-1)C_{\varepsilon,1}\,.\end{aligned}$$

Decreasing ε to zero, we have

$$0\leqq C_2-C_1\leqq(\bar{\alpha}_2/\bar{\alpha}_1-1)C_1\,.$$

Thus, if there exists $\bar{\alpha}_1>0$ such that $C_1=0$, then, for any $\bar{\alpha}_2\geqq\bar{\alpha}_1$, we have $C_2=0$. This contradicts iii (we assume iii for a moment). Thus, for any $\bar{\alpha}_1>0$, we have $C_1>0$. Besides, we show $C_2-C_1>0$. Since

$$(\bar{\alpha}_2-\bar{\alpha}_1)\liminf_{\varepsilon\to 0}\int_{\partial T}|z_{\varepsilon,2}|^2dS\leqq C_2-C_1\,,$$

it suffices to show

$$\liminf_{\varepsilon\to 0}\int_{\partial T}|z_{\varepsilon,2}|^2dS^T>0\,.$$

If $\|z_{\varepsilon,2}\|_{\partial T}\to 0$ as $\varepsilon\to 0$, then $\int|z_{\varepsilon,2}|dS^T\to 0$. Since $z_{\varepsilon,2}=|z_{\varepsilon,2}|$, we get $\bar{\alpha}_2\int_{\partial T}z_{\varepsilon,2}dS$ $\to 0$ as $\varepsilon\to 0$. By (3.1) we get $\int_{\partial T}(\partial z_{\varepsilon,2}/\partial\nu)dS\to 0$ as $\varepsilon\to 0$. By (3.1) and (3.2)* we get

$$-\int_{\partial T}\frac{\partial z_{\varepsilon,2}}{\partial\nu}dS=\int_{\partial B_{\varepsilon}^{0}}\frac{\partial z_{\varepsilon,2}}{\partial r}dS=C_{\varepsilon,2}\to 0.$$

This contradicts that $C_2>0$, and we have $C_2-C_1>0$. Q.E.D.

THE PROOF OF iii. Let $z_{\varepsilon,2}^{D}$ and $C_{\varepsilon,2}^{D}$ be the function in (3.3) and the constant (3.6) with $\varepsilon=\varepsilon_2$. Let $z_{\varepsilon,1}$ and $C_{\varepsilon,1}$ be the solution of (3.1) and the constant associated with $\varepsilon=\varepsilon_1$. By an integrations by parts we get

$$-\int_{\partial T}z_{\varepsilon,1}(\partial z_{\varepsilon,2}^{D}/\partial\nu)dS=\int_{\partial B_2^0}(\partial z_{\varepsilon,2}^{D}/\partial r)dS$$
$$+\int_{\partial B_1^0}(1-z_{\varepsilon,2}^{D})(\partial z_{\varepsilon,1}/\partial r)dS-\int_{\partial B_1^0}(\partial z_{\varepsilon,1}/\partial r)dS,$$

where $B_i^0=B^0(R(\varepsilon_i)), i=1,2$. By (3.2)* and (3.6) we get

$$|C_{\varepsilon,2}^{D}-C_{\varepsilon,1}|\leqq\|\partial z_{\varepsilon,2}^{D}/\partial\nu\|_{\infty,\partial T}\|z_{\varepsilon,1}\|_{\partial T}|\partial T|^{1/2}+\sup_{\partial B_1^0}(1-z_{\varepsilon,2}^{D})C_{\varepsilon,1}.$$

By the weak minimum principle we get $z_{\varepsilon,2}^{D}\geqq\bar{z}_{\varepsilon,2}^{D}$, where $\bar{z}_{\varepsilon,2}^{D}$ is a function $\bar{z}_{\varepsilon}^{D}$ in (3.14) with $\underline{R}(\varepsilon)=R(\varepsilon_2)$. We get the inequality induced from (3.15) by putting $z_{\varepsilon,2}^{D}$ in the place of $\bar{z}_{\varepsilon}$. By (3.9), (3.11) and (3.15) with $R(\varepsilon_1)$ and $R(\varepsilon_2)$, we get

$$|C_T-C_{\varepsilon,1}|=\lim_{\varepsilon_2\to 0}|C_{\varepsilon,2}^{D}-C_{\varepsilon,1}|$$
$$\leqq k_1(C_{\varepsilon,1}^{D}|\partial T|/\bar{\alpha}_{\varepsilon,1})^{1/2}+(R(\varepsilon_1)/\kappa)^{2-N}C_{\varepsilon,1}.$$

Thus, by (3.4) and $R(\varepsilon_1)\uparrow\infty$ as $\varepsilon_1\downarrow 0$ we get

$$|C_T-C_{\bar{\alpha},T}|=\lim_{\varepsilon_1\to 0}|C_T-C_{\varepsilon,1}|$$
$$\leqq k_1(C_T|\partial T|/\bar{\alpha})^{1/2}.$$

THE PROOF OF iv. For any harmonic function u vanishing at infinity we have well known facts as follows (cf. Theorems (2.69)-(2.73), pp. 144-149, of Folland [9]).

$$(3.16)\quad\begin{cases}\text{(i)}\quad |u(x)|=O(|x|^{2-N}) & \text{as } |x|\to\infty,\\ \text{(ii)}\quad |\partial u(x)/\partial r|=O(|x|^{1-N}) & \text{as } |x|\to\infty.\end{cases}$$

Let X be the space of locally summable functions u such that $\nabla u\in L^2(\boldsymbol{R}^N\setminus T)^N$ and $u(x)\to 0$ as $|x|\to\infty$, with the norm

$$\|v\|_X=\{\|\nabla v\|_{\boldsymbol{R}^N\setminus T}^2+\|v\|_{\partial T}^2\}^{1/2}.$$

The space X is a Banach space containing z_ε, 1, by (3.2), (3.4) and (3.8). By (3.4), (3.8) $\{z_\varepsilon\}_\varepsilon$ is bounded in X. We set $v_\varepsilon=1-z_\varepsilon$. We can take a subsequence, still denoted by $\{v_\varepsilon\}_\varepsilon$, convergent weakly in X to v_0. The function v_0 is harmonic on $\boldsymbol{R}^N\backslash T$. By (3.4), (3.11) and $\bar{\alpha}=\infty$, we see $v_0=1$ on ∂T. By (3.16) and an integration by parts we see that v_0 is uniquely determined in X. So, we have $v_\varepsilon \xrightarrow{w} v_0$ in X as $\varepsilon\to 0$. By Lemma 3.2 it suffices to show the first equality of iv. We set

$$Q=\int_{R^N\backslash T}|\nabla v_0|^2dx\,.$$

Using the Fatou lemma and also by Lemma 3.2 we see

$$Q\leqq\liminf_{\varepsilon\to 0}\int_{T^C}|\nabla v_\varepsilon|^2dx\leqq C_T\,.$$

Thus, it suffices to show that $Q=C_T$. Next we show

$$Q=\int_{\partial T}\frac{\partial v_0}{\partial\nu}dS\,. \tag{3.17}$$

Actually, by $v_0=1$ on ∂T and an integration by parts, we have

$$\int_{\partial T}\frac{\partial v_0}{\partial\nu}dS=\int_{D_\varepsilon}|\nabla v_0|^2dx-\int_{\partial B_\varepsilon^0}v_0\frac{\partial v_0}{\partial r}dS\,.$$

By (i) and (ii) of (3.16) we see that the second term of the right side of the above equality tends to zero. We get (3.17). We show $Q=C_T$. Let $v_\varepsilon^D=1-z_\varepsilon^D$. By $v_0=v_\varepsilon^D=1$ on ∂T and an integration by parts we have

$$\int_{\partial T}\left[\frac{\partial v_\varepsilon^D}{\partial\nu}-\frac{\partial v_0}{\partial\nu}\right]dS=\int_{\partial B_\varepsilon^0}\left[v_\varepsilon^D\frac{\partial v_0}{\partial r}-v_0\frac{\partial v_\varepsilon^D}{\partial r}\right]dS=-\int_{\partial B_\varepsilon^0}v_0\frac{\partial v_\varepsilon^D}{\partial r}dS\,. \tag{3.18}$$

By (3.6) and an integration by parts we get

$$\int_{\partial T}\frac{\partial v_\varepsilon^D}{\partial\nu}dS=-\int_{\partial B_\varepsilon^0}\frac{\partial v_\varepsilon^D}{\partial r}dS=\int_{\partial T}\frac{\partial z_\varepsilon^D}{\partial\nu}dS=-C_\varepsilon^D\,.$$

By $v_\varepsilon^D\geqq 0$ on D_ε and $v_\varepsilon^D=0$ on ∂B_ε^0, we see $\partial v_\varepsilon^D/\partial r=-|\partial v_\varepsilon^D/\partial r|$ on ∂B_ε^0. By (3.17), (3.18) and $v_0\partial v_\varepsilon^D/\partial r\leqq 0$ on ∂B_ε^0, we get

$$|C_\varepsilon^D-Q|\leqq C_\varepsilon^D\sup_{\partial B_\varepsilon^0}v_0\,.$$

By (3.4) and (3.16)-(i) we get the desired conclusion. Q.E.D.

§ 3.2. Semilinear boundary condition

We extend Proposition 3.4 to harmonic functions satisfying a semilinear boundary condition on ∂T. We consider the problem:

$$(3.19)\qquad \begin{cases} -\Delta w_\varepsilon^* = 0 & \text{on } D_\varepsilon, \\ \partial w_\varepsilon^*/\partial\nu + \bar{\alpha}_\varepsilon g_\varepsilon^*(x, w^*) = 0 & \text{on } \partial T, \\ w_\varepsilon^* = 1 & \text{on } \partial B^0(R(\varepsilon)). \end{cases}$$

Here, g_ε^* satisfies the hypothesis (m. 1). We set

$$(3.20)\qquad C_{R(r)}^* = \int_{D_\varepsilon} |\nabla w_\varepsilon^*|^2 dx + \bar{\alpha}_\varepsilon \int_{\partial T} w_\varepsilon^* g_\varepsilon^*(x, w_\varepsilon^*) dS.$$

We often write $C_\varepsilon^* = C_{R(r)}^*$. By an integration by parts with (3.19) we have

$$(3.20)^*\qquad C_{R(\varepsilon)}^* = \int_{\partial B_\varepsilon^0} \frac{\partial w_\varepsilon^*}{\partial r} dS.$$

PROPOSITION 3.5. *We suppose the hypothesis* $\bar{\alpha}_\varepsilon \to \bar{\alpha}$, $0 \leqq \bar{\alpha} \leqq \infty$. *Then, we have*

$$(3.21)\qquad 0 \leqq \limsup_{\varepsilon\to 0} C_\varepsilon^* \leqq C_T.$$

Let $h_\varepsilon = \| g_\varepsilon^*(x, v) - v \|_\infty$. *Then we have*

$$(3.22)\qquad \limsup_{\varepsilon\to 0} |C_\varepsilon - C_\varepsilon^*| \leqq \bar{\alpha} |\partial T| \limsup_{\varepsilon\to 0} h_\varepsilon,$$

for $0 \leqq \bar{\alpha} < \infty$, *and*

$$(3.23)\qquad \limsup_{\varepsilon\to 0} |C_\varepsilon^D - C_\varepsilon^*| \leqq 4k_1 |\partial T|^{1/2} \limsup_{\varepsilon\to 0} \max\{(C_T/\bar{\alpha})^{1/2}, |\partial T|^{1/2} h_\varepsilon\},$$

for $0 < \bar{\alpha} \leqq \infty$, *where* k_1 *denotes a constant in* (3.9). *In the case* $\bar{\alpha} = \infty$ *with the additional condition* $h_\varepsilon \to 0$ *we have*

$$(3.24)\qquad \begin{cases} C_T = \lim\limits_{\bar{\alpha}_\varepsilon\to\infty} \int_{D_\varepsilon} |\nabla w_\varepsilon^*|^2 dx, \\ 0 = \lim\limits_{\bar{\alpha}_\varepsilon\to\infty} \bar{\alpha}_\varepsilon \int_{\partial T} w_\varepsilon^* g_\varepsilon(x, w_\varepsilon^*) dS. \end{cases}$$

PROOF. Let z_ε^D and C_ε^D be the function and the constant in the proof of Lemma 3.2. By an integration by parts and the boundary condition on ∂B_ε^0, (3.6) and (3.20)* we get

$$(3.25)\qquad 0\leqq-\int_{\partial T}w_{\varepsilon}^{*}(\partial z_{\varepsilon}^{D}/\partial\nu)dS$$

$$=\int_{\partial B_{\varepsilon}^{0}}[w_{\varepsilon}^{*}\partial z^{D}/\partial r-z_{\varepsilon}^{D}\partial w_{\varepsilon}^{*}/\partial r]dS=C_{\varepsilon}^{D}-C_{\varepsilon}^{*}.$$

The first assertion follows from (3.25). By an integration by parts, we get

$$\bar{\alpha}_{\varepsilon}\int_{\partial T}z_{\varepsilon}[g_{\varepsilon}^{*}(x, w_{\varepsilon}^{*})-w_{\varepsilon}^{*}]dS=\int_{\partial T}[w_{\varepsilon}^{*}\partial z_{\varepsilon}/\partial\nu-z_{\varepsilon}\partial w_{\varepsilon}^{*}/\partial\nu]dS$$

$$=\int_{\partial B_{\varepsilon}^{0}}[\partial w_{\varepsilon}^{*}/\partial r-\partial z_{\varepsilon}/\partial r]dS=C_{R(\varepsilon)}^{*}-C_{R(\varepsilon)}.$$

By these equalities we get (3.22) for $0\leqq\bar{\alpha}<\infty$. We see that (3.23) stands by (3.25) and the lemma below.

LEMMA 3.6. *We assume* $0<\bar{\alpha}\leqq\infty$. *Then*

$$\limsup_{\varepsilon\to 0}\|w_{\varepsilon}^{*}\|_{\partial T}\leqq 2\max\{(2C_{T}/\bar{\alpha})^{1/2}, 2|\partial T|^{1/2}\limsup_{\varepsilon\to 0}h_{\varepsilon}\}.$$

PROOF. We divide ∂T into the following two parts:

$$\partial T_{1}=\{x\in\partial T\,;\ |g_{\varepsilon}^{*}(x, w_{\varepsilon}^{*})|\geqq|w_{\varepsilon}^{*}|/2\}$$

and

$$\partial T_{2}=\{x\in\partial T\,;\ |g_{\varepsilon}^{*}(x, w_{\varepsilon}^{*})|<|w_{\varepsilon}^{*}|/2\}.$$

By the definition of the constant C_{ε}^{*} we have

$$\limsup_{\varepsilon}\int_{\partial T_{1}}|w_{\varepsilon}^{*}|^{2}dS\leqq 2C_{T}\limsup_{\varepsilon}\bar{\alpha}_{\varepsilon}^{-1},$$

while we have $0\leqq w_{\varepsilon}^{*}\leqq 2|g_{\varepsilon}^{*}(x, w_{\varepsilon}^{*})-w_{\varepsilon}^{*}|$ for $x\in\partial T_{2}$. Thus,

$$\limsup_{\varepsilon}\int_{\partial T_{2}}|w_{\varepsilon}^{*}|^{2}dS\leqq 4|\partial T|\limsup_{\varepsilon}h_{\varepsilon}^{2}.$$

These inequalities show the assertion. Q. E. D.

Finally, we show (3.24). It suffices to show the first equality. By the above lemma we see $\|w_{\varepsilon}^{*}\|_{\partial T}\to 0$ as $\varepsilon\to 0$ for $\bar{\alpha}=\infty$. We regard $v_{\varepsilon}^{*}(=1-w_{\varepsilon}^{*})$ as an element of X with zero extension outside B_{ε}^{0}, where X is the same space as in the proof of Proposition 3.4-iv. Then we see that $v_{\varepsilon}^{*}\overset{w}{\to}v_{0}$ in X, where v_{0} is the same one as in the proof of Proposition 3.4-iv. By the Fatou lemma, $Q=C_{T}$ and (3.21) we see

$$C_T=\int_{R^N\setminus T}|\nabla v_0|^2dx\leqq\liminf_{\varepsilon\to 0}\int_{R^N\setminus T}|\nabla v_\varepsilon^*|^2dx\leqq C_T.$$

Thus, we have proved the first equality of (3.24), so, the second equality of (3.24). Q. E. D.

REMARK 3.7. By the proofs of Lemma 3.6 and (3.24) we see that the following statement is true (we omit to prove it):

Suppose that there exists a positive constant k such that

$$k|v|\leqq|g_\varepsilon^*(x, v)| \qquad \text{for all } \varepsilon, x \text{ and } v.$$

Then we have

$$\limsup|C_\varepsilon^D-C_\varepsilon^*|\leqq k_1(C_T|\partial T|/(k\bar{\alpha}))^{1/2} \tag{3.26}$$

for $0<\bar{\alpha}\leqq\infty$. *Further, we also have* (3.24) *for* $\bar{\alpha}=\infty$.

§ 4. Convergence of measures with fragmented supports II

Let $w_\varepsilon\in H^1(\Omega_\varepsilon)$ be the function defined by

$$\begin{cases} \Delta w_\varepsilon=0 & \text{on } B_\varepsilon\setminus T_\varepsilon, \\ \partial w_\varepsilon/\partial\nu+\alpha_\varepsilon g_\varepsilon(x, w_\varepsilon)=0 & \text{on } \partial T_\varepsilon, \\ w_\varepsilon=1 & \text{on } \Omega\setminus B_\varepsilon, \end{cases} \tag{4.1}$$

where $\alpha_\varepsilon\in(0, \infty)$ and g_ε satisfies the hypothesis (m. 1). In this section we write $\theta=\theta_{N-2}$ and $\theta_\varepsilon=\theta_{N-2,\varepsilon}$ for simplicity.

LEMMA 4.1. *We suppose* (m. 1) *and* $0\leqq\theta\leqq\infty$. *Then we get*

$$\begin{cases} \limsup_{\varepsilon\to 0}\|\nabla w_\varepsilon\|_{\Omega_\varepsilon}^2\leqq\theta C_T|\Omega'| & \text{if } \theta<\infty, \\ \limsup_{\varepsilon\to 0}\theta_\varepsilon^{-1}\|\nabla w_\varepsilon\|_{\Omega_\varepsilon}^2\leqq C_T|\Omega'| & \text{if } \theta=\infty. \end{cases}$$

PROOF. Let $x^*=(x-p_\varepsilon^i)/r_\varepsilon$ and $w_{\varepsilon,i}^*(x^*)=w_\varepsilon(x)$, $x\in B^i(\varepsilon)$. Then, $w_{\varepsilon,i}^*$ satisfies (3.19) with $g_{\varepsilon,i}^*(x^*, v)=g_\varepsilon(x, v)$, $\bar{\alpha}_\varepsilon=\alpha_\varepsilon r_\varepsilon$ and D_ε with $R(\varepsilon)=\varepsilon/r_\varepsilon$. By (3.21) we get

$$\begin{aligned}\limsup_{\varepsilon\to 0}\int_{\Omega_\varepsilon}|\nabla w_\varepsilon|^2dx&=\limsup_{\varepsilon\to 0}\sum_{i=1}^{n_\varepsilon}\int_{B^i(\varepsilon)\setminus T_\varepsilon^i}|\nabla w_\varepsilon|^2dx\\&=\limsup_{\varepsilon\to 0} r_\varepsilon^{N-2}\sum_i\int_{D_\varepsilon}|\nabla w_{\varepsilon,i}^*|^2dx^*\leqq|\Omega'|\theta C_T.\end{aligned}$$

Here we have used

$$\lim_{\varepsilon} r_{\varepsilon}^{N-2} \sum_{i=1}^{n_{\varepsilon}} 1 = |\Omega'|\theta . \tag{4.2}$$

This shows the first assertion. For $\theta=\infty$, we divide these inequalities by θ_{ε}, then we decrease ε to zero. The second assertion is proved. Q. E. D.

LEMMA 4.2. *There exists a constant c_1 such that*

i. $$\|\nabla\tilde{v}\|_T \leqq c_1\|\nabla v\|_{Y\setminus T} ,$$

ii. $$\|\tilde{v}\|_T \leqq c_1(\|\nabla v\|_{Y\setminus T} + \|v\|_{Y\setminus T}) ,$$

for all $v\in H^1(Y\setminus T)$, where $\tilde{v}\in H^1(Y)$, is defined by

$$\Delta\tilde{v}=0 \quad on\ T \quad and \quad \tilde{v}=v \quad on\ \partial T .$$

This lemma is proved by the standard theory of the elliptic operators.

COROLLARY 4.3 (Uniform extension property). *There exists a constant c such that, for any $v\in H^1(\Omega_{\varepsilon})$, the extension $\tilde{v}\in H^1(\Omega)$ introduced in* § 0, *satisfies the following inequalities.*

$$\|\nabla\tilde{v}\|_{T_{\varepsilon}} \leqq c\|\nabla v\|_{\Omega_{\varepsilon}} ,$$

$$\|\tilde{v}\|_{T_{\varepsilon}} \leqq c(\|\nabla v\|_{\Omega_{\varepsilon}} + \|v\|_{\Omega_{\varepsilon}}) .$$

PROOF. By the method of scaling (Example 1, p. 40, of Rauch-Taylor [21]) and Lemma 4.2 we get

$$\int_{T_{\varepsilon}^i} |\nabla\tilde{v}|^2 dx \leqq c_1^2 \int_{p_{\varepsilon}^i + r_{\varepsilon}(Y\setminus T)} |\nabla v|^2 dx \leqq c_1^2 \int_{Y_{\varepsilon}^i\setminus T_{\varepsilon}^i} |\nabla v|^2 dx ,$$

$$\int_{T_{\varepsilon}^i} |\tilde{v}|^2 dx \leqq c_1^2 \int_{p_{\varepsilon}^i + r_{\varepsilon}(Y\setminus T)} (r_{\varepsilon}^2 |\nabla v|^2 + |v|^2) dx$$

$$\leqq c_1^2 \int_{Y_{\varepsilon}^i\setminus T_{\varepsilon}^i} (r_{\varepsilon}^2 |\nabla v|^2 + |v|^2) dx ,$$

where this constant c_1 is the same one as in Lemma 4.2. Summing up these inequalities for $i=1, 2, \cdots, n_{\varepsilon}$, we get the result. Q. E. D.

We note that the space $C_0^{\infty}(\Omega'\cup\Omega'')$ is dense in $L^1(\Omega)$ provided that the Lebesgue measure of $\partial\Omega'$ is zero. So, by the method in Cioranescu and Murat [6], combined with Lemma 4.1 we get the following lemma.

PROPOSITION 4.4. *We suppose* (m. 1), $0\leqq\theta\leqq\infty$ *and that the Lebesgue measure of* $\partial\Omega'$ *is zero. Then we have*

$$\begin{cases} \tilde{w}_\varepsilon \xrightarrow{w} 1 \quad in\ H^1(\Omega), & if\ \theta<\infty, \\ \theta_\varepsilon^{-1}\tilde{w}_\varepsilon \xrightarrow{w} 0 \quad in\ H^1(\Omega), & if\ \theta=\infty. \end{cases}$$

Let $\partial/\partial r$ denotes the outer normal derivative on the boundary $\partial B(\varepsilon)$.

THEOREM 4.5. *We suppose* (m. 1) *and* $0\leqq\theta\leqq\infty$. *If*

(i) *the conditions* $0\leqq\bar{\alpha}\leqq\infty$ *and* $(\text{m}.3)_v$ *with the exponent* $t=0$ *are satisfied,*

or

(ii) *the conditions* $\bar{\alpha}=\infty$ *and* (L.2) *are satisfied.*

Then we have

$$\frac{\partial w_\varepsilon}{\partial r}\delta_\varepsilon^B \xrightarrow{s} \theta C_{\bar{\alpha},T}\chi_{\Omega'}dx \quad in\ W^{-1,\infty}(\Omega), \quad if\ 0\leqq\theta<\infty,$$

$$\theta_\varepsilon^{-1}\frac{\partial w_\varepsilon}{\partial r}\delta_\varepsilon^B \xrightarrow{s} C_{\bar{\alpha},T}\chi_{\Omega'}dx \quad in\ W^{-1,\infty}(\Omega) \quad if\ \theta=\infty.$$

PROOF. For $0\leqq\theta<\infty$ we define $w_\varepsilon^D\in H^1(\Omega_\varepsilon)$ by

$$\begin{cases} \Delta w_\varepsilon^D=0 \quad \text{on } B(\varepsilon)\backslash B(r_\varepsilon\kappa), \\ w_\varepsilon^D=1 \quad \text{on } \Omega\backslash B(\varepsilon) \quad \text{and} \quad w_\varepsilon^D=0 \quad \text{on } \partial B(r_\varepsilon\kappa), \end{cases}$$

where κ is a constant such that $B^0(\kappa)\supset T$. We see exactly w_ε^D by a calculation (see (3.14)). By the minimum principle (cf. Appendix 1) we get $0\leqq w_\varepsilon^D\leqq w_\varepsilon\leqq 1$ on $B(\varepsilon)\backslash B(r_\varepsilon\kappa)$. Combining this fact with the fact that $w_\varepsilon=w_\varepsilon^D=1$ on $\partial B(\varepsilon)$ we get

$$0\leqq\frac{\partial\tilde{w}_\varepsilon}{\partial r}\delta_\varepsilon^B\leqq\frac{\partial\tilde{w}_\varepsilon^D}{\partial r}\delta_\varepsilon^B = 2(N-2)\kappa^{N-2}\theta_\varepsilon(1-(r_\varepsilon\kappa/\varepsilon)^{N-2})^{-1}\varepsilon\delta_\varepsilon^B. \tag{4.3}$$

In (4.3) note that parameters ε, $r_\varepsilon\kappa$ denote diameters of $B^i(\varepsilon)$ and $B^i(r_\varepsilon\kappa)$. By Lemma 2.6-i, Theorem 2.1 and $\theta<\infty$, $\{(\partial w/\partial r)\varepsilon\delta_\varepsilon^B\}_\varepsilon$ is bounded in $W^{-1,\infty}(\Omega)$. By Lemma 2.6-ii it suffices to show

$$\left\langle \frac{\partial w_\varepsilon}{\partial r}\delta_\varepsilon^B, \zeta\right\rangle \longrightarrow \theta C_{\bar{\alpha},T}\int_{\Omega'}\zeta dx, \quad \zeta\in C_0^\infty(\Omega).$$

The difficulty lies in the point that g_ε depends on $x\in \boldsymbol{R}^N$. We have

$$\left|\left\langle \frac{\partial w_\varepsilon}{\partial r}\delta_\varepsilon^B, \zeta\right\rangle - \theta C_{\bar{\alpha},T}\int_{\Omega'}\zeta dx\right| \leqq I_0+I_1+I_2+I_3,$$

where

$$I_0=\sum_i\int_{\partial B_\varepsilon^i}|\zeta(x)-\zeta(p_\varepsilon^i)|\frac{\partial w_\varepsilon}{\partial r}dS,$$

$$I_1=|\theta_\varepsilon-\theta|\varepsilon^N\sum_i|\zeta(p_\varepsilon^i)|C_\varepsilon^{*,i},$$

$$I_2=\theta\varepsilon^N\sum_i|\zeta(p_\varepsilon^i)(C_\varepsilon^{*,i}-C_{\bar{\alpha},T})|,$$

$$I_3=\theta C_{\bar{\alpha},T}\left|\varepsilon^N\sum_i\zeta(p_\varepsilon^i)-\int_{\Omega'}\zeta dx\right|,$$

and $C_\varepsilon^{*,i}$ is given by

$$\int_{\partial B_\varepsilon^i}\frac{\partial w_\varepsilon}{\partial r}dS=r_\varepsilon^{N-2}\left\{\int_{D_\varepsilon}|\nabla w_\varepsilon^*|^2dx+\bar{\alpha}_\varepsilon\int_{\partial T}w_\varepsilon^*g_\varepsilon^*(x^*, w_\varepsilon^*)dS\right\} \tag{4.4}$$
$$=\varepsilon^N\theta_\varepsilon C_\varepsilon^{*,i},$$

where w_ε^* and g_ε^* may be different each other for $i=1, 2, \cdots, n_\varepsilon$. Here we use the notations w_ε^*, x^*, in the proof of Lemma 4.1. $I_0\to 0$ as $\varepsilon\to 0$ follows from (4.4), (3.21) and the uniform continuity of ζ. By (3.21), $\|\zeta\|_\infty<\infty$ and the definition of θ, we see $I_1\to 0$ as $\varepsilon\to 0$. By the uniform continuity of ζ, and θ, $C_{\bar{\alpha},T}<\infty$, we also see $I_3\to 0$.

We consider I_2 under the hypothesis (ii). By (3.26) we see that, for any $\delta>0$, there exists $c_\delta>0$ such that

$$\max\{|C_{\bar{\alpha},T}-C_\varepsilon^{*,i}|;\ 1\leqq i\leqq n_\varepsilon\}\leqq\delta \tag{4.5}$$

for all ε satisfying

$$c_\delta^{-1}\leqq\bar{\alpha}_\varepsilon. \tag{4.6}$$

Then we get

$$I_2\leqq\delta\theta\|\zeta\|_\infty|\Omega'|. \tag{4.7}$$

We consider I_2 under (i). By (3.22) and (3.23) we also see that, for any $\delta>0$, there exists $c_\delta>0$ such that (4.5) stands for all ε satisfying $\|g_\varepsilon^*(x, v)-v\|_\infty\leqq c_\delta$ and

$$\begin{cases} (4.6) \quad \text{for } \bar{\alpha}=\infty\,, \\ |\bar{\alpha}_\varepsilon-\bar{\alpha}|\leqq c_\delta \quad \text{for } 0\leqq\bar{\alpha}<\infty\,. \end{cases}$$

Thus, under (i), we also get (4.7). Since δ is arbitrary, we see $I_2\to 0$. The assertion is true.

For $\theta=\infty$, by (4.3) we have

$$0\leqq\theta_\varepsilon^{-1}\frac{\partial\tilde{w}_\varepsilon}{\partial r}\delta_\varepsilon^B$$
$$\leqq 2(N-2)\kappa^{N-2}(1-(r_\varepsilon\kappa/\varepsilon)^{N-2})^{-1}\varepsilon\delta_\varepsilon^B\,.$$

By Lemma 2.6-i and Theorem 2.1, $\{\theta_\varepsilon^{-1}(\partial w_\varepsilon/\partial r)\delta_\varepsilon^B\}$ is bounded in $W^{-1,\infty}(\Omega)$. By Lemma 2.6-ii it suffices to show

$$\theta_\varepsilon^{-1}\Big\langle\frac{\partial w_\varepsilon}{\partial r}\delta_\varepsilon^B,\, \zeta\Big\rangle\longrightarrow C_{\bar{\alpha},T}\int_{\Omega'}\zeta dx$$

for $\zeta\in C_0^\infty(\Omega)$. In fact, we get

$$\left|\theta_\varepsilon^{-1}\Big\langle\frac{\partial w_\varepsilon}{\partial r}\delta_\varepsilon^B,\, \zeta\Big\rangle-C_{\bar{\alpha},T}\int_{\Omega'}\zeta dx\right|\leqq\theta_\varepsilon^{-1}I_0+I_2'+I_3'\,,$$

where $I_2'=\varepsilon^N\sum_i|\zeta(p_\varepsilon^i)(C_\varepsilon^{*,i}-C_{\bar{\alpha},T})|$ and $I_3'=C_{\bar{\alpha},T}\left|\varepsilon^N\sum_i\zeta(p_\varepsilon^i)-\int_{\Omega'}\zeta dx\right|$. By (4.4), (3.21) and the uniform continuity of ζ we see $I_0/\theta_\varepsilon\to 0$. By the uniform continuity of ζ we have $I_3'\to 0$. By (3.22), (3.23) and (3.26) we see $I_2'\to 0$. Thus, the proof is completed. Q. E. D.

THEOREM 4.6. *We suppose* (m. 1), $\bar{\alpha}=\infty$ *and* $0\leqq\theta\leqq\infty$. *If*

(i) *the hypothesis* $(\text{m. }3)_v$ *with the exponent* $t=0$ *is satisfied, or*

(ii) *the hypothesis* (L.2) *is satisfied.*

Then we have

$$(\mathrm{I})\quad\begin{cases}\displaystyle\int_{\Omega_\varepsilon}|\nabla w_\varepsilon|^2dx\longrightarrow\theta C_T|\Omega'|\,, & \textit{if } 0\leqq\theta<\infty\,,\\ \displaystyle\theta_\varepsilon^{-1}\int_{\Omega_\varepsilon}|\nabla w_\varepsilon|^2dx\longrightarrow C_T|\Omega'|\,, & \textit{if } \theta=\infty\,,\end{cases}$$

$$(\mathrm{II})\quad\begin{cases}\displaystyle\alpha_\varepsilon\int_{\partial T_\varepsilon}w_\varepsilon g_\varepsilon(x,w_\varepsilon)dS\longrightarrow 0\,, & \textit{if } 0\leqq\theta<\infty\,,\\ \displaystyle\alpha_\varepsilon\theta_\varepsilon^{-1}\int_{\partial T_\varepsilon}w_\varepsilon g_\varepsilon(x,w_\varepsilon)dS\longrightarrow 0 & \textit{if } \theta=\infty\,.\end{cases}$$

Furthermore, under the hypothesis (ii), *we have*

$$\text{(III)}\quad \begin{cases} \alpha_\varepsilon\int_{\partial T_\varepsilon}[|w_\varepsilon|^2+|g_\varepsilon(x, w_\varepsilon)|^2]dS\longrightarrow 0\,, & if\ 0\leqq\theta<\infty\,. \\ \alpha_\varepsilon\theta_\varepsilon^{-1}\int_{\partial T_\varepsilon}[|w_\varepsilon|^2+|g_\varepsilon(x, w_\varepsilon)|^2]dS\longrightarrow 0\,, & if\ \theta=\infty\,. \end{cases}$$

Under the hypothesis (i) *with the additional assumption*

$$\text{(4.8)}\quad \begin{cases} \alpha_\varepsilon\|g_\varepsilon(x, v)-v\|_\infty^2\longrightarrow 0\,, & if\ 0\leqq\theta<\infty\,, \\ \bar{\alpha}_\varepsilon\|g_\varepsilon(x, v)-v\|_\infty^2\longrightarrow 0\,, & if\ \theta=\infty\,, \end{cases}$$

the relation (III) *holds again.*

PROOF. Let $h_\varepsilon=\|g_\varepsilon(x, v)-v\|_\infty$. We prove for $0\leqq\theta<\infty$. We see as in the proof of Lemma 4.1 that the superior limit of the sum of the left sides of (I) and (II) is not larger than $\theta C_T|\Omega|$. It suffices to show either (I) or (II). We show (I). We use the notation of the proof of Lemma 4.1.

Let

$$I_\varepsilon=\min\left\{\int_{D_\varepsilon}|\nabla w_{\varepsilon,i}^*|^2dx^*\,;\ i=1, 2, \cdots, n_\varepsilon\right\}.$$

Let $\bar{\alpha}_\varepsilon=\alpha_\varepsilon r_\varepsilon$ and $D_\varepsilon=B^0(\varepsilon/r_\varepsilon)\backslash T$. By $\bar{\alpha}_\varepsilon\to\infty$ and $h_\varepsilon\to 0$, (3.24) and Remark 3.7, we see $I_\varepsilon\to C_T$ under any hypothesis (i) or (ii). By Lemma 4.1 we get

$$\begin{aligned}\theta C_T|\Omega'| &= \lim_\varepsilon I_\varepsilon|\Omega'|r_\varepsilon^{N-2}/\varepsilon^N \\ &\leqq \liminf_\varepsilon\int_{\Omega_\varepsilon}|\nabla w_\varepsilon|^2dx\leqq\theta C_T|\Omega'|\,.\end{aligned}$$

So, we see the first part of (I). Thus, we have the first part of (II). For $\theta=\infty$, by Lemma 4.1 and the same argument as in the above we have

$$C_T|\Omega'|=\lim_\varepsilon I_\varepsilon|\Omega'|\leqq\liminf_\varepsilon\theta_\varepsilon^{-1}\int_{\Omega_\varepsilon}|\nabla w_\varepsilon|^2dx\leqq C_T|\Omega'|\,.$$

Thus, the latter part of (I) is true and the remaining part of (II) is also true.

We show (III). Under (ii) we see that (II) implies (III). We show (III) under (i) and (4.8). Let $\partial T_{\varepsilon,1}=\{x\in\partial T_\varepsilon\,;\ |g_\varepsilon(x, w_\varepsilon)|\geqq|w_\varepsilon|/2\}$, $\partial T'_{\varepsilon,1}=\{x\in\partial T_\varepsilon\,;\ |w_\varepsilon|\geqq|g_\varepsilon(x, w_\varepsilon)|/2\}$ and $\partial T_{\varepsilon,2}=\{x\in\partial T_\varepsilon\,;\ |g_\varepsilon(x, w_\varepsilon)|<|w_\varepsilon|/2\}$, $\partial T'_{\varepsilon,2}=\{x\in\partial T_\varepsilon\,;\ |w_\varepsilon|<|g_\varepsilon(x, w_\varepsilon)|/2\}$. Note $w_\varepsilon\geqq 0$. We consider the case $\theta<\infty$. By (II) we get

$$\max\left\{\alpha_\varepsilon\int_{\partial T_{\varepsilon,1}}|w_\varepsilon|^2dS,\ \alpha_\varepsilon\int_{\partial T'_{\varepsilon,1}}|g_\varepsilon(x,w_\varepsilon)|^2dS\right\}\leqq 2\alpha_\varepsilon\int_{\partial T_\varepsilon}w_\varepsilon g_\varepsilon(x,w_\varepsilon)dS\longrightarrow 0\,.$$

On $\partial T_{\varepsilon,2}$ and $\partial T'_{\varepsilon,2}$ we get $|w_\varepsilon|\leqq 2h_\varepsilon$ and $|g_\varepsilon(x,w_\varepsilon)|\leqq 2h_\varepsilon$, respectively. We have $\alpha_\varepsilon|\partial T_\varepsilon|\sim\alpha_\varepsilon|\partial T||\Omega'|$ as $\varepsilon\to 0$. By the first part of (4.8) we have

$$\max\left\{\alpha_\varepsilon\int_{\partial T_{\varepsilon,2}}|w_\varepsilon|^2dS,\ \alpha_\varepsilon\int_{\partial T'_{\varepsilon,2}}|g_\varepsilon(x,w_\varepsilon)|^2dS\right\}$$
$$\leqq 4\alpha_\varepsilon h_\varepsilon^2\,|\partial T_\varepsilon|\leqq 5\alpha_\varepsilon h_\varepsilon^2\,|\partial T||\Omega'|\longrightarrow 0\,.$$

For $\theta=\infty$ by the same argument using the latter part of (4.5) with $\alpha_\varepsilon=\theta_{N-2,\varepsilon}\bar{\alpha}_\varepsilon$, we get the assertion. Q. E. D.

§ 5. Proofs of Theorems

§ 5.1. Proofs of Theorems A.1 and A.2

We prove them by three steps. First step is common to the other two Theorems B, C.

Preliminary step. We write $\theta=\theta_{N-2}$ and $\theta_\varepsilon=\theta_{N-2,\varepsilon}$ only in § 5.1. Let $V_\varepsilon=\{v\in H^1(\Omega_\varepsilon)\,;\,v|\partial\Omega=0\}$ and for $v\in V_\varepsilon$ we have

$$\int_{\Omega_\varepsilon}\{\nabla u_\varepsilon\cdot\nabla v-fv\}dx+\alpha_\varepsilon\langle\delta_\varepsilon^T,\,vg_\varepsilon(x,\tilde{u}_\varepsilon)\rangle=0\,. \tag{A.1}$$

After putting $v=u_\varepsilon$ into (A.1), by (m. 1) and the Schwarz inequality we get $\|\nabla u_\varepsilon\|_{\Omega_\varepsilon}^2\leqq\|f\|_\Omega\|u_\varepsilon\|_{\Omega_\varepsilon}$. By Corollary 4.3, we get c, which does not depend on ε, such that

$$\|\nabla\tilde{u}_\varepsilon\|_\Omega^2\leqq c\|f\|_\Omega\|\tilde{u}_\varepsilon\|_\Omega\,.$$

By the Poincaré inequality in $H_0^1(\Omega)$, we have

$$\sup_\varepsilon\|\nabla\tilde{u}_\varepsilon\|_\Omega=k_1<\infty\,. \tag{A.2}$$

By (A.2) we can take a weakly convergent subsequence in $H_0^1(\Omega)$, still denoted by $\{\tilde{u}_\varepsilon\}_\varepsilon$, such that

$$\tilde{u}_\varepsilon\xrightarrow{w}u\quad\text{in }H_0^1(\Omega)\,.$$

Let $L_\varepsilon(v_\varepsilon)=\int_{\Omega_\varepsilon}\{\nabla u_\varepsilon\cdot\nabla\zeta-f\zeta\}v_\varepsilon dx$ and $L(v)=\int_\Omega\{\nabla u\cdot\nabla\zeta-f\zeta\}vdx$ for $v_\varepsilon, v\in H^1(\Omega)\cap L^\infty(\Omega)$.

LEMMA A.1. *Suppose $r_\varepsilon\to 0$ as $\varepsilon\to 0$ with $\theta_N=0$. Let $\{v_\varepsilon\}_\varepsilon\subset H^1(\Omega)\cap L^\infty(\Omega)$*

such that $\sup_\varepsilon \|v_\varepsilon\|_\infty = k_2 < \infty$ *and* $v_\varepsilon \overset{w}{\to} v$ *in* $H^1(\Omega)$. *For any* $\zeta \in C_0^\infty(\Omega)$ *we have*

$$L_\varepsilon(v_\varepsilon) \longrightarrow L(v) \quad as\ \varepsilon \to 0.$$

PROOF. Note that $v_\varepsilon \nabla \tilde{u}_\varepsilon \cdot \nabla \zeta$, $v_\varepsilon f \zeta$, $v \nabla u \cdot \nabla \zeta$, $v f \zeta \in L^1(\Omega)$. By the Schwarz inequality and (A.2) we have

$$\left| \int_{T_\varepsilon} (\nabla \tilde{u}_\varepsilon \cdot \nabla \zeta - f\zeta) v_\varepsilon dx \right| \leqq (k_1 + \|f\|_\Omega) k_2 (\|\nabla \zeta\|_\infty + \|\zeta\|_\infty) |T_\varepsilon|^{1/2} \longrightarrow 0$$

as $\varepsilon \to 0$. By $r_\varepsilon \to 0$ as $\varepsilon \to 0$ with $\theta_N = 0$ we get $|T_\varepsilon| \to 0$. We have $v_\varepsilon \nabla \zeta \overset{s}{\to} v \nabla \zeta$ in $L^2(\Omega)^N$, $v_\varepsilon \zeta \overset{s}{\to} v \zeta$ in $L^2(\Omega)$ by the Rellich theorem. Thus, we have

$$L_\varepsilon(v_\varepsilon) = \int_\Omega (\nabla \tilde{u}_\varepsilon \cdot \nabla \zeta - f\zeta) v_\varepsilon dx - \int_{T_\varepsilon} (\nabla \tilde{u}_\varepsilon \cdot \nabla \zeta - f\zeta) v_\varepsilon dx \longrightarrow L(v)$$

$$\text{as } \varepsilon \to 0. \qquad \text{Q. E. D.}$$

Second step. Let w_ε be the function introduced in § 4. Take $\zeta \in C_0^\infty(\Omega)$ arbitrarily. Let $Q_\varepsilon = B_\varepsilon \backslash T_\varepsilon$. Since $\nabla w_\varepsilon = 0$ a. e. on $\Omega_\varepsilon \backslash B_\varepsilon$, after integrating by parts over Q_ε, using (4.1), we have

$$\int_{Q_\varepsilon} \zeta \nabla u_\varepsilon \cdot \nabla w_\varepsilon dx = -\int_{\Omega_\varepsilon} u_\varepsilon \nabla \zeta \cdot \nabla w_\varepsilon dx + \left\langle \frac{\partial w_\varepsilon}{\partial r} \delta_\varepsilon^B, \zeta \tilde{u}_\varepsilon \right\rangle - \alpha_\varepsilon \langle \delta_\varepsilon^T, \zeta \tilde{u}_\varepsilon g_\varepsilon(x, \tilde{w}_\varepsilon) \rangle .$$

Putting $v = \zeta w_\varepsilon$ into (A.1) and using the above equality, we get

$$\text{(A.3)} \qquad L_\varepsilon(w_\varepsilon) + \left\langle \frac{\partial w_\varepsilon}{\partial r} \delta_\varepsilon^B, \zeta \tilde{u}_\varepsilon \right\rangle + \alpha_\varepsilon \langle \delta_\varepsilon^T, \zeta (g_\varepsilon(x, \tilde{u}_\varepsilon) w_\varepsilon - \tilde{u}_\varepsilon g_\varepsilon(x, \tilde{w}_\varepsilon)) \rangle$$

$$- \int_{\Omega_\varepsilon} u_\varepsilon \nabla \zeta \cdot \nabla w_\varepsilon dx = 0 .$$

By Proposition 4.4, $0 \leqq w_\varepsilon \leqq 1$ and Lemma A.1 with $v_\varepsilon = \tilde{w}_\varepsilon$ or $\tilde{w}_\varepsilon / \theta_\varepsilon$ we see

$$L_\varepsilon(w_\varepsilon) \longrightarrow L(1) \quad \text{for } 0 \leqq \theta < \infty$$

and

$$L_\varepsilon(w_\varepsilon)/\theta_\varepsilon \longrightarrow 0 \quad \text{for } \theta = \infty$$

as $\varepsilon \to 0$. Let $q = 2N/(N-2)$. By the Hölder inequality we have

$$\|\tilde{u}_\varepsilon\|_{T_\varepsilon} \leqq \|\tilde{u}_\varepsilon\|_{q, T_\varepsilon} |T_\varepsilon|^{1/N} \leqq k_3 |T_\varepsilon|^{1/N} ,$$

where $k_3 = \sup_\varepsilon \|\tilde{u}_\varepsilon\|_{q,\Omega} < \infty$ (by the Sobolev imbedding theorem $k_3 < \infty$). By these inequalities and Lemma 4.1, we have

$$\left|\int_{T_\varepsilon}\tilde{u}_\varepsilon\nabla\zeta\cdot\nabla\tilde{w}_\varepsilon dx\right|\leqq 2[\theta_\varepsilon C_T|\Omega'|]^{1/2}\|\nabla\zeta\|_\infty\|\tilde{u}_\varepsilon\|_{T_\varepsilon}$$
$$\leqq 2[\theta_\varepsilon C_T|\Omega'|]^{1/2}\|\nabla\zeta\|_\infty k_3|T_\varepsilon|^{1/N}.$$

By the assumption $\theta_N=0$ we have $|T_\varepsilon|\to 0$ as $\varepsilon\to 0$. We get

$$\int_{T_\varepsilon}\tilde{u}_\varepsilon\nabla\zeta\cdot\nabla\tilde{w}_\varepsilon dx\ (=B_\varepsilon)\longrightarrow 0 \quad \text{for } 0\leqq\theta<\infty$$

and

$$B_\varepsilon/\theta_\varepsilon\longrightarrow 0 \qquad \text{for } \theta=\infty.$$

Thus, we get

$$\int_{\Omega_\varepsilon}u_\varepsilon\nabla\zeta\cdot\nabla w_\varepsilon dx=C_\varepsilon=\int_\Omega\tilde{u}_\varepsilon\nabla\zeta\cdot\nabla\tilde{w}_\varepsilon dx-B_\varepsilon$$
$$=D_\varepsilon-B_\varepsilon\longrightarrow 0 \quad \text{for } 0\leqq\theta<\infty$$

and

$$C_\varepsilon/\theta_\varepsilon=(D_\varepsilon-B_\varepsilon)/\theta_\varepsilon\longrightarrow 0 \qquad \text{for } \theta=\infty.$$

We have $\zeta\tilde{u}_\varepsilon\overset{w}{\to}\zeta u$ in $H_0^1(\Omega)$. By Theorem 4.5 we have

$$\left\langle\frac{\partial w_\varepsilon}{\partial r}\delta_\varepsilon^B,\zeta\tilde{u}_\varepsilon\right\rangle\longrightarrow\theta C_{\bar{\alpha},T}\int_{\Omega'}\zeta u dx \qquad \text{for } 0\leqq\theta<\infty$$

and

$$\left\langle\frac{\partial w_\varepsilon}{\partial r}\delta_\varepsilon^B,\zeta\tilde{u}_\varepsilon\right\rangle/\theta_\varepsilon\longrightarrow C_{\bar{\alpha},T}\int_{\Omega'}\zeta u dx \quad \text{for } \theta=\infty.$$

Let

$$I_\varepsilon=\alpha_\varepsilon\int_{\partial T_\varepsilon}\zeta[w_\varepsilon g_\varepsilon(x,u_\varepsilon)-u_\varepsilon g_\varepsilon(x,w_\varepsilon)]dS_\varepsilon^T.$$

Therefore, if

$$\text{(A.4)}\qquad \begin{cases} I_\varepsilon\longrightarrow 0 & \text{for } 0\leqq\theta<\infty, \\ I_\varepsilon/\theta_\varepsilon\longrightarrow 0 & \text{for } \theta=\infty, \end{cases}$$

is shown, then, by (A.3), we obtain

$$L(1)+\theta C_{\bar{\alpha},T}\int_\Omega\chi_{\Omega'}u\zeta dx=0 \quad \text{for } 0\leqq\theta<\infty$$

and

$$C_{\bar{\alpha},T}\int_{\Omega'}u\zeta dx=0 \qquad \text{for } \theta=\infty.$$

For $0\leqq\theta<\infty$ in Theorems A.1 and A.2 we have proved (1.1) for u. The remaining parts of the proofs for $\theta=\infty$, are given as follows. By $\bar{\alpha}>0$ and Proposition 3.4-ii, we see $C_{\bar{\alpha},T}>0$. Since ζ is arbitrary, we get $u=0$ a.e. on Ω'. For $\Omega'=\Omega$ with $\theta=\infty$, the desired relation (1.3) have been obtained. For $\Omega'\neq\Omega$, putting ζ with $\operatorname{supp}\zeta\subset\Omega''$ into (A.1) we get (1.2). The proof is completed. We prove (A.4) in the final step below.

Third step. We consider the case (I) $\bar{\alpha}<\infty$ for Theorem A.1. Then we consider the case (II) $\bar{\alpha}=\infty$ for Theorem A.2. Finally, (III) $\bar{\alpha}=\infty$ for Theorem A.1, is considered.

(I) By the condition $(\mathrm{m}.3)_v$ with the exponent $t=0$, we have

$$|g_\varepsilon(x,u_\varepsilon)w_\varepsilon-u_\varepsilon g_\varepsilon(x,w_\varepsilon)|\leqq 2c_\varepsilon(1+|u_\varepsilon|)\quad\text{and}\quad |u_\varepsilon|\leqq|g_\varepsilon(x,u_\varepsilon)|+2c_\varepsilon\,. \tag{A.5}$$

Let $A_\varepsilon=\alpha_\varepsilon|\partial T_\varepsilon|$ and $U_\varepsilon=\alpha_\varepsilon\|u_\varepsilon\|_{\partial T_\varepsilon}|\partial T_\varepsilon|^{1/2}$. Then, by the Schwarz inequality and the first inequality of (A.5) we have

$$\begin{aligned}|I_\varepsilon|&\leqq 2\|\zeta\|_\infty\alpha_\varepsilon c_\varepsilon(|\partial T_\varepsilon|+\|u_\varepsilon\|_{\partial T_\varepsilon}|\partial T_\varepsilon|^{1/2})\\&=2\|\zeta\|_\infty c_\varepsilon(A_\varepsilon+U_\varepsilon)\,.\end{aligned} \tag{A.6}$$

By $\bar{\alpha}<\infty$ and that n_ε behaves like $|\Omega'|/\varepsilon^N$ as $\varepsilon\to 0$, we get

$$\lim_\varepsilon A_\varepsilon=|\Omega'||\partial T|\bar{\alpha}\theta<\infty\qquad\text{for }\theta<\infty$$

and

$$\lim_\varepsilon A_\varepsilon/\theta_\varepsilon=|\Omega'||\partial T|\bar{\alpha}<\infty\quad\text{for }\theta=\infty\,.$$

By the second inequality of (A.5) we can derive the inequality:

$$\|u_\varepsilon\|_{\partial T_\varepsilon}\leqq\left(\int_{\partial T_\varepsilon}u_\varepsilon g_\varepsilon(x,u_\varepsilon)dS_\varepsilon^T\right)^{1/2}+(2c_\varepsilon|\partial T_\varepsilon|^{1/2}\|u_\varepsilon\|_{\partial T_\varepsilon})^{1/2}\,.$$

By putting $v=u_\varepsilon$ into (A.1) we have a constant k_4 such that

$$\sup_\varepsilon\left(\alpha_\varepsilon\int_{\partial T_\varepsilon}u_\varepsilon g_\varepsilon(x,u_\varepsilon)dS_\varepsilon^T\right)^{1/2}=k_4<\infty\,. \tag{A.7}$$

Thus we get

$$U_\varepsilon\leqq k_4A_\varepsilon^{1/2}+(2c_\varepsilon U_\varepsilon A_\varepsilon)^{1/2}$$

and

$$U_\varepsilon/(\theta_\varepsilon)^{1/2}\leqq k_4(A_\varepsilon/\theta_\varepsilon)^{1/2}+(2c_\varepsilon U_\varepsilon A_\varepsilon/\theta_\varepsilon)^{1/2}\,.$$

For the case $0\leqq\theta<\infty$ we see that $\{U_\varepsilon\}_\varepsilon$ is bounded. By (A.6) we have $I_\varepsilon\to 0$ as $\varepsilon\to 0$. For $\theta=\infty$, we have a constant k_5 such that

$$U_\varepsilon/(\theta_\varepsilon)^{1/2}\leqq k_5(1+(\theta_\varepsilon)^{1/2}(U_\varepsilon/\theta_\varepsilon)^{1/2})$$

for sufficiently small $\varepsilon>0$. So, we get

$$U_\varepsilon/\theta_\varepsilon\leqq k_5(1+(U_\varepsilon/\theta_\varepsilon)^{1/2}).$$

This means that $\{U_\varepsilon/\theta_\varepsilon\}_\varepsilon$ is bounded. Thus, by (A.6) we see $I_\varepsilon/\theta_\varepsilon\to 0$ as $\varepsilon\to 0$ for $\theta=\infty$.

(II) We consider the remaining part of the proof of Theorem A.2. It suffices to show

$$\text{(A.8)}\qquad \alpha_\varepsilon\int_{\partial T_\varepsilon}\zeta g_\varepsilon(x, u_\varepsilon)w_\varepsilon dS_\varepsilon^T\longrightarrow 0 \quad\text{and}\quad \alpha_\varepsilon\int_{\partial T_\varepsilon}\zeta u_\varepsilon g_\varepsilon(x, w_\varepsilon)dS_\varepsilon^T\longrightarrow 0$$

for $0\leqq\theta<\infty$, and

$$\text{(A.9)}\qquad \alpha_\varepsilon\int_{\partial T_\varepsilon}\zeta g_\varepsilon(x, u_\varepsilon)w_\varepsilon dS_\varepsilon^T/\theta_\varepsilon\longrightarrow 0 \quad\text{and}\quad \alpha_\varepsilon\int_{\partial T_\varepsilon}\zeta u_\varepsilon g_\varepsilon(x, w_\varepsilon)dS_\varepsilon^T/\theta_\varepsilon\longrightarrow 0$$

for $\theta=\infty$. Let

$$J_\varepsilon=\alpha_\varepsilon\int_{\partial T_\varepsilon}[|g_\varepsilon(x, u_\varepsilon)|^2+|u_\varepsilon|^2]dS_\varepsilon^T$$

and

$$K_\varepsilon=\alpha_\varepsilon\int_{\partial T_\varepsilon}[|w_\varepsilon|^2+|g_\varepsilon(x, w_\varepsilon)|^2]dS_\varepsilon^T.$$

Then the relations (A.8) and (A.9) are implied by

$$\text{(A.10)}\qquad \text{(i)}\ \begin{cases}\sup_\varepsilon J_\varepsilon<\infty & \text{for } 0\leqq\theta<\infty,\\ \sup_\varepsilon J_\varepsilon/\theta_\varepsilon<\infty & \text{for } \theta=\infty\end{cases}$$

and

$$\text{(ii)} \quad \begin{cases} K_\varepsilon \longrightarrow 0 & \text{as } \varepsilon \to 0 \quad \text{for } 0 \leqq \theta < \infty, \\ K_\varepsilon/\theta_\varepsilon \longrightarrow 0 & \text{as } \varepsilon \to 0 \quad \text{for } \theta = \infty. \end{cases}$$

By (A.7) and (L.2), we see $\sup_\varepsilon' J_\varepsilon < \infty$ for any θ such that $0 \leqq \theta \leqq \infty$. Thus, we have (A.10)-(i). Also, by (L.2) and Theorem 4.6-(III) we see that (A.10)-(ii) holds. Q. E. D.

(III) We treat the proof of Theorem A.1 here. By the same argument as in the first paragraph of (II) it suffices to show (A.10). By Theorem 4.6-(III) with (L.1) and $(\mathrm{m.}\,3)_v$ with $t=0$ we see (A.10)-(ii). We show (A.10)-(i).

By the definition of J_ε, (A.7), the Schwarz inequality, the latter part of (A.5) and $|g_\varepsilon(x, u_\varepsilon)| \leqq |u_\varepsilon| + 2c_\varepsilon$ we have

$$\begin{aligned} J_\varepsilon &\leqq 2\alpha_\varepsilon \int_{\partial T_\varepsilon} [u_\varepsilon g_\varepsilon(x, u_\varepsilon) + 2c_\varepsilon(|u_\varepsilon| + |g_\varepsilon(x, u_\varepsilon)|)] dS_\varepsilon^T \\ &\leqq 2k_4^2 + 4\alpha_\varepsilon c_\varepsilon |\partial T_\varepsilon|^{1/2} \left\{ \left[\int_{\partial T_\varepsilon} |u_\varepsilon|^2 dS_\varepsilon^T \right]^{1/2} + \left[\int_{\partial T_\varepsilon} |g_\varepsilon(x, u_\varepsilon)|^2 dS_\varepsilon^T \right]^{1/2} \right\} \\ &\leqq 2k_4^2 + 8c_\varepsilon (J_\varepsilon \alpha_\varepsilon |\partial T_\varepsilon|)^{1/2}. \end{aligned}$$

By (L.1) we have a constant k_6 such that

$$\limsup_\varepsilon \alpha_\varepsilon c_\varepsilon |\partial T_\varepsilon| < k_6 \theta_\varepsilon.$$

Thus, we have proved (A.10)-(i) and so, Theorem A.1. Q. E. D.

§ 5.2. Proof of Theorem B.

We prove Theorem B in two steps.

Preliminary step. Let V_ε be the same space as in the proofs of Theorems A.1, A.2. The preliminary step for the proofs of Theorems A.1, A.2, is available for the proof of Theorem B. As in § 5.1 we have a weakly convergent subsequence in $H_0^1(\Omega)$, still denoted by $\{\tilde{u}_\varepsilon\}_\varepsilon$, such that

$$\tilde{u}_\varepsilon \xrightarrow{w} u \quad \text{in } H_0^1(\Omega).$$

We have the equality for $\zeta \in C_0^\infty(\Omega)$.

$$\text{(B.1)} \qquad L_\varepsilon(1) + \alpha_\varepsilon \langle \delta_\varepsilon^T, \zeta g_\varepsilon(x, \tilde{u}_\varepsilon) \rangle = 0.$$

By Lemma A.1 with $v_\varepsilon=1$ for all ε, we have

$$L_\varepsilon(1)\longrightarrow L(1) \tag{B.2}$$

as $\varepsilon\to 0$ with $\theta_N=0$. We need the following lemma.

LEMMA B.1. *We suppose* (m. 2) *and* (m. 3)$_g$. *Suppose* $v_\varepsilon \xrightarrow{w} v$ *in* $H_0^1(\Omega)$ *as* $\varepsilon\to 0$. *Then, for* $\zeta\in C_0^\infty(\Omega)$, *we have*

$$\zeta g_\varepsilon(x, v_\varepsilon)\xrightarrow{w}\zeta g(x, v) \quad in \ W_0^{1,\bar p}(\Omega),$$

where $\bar p=\dfrac{2N}{r(N-2)+N}$.

PROOF. We show only $g_\varepsilon(x, v_\varepsilon)\xrightarrow{w} g(x, v)$ in $W_0^{1,\bar p}(\Omega)$. For this aim it suffices to show

(i) $\sup_\varepsilon\{\|\nabla g_\varepsilon(x, v_\varepsilon)\|_{\bar p}+\|\nabla g(x, v_\varepsilon)\|_{\bar p}\}<\infty$,

(ii) $g(x, v_\varepsilon)\xrightarrow{s} g(x, v)$ in $L^{\bar p}(\Omega)$,

(iii) $g_\varepsilon(x, v_\varepsilon)-g(x, v_\varepsilon)\xrightarrow{s} 0$ in $L^{\bar p}(\Omega)$.

We show (i). For simplicity we show only $\sup_\varepsilon\|\nabla g_\varepsilon(x, v_\varepsilon)\|_{\bar p}<\infty$. By (m. 2) we get a constant k_1 such that

$$|\partial g_\varepsilon(x, v_\varepsilon)/\partial x_i|\leqq k_1(1+|v_\varepsilon|^s+|v_\varepsilon|^r|\partial v_\varepsilon/\partial x_i|).$$

Thus, by the Hölder inequality we have k_2, k_3 such that

$$\int_\Omega|\partial g_\varepsilon(x, v_\varepsilon)/\partial x_i|^{\bar p}dx\leqq k_2\Big(1+\int_\Omega\Big[|v_\varepsilon|^{\bar p s}+|v_\varepsilon|^{\bar p r}\Big|\frac{\partial v_\varepsilon}{\partial x_i}\Big|^{\bar p}\Big]dx\Big)$$

$$\leqq k_3\Big(1+\int_\Omega|v_\varepsilon|^q dx+\Big[\int_\Omega|v_\varepsilon|^{\bar p rb}dx\Big]^{1/b}\Big[\int_\Omega|\nabla v_\varepsilon|^{\bar p c}dx\Big]^{1/c}\Big),$$

where $q=2N/(N-2)$, $\bar p rb=q$, $\bar p c=2$ and $1/b+1/c=1$. By the Sobolev theorem we have $\sup_\varepsilon\|v_\varepsilon\|_q<\infty$. Thus, we have shown

$$\sup_\varepsilon\|\nabla g_\varepsilon(x, v_\varepsilon)\|_{\bar p}<\infty.$$

Next we show (ii). By (m. 2) we have k_4 such that

$$|g_\varepsilon(x, v)|\leqq k_4(1+|v|^s+|v|^{r+1}).$$

We need the following lemma (cf. p. 30, Krasnosel'skii [19]).

LEMMA B.2. *Let $G:\Omega\times\boldsymbol{R}\to\boldsymbol{R}$ be a function satisfying the Carathéodory condition:*

(a) *for each v, $G(x,v)$ is measurable with respect to x,*

(b) *for each x, $G(x,v)$ is continuous with respect to v.*

Besides we suppose

$$|G(x,v)|\leqq c(1+|v|^{\gamma/\delta})$$

with a certain constant c and $1\leqq\gamma,\ \delta<\infty$. Then the mapping: $v\in L^{\gamma}(\Omega)\to G(x,v(x))\in L^{\delta}(\Omega)$ is continuous with respect to the strong topologies.

We apply Lemma B.2 with $\gamma=q-\delta'$, $\delta=\bar{p}$ and $G(x,v)=g(x,v)$, where δ' denotes a small positive constant such that

$$r+1,\ s<\{q-\delta'\}/\bar{p}\,.$$

Then, by the compact injection: $H^1(\Omega)\to L^{q-\delta'}(\Omega)$ we get (ii).

Finally we show (iii). Set $D_\varepsilon=\{x\in\Omega\,;\,|v_\varepsilon|\leqq 1)$ and $E_\varepsilon=\Omega\backslash D_\varepsilon$. By $(\mathrm{m}.3)_g$ and $\bar{p}t\leqq q$ we have k_5 such that

$$\int_{D_\varepsilon\cup E_\varepsilon}|g_\varepsilon(x,v_\varepsilon)-g(x,v_\varepsilon)|^{\bar{p}}dx\leqq k_5c_\varepsilon^{\bar{p}}\left(|\Omega|+\int_\Omega|v_\varepsilon|^q dx\right).$$

Thus, (iii) is implied by the above inequality and the Sobolev imbedding theorem. Q. E. D.

Second step. By Lemma B.1, (m.2) and $(\mathrm{m}.3)_g$, we get

$$\zeta g_\varepsilon(x,\tilde{u}_\varepsilon)\xrightarrow{w}\zeta g(x,u)\quad\text{as }\varepsilon\to 0\quad\text{in }W_0^{1,\bar{p}}(\Omega)\,.$$

Because of $N-\bar{p}=N(1-2/(r(N-2)+N))=\Lambda(r)$, by Theorem 2.2 with $\theta_{\Lambda(r)}=\infty$, we get

$$\varepsilon^N/r_\varepsilon^{N-1}\delta_\varepsilon^T\xrightarrow{w}|\partial T|\chi_{\Omega'}dx\quad\text{in }W^{-1,\bar{p}^*}(\Omega)\,.$$

Thus, we have

$$\text{(B.3)}\qquad\begin{cases}\alpha_\varepsilon\langle\delta_\varepsilon^T,\zeta g_\varepsilon(x,\tilde{u}_\varepsilon)\rangle\longrightarrow a|\partial T|\displaystyle\int_{\Omega'}\zeta g(x,u)dx & \text{for } 0\leqq a<\infty\,,\\[2ex] \dfrac{\varepsilon^N}{r_\varepsilon^{N-1}}\langle\delta_\varepsilon^T,\zeta g_\varepsilon(x,\tilde{u}_\varepsilon)\rangle\longrightarrow|\partial T|\displaystyle\int_{\Omega'}\zeta g(x,u)dx & \text{for } a=\infty\,.\end{cases}$$

For $a=\infty$, dividing both sides of (B.1) by a_ε we have

$$a_\varepsilon^{-1}L_\varepsilon(1)+\varepsilon^N r_\varepsilon^{1-N}\langle\delta_\varepsilon^T,\zeta g_\varepsilon(x,\tilde{u}_\varepsilon)\rangle=0\,.$$

Thus, by the weak convergence of $\{\tilde{u}_\varepsilon\}_\varepsilon$ and (B.3) we have $\int_{\Omega'}\zeta g(x,u)dx=0$

for all $\zeta\in C_0^\infty(\Omega)$. We get $g(x,u)=0$, a.e. in Ω'. By (m.4) we get $u=0$ a.e. on Ω'. For $\Omega'=\Omega$ we get (1.3). For $\Omega'\neq\Omega$ we already have $u|\Omega''\in H_0^1(\Omega'')$. We have (1.2) from (B.1) with $\operatorname{supp}\zeta\subset\Omega''$. For $0\leqq a<\infty$, we have (1.4) by (B.1), (B.2) and (B.3). Q.E.D.

§ 5.3. Proof of Theorem C.

We divide the proof into three steps.

Preliminary step. The argument of the preliminary step of Theorem A is also available in this section. We have the equality below.

$$\int_{\Omega_\varepsilon}[\nabla u_\varepsilon\cdot\nabla\zeta-f\zeta]dx+\alpha_\varepsilon\langle\delta_\varepsilon^T,\zeta g_\varepsilon(x,\tilde{u}_\varepsilon)\rangle=0\,. \tag{C.1}$$

We choose a subsequence, still denoted by $\{u_\varepsilon\}_\varepsilon$, such that

$$\tilde{u}_\varepsilon\xrightarrow{w}u\quad\text{in } H_0^1(\Omega)\quad\text{and}\quad q_\varepsilon\xrightarrow{w}q\quad\text{in } L^2(\Omega)^N, \tag{C.2}$$

where $q_\varepsilon=\chi_\varepsilon\nabla\tilde{u}_\varepsilon$, and χ_ε denotes the characteristic function of Ω_ε. If the measure of $\partial\Omega'$ is zero, $C_0^\infty(\Omega'\cup\Omega'')$ is dense in $L^1(\Omega)$. So, according to Bensoussan, Lions nnd Papanicolaou [3], p. 26, we get the following lemma.

LEMMA C.1. *We assume that the Lebesgue measure of $\partial\Omega'$ is zero. Let*

$$\chi_0(x)=\begin{cases}1 & \text{for } x\in\Omega''\,.\\ |Y\backslash\theta_N^{1/N}T| & \text{for } x\in\overline{\Omega'}\,.\end{cases}$$

Then we have $\chi_\varepsilon\overset{w^*}{\to}\chi_0$ in $L^\infty(\Omega)$, as $\varepsilon\to0$, with $0<\theta_N\leqq1$.

Now, for $a=\infty$, dividing (C.1) by α_ε, we get

$$\alpha_\varepsilon^{-1}\int_\Omega\chi_\varepsilon[q_\varepsilon\cdot\nabla\zeta-f\zeta]dx+\varepsilon^N r_\varepsilon^{1-N}\langle\delta_\varepsilon^T,\zeta g_\varepsilon(x,\tilde{u}_\varepsilon)\rangle=0\,.$$

Since $\{q_\varepsilon\}$ is bounded in $L^2(\Omega)^N$ and $\alpha_\varepsilon\to\infty$, the first term of the above equality tends to zero. By Lemma B.1, (m.2) and $(\mathrm{m}.3)_g$ we get $\zeta g_\varepsilon(x,\tilde{u}_\varepsilon)\overset{w}{\to}\zeta g(x,u)$ in $W_0^{1,\bar{p}}(\Omega)$ with $\bar{p}=2N/[r(N-2)+N]^{-1}$. We note that $0<\theta_N$ implies $\theta_{N-\bar{p}}=\infty$. By Theorem 2.2, we have $\varepsilon^N/r_\varepsilon^{N-1}\delta_\varepsilon^T\overset{s}{\to}|\partial T|\chi_{\Omega'}dx$ in $W^{-1,\bar{p}^*}(\Omega)$. We get

$$\varepsilon^N r_\varepsilon^{1-N}\langle\delta_\varepsilon^T,\zeta g_\varepsilon(x,\tilde{u}_\varepsilon)\rangle\longrightarrow|\partial T|\int_{\Omega'}\zeta(x)g(x,u)dx\quad\text{as }\varepsilon\to0\,. \tag{C.3}$$

We see $g(x, u)=0$, $x\in\Omega'$. By (m. 4) we have $u=0$ on Ω'. We get (1.3) for $\Omega'=\Omega$. For $\Omega'\neq\Omega$, putting $\zeta|\Omega_\varepsilon$ such that $\operatorname{supp}\zeta\subset\Omega''$, into (C.1) we get (1.2).

The equation (1.5) remains to be proved. From here we assume $\Omega'=\Omega$. For $0\leqq a<\infty$, we get (C.3) too. In (C.1), decreasing ε to zero and by Lemma C.1 we get

$$\text{(C.4)}\qquad \int_\Omega [q\cdot\nabla\zeta+a|\partial T|g(x, u)\zeta-\chi_0 f\zeta]dx=0\,.$$

We have to determine $I_q(\zeta)=\int_\Omega q\cdot\nabla\zeta dx$ for all $\zeta\in C_0^\infty(\Omega)$.

(The value $I_q(\zeta)$ is considered in the third step. The second step is a preparation for the third step.)

Second step. Let $\tau_\varepsilon=r_\varepsilon/\varepsilon$ $(=\theta_{N,\varepsilon}^{1/N})$. We introduce auxiliary functions $\kappa_\varepsilon^j(y)$, $y\in Y\backslash\tau_\varepsilon T$, $1\leqq j\leqq N$. Let W_ε be the space of all $v\in H^1(Y\backslash\tau_\varepsilon T)$ such that there exist extensions $\bar{v}\in H^1(Y)$ which have period 1 in each variable y_j, $1\leqq j\leqq N$. Let $W_{0,\varepsilon}=\left\{v\in W_\varepsilon\,;\int_{Y\backslash\tau_\varepsilon T} vdy=0\right\}$. Let $\nu_\varepsilon=\{\nu_{\varepsilon,j}\}_{j=1}^N$ be the outer unit normal to $\partial(Y\backslash\tau_\varepsilon T)$. We define $\kappa_\varepsilon^j\in W_{0,\varepsilon}$ by the variational problem:

$$\text{(C.5)}\qquad \int_{Y\backslash\tau_\varepsilon T}\nabla\kappa_\varepsilon^j\cdot\nabla v dy+\int_{\tau_\varepsilon\partial T}\nu_{\varepsilon,j} v dS=0$$

for all $v\in W_{0,\varepsilon}$. Since ∂T is smooth, we can assume $\nu_{\varepsilon,j}=-\phi_{\varepsilon,j}|(\tau_\varepsilon\partial T)$ with $\phi_{\varepsilon,j}\in C^\infty(\boldsymbol{R}^N)$ and $\sup_\varepsilon\|D^\alpha\phi_{\varepsilon,i}\|_{\infty,\boldsymbol{R}^N}<\infty$, $1\leqq j\leqq N$, $|\alpha|\leqq[N/2]+1$, where $D^\alpha\phi$ are derivatives of ϕ. By Proposition 2 of Appendix 2, with $(F_\varepsilon, \phi_\varepsilon)=(0, \phi_{\varepsilon,j})$, we get a family of extensions $\bar{\kappa}_\varepsilon^j$ of κ_ε^j to $\boldsymbol{R}^N$ such that $\{\bar{\kappa}_\varepsilon^j\}_\varepsilon$ is bounded in $H^{s_0}(\boldsymbol{R}^N)$, $s_0=[N/2]+2$, so,

$$\text{(C.6)}\qquad \limsup_\varepsilon\|\nabla\bar{\kappa}_\varepsilon^j\|_{\infty,\boldsymbol{R}^N}=E<\infty\,.$$

By (C.6) we have

$$\text{(C.7)}\qquad \lim_\varepsilon\int_{Y\backslash\tau_\varepsilon T}\frac{\partial\kappa_\varepsilon^j}{\partial y_k}dy=\int_{Y\backslash\tau_0 T}\frac{\partial\kappa_0^j}{\partial y_k}dy\,,$$

where $\tau_0=\theta_N^{1/N}$. We show (C.7). We have

$$\int_{Y\backslash\tau_\varepsilon T}\partial\kappa_\varepsilon^j/\partial y_k dy=\int_{\partial Y}\kappa_\varepsilon^j\nu_{\varepsilon,k}dS+\int_{\tau_\varepsilon\partial T}\kappa_\varepsilon^j\nu_{\varepsilon,k}dS\,.$$

By $\tau_0=\lim_\varepsilon\tau_\varepsilon$, we have $\phi_{\varepsilon,k}\to\phi_{0,k}$ on $\boldsymbol{R}^N$ and $\kappa_\varepsilon^j\to\kappa_0^j$ on $\overline{Q_0}$ (cf. Proposition 2,

Appendix 2). Then, we show $\int_{\partial Y}\kappa_\varepsilon^j\phi_{\varepsilon,k}dS\to\int_{\partial Y}\kappa_0^j\phi_{0,k}dS$ as $\varepsilon\to 0$. Setting $R_\varepsilon=\tau_\varepsilon T$, we have

$$\begin{aligned}&\left|\int_{\partial R_\varepsilon}\kappa_\varepsilon^j\phi_{\varepsilon,k}dS-\int_{\partial R_0}\kappa_0^j\phi_{0,k}dS\right|\\&\leqq\left|\left(\int_{\partial R_\varepsilon}dS-\int_{\partial R_0}dS\right)\bar\kappa_\varepsilon^j\phi_{\varepsilon,k}\right|+\left|\int_{\partial R_0}(\bar\kappa_\varepsilon^j\phi_{\varepsilon,k}-\bar\kappa_0^j\phi_{0,k})dS\right|\\&=I_\varepsilon+J_\varepsilon\,.\end{aligned}$$

By the uniform convergence of $\{\bar\kappa_\varepsilon^j\}_\varepsilon$ and $\{\phi_{\varepsilon,k}\}_\varepsilon$ on $\overline{Q_0}$ we have $J_\varepsilon\to 0$ as $\varepsilon\to 0$. By (C.6) we have $\limsup_\varepsilon\|\nabla(\bar\kappa_\varepsilon^j\phi_{\varepsilon,k})\|_\infty=c_1<\infty$. For $x^*\in\tau_0\partial T$ we set $x=\tau_0^{-1}\tau_\varepsilon x^*$. Then we get $|x-x^*|\leqq|\tau_\varepsilon-\tau_0|R(T)$ and $|h(x)-h(x^*)|\leqq 2c_1|\tau_\varepsilon-\tau_0|R(T)$ for small ε, with $h(x)=\bar\kappa_\varepsilon^j\phi_{\varepsilon,k}$, where $R(T)=\sup\{|x|\,;\,x\in T\}$. The relation (C.7) follows from

$$I_\varepsilon\leqq 3c_1|\tau_\varepsilon-\tau_0||\tau_0\partial T|R(T)\longrightarrow 0\quad\text{as } \varepsilon\to 0\,.$$

According to Bensoussan, Lions and Papanicolaou [3], we give approximate sequences $\{\hat w_\varepsilon^j\in W^{1,\infty}(\boldsymbol{R}^N)\}_\varepsilon$ to the "coordinate functions", $x_1, x_2, \cdots, x_N$, by the above auxiliary functions κ_ε^j, $1\leqq j\leqq N$, as follows:

$$\hat w_\varepsilon^j(x)=\begin{cases}x_j+\varepsilon\kappa_\varepsilon^j((x-p_\varepsilon^i)/\varepsilon), & x\in Y_\varepsilon^i\backslash T_\varepsilon^i\,,\\ x_j+\varepsilon\bar\kappa_\varepsilon^j((x-p_\varepsilon^i)/\varepsilon), & x\in T_\varepsilon^i,\ i\in\boldsymbol{N}\,.\end{cases}$$

We still denote by $\hat w_\varepsilon^j$, the restriction of $\hat w_\varepsilon^j$ to Ω. Let w_ε^j be the restriction of $\hat w_\varepsilon^j$ to Ω_ε.

LEMMA C.2. *Let $p\in[1,\infty)$. Then, for any v_ε, $v\in W^{1,p}(\Omega)$ such that $v_\varepsilon\overset{w}{\to}v$ in $W^{1,p}(\Omega)$, we have*

$$v_\varepsilon\hat w_\varepsilon^j\overset{w}{\longrightarrow}vx_j\quad\text{in } W^{1,p}(\Omega)\quad\text{as } \varepsilon\to 0\,.$$

PROOF. By (C.6), $\{v_\varepsilon\hat w_\varepsilon^j\}_\varepsilon$ is bounded in $W^{1,p}(\Omega)$, while $v_\varepsilon\hat w_\varepsilon^j\overset{s}{\to}vx_j$ in $L^p(\Omega)$, because $\hat w_\varepsilon^j\overset{s}{\to}x_j$ in $L^\infty(\Omega)$. Q.E.D.

LEMMA C.3. *For any $v\in H^1(\Omega)$ such that* $\operatorname{supp}v\subset\Omega$, *there exists a small number ε_v such that*

$$\int_{\Omega_\varepsilon}\nabla w_\varepsilon^j\cdot\nabla v\,dx=0\quad\textit{for all } \varepsilon,\ 0<\varepsilon\leqq\varepsilon_v\,.$$

This lemma is proved by the same way as in the proof of (3.17), p. 26, of Bensoussan, Lions and Papanicolaou [3] with the boundary condition of κ_ε^j.

LEMMA C.4. *Suppose* $\varepsilon\to 0$ *with* $0<\theta_N\leqq 1$. *Then we have*

$$\chi_\varepsilon\frac{\partial\hat{w}_\varepsilon^j}{\partial x_k}\xrightarrow{\hat{w}^*}\int_{Y\setminus\tau_0 T}\left[\delta_{jk}+\frac{\partial\kappa_0^j}{\partial y_k}\right]dy,\ \text{a constant function}$$

in $L^\infty(\Omega)$, *where* δ_{jk} *denotes Kronecker's delta.*

PROOF. By (C.6), $\{\chi_\varepsilon(\partial\hat{w}_\varepsilon^j/\partial x_k)\}_\varepsilon$ is bounded in $L^\infty(\Omega)$. It suffices to search the limit of $\int_{\Omega_\varepsilon}\zeta(\partial w_\varepsilon^j/\partial x_k)dx$ as $\varepsilon\to 0$ for an arbitrary $\zeta\in C_0^\infty(\Omega)$. Let $\Omega_\varepsilon^0=\cup\{Y_\varepsilon^i\,;\,Y_\varepsilon^i\subset\overline{\Omega'}\}$. For any $\delta>0$, using the uniform continuity of ζ, there exists an $\varepsilon_\delta>0$ such that $\sup\{|\zeta(x)-\zeta(p_\varepsilon^i)|\,;\,x\in Y_\varepsilon^i,\ 1\leqq i\leqq n_\varepsilon\}\leqq\delta$ and $\operatorname{supp}\zeta\subset\Omega_\varepsilon^0$ for $0<\varepsilon\leqq\varepsilon_\delta$. We have

$$\varepsilon^{-N}\int_{Y_\varepsilon^i\setminus T_\varepsilon^i}\frac{\partial\hat{w}_\varepsilon^j}{\partial x_k}dx=\int_{Y\setminus\tau_\varepsilon T}\left(\delta_{jk}+\frac{\partial\kappa_\varepsilon^j}{\partial y_k}\right)dy\quad(=I_{jk,\varepsilon})\,.$$

We set $I_{jk}=\int_{Y\setminus\tau_0 T}\left(\delta_{jk}+\dfrac{\partial\kappa_0^j}{\partial y_k}\right)dy$. By (C.6) we get

$$\left|\left\langle\chi_\varepsilon\frac{\partial\hat{w}_\varepsilon^j}{\partial x_k},\zeta\right\rangle-\sum_{i=1}^{n_\varepsilon}\zeta(p_\varepsilon^i)\int_{Y\setminus\tau_\varepsilon T}\frac{\partial w_\varepsilon^j}{\partial y_k}dy\varepsilon^N\right|\leqq\delta(1+E)|\Omega|\,,$$

and

$$\left|I_{jk,\varepsilon}\left(\sum_{i=1}^{n_\varepsilon}\zeta(p_\varepsilon^i)\varepsilon^N-\int_\Omega\zeta dx\right)\right|\leqq\delta(1+E)|\Omega|\,.$$

By (C.7) we also have

$$\left|(I_{jk,\varepsilon}-I_{jk})\int_\Omega\zeta dx\right|\longrightarrow 0\quad\text{as }\tau_\varepsilon\to\tau_0\,.\qquad\text{Q.E.D.}$$

By Lemma C.1–Lemma C.4 with the proof of energy for the convergence theorem in the homogenization theory (cf. p. 24–p. 28 of Bensoussan, Lions and Papanicolaou [3]), we get the following formula for $q=(q_j\,;\,1\leqq j\leqq N)$.

$$\text{(C.8)}\qquad\begin{cases} q_{ij}=\delta_{ij}|Y\setminus\theta^{1/N}T|-\displaystyle\int_{Y\setminus\theta^{1/N}T}\nabla\kappa_0^i\cdot\nabla\kappa_0^j dy\,,\\[2ex] q_j=\displaystyle\sum_{i=1}^N q_{ij}\,\partial u/\partial x_i\,.\end{cases}$$

Now, by (C.4) and (C.8) we have

$$\int_{\Omega}\left[\sum_{i,j=1}^{N} q_{ij}(\partial u/\partial x_i)(\partial\zeta/\partial x_j)+a|\partial T|g(x,u)\zeta-\chi_0 f\zeta\right]dx=0\,,$$

Since ζ is arbitrary, this equality shows (1.5). Q. E. D.

Appendix 1. The weak minimum principle

In this paper, the weak minimum principle for a supersolution to $-\Delta$ is used frequently, so, we give it here.

Let $Lu=-\sum\partial(a_{ij}(x)\partial u/\partial x_j)/\partial x_i$ with the condition: $a_{ij}(x)\in L^\infty(\boldsymbol{R}^N)$, $A^{-1}|\xi|^2\leqq\sum_{i,j=1}^{N}a_{ij}(x)\xi_i\xi_j\leqq A|\xi|^2$ for $\xi\in\boldsymbol{R}^N$ and $x\in\boldsymbol{R}^N$, with a constant $A\geqq 1$. Let G be a bounded domain of $\boldsymbol{R}^N$, $N\geqq 1$. We set

$$a(u,v)=\int_G a_{ij}(x)(\partial u/\partial x_j)(\partial v/\partial x_i)dx,\qquad u,v\in H^1(G)\,.$$

A function $u\in H^1(G)$ is a supersolution to L, if

$$a(u,\zeta)\geqq 0 \quad \text{for } \zeta\in H_0^1(G) \text{ with } \zeta\geqq 0 \text{ a.e. in } G\,.$$

THEOREM 1. *Let $u\in H^1(G)$ be a supersolution to L. Then*

$$u(x)\geqq\inf_{\partial G}u \quad \text{a.e. in } G.$$

For the proof of this theorem and applications see Theorem 5.7, p. 39, of Kinderlehrer and Stampacchia [18].

Appendix 2. Uniform estimates of periodic solutions

We show (C.6), § 5.3, in a general way. Let T be a compact subset of Y such that $T\Subset Y$, $T^0\neq\emptyset$ and ∂T is of class C^s (cf. p. 756 of Agmon, Douglis and Nirenberg [1]).

Let $Q_\varepsilon=Y\backslash\tau_\varepsilon T_\varepsilon$, where $1\geqq\tau_\varepsilon>0$, $1\geqq\varepsilon\geqq 0$ and $\tau_\varepsilon T\Subset Y$. We adopt the space $W_{0,\varepsilon}$ introduced in § 5.3. For $t\in\{0\}\cup\boldsymbol{N}$, $C_\#^t$ is the space of functions v defined on $\boldsymbol{R}^N$ which have period 1 in each variable and t times continuously differentiable. $H_\#^t$ is the completion of $C_\#^t$ with respect to the norm $\|\cdot\|_{t,Y}$, where $\|\cdot\|_{t,Y}$ is the standard norm of the Sobolev space $H^t(Y)$ of order t.

Let $F_\varepsilon\in H_\#^{s-2}$ and $\psi_\varepsilon\in C^{s-1}(\boldsymbol{R}^N)$. We consider the following boundary value problem:

$$-\Delta v_\varepsilon = F_\varepsilon \quad \text{a.e. in } Q_\varepsilon,$$

$$\frac{\partial v_\varepsilon}{\partial \nu_\varepsilon} = \phi_\varepsilon \quad \text{on } \tau_\varepsilon \partial T,$$

where ν_ε denotes the outer normal to ∂Q_ε.

Using Theorem 15.2, p. 704, and the definition of the trace norm in p. 699 of [1] together with a suitable family of partition of unity for $\{Q_\varepsilon\}_\varepsilon$ we get the following proposition.

PROPOSITION 2. *Let* $s_0=[N/2]+2$ *and* $s=s_0, s_0+1, \cdots$. *Suppose that* $\tau_0=\lim_{\varepsilon\to 0}\tau_\varepsilon$, $F_\varepsilon \in H_\#^{s-2}$,

$$\sup_\varepsilon \|F_\varepsilon\|_{s-2} < \infty \quad \text{and} \quad \limsup_{\varepsilon\to 0} \sum_{|\alpha|\le s-1} \|D^\alpha \phi_\varepsilon\|_{\infty, \boldsymbol{R}^N} < \infty .$$

Then there exist extensions $\bar{v}_\varepsilon \in C^1(\boldsymbol{R}^N)$ *of* v_ε *such that*

$$\limsup_{\varepsilon\to 0} \|\nabla \bar{v}_\varepsilon\|_{\infty, \boldsymbol{R}^N} < \infty .$$

We further suppose that $F_\varepsilon \stackrel{s}{\to} F_0$ *in* $L^2(Y)$, $\sup_\varepsilon \|F_\varepsilon\|_\infty < \infty$ *and* $\phi_\varepsilon \to \phi_0$ *uniformly on* Y. *Then we have*

$$\bar{v}_\varepsilon | Q_0 \longrightarrow v_0 \quad \text{uniformly } \bar{Q}_0 \text{ as } \varepsilon \to 0 .$$

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Department of Computer Science
and Information Mathematics
Faculty of Electro-Communications
The University of Electro-Communications
1-5-1, Chofugaoka, Chofu-shi, Tokyo
182 Japan