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On the existence of weak solutions of equation of natural convection

Dedicated to Professor Hiroshi Fujita on his sixtieth birthday

By Hiroko MORIMOTO

§1. Introduction.

There is a lot of literature for convection problem. In certain cases, so called trivial regular solutions can be found. Therefore the bifurcation problem is studied. For the Bénard problem, see Rabinowitz [13], Fife [5], Iudovich [6]. See also Velte [15] for the convection problem in tubular domain. Joseph [9] studied the stability problem and obtained a uniqueness result for steady convective flow in a bounded region. The condition is given under the form of relation between the Rayleigh number and the Reynolds number.

In this paper, we discuss the existence of weak solutions of equations which describe the motion of fluid of natural convection (Boussinesq approximation), and the interior regularity. Let Ω be a bounded domain in R^3. We consider the following system of differential equations:

$$(1\text{-}1)\qquad \begin{cases} (u\cdot\nabla)u=-\dfrac{1}{\rho}\nabla p+\Delta\nu u+\beta g\theta\,, & \\ \operatorname{div} u=0\,, & \text{in } \Omega \\ (u\cdot\nabla)\theta=\kappa\Delta\theta\,, & \end{cases}$$

where $u\cdot\nabla=\sum_{j=1}^{3}u_j\,\partial/\partial x_j$. Here $u=(u_1, u_2, u_3)$ is the fluid velosity, p is the pressure, θ is the temperature, g is the gravitational vector function, and ρ (density), ν (kinematic viscosity), β (coefficient of volume expansion), κ (thermal conductivity) are positive constants.

The boundary conditions are as follows. Let $\partial\Omega$ (the boundary of Ω) be divided into two parts Γ_1, Γ_2 such that

$$\partial\Omega=\Gamma_1\cup\Gamma_2\,,\qquad \Gamma_1\cap\Gamma_2=\emptyset\,.$$

$$(1\text{-}2)\qquad \begin{cases} u=0, \quad \theta=\xi, & \text{on } \Gamma_1\,, \\ u=0, \quad \dfrac{\partial\theta}{\partial n}=0, & \text{on } \Gamma_2\,, \end{cases}$$

where ξ is a given function on Γ_1, n is the outer normal vector to $\partial\Omega$. If we can find a function θ_0 defined on Ω, of class $C^2(\Omega)\cap C^1(\bar{\Omega})$, satisfying $\theta_0=\xi$ on Γ_1 and $(\partial/\partial n)\theta_0=0$ on Γ_2, then we can transform the equations (1-1), (1-2) for u and $\tilde{\theta}=\theta-\theta_0$ and we obtain the following:

$$
(1\text{-}3)\qquad \begin{cases} (u\cdot\nabla)u=-\dfrac{1}{\rho}\nabla p+\nu\Delta u+\beta g\tilde{\theta}+\beta g\theta_0\,, & \text{in } \Omega\,,\\ \operatorname{div} u=0\,, & \text{in } \Omega\,,\\ (u,\nabla)\tilde{\theta}=\kappa\Delta\tilde{\theta}-(u\cdot\nabla)\theta_0+\kappa\Delta\theta_0\,, & \text{in } \Omega\,,\\ u=0,\quad \tilde{\theta}=0\,, & \text{on } \Gamma_1\,,\\ u=0,\quad \dfrac{\partial\tilde{\theta}}{\partial n}=0\,, & \text{on } \Gamma_2\,. \end{cases}
$$

After studying this auxiliary equation, we shall show the existence of a weak solution of the problem (1-1), (1-2) under certain conditions, and the interior regularity of the solution. Uniqueness of the weak solution of the system (1-3) is also studied.

We state the notations and the results in § 2, preliminary lemmas in § 3, proof of Theorem 1 (existence) in § 4, proof of Theorem 2 (interior regularity) in § 5, proof of a uniqueness result in § 6.

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§ 2. Notations and results.

Ω stands for a bounded domain in R^3 and its boundary is of class C^2.

CONDITION (H). The boundary $\partial\Omega$ of Ω is divided as follows:

$$\partial\Omega=\Gamma_1\cup\Gamma_2,\qquad \Gamma_1\cap\Gamma_2=\emptyset\,,$$

$$\text{measure of } \Gamma_1\neq 0\,,$$

and the intersection $\bar{\Gamma}_1\cap\bar{\Gamma}_2$ is a C^1-curve.

The functions considered in this paper are all real valued. $L^p(\Omega)$ and the Sobolev space $W_p^m(\Omega)$ are defined as usual. We also denote $H^m(\Omega)=W_2^m(\Omega)$. Whether the elements of the spaces are scalar or vector functions is understood from the contexts unless stated explicitly.

We define the inner product and the norm of $L^2(\Omega)$ as follows:

$$(u,v)=\int_\Omega \sum u_j(x)v_j(x)dx\,,\qquad \|u\|=\sqrt{(u,u)}\,,$$

for vector $L^2(\Omega)$ functions $u=(u_1, u_2, u_3)$, $v=(v_1, v_2, v_3)$,

$$(\theta, \tau)=\int_\Omega \theta(x)\tau(x)dx\,, \qquad \|\theta\|=\sqrt{(\theta, \theta)}\,,$$

for scalar $L^2(\Omega)$ functions θ, τ.

Now we define the solenoidal function spaces as follows:

$D_\sigma=\{$vector function $\varphi\in C^\infty(\Omega)\mid \operatorname{supp}\varphi\subset\Omega$, $\operatorname{div}\varphi=0$ in $\Omega\}$,
$H=$completion of D_σ under the $L^2(\Omega)$-norm,
$V=$completion of D_σ under the $H^1(\Omega)$-norm.

It is well known that $V=H_0^1(\Omega)\cap H$, where $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under the $H^1(\Omega)$ norm (see, e.g., [14]). We define another basic function spaces as follows:

$D_0=\{$scalar function $\varphi\in C^\infty(\bar{\Omega})\mid \varphi\equiv 0$ in a neighborhood of $\Gamma_1\}$,
$W=$completion of D_0 under the $H^1(\Omega)$-norm.

We recall the completion of D_0 under the $L^2(\Omega)$-norm coincides exactly with $L^2(\Omega)$.

Now let us define two trilinear forms $B(u, v, w)$ and $b(u, \theta, \tau)$ as follows:

$$\begin{cases} B(u, v, w)=((u\cdot\nabla)v, w) \\ \qquad\qquad =\displaystyle\int_\Omega \sum_{i,j} u_j(x)\frac{\partial v_i(x)}{\partial x_j}w_i(x)dx\,, \qquad u, v, w\in H^1(\Omega)\,, \\ b(u, \theta, \tau)\ =((u\cdot\nabla)\theta, \tau) \\ \qquad\qquad =\displaystyle\int_\Omega \sum_j u_j(x)\frac{\partial\theta(x)}{\partial x_j}\tau(x)dx, \quad u\in V, \quad \theta, \tau\in H^1(\Omega)\,. \end{cases}$$

First, we study the auxiliary problem:

Find $u\in V$ and $\tilde{\theta}\in W$ satisfying

$$\text{(2-1)} \quad \begin{cases} \nu(\nabla u, \nabla v)+B(u, u, v)-(\beta g\tilde{\theta}, v)-(\beta g\theta_0, v)=0, & \text{for all } v \text{ in } V, \\ \kappa(\nabla\tilde{\theta}, \nabla\tau)+b(u, \tilde{\theta}, \tau)+b(u, \theta_0, \tau)+\kappa(\nabla\theta_0, \nabla\tau)=0, & \text{for all } \tau \text{ in } W, \end{cases}$$

where ν, κ, β are given positive constants, g, θ_0 are given functions.

DEFINITION. The pair of functions $\{u, \tilde{\theta}\}$ is called a weak solution of (1-3) if $(u, \tilde{\theta})$ belongs to $V\times W$ and satisfies (2-1).

Now we define the weak solution of (1-1), (1-2).

DEFINITION. The pair of functions $\{u, \theta\}$ is called a weak solution of

(1-1), (1-2) if there exists a function θ_0 in $C^1(\bar{\Omega})$ such that $u \in V$, $\theta-\theta_0 \in W$, $\theta_0=\xi$ on Γ_1, $(\partial/\partial n)\theta_0=0$ on Γ_2, and, $\{u, \tilde{\theta}\}$ $(\tilde{\theta}=\theta-\theta_0)$ is a weak solution of (1-3).

It is well known that the eigenvalue problem

$$(\nabla u, \nabla v)=\lambda(u, v)\,, \qquad \text{for any } v \text{ in } V$$

has a countable set of positive eigenvalues with finite multiplicities [10]. Let λ_1 be the first eigenvalue of this problem. We have

$$\|u\| \leqq \|\nabla u\|/\sqrt{\lambda_1}\,, \qquad \text{for any } u \text{ in } V. \tag{2-2}$$

Under the assumption

$$\text{measure of } \Gamma_1 \neq 0\,,$$

W is a closed subspace of $L^2(\Omega)$ and the canonical injection of W into $L^2(Q)$ is compact ([4], p. 115). Therefore, the eigenvalue problem

$$(\nabla\theta, \nabla\tau)=\mu(\theta, \tau)\,, \qquad \text{for any } \tau \text{ in } W$$

has also a countable set of positive eigenvalues with finite multiplicities [10]. Let μ_1 be the first eigenvalue of this problem. It is well known

$$\|\theta\| \leqq \|\nabla\theta\|/\sqrt{\mu_1}\,, \qquad \text{for any } \theta \text{ in } W. \tag{2-3}$$

A priori estimate. Let $\{u, \tilde{\theta}\}$ be a weak solution of (1-3). We have

$$\nu\|\nabla u\|^2+B(u, u, u)-(\beta g\tilde{\theta}, u)-(\beta g\theta_0, u)=0\,,$$

$$\kappa\|\nabla\tilde{\theta}\|^2+b(u, \tilde{\theta}, \tilde{\theta})+b(u, \theta_0, \tilde{\theta})+\kappa(\nabla\theta_0, \nabla\tilde{\theta})=0\,.$$

Note that $B(u, u, u)=0$, $b(u, \tilde{\theta}, \tilde{\theta})=0$ (Lemma 4 in § 3). We have

$$\nu\|\nabla u\|^2=(\beta g\tilde{\theta}, u)+(\beta g\theta_0, u) \leqq \beta\|g\|_\infty(\|\tilde{\theta}\|+\|\theta_0\|)\|u\|\,.$$

Therefore, using the inequalities (2-2), (2-3), we have

$$\begin{aligned} \|\nabla u\| &\leqq \frac{\beta}{\nu\sqrt{\lambda_1}}\|g\|_\infty(\|\tilde{\theta}\|+\|\theta_0\|) \\ &\leqq \frac{\beta}{\nu\sqrt{\lambda_1\mu_1}}\|g\|_\infty(\|\nabla\tilde{\theta}\|+\sqrt{\mu_1}\|\theta_0\|)\,. \end{aligned} \tag{2-4}$$

Similarly,

$$\kappa\|\nabla\tilde{\theta}\|^2=-b(u,\theta_0,\tilde{\theta})-\kappa(\nabla\theta_0,\nabla\tilde{\theta})$$

$$=b(u,\tilde{\theta},\theta_0)-\kappa(\nabla\theta_0,\nabla\tilde{\theta})$$

$$\leqq\|u\|_6\|\nabla\tilde{\theta}\|\|\theta_0\|_3+\kappa\|\nabla\theta_0\|\|\nabla\tilde{\theta}\|$$

$$\leqq c_0\|\nabla u\|\|\nabla\tilde{\theta}\|\|\theta_0\|_3+\kappa\|\nabla\theta_0\|\|\nabla\tilde{\theta}\|\ ,$$

where c_0 is the constant $c_V(6)$ defined in Lemma 6 (§ 3). So,

$$\|\nabla\tilde{\theta}\|\leqq\frac{c_0}{\kappa}\|\theta_0\|_3\|\nabla u\|+\|\nabla\theta_0\|\ . \tag{2-5}$$

Combining (2-4) with (2-5), we have

$$\|\nabla u\|\leqq\frac{\beta c_0}{\nu\kappa\sqrt{\lambda_1\mu_1}}\|g\|_\infty\|\theta_0\|_3\|\nabla u\|+\frac{\beta}{\nu\sqrt{\lambda_1\mu_1}}\|g\|_\infty\{\|\nabla\theta_0\|+\sqrt{\mu_1}\|\theta_0\|\}\ .$$

Suppose

$$r\equiv\frac{\beta c_0}{\nu\kappa\sqrt{\lambda_1\mu_1}}\|g\|_\infty\|\theta_0\|_3<1\ . \tag{2-6}$$

Then we have

$$\|\nabla u\|\leqq\frac{\beta}{\nu\sqrt{\lambda_1\mu_1}(1-r)}\|g\|_\infty\{\|\nabla\theta_0\|+\sqrt{\mu_1}\,\|\theta_0\|\}, \tag{2-7}$$

$$\|\nabla\tilde{\theta}\|\leqq\frac{1}{1-r}\{\|\nabla\theta_0\|+r\sqrt{\mu_1}\,\|\theta_0\|\}\ . \tag{2-8}$$

LEMMA 1. *Suppose*

(i) $g\in L^\infty(\Omega)$

(ii) $\theta_0\in H^1(\Omega)$

(iii) $r\equiv\frac{\beta c_0}{\nu\kappa\sqrt{\lambda_1\mu_1}}\|g\|_\infty\|\theta_0\|_3<1\ .$

Then, there exists a weak solution $\{u,\tilde{\theta}\}$ *of* (1-3).

Our main results are the following:

THEOREM 1. *Let* Ω *be a bounded domain in* R^3 *with* C^2 *boundary satisfying the condition* (H). *If the function* $g(x)$ *is in* $L^\infty(\Omega)$ *and* ξ *is of class* $C^1(\bar{\Gamma}_1)$, *then there exists a weak solution of* (1-1), (1-2).

THEOREM 2. *Let* Ω *be a bounded domain in* R^3 *with* C^∞ *boundary satisfying Condition* (H). *We suppose further* $\bar{\Gamma}_1\cap\bar{\Gamma}_2$ *is a* C^∞ *curve,* ξ *is* *of class* $C^{\infty}(\bar{\Gamma}_1)$, *and* g *is of class* $C^{\infty}(\bar{\Omega})$. *Then the weak solution of* (1-1), (1-2) *is of class* $C^{\infty}(\overline{\Omega'})$ *for any subdomain* Ω' *such that* $\overline{\Omega'}\subset\Omega$.

REMARK. Suppose all the conditions of Lemma 1 are satisfied and the inequality

$$\text{(iv)}\quad \frac{c_B\beta\|g\|_{\infty}}{(1-r)\nu\kappa\sqrt{\lambda_1\mu_1}}\left\{\left(\frac{\kappa}{\nu}+2-r\right)\|\nabla\theta_0\|+\sqrt{\mu_1}\left(\frac{\kappa}{\nu}+r\right)\|\theta_0\|\right\}<1$$

holds. Then the weak solution of (1-3) is unique.

§ 3. Preliminary lemmas.

In this section, we list up lemmas necessary for proving Lemma 1 and Theorems.

LEMMA 2 (Sobolev). *Let* Ω *be a bounded domain in* R^3 *with* C^1 *boundary. The following imbeddings are hold.*

(i) *If* $\frac{1}{p}-\frac{m}{3}=\frac{1}{q}$ (>0), *then* $W_p^m(\Omega)\subset L^q(\Omega)$.

(ii) *If* $\frac{1}{p}-\frac{m}{3}=0$, *then* $W_p^m(\Omega)\subset L^{\alpha}(\Omega)$, *for any* α *in* $[1,\infty)$.

(iii) *If* $\frac{1}{p}-\frac{m}{3}<0$, *then* $W_p^m(\Omega)\subset L^{\infty}(\Omega)$.

This is well known and the proof is omitted (cf. [1]).

LEMMA 3.

$$\|\varphi\|_{L^4(\Omega)}\leqq c\|\varphi\|^{1/4}\|\nabla\varphi\|^{3/4},\quad \varphi\in V \text{ or } W,$$

where c *is a constant depending only on* Ω.

PROOF. By Cauchy's inequality, we have

$$\int\varphi(x)^4dx\leqq\left\{\int\varphi(x)^2dx\right\}^{1/2}\left\{\int\varphi(x)^6dx\right\}^{1/2}.$$

Desired inequality follows from Lemma 1, and the inequalities (2-2), (2-3).

LEMMA 4.

(i) $B(u,v,w)=-B(u,w,v)$ *for* $u\in V$, $v,w\in H^1(\Omega)$.

In particular,

$B(u, v, v)=0,$ *for* $u\in V,\ v\in H^1(\Omega).$

(ii) $b(u, \theta, \tau)=-b(u, \tau, \theta),$ *for* $u\in V,\ \theta, \tau\in H^1(\Omega).$

$b(u, \theta, \theta)=0,$ *for* $u\in V,\ \theta\in H^1(\Omega).$

PROOF. (i) We use the integration by parts. Since $u=0$ on $\partial\Omega$,

$$\begin{aligned} B(u, v, w) &= ((u\cdot\nabla)v, w) \\ &= \int_\Omega \sum u_j(x)\left\{\frac{\partial}{\partial x_j}v_i(x)\right\}w_i(x)dx \\ &= -\int_\Omega \sum v_i(x)\frac{\partial}{\partial x_j}\{u_j(x)w_i(x)\}dx \\ &= -\int_\Omega \sum v_i(x)\left\{\frac{\partial}{\partial x_j}u_j(x)\right\}w_i(x)dx \\ &\quad -\int_\Omega \sum v_i(x)u_j(x)\frac{\partial}{\partial x_j}w_i(x)dx\,. \end{aligned}$$

Since $\operatorname{div} u=0$, the first term of the right hand side vanishes, and the required equality holds. The second equality follows by putting $v=w$. (ii) is proved in a similar way, and the proof is omitted.

LEMMA 5. *There exists a constant* c_B *depending only on* Ω *such that*

$$|B(u, v, w)|\leqq c_B\|\nabla u\|\,\|\nabla v\|\,\|\nabla w\|, \qquad \text{for } u, v, w\in V,$$

$$|b(u, \theta, \tau)|\leqq c_B\|\nabla u\|\,\|\nabla\theta\|\,\|\nabla\tau\|, \qquad \text{for } u\in V,\ \theta, \tau\in W.$$

PROOF. By Hölder's inequality, we have

$$|B(u, v, w)|\leqq c\|u\|_{L^4(\Omega)}\|\nabla v\|\,\|w\|_{L^4(\Omega)}, \tag{3-1}$$

where c is a constant depending only on the dimension ($n=3$). Using Lemma 6 below, we have the inequality for $B(u, v, w)$. The inequality for $b(u, \theta, \tau)$ is proved in a similar way.

LEMMA 6. *For* $2\leqq p\leqq 6$, *we have*

(i) $\|u\|_{L^p(\Omega)}\leqq c_V(p)\|\nabla u\|,$ *for any* u *in* V,

(ii) $\|\theta\|_{L^p(\Omega)}\leqq c_W(p)\|\nabla\theta\|,$ *for any* θ *in* W,

where $c_V(p)$, $c_W(p)$ *are constants depending only on* Ω *and* p.

PROOF. By Lemma 2, we find a constant $c_1(p)$ depending only on Ω and p, such that the inequality

$$\|u\|_{L^p(\Omega)} \leqq c_1(p)\{\|u\| + \|\nabla u\|\}$$

holds for any u in V. From (2-2) the right hand side is bounded by

$$c_1(p)\left\{\frac{1}{\sqrt{\lambda_1}} + 1\right\}\|\nabla u\| .$$

Taking $c_V(p) = c_1(p)\{(1/\sqrt{\lambda_1}) + 1\}$, the inequality (i) is obtained. The inequality (ii) is obtained in a similar way and the proof is omitted.

§ 4. Proof of Theorem 1.

We begin by proving Lemma 1. We use the Leray-Schauder principle.

Let A be a completely continuous (nonlinear) operator in a separable Hilbert space $\mathcal{H}$. If all possible solutions $x = x(\lambda)$ of the equation

$$x = \lambda A x$$

for $\lambda \in [0, 1]$ lie within some ball $\|x\| \leqq \rho$, then the equation

$$x = Ax$$

has at least one solution inside this ball (cf. [7] Theorem 11.3, [11]).

Let $\mathcal{H} = V \times W$. The inner product and the norm in $\mathcal{H}$ are defined as follows:

$$[(u, \theta), (v, \tau)] = (\nabla u, \nabla v) + (\nabla\theta, \nabla\tau), \qquad \text{for } (u, \theta),\ (v, \tau) \in \mathcal{H},$$

$$\|(u, \theta)\|_{\mathcal{H}} = \sqrt{\|\nabla u\|^2 + \|\nabla\theta\|^2}.$$

Let (u, θ) be fixed in $\mathcal{H}$ and consider the linear functional on $\mathcal{H}$:

$$\mathcal{F}_1(v, \tau) = \frac{1}{\nu} B(u, u, v) + \frac{1}{\kappa} b(u, \theta, \tau) . \tag{4-1}$$

Using Lemma 5, we have the following estimate.

$$\begin{aligned} |\mathcal{F}_1(v, \tau)| &\leqq \frac{c_B}{\nu}\|\nabla u\|^2\|\nabla v\| + \frac{c_B}{\kappa}\|\nabla u\|\,\|\nabla\theta\|\,\|\nabla\tau\| \\ &\leqq c\|(u, \theta)\|_{\mathcal{H}}^2\|(v, \tau)\|_{\mathcal{H}}, \end{aligned}$$

where c stands for a constant independent of (u, θ), (v, τ). Therefore the functional defined by (4-1) is continuous in $\mathcal{H}$. By Riesz' theorem, there exists a bounded operator $\mathscr{B}: \mathscr{H} \to \mathscr{H}$ such that

$$[\mathscr{B}(u, \theta), (v, \tau)] = \frac{1}{\nu} B(u, u, v) + \frac{1}{\kappa} b(u, \theta, \tau).$$

Next, let us consider the linear functional

$$\mathscr{F}_2(v, \theta) = -\frac{1}{\nu}(\beta g \theta, v) + \frac{1}{\kappa} b(u, \theta_0, \tau). \tag{4-2}$$

We have the estimate

$$|\mathscr{F}_2(v, \tau)| \leqq \frac{\beta}{\nu} \|g\|_\infty \|\theta\| \|v\| + \frac{1}{\kappa} \|\nabla\theta_0\| \|u\|_4 \|\tau\|_4$$

$$\leqq \frac{1}{\sqrt{\lambda_1 \mu_1}} \left\{ \frac{\beta}{\nu} \|g\|_\infty + \frac{c}{\kappa} \|\nabla\theta_0\| \right\} \|(u, \theta)\|_{\mathscr{H}} \|(v, \tau)\|_{\mathscr{H}},$$

where c is a constant depending only on Ω (cf. Lemma 6). Therefore, the functional $\mathscr{F}_2$ is continuous in $\mathscr{H}$, and there exists a bounded operator $C: \mathscr{H} \to \mathscr{H}$ such that

$$[C(u, \theta), (v, \tau)] = -\frac{1}{\nu}(\beta g \theta, v) + \frac{1}{\kappa} b(u, \theta_0, \tau).$$

Consider the linear functional

$$\mathscr{F}_3(v, \tau) = -\frac{1}{\nu}(\beta g \theta_0, v) + (\nabla\theta_0, \nabla\tau).$$

Since $\mathscr{F}_3$ is continuous in $\mathscr{H}$, there exists $F \in \mathscr{H}$ such that

$$[F, (v, \tau)] = -\frac{1}{\nu}(\beta g \theta_0, v) + (\nabla\theta_0, \nabla\tau).$$

Suppose (u, θ) in $\mathscr{H}$ satisfy (2-1) for any (v, τ) in $\mathscr{H}$. Then we have

$$[(u, \theta) + \mathscr{B}(u, \theta) + C(u, \theta) + \mathscr{F}, (v, \tau)] = 0. \tag{4-3}$$

Conversely, if (u, θ) satisfies (4-3) for any (v, τ) in $\mathscr{H}$, then the equation (2-1) holds.

We now show the operator $\mathscr{B}$, C are completely continuous in $\mathscr{H}$. Let (v_m, τ_m) converge weakly in $\mathscr{H}$. So, v_m (resp. τ_m) converges weakly in V (resp. W). By Rellich's theorem [1], there exists a subsequence, which we denote by the same symbol, converging strongly in $L^2(\Omega)$. According to Lemma 3, v_m (resp. τ_m) converges strongly in $L^4(\Omega)$. Using Lemma 4, we have

$$
\begin{aligned}
&[\mathscr{B}(v_m, \tau_m) - \mathscr{B}(v_n, \tau_n), (v, \tau)] \\
&= \frac{1}{\nu} B(v_m, v_m, v) + \frac{1}{\kappa} b(v_m, \tau_m, \tau) - \frac{1}{\nu} B(v_n, v_n, v) - \frac{1}{\kappa} b(v_n, \tau_n, \tau) \\
&= \frac{1}{\nu} \{B(v_m - v_n, v_m, v) + B(v_n, v_m - v_n, v)\} \\
&\qquad + \frac{1}{\kappa} \{b(v_m - v_n, \tau_m, \tau) + b(v_n, \tau_m - \tau_n, \tau)\} \\
&= \frac{1}{\nu} \{B(v_m - v_n, v_m, v) - B(v_n, v, v_m - v_n)\} \\
&\qquad + \frac{1}{\kappa} \{b(v_m - v_n, \tau_m, \tau) - b(v_n, \tau, \tau_m - \tau_n)\} .
\end{aligned}
$$

Similarly to the proof of Lemma 5, the following estimate holds.

$$
\begin{aligned}
&|[\mathscr{B}(v_m, \tau_m) - \mathscr{B}(v_n, \tau_n), (v, \tau)]| \\
&\leqq \frac{1}{\nu} \{\|v_m - v_n\|_{L^4} \|\nabla v_m\| \|v\|_{L^4} + \|v_n\|_{L^4} \|\nabla v\| \|v_m - v_n\|_{L^4}\} \\
&\quad + \frac{1}{\kappa} \{\|v_m - v_n\|_{L^4} \|\nabla \tau_m\| \|\tau\|_{L^4} + \|v_n\|_{L^4} \|\nabla \tau\| \|\tau_m - \tau_n\|_{L^4}\} \\
&\leqq \frac{c}{\nu} \|v_m - v_n\|_{L^4} \{\|\nabla v_m\| + \|\nabla v_n\|\} \|\nabla v\| \\
&\quad + \frac{c}{\kappa} \{\|v_m - v_n\|_{L^4} \|\nabla \tau_m\| + \|\nabla v_n\| \|\tau_m - \tau_n\|_{L^4}\} \|\nabla \tau\| ,
\end{aligned}
$$

where c is a constant depending only on Ω. Therefore

$$
\begin{aligned}
\|\mathscr{B}(v_m, \tau_m) - \mathscr{B}(v_n, \tau_n)\|_{\mathscr{H}} &\leqq \frac{c}{\nu} \|v_m - v_n\|_{L^4} \{\|\nabla v_m\| + \|\nabla v_n\|\} \\
&\quad + \frac{c}{\kappa} \{\|v_m - v_n\|_{L^4} \|\nabla \tau_m\| + \|\nabla v_n\| \|\tau_m - \tau_n\|_{L^4}\} .
\end{aligned}
$$

Therefore, $\mathscr{B}(v_m, \tau_m)$ converges strongly in $\mathscr{H}$ and the operator $\mathscr{B}$ is completely continuous in $\mathscr{H}$. As for the operator $\mathcal{C}$, we have

$$
\begin{aligned}
&|[\mathcal{C}(v_m, \tau_m) - \mathcal{C}(v_n, \tau_n), (v, \tau)]| \\
&\qquad \leqq \left| \frac{1}{\nu} (\beta g(\tau_m - \tau_n), v) + \frac{1}{\kappa} ((v_m - v_n) \cdot \nabla \theta_0, \tau) \right|
\end{aligned}
$$

$$\leqq \frac{\beta}{\nu}\|g\|_{\infty}\|\tau_m-\tau_n\|\|v\|+\frac{1}{\kappa}\|\nabla\theta_0\|\|v_m-v_n\|_4\|\tau\|_4 .$$

Therefore we have

$$\|\mathcal{C}(v_m,\tau_m)-\mathcal{C}(v_n,\tau_n)\|_{\mathcal{H}}\leqq\frac{c\beta}{\nu}\|g\|_{\infty}\|\tau_m-\tau_n\|+\frac{c}{\kappa}\|\nabla\theta_0\|\|v_m-v_n\|_4$$

where c is a domain constant. Since v_n (resp. τ_n) converges strongly in $L^4(\Omega)$ (resp. $L^2(\Omega)$), $\mathcal{C}(v_n,\tau_n)$ converges strongly in $\mathcal{H}$, and $\mathcal{C}$ is also completely continuous.

Let $0\leqq\lambda\leqq 1$, and suppose $(u,\theta)=(u(\lambda),\theta(\lambda))$ satisfy

$$(u,\theta)+\lambda\{\mathcal{B}(u,\theta)+\mathcal{C}(u,\theta)+F\}=0 . \tag{4-4}$$

This is equivalent that the following two equations hold:

$$\nu(\nabla u,\nabla v)+\lambda\{B(u,u,v)-(\beta g\theta,v)-(\beta g\theta_0,v)\}=0,\quad \text{for any } v\in V,$$

$$\kappa(\nabla\theta,\nabla\tau)+\lambda\{b(u,\theta,\tau)+b(u,\theta_0,\tau)+\kappa(\nabla\theta_0,\nabla\tau)\}=0,\quad \text{for any } \tau\in W.$$

Putting $v=u$ and $\tau=\theta$, and calculating in a similar way when we obtained (2-7), (2-8), we have

$$\|\nabla u\|\leqq\frac{1}{\{1-\lambda^2 r\}\nu\sqrt{\lambda_1\mu_1}}\beta\|g\|_{\infty}\{\lambda^2\|\nabla\theta_0\|+\lambda\sqrt{\mu_1}\|\theta_0\|\},$$

$$\|\nabla\theta\|\leqq\frac{1}{1-\lambda^2 r}\{\lambda\|\nabla\theta_0\|+\lambda^2\sqrt{\mu_1}r\|\theta_0\|\},$$

where we have used $0<r<1$ and $\lambda\in[0,1]$. Therefore the solutions to (4-4) are bounded in $\lambda\in[0,1]$, and the Leray-Schauder principle shows us the equation

$$(u,\theta)+\mathcal{B}(u,\theta)+\mathcal{C}(u,\theta)+F=0$$

has a solution and Lemma 1 is proved.

PROOF OF THEOREM 1. If we succeed in constructing a function $\theta_0\in C^1(\bar{\Omega})$ which satisfies $\theta_0=\xi$ on $\bar{\Gamma}_1$, $\partial\theta_0/\partial n=0$ on Γ_2, and the condition (iii) in Lemma 1, then we can apply Lemma 1 and Theorem 1 is proved. By Whitney's extension theorem [12], we can obtain an extension θ satisfying $\theta\in C^1(R^3)$, $\theta=\xi$ on $\bar{\Gamma}_1$ and $\partial\theta/\partial n=0$ on $\partial\Omega$. Let $d(x)$ be the distance of x and $\partial\Omega$, and $\Omega_\delta=\{x\in R^3 \mid d(x)<\delta\}$ and $\alpha(x)\in C_0^1(\Omega_\delta)$ such that $0\leqq\alpha(x)\leqq 1$, $\alpha(x)\equiv 1$ in $\Omega_{\delta/2}$. Put $\theta_0(x)=\alpha(x)\theta(x)$. Then θ_0 is a required extension. Because, $\theta_0\in C_0^1(\Omega_\delta)$ and

$$\|\theta_0\|_{L^p(\Omega)} \leqq \|\theta_0\|_{L^p(\Omega_\delta)}$$
$$\leqq \sup_{x \ni \Omega_\delta} |\theta(x)| |\Omega_\delta|^{1/p}, \qquad 1 \leqq p < \infty,$$

where $|\Omega_\delta|$ is the volume of Ω_δ. For a given positive number ε, we can choose $\delta > 0$ sufficiently small so that the right hand side of the above inequality is less than ε.

In the case $\partial\Omega$ is of class C^2, however, its construction is direct and easy. Therefore, we give it in the following.

Let $\partial\Omega$ be of class C^2 and $\bar{\Gamma}_1 \cap \bar{\Gamma}_2$ be C^1 curve. Then ξ has a C^1-extension on whole $\partial\Omega$ ([7], p. 136). We denote this extension by $\tilde{\xi}$. Since Ω is a bounded domain with C^2 boundary $\partial\Omega$, we find a positive constant δ such that, for any $x \in \Omega_\delta$, there exists a unique point $y = y(x) \in \partial\Omega$ such that $|x - y| = d(x)$. The points x and y are related by

$$x = y + n(y)d(x). \tag{4-5}$$

We fix such δ. Now we define an extension $\tilde{\theta}$ of $\tilde{\xi}(x)$ $(x \in \partial\Omega)$ as follows: Let x be in Ω_δ. We define

$$\tilde{\theta}(x) = \tilde{\xi}(y),$$

where x and y are related by (4-5). Clearly $\tilde{\theta}$ is an extension of $\tilde{\xi}(x)$, $\tilde{\theta}$ is in $C^1(\overline{\Omega_\delta})$ and independent of d.

Now we can easily see that

$$\frac{\partial\tilde{\theta}}{\partial n} = 0 \qquad \text{on } \partial\Omega,$$

because $\tilde{\theta}$ is independent of d.

Let $\alpha(x) \in C_0^1(\Omega_\delta)$ and $0 \leqq \alpha(x) \leqq 1$ $\alpha(x) \equiv 1$ in $\Omega_{\delta/2}$. We put $\theta_0(x) = \tilde{\theta}(x)\alpha(x)$. It is easy to verify that $\theta_0(x) \in C_0^1(\Omega_\delta)$, $\theta_0(x)$ is an extension of $\tilde{\xi}$ satisfying $\partial\theta_0/\partial n = 0$ on $\partial\Omega$, and

$$\sup_{x \in \Omega_\delta} |\theta_0(x)| \leqq \sup_{x' \in \partial\Omega} |\xi(x')|.$$

Now we calculate the L^p-norm of θ_0, $1 \leqq p < \infty$:

$$\iiint_\Omega |\theta_0(x)|^p dx \leqq \sup_{x \in \Omega_\delta} |\theta_0(x)|^p |\Omega_\delta|.$$

For any positive ε, we can choose δ such that the volume $|\Omega_\delta|$ is smaller than ε. Thereby, θ_0 is a desired extension of $\tilde{\xi}$, and Theorem 1 is proved.

§ 5. Proof of Theorem 2.

Under our hypothesis, we can find θ_0 belonging to $C^\infty(\bar{\Omega})$, which satisfies $\theta_0=\xi$ on Γ_1, $\partial\theta_0/\partial n=0$ on Γ_2 and the condition (iii) of Lemma 1. According to Lemma 1, there exists a weak solution $\{u, \tilde{\theta}\}$ of (1-3).

It is known that there exists a distribution p such that

$$\nu\Delta u - \frac{1}{\rho}\nabla p=(u\cdot\nabla)u-\beta g\tilde{\theta}-\beta g\tilde{\theta}_0, \tag{5-1}$$

$$\kappa\Delta\tilde{\theta}=(u\cdot\nabla)\tilde{\theta}+(u\cdot\nabla)\theta_0-\kappa\Delta\theta_0, \tag{5-2}$$

are satisfied in distribution sense (e.g. Temam [14]).

We shall show u, θ belong to $C^\infty(\overline{\Omega'})$, for any subdomain Ω' such that $\overline{\Omega'}\subset\Omega$. Since u is in $H^1(\Omega)$, we find $(u\cdot\nabla)u$ is in $L^{3/2}(\Omega)$. By Lemma 2, $H^1(\Omega)\subset L^6(\Omega)$, and we see $\beta g\tilde{\theta}\in L^6(\Omega)$. From the hypothesis, $\beta g\theta_0$ is in $C^\infty(\bar{\Omega})$. Therefore the left hand side of (5-1) is in $L^{3/2}(\Omega)$. By the theorem of Cattabriga [3] (see also Temam [14]), we find

$$\begin{cases} u\in W^2_{3/2}(\Omega), \\ p\in W^1_{3/2}(\Omega). \end{cases} \tag{5-3}$$

From Lemma 2, we have $W^2_{3/2}(\Omega)\subset L^\alpha(\Omega)$ for $\forall\alpha\in[1,\infty)$. Hence u is in $L^\alpha(\Omega)$ and $(\partial/\partial x_j)(u_iu_j)$ is in $W^{-1}_\alpha(\Omega)$. Since $\operatorname{div} u=0$, $(u\cdot\nabla)u\in W^{-1}_\alpha(\Omega)$, and we find the left hand side of (5-1) is in $W^{-1}_\alpha(\Omega)$. Using again the theorem of Cattabriga, we find

$$\begin{cases} u\in W^1_\alpha(\Omega), \\ p\in L^\alpha(\Omega), \qquad \text{for } \forall\alpha\in[1,\infty). \end{cases}$$

Since u and $\partial u/\partial x_j$ belong to $L^\alpha(\Omega)$, we see $(u\cdot\nabla)u$ is also in that space, and considering $\tilde{\theta}\in L^6(\Omega)$, we know the left hand side of (5-1) is in $L^6(\Omega)$. Cattabriga's theorem gives us the following result:

$$\begin{cases} u\in W^2_6(\Omega), \\ p\in W^1_6(\Omega). \end{cases} \tag{5-4}$$

By Lemma 2, the inclusions

$$W^2_6(\Omega)\subset W^1_6(\Omega)\subset L^\infty(\Omega)$$

hold. Therefore $(u\cdot\nabla)\tilde{\theta}\in L^2(\Omega)$. Since θ_0 is smooth, the left hand side of (5-2) is in $L^2(\Omega)$, and from the result concerning the interior regularity of the solution to the elliptic equation (e.g., Grisvard [8]), we find $\tilde{\theta}\in$

$W_2^2(\Omega')$ for any Ω' such that $\overline{\Omega'}\subset\Omega$. Since $W_2^2(\Omega')\subset W_6^1(\Omega')$, it follows $(u\cdot\nabla)\tilde{\theta}\in L^6(\Omega')$. Using a similar argument, we find $\tilde{\theta}$ belongs to $W_6^2(\Omega'')$ for any Ω'' such that $\overline{\Omega''}\subset\Omega'$. Since Ω' and Ω'' are arbitrary, we find

$$\tilde{\theta}\in W_6^2(\Omega') \tag{5-5}$$

for any Ω' such that $\overline{\Omega'}\subset\Omega$.

Now, we shall show $u, \tilde{\theta}\in W_6^3(\Omega')$. The first derivatives of u are in $W_6^1(\Omega)\subset L^\infty(\Omega)$, and the second derivatives are in $L^6(\Omega)$. Therefore $(\partial/\partial x_i)(u\cdot\nabla)u$ is in $L^6(\Omega)$, that is, $(u\cdot\nabla)u$ belongs to $W_6^1(\Omega)$. As is already shown, $\tilde{\theta}$ is in $W_6^2(\Omega')$. Therefore the left hand side of (5-1) is in $W_6^1(\Omega')$. In consideration of (5-4), we can use Theorem 10.1 of Agmon-Douglis-Nirenberg [2], and we obtain

$$\begin{cases} u\in W_6^3(\Omega''), \\ p\in W_6^2(\Omega''), \end{cases}$$

for $\forall\Omega''$ such that $\overline{\Omega''}\subset\Omega'$. Since Ω' and Ω'' are arbitrary, we have

$$u\in W_6^3(\Omega'),$$

$$p\in W_6^2(\Omega'),$$

for $\forall\Omega'$ such that $\overline{\Omega'}\subset\Omega$. In a similar way, we can show

$$\tilde{\theta}\in W_6^3(\Omega'),$$

for $\forall\Omega'$ such that $\overline{\Omega'}\subset\Omega$, and for any $m=1,2,3,\cdots$,

$$u\in W_6^m(\Omega'),$$

$$p\in W_6^m(\Omega'),$$

$$\tilde{\theta}\in W_6^m(\Omega'),$$

for $\forall\Omega'$ such that $\overline{\Omega'}\subset\Omega$. Theorem 2 is proved.

§6. Proof of Remark.

Let (u_1, θ_1), (u_2, θ_2) be weak solutions of (1-3). We subtract the equations corresponding to u_1 and u_2 and we obtain

$$\nu(\nabla(u_1-u_2), \nabla v)+B(u_1-u_2, u_1, v)+B(u_2, u_1-u_2, v)-(\beta g(\theta_1-\theta_2), v)=0,$$

$$\text{for } v\in V.$$

Similarly we have

$$\kappa(\nabla(\theta_1-\theta_2), \nabla\tau)+b(u_1-u_2, \theta_1, \tau)+b(u_2, \theta_1-\theta_2, \tau)+b(u_1-u_2, \theta_0, \tau)=0\,,$$

$$\text{for } \tau\in W\,.$$

We put $u=u_1-u_2$, $\theta=\theta_1-\theta_2$, and take $(v, \tau)=(u, \theta)$. Then, according to Lemma 4, we have

$$\nu\|\nabla u\|^2+B(u, u_1, u)-(\beta g\theta, u)=0\,,$$

$$\kappa\|\nabla\theta\|^2+b(u, \theta_1, \theta)+b(u, \theta_0, \theta)=0\,.$$

Using Lemma 5 and the a priori estimate (2-7), (2-8), we have

(6-1)
$$\begin{aligned}\nu\|\nabla u\|^2 &\leqq c_B\|\nabla u_1\|\,\|\nabla u\|^2+\beta\|g\|_\infty\|\theta\|\,\|u\| \\ &\leqq \frac{c_B\beta}{\nu\sqrt{\lambda_1\mu_1}(1-r)}\|g\|_\infty\{\|\nabla\theta_0\|+\sqrt{\mu_1}\|\theta_0\|\}\|\nabla u\|^2 \\ &\quad+\frac{\beta}{\sqrt{\lambda_1\mu_1}}\|g\|_\infty\|\nabla\theta\|\,\|\nabla u\|\,,\end{aligned}$$

and

(6-2)
$$\begin{aligned}\kappa\|\nabla\theta\|^2 &\leqq c_B\{\|\nabla\theta_1\|\,\|\nabla u\|\,\|\nabla\theta\|+\|\nabla\theta_0\|\,\|\nabla u\|\,\|\nabla\theta\|\} \\ &\leqq c_B\left\{\frac{2-r}{1-r}\|\nabla\theta_0\|+\frac{r}{1-r}\sqrt{\mu_1}\|\theta_0\|\right\}\|\nabla u\|\,\|\nabla\theta\|\,.\end{aligned}$$

Therefore, if $\|\nabla\theta\|, \|\nabla u\|\neq 0$, then, the inequalities

(6-3)
$$\left\{\nu-\frac{c_B\beta}{\nu\sqrt{\lambda_1\mu_1}(1-r)}\|g\|_\infty(\|\nabla\theta_0\|+\sqrt{\mu_1}\|\theta_0\|)\right\}\|\nabla u\|\leqq\frac{\beta}{\sqrt{\lambda_1\mu_1}}\|g\|_\infty\|\nabla\theta\|,$$

and

(6-4)
$$\kappa\|\nabla\theta\|\leqq c_B\left\{\frac{2-r}{1-r}\|\nabla\theta_0\|+\frac{r}{1-r}\sqrt{\mu_1}\|\theta_0\|\right\}\|\nabla u\|$$

hold. Substituting (6-4) into (6-3), we have

$$\begin{aligned}&\left\{\nu-\frac{c_B\beta}{\nu\sqrt{\lambda_1\mu_1}(1-r)}\|g\|_\infty(\|\nabla\theta_0\|+\sqrt{\mu_1}\|\theta_0\|)\right\}\|\nabla u\| \\ &\leqq\frac{c_B\beta}{\kappa\sqrt{\lambda_1\mu_1}(1-r)}\|g\|_\infty\{(2-r)\|\nabla\theta_0\|+r\sqrt{\mu_1}\|\theta_0\|\}\|\nabla u\|\,.\end{aligned}$$

Therefore, if

$$\frac{c_B\beta}{\kappa\sqrt{\lambda_1\mu_1}(1-r)}\|g_\infty\|\{(2-r)\|\nabla\theta_0\|+r\sqrt{\mu_1}\|\theta_0\|\}$$

$$<\nu-\frac{c_B\beta}{\nu\sqrt{\lambda_1\mu_1}(1-r)}\|g\|_\infty\{\|\nabla\theta_0\|+\sqrt{\mu_1}\|\theta_0\|\}$$

holds, that is,

$$\frac{c_B}{1-r}\frac{\beta}{\nu\kappa\sqrt{\lambda_1\mu_1}}\|g\|_\infty\left\{\left(\frac{\kappa}{\nu}+2-r\right)\|\nabla\theta_0\|+\sqrt{\mu_1}\left(\frac{\kappa}{\nu}+r\right)\|\theta_0\|\right\}<1, \tag{6-5}$$

then, this contradicts $\|\nabla u\|$, $\|\nabla\theta\|\neq 0$, and the uniqueness holds.

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Faculty of Engineering
Meiji University
Tama-ku, Kawasaki
214 Japan