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On the Stokes operator in exterior domains

Dedicated to Professor Hiroshi Fujita on his sixtieth birthday

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§1. Introduction

We consider the Stokes approximation of a stationary viscous incompressible flow past a body. The region Ω occupied by the fluid is assumed to be an exterior domain in $\boldsymbol{R}^n$ with $n\geqq 3$, *i.e.*, a domain whose complement is compact. The velocity field $u=(u^1,\cdots,u^n)$ and the pressure p solve the Stokes system in Ω

$$-\Delta u+\nabla p=f,\quad \operatorname{div} u=0,\quad u|_{\partial\Omega}=0 \tag{1.1}$$

with some external force f, where the boundary $\partial\Omega$ is supposed to be of class $C^{2+\mu}$ with $0<\mu<1$. We consider this problem in the Lebesgue space $L^r(\Omega)$ with $1<r<\infty$. This imposes a decay condition on u at infinity.

It is convenient to eliminate the pressure term ∇p by introducing the Stokes operator. Let $L^r_\sigma(\Omega)$ denote the closure in $L^r(\Omega)^n$ of all solenoidal vector fields with compact support, where $L^r(\Omega)^n$ denotes the corresponding space of L^r vector fields in Ω. There is a bounded projection P_r from $L^r(\Omega)^n$ to $L^r_\sigma(\Omega)$ such that $P_r(\nabla p)=0$ (cf. [9, 19]). The Stokes operator A_r is defined by $A_r=-P_r\Delta$ with domain

$$D(A_r)=\{u\in L^r_\sigma(\Omega)\,;\,\partial_i\partial_j u\in L^r(\Omega)^n,\ 1\leqq i,j\leqq n,\ u|_{\partial\Omega}=0\}$$

which is dense in $L^r_\sigma(\Omega)$, where $\partial_i=\partial/\partial x_i$ and $\Delta=\sum_{i=1}^n\partial_i^2$. Applying P_r to the first equation in (1.1) yields

$$A_r u=P_r f. \tag{1.2}$$

This is equivalent to the Stokes system (1.1) provided that u is in $D(A_r)$.

Our main goal is to prove a local bound on pure imaginary powers A_r^{iy} of the Stokes operator A_r.

THEOREM A. *Assume that $n\geqq 3$ and $1<r<\infty$. Then, for every $\gamma>0$ there is a constant C such that*

$$\|A_r^{iy}f\|_r \leqq Ce^{\gamma|y|}\|f\|_r \tag{1.3}$$

holds for all $f\in L_\sigma^r(\Omega)$ *and* $y\in \boldsymbol{R}$, *where* $\|f\|_r$ *denotes the norm of* f *in* $L^r(\Omega)^n$.

When $r=2$, it is easy to see A_2 is a nonnegative self-adjoint operator (cf. [19, p. 122]). So (1.3) holds with $C=1$, since A_2^{iy} is unitary. The estimate (1.3) is useful to compare the norm $\|A_r^\alpha u\|_r$, $0\leqq\alpha\leqq1$ with other norms. Let D_r^α be the completion of $D(A_r)$ in the norm $\|A_r^\alpha u\|_r$. Since A_r has no bounded inverse, D_r^α is larger than the domain $D(A_r^\alpha)$ of the fractional power A_r^α. Applying Theorem A, we shall prove:

THEOREM B. *Assume that* $n\geqq3$ *and* $1<r<\infty$. *Then,* D_r^α *agrees with the complex interpolation space* $[L_\sigma^r(\Omega), D_r^1]_\alpha$, $0\leqq\alpha\leqq1$ *as Banach spaces.*

By an a priori estimate $\|\nabla^2 u\|_r\leqq C\|A_r u\|_r$ $(1<r<n/2)$ for (1.2) [6, 21], Theorem B yields a comparison of two norms $\|A_r^{1/2}u\|_r$ and $\|\nabla u\|_r$. We shall prove

$$\|\nabla u\|_r\leqq C\|A_r^{1/2}u\|_r \qquad 1<r<\max(n/2, 2) \text{ or } r=2 \tag{1.4}$$

holds for all $u\in D_r^{1/2}$ with C independent of u. Combining the Sobolev inequality in Ω, we also prove

$$\|u\|_\rho\leqq C\|A_r^\alpha u\|_r, \qquad 1/\rho=1/r-2\alpha/n \tag{1.5}$$

when $1<r<n/2$, $0\leqq\alpha\leqq1$ or $1<r\leqq2$, $0\leqq\alpha\leqq1/2$.

Estimates like (1.4) and (1.5) are useful to study existence, regularity and large time behavior of solutions of the nonstationary Navier-Stokes system. Although this was noticed by Kato and Fujita [15] long time ago (cf. [12, 24]), so far Theorem A was proved only when Ω is a bounded domain [11] or a half space [5]. When Ω is an exterior domain, recently Borchers and Sohr [6] proved a resolvent estimate

$$\|(\lambda+A_r)^{-1}f\|_r\leqq C_\varepsilon\|f\|_r/|\lambda|, \tag{1.6}$$

with C_ε independent of λ and f, where λ is a complex number with $|\arg\lambda|\leqq\pi-\varepsilon$, $\lambda\neq0$, $0<\varepsilon<\pi/2$. Such estimate is known by [10, 21] for large λ, so they studied the behavior of the resolvent near $\lambda=0$, which corresponds to the large time behavior of the semigroup e^{-tA_r} generated by A_r. When Ω is bounded, estimate (1.6) for small λ is clear since A_r^{-1} is bounded. The estimate (1.6) means that e^{-tA_r} is a bounded analytic semigroup on $L_\sigma^r(\Omega)$, so for $\alpha\geqq0$ we have decay estimates

$$\|A_r^\alpha e^{-tA_r}f\|_r \leqq C\|f\|_r/t^\alpha, \qquad t>0.$$

Applying (1.4) and (1.5) yields $L^\rho-L^r$ decay estimates

$$\|e^{-tA_r}f\|_\rho \leqq C\|f\|_r/t^\alpha, \quad \|\nabla e^{-tA_r}f\|_\rho \leqq C\|f\|_r/t^{\alpha+1/2}, \tag{1.7}$$

$$\alpha=(1/r-1/\rho)n/2,$$

where $1<r<n/2$, $0\leqq\alpha\leqq 1$ or $1<r\leqq 2$, $0\leqq\alpha\leqq 1/2$. Such estimates are useful to study the large time behavior of solutions of the nonstationary Navier-Stokes system in exterior domains (cf. [14] for $\boldsymbol{R}^n$). For recent progress we refer to [4]. We note that the estimate (1.6) for small λ remains unproved when the space dimension n is two unless Ω is a half plane [5].

Although (1.6) is enough to define fractional powers A_r^z for all complex numbers z according to Komatsu [16], more analysis for the resolvent is necessary to prove Theorem A. In this paper we approximate the resolvent $(\lambda+A_r)^{-1}$ with the resolvent of the Stokes operator in the entire space $\boldsymbol{R}^n$. The error is well studied for large λ in [11]. The crucial step is to show that the error is actually minor compared with the behavior of both resolvents when λ is small. We study behaviors near space infinity and near $\partial\Omega$ separately by usual cutting-off procedures. However, $\operatorname{div} u=0$ is not compatible with cutting-off procedures. We use a priori estimates for the Stokes system with $\operatorname{div} u\neq 0$, which is essentially proved in [6] by applying Bogovski's results [3] to $\operatorname{div} u=g$. Once Theorem A is proved, Theorem B follows from an abstract theory. On our way to prove Theorem A, we give in § 3 a proof of (1.6) which simplifies that of [6].

This paper is organized as follows. In § 2 we review Bogovski's results on $\operatorname{div} u=g$ and, as an application, the Sobolev type inequalities in the exterior domains. We show higher regularity properties of the projection P_r for later use. § 3 is devoted to deriving a priori estimates for the Stokes system with $\operatorname{div} u\neq 0$, which is essentially known in [6]. We give here a simpler proof. We also show that A_r is injective, *i. e.*, the solution of (1.2) is unique. § 4 estimates the difference of $(\lambda+A_r)^{-1}$ and the resolvent of the Stokes operator in $\boldsymbol{R}^n$. Here we use estimates in § 3. In § 5 we prove Theorem A. Since A_r does not have bounded inverse, we are forced to use Komatsu's general theory [16] of fractional powers of operators. We compare A_r^{iy} with the corresponding powers of the Stokes operator in $\boldsymbol{R}^n$, using results in § 4. The final section is devoted to proofs of Theorem B and its corollaries (1.4) and (1.5). We prepare a new abstract interpolation theorem so that we can also calculate complex interpolation spaces of homogeneous Sobolev spaces in $\boldsymbol{R}^n$ which often appear in this paper.

After we completed this work, we learned that Borchers and Miyakawa [4] show that (1.4) holds for all r, $1<r<n$. We also learned that Iwashita [13] proves (1.7) without using (1.3).

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§ 2. Preliminaries

This section establishes conventions of notation, reviews some properties of the equation $\operatorname{div} u=f$ with zero boundary condition and derives some calculus inequalities on an exterior domain. This section also reviews the Helmholtz decomposition on an exterior domain.

Throughout, Ω is a domain in $\boldsymbol{R}^n$ with compact boundary $\partial\Omega$. As usual, $C_0^\infty(\Omega)$ denotes the space of smooth functions defined in Ω with compact support and $L^r(\Omega)$ is the Lebesgue space equipped with norm

$$\|f\|_r=\|f\|_{r\Omega}=\left(\int_\Omega |f(x)|^r dx\right)^{1/r}.$$

We shall denote $\|f\|_r$ by $|f|_r$ if $\Omega=\boldsymbol{R}^n$. Usual Sobolev space $H^{k,r}(\Omega)$ is the space of functions f such that $\partial^\alpha f\in L^r(\Omega)$ for all $|\alpha|\leqq k$; the norm of f in $H^{k,r}(\Omega)$ is

$$\left(\sum_{|\alpha|\leqq k}\|\partial^\alpha f\|_r^r\right)^{1/r}, \qquad \partial^\alpha=\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n},$$

where $\partial_j=\partial/\partial x_j$ $(1\leqq j\leqq n)$ denotes the distribution derivative and $|\alpha|=\sum_{j=1}\alpha_j$. The completion of $C_0^\infty(\Omega)$ in $H^{k,r}(\Omega)$ is written by $H_0^{k,r}(\Omega)$. We often use homogeneous Sobolev spaces $\hat{H}_0^{k,r}(\Omega)$ defined as the completion of $C_0^\infty(\Omega)$ in the norm

$$\|\nabla^k f\|_r=\left(\sum_{|\alpha|=k}\|\partial^\alpha f\|_r^r\right)^{1/r}.$$

We shall use this space only when Ω is unbounded, since otherwise it follows from the Poincaré inequality that this space $\hat{H}_0^{k,r}(\Omega)$ agrees with $H_0^{k,r}(\Omega)$ as Banach spaces. We also see the element $f\in\hat{H}_0^{k,r}(\Omega)$ is identified with some f in $L_{\mathrm{loc}}^r(\bar{\Omega})$ with $\|\nabla^k f\|_r<\infty$ provided that Ω has the boundary, where $f\in L_{\mathrm{loc}}^r(\bar{\Omega})$ means that $f\in L^r(\Omega\cap B)$ for every ball $B\subset\boldsymbol{R}^n$; $\bar{\Omega}$ is the closure of Ω in $\boldsymbol{R}^n$. Throughout, we assume that $1<r<\infty$ and that k is a nonnegative integer. All spaces are considered to be complex vector spaces. The space $C_0^\infty(\Omega)^m$, $L^r(\Omega)^m$, $\hat{H}_0^{k,r}(\Omega)^m,\cdots$ denote the corresponding space of m-vector valued functions.

We consider

$$\text{div}\, u=f \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega, \tag{2.1}$$

where $\text{div}\, u=\sum_{j=1}^{n}\partial_j u^j$ and $u=(u^1, \cdots, u^n)$. The solution u is not unique but one can choose a good representative such that the mapping $f\mapsto u$ is bounded linear in appropriate Banach spaces. Such results are due to Bogovski [3]; see also [7] for recent progress. We shall only use the following special cases.

LEMMA 2.1 (Bogovski [3]). (i) *Let Ω be an exterior domain, i.e. a domain whose complement in $\boldsymbol{R}^n$ is compact in $\boldsymbol{R}^n$. Assume that $\partial\Omega$ is C^1 and $n\geqq 2$. Then, there is a bounded linear operator $f\mapsto u$ from $L^r(\Omega)$ to $\hat{H}_0^{1,r}(\Omega)^n$ such that* (2.1) *holds.*

(ii) *Let D be a bounded domain in $\boldsymbol{R}^n$ $(n\geqq 2)$ with C^1 boundary. There is a bounded linear operator $f\mapsto u$ from*

$$\left\{f\in L^r(D)\,;\ \int_D f(x)dx=0\right\}$$

to $H_0^{1,r}(D)$ such that (2.1) *holds with $\Omega=D$. Moreover, $f\in H_0^{1,r}(D)$ implies $u\in H_0^{2,r}(D)^n$ and*

$$\|\nabla^2 u\|_r\leqq C\|\nabla f\|_r \tag{2.2}$$

with C independent of f.

Outline of the proof. We first discuss (ii). Suppose that D is starlike with respect to some open ball B with $\bar{B}\subset D$. We take $h\in C_0^\infty(B)$ with $\int_B h(x)dx=1$. Using this h, Bogovski [3] gave an explicit formula for u:

$$u(x)=\int_D K(x, y)f(y)dy\,, \qquad K(x, y)=\frac{x-y}{|x-y|^n}\int_{|x-y|}^{\infty} h\left(y+t\frac{x-y}{|x-y|}\right)t^{n-1}dt,$$

where $|x|^2=\sum_{i=1}^{n}x_i^2$. A direct calculation shows that u solves (2.1). Applying the Calderón-Zygmund inequality [23, § 2.2], we have $\|\nabla u\|_r\leqq C\|f\|_r$. Similarly, one can show that $u\in H_0^{2,r}(D)^n$ and (2.2) if $f\in H_0^{1,r}(D)$ at least when D is starlike with respect to B. For general D, (ii) follows from a localization procedure.

It remains to prove (i). We first show the corresponding property when Ω is the entire or half space. By a localization, (i) for general Ω can be reduced to these special cases and (ii). □

As an simple application of Lemma 2.1, we have the Poincaré-Sobolev and the Gagliardo-Nirenberg inequalities in *exterior domains.* This fact is pointed out by [6]. We give them below for the reader's convenience and completeness.

Let $\hat{H}^{-1,r'}(\Omega)$ denote the dual of $\hat{H}_0^{1,r}(\Omega)$ with $1/r+1/r'=1$. When Ω has the boundary, $\hat{H}^{-1,r'}(\Omega)$ is a space of distributions u on Ω such that

$$\|u\|_{-1,r'}=\sup_{\substack{\varphi\in C_0^\infty(\Omega)\\ \|\nabla\varphi\|_r\neq 0}}|\langle u,\varphi\rangle|/\|\nabla\varphi\|_r<\infty,$$

where $\langle u,\varphi\rangle$ denotes the value of u at φ. When u is locally integrable on Ω, $\langle u,\varphi\rangle=\int_\Omega u\varphi dx$. We shall also use the notation $\langle u,\varphi\rangle$ even if $u\in\hat{H}^{-1,r'}(\Omega)^m$ by understanding $\langle u,\varphi\rangle=\sum_{j=1}^m\langle u^j,\varphi^j\rangle$, where $u=(u^1,\cdots,u^m)$, $\varphi=(\varphi^1,\cdots,\varphi^m)$ and $\varphi\in C_0^\infty(\Omega)^m$.

COROLLARY 2.2. *Let Ω be an exterior domain in $\boldsymbol{R}^n$ $(n\geqq 2)$ with C^1 boundary.*

(i) *Suppose that $w\in\hat{H}^{-1,r'}(\Omega)^n$ satisfies $\langle w,v\rangle=0$ for all $v\in\hat{H}_0^{1,r}(\Omega)$ with* $\operatorname{div} v=0$. *Then, there is a unique $u\in L^{r'}(\Omega)$ such that $w=\operatorname{grad} u$ $(=\nabla u)$ and*

$$\|u\|_{r'}\leqq C\|\nabla u\|_{-1,r'} \tag{2.3}$$

with C independent of u.

(ii) *Assume that $1<r<n$ and that $u\in L_{\mathrm{loc}}^1(\bar{\Omega})$ and $\nabla u\in L^r(\Omega)^n$. Then, there is a constant K_u such that $u+K_u\in L^q(\Omega)$ with $1/n+1/q=1/r$ and*

$$\|u+K_u\|_q\leqq C\|\nabla u\|_r \tag{2.4}$$

with C independent of u.

(iii) *Assume that $1<r<n/2$ and that $u\in L^r(\Omega)$ with $\|\nabla^2u\|_r<\infty$. Then $u\in L^q(\Omega)$ and $\nabla u\in L^q(\Omega)^n$ with $2/n+1/s=1/r$ and $1/n+1/s=1/q$, and*

$$\|u\|_s\leqq C_1\|\nabla u\|_q\leqq C_2\|\nabla^2u\|_r\,. \tag{2.5}$$

Moreover, $u\in L^\rho(\Omega)$ with $\alpha/s+(1-\alpha)/r=1/\rho$ and $0<\alpha<1$, and

$$\|u\|_\rho\leqq C_3\|\nabla^2u\|_r^\alpha\|u\|_r^{1-\alpha}\,. \tag{2.6}$$

Here C_j $(j=1,2,3)$ is a positive constant independent of u.

PROOF. (i) From Lemma 2.1 (i) it follows that the operator

$$\operatorname{div}:\hat{H}_0^{1,r}(\Omega)^n\longrightarrow L^r(\Omega)$$

is surjective. Applying the closed range theorem [25] to the dual operator,

$$\operatorname{div}^* = -\nabla : L^{r'}(\Omega) \longrightarrow \hat{H}^{-1,r'}(\Omega)^n$$

we see $-\nabla$ has the closed range. Since $-\nabla$ is injective, the closed graph theorem [25] yields the desired results.

(ii) By the Hölder inequality we have

$$|\langle \nabla u, \varphi \rangle| \leqq \|\nabla u\|_r \|\varphi\|_{r'}, \qquad \varphi \in C_0^\infty(\Omega)^n .$$

The Sobolev inequality in $\boldsymbol{R}^n$ [8, p. 24]:

$$|\varphi|_{r'} \leqq C |\nabla \varphi|_{q'}$$

now yields $\nabla u \in \hat{H}^{-1,q}(\Omega)^n$ and

$$\|\nabla u\|_{-1,q} \leqq C \|\nabla u\|_r .$$

Applying (i), we now obtain (2.4).

(iii) It suffices to prove (2.5) and (2.6). The estimate (2.5) follows directly from (2.4). The estimate (2.6) follows from (2.5) and the interpolation inequality:

$$\|u\|_p \leqq \|u\|_s^\alpha \|u\|_r^{1-\alpha} . \qquad \square$$

The rest of this section is devoted to the regularity property for the projection P to divergence free vector fields. Let $L_\sigma^r(\Omega)$ denote the closure of $C_{0,\sigma}^\infty(\Omega)$ in $L^r(\Omega)^n$, where

$$C_{0,\sigma}^\infty(\Omega) = \{u \in C_0^\infty(\Omega)^n \,;\ \operatorname{div} u = 0\} .$$

Each vector field $f \in L^r(\Omega)^n$ is decomposed as

$$f = f_0 + \nabla p \tag{2.7}$$

with some $f_0 \in L_\sigma^r(\Omega)$, $p \in L_{\mathrm{loc}}^r(\bar{\Omega})$ and $\nabla p \in L^r(\Omega)^n$ and

$$\|\nabla p\|_r \leqq C \|f\|_r \quad \text{and} \quad \|p_0\|_{r, B\cap\Omega} \leqq C \|f\|_r \tag{2.8}$$

where B is an open ball in $\boldsymbol{R}^n$ and C is a constant independent of f. This decomposition is unique and called the Helmholtz decomposition. The mapping $f \mapsto f_0$ defines a continuous projection P_r from $L^r(\Omega)^n$ to $L_\sigma^r(\Omega)$. Since P_r is independent of r on $C_{0,\sigma}^\infty(\Omega)$, we often supress subscript r. Results (2.7) and (2.8) are proved by [9] when Ω is bounded and by [19] when Ω is an exterior domain when $n=3$; the proof works for $n \geqq 3$. For regularity of $\partial\Omega$ we only need to assume C^1, although they do not mention explicitly. We shall prove that P is bounded in Sobolev spaces. This is proved by [12] when Ω is a smoothly bounded domain.

LEMMA 2.3. *Let k be a positive integer and let Ω be an exterior domain in $\boldsymbol{R}^n$ $(n \geqq 3)$ with $C^{k+\mu}$ boundary, where $0<\mu<1$. Assume that $1<r\leqq q<\infty$. Then*

$$\|\nabla^k Pf\|_r \leqq C(\|\nabla^k f\|_r + \|f\|_q) \tag{2.9}$$

with C independent of $f \in L^q(\Omega)^n$.

PROOF. We may assume that $\|\nabla^k f\|_r<\infty$. Since $f \in L^q(\Omega)^n$, (2.7) holds with $p \in L^q_{\mathrm{loc}}(\bar{\Omega})$. By (2.7) we see

$$\Delta p = \operatorname{div} f \quad \text{in } \Omega, \qquad N\cdot\nabla\phi|_{\partial\Omega} = N\cdot f|_{\partial\Omega} \tag{2.10}$$

where N is the normal vector to $\partial\Omega$ and $\cdot$ denotes the standard inner product in $\boldsymbol{R}^n$. The estimate (2.9) will follow from L^p regularity theory for the elliptic boundary value problem (2.10). When Ω is bounded, (2.9) follows directly (cf. [12]). However, since Ω is unbounded, we shall reduce our problems to cases when Ω is bounded and $\Omega=\boldsymbol{R}^n$ by multiplying cut-off functions $\varphi \in C^\infty(\boldsymbol{R}^n)$ with p. By (2.10) we have

$$\Delta(\varphi p) = F = \varphi \operatorname{div} f + 2\nabla p\cdot\nabla\varphi + p\Delta\varphi \quad \text{in } \Omega\,. \tag{2.11}$$

We assume that $\nabla\varphi=0$ for large $|x|$ so that (2.8) implies

$$\|F\|_r \leqq C_1(\|\nabla f\|_r + \|f\|_q)\,, \quad r\leqq q\,, \tag{2.12}$$

with C_l $(l=1, 2, \cdots)$ independent of f.

For a given open ball $B(R)$ of radius R with center zero we take $\varphi=\varphi_R$ such that $\varphi_R=1$ on $B(R)$ and $\operatorname{supp}\varphi_R \subset B(R+1)$. Taking R large, we may assume $B(R)\supset\partial\Omega$. Applying L^p theory for (2.11) on a bounded domain $\Omega(R+1)=B(R+1)\cap\Omega$ with boundary conditions

$$N\cdot\nabla(\varphi p)|_{\partial\Omega} = N\cdot f|_{\partial\Omega} \quad \text{and} \quad \varphi p=0 \quad \text{on } \partial B(R+1)\,,$$

we obtain, by (2.12),

$$\|\nabla^2 p\|_{r\Omega(R)} \leqq C_2\|F\|_r \leqq C_3(\|\nabla f\|_r + \|f\|_q)\,.$$

This yields

$$\|\nabla F\|_r \leqq C_4(\|\nabla^2 f\|_r + \|\nabla f\|_r + \|f\|_q) \leqq C_5(\|\nabla^2 f\|_r + \|f\|_q)\,.$$

Repeating this argument with $\varphi=\varphi_{R-j+1}$ $(j=2, 3, \cdots, k)$ successively, we have

$$\|\nabla^{j+1} p\|_{r\Omega(R-j+1)} \leqq C_6(\|\nabla^j f\|_r + \|f\|_q)\,, \qquad 1\leqq j\leqq k \tag{2.13}$$

for sufficiently large R; at this stage C_5 may depend on R.

It remains to estimate $\partial^\alpha p$ near infinity. We take $\varphi\in C^\infty(\boldsymbol{R}^n)$ such that $\varphi=0$ near $\partial\Omega$ and $\varphi=1$ for large $|x|$. This time (2.11) becomes an equation of distribution on $\boldsymbol{R}^n$

$$\Delta U=F \quad \text{in } \boldsymbol{R}^n \quad \text{with } U=\varphi p. \tag{2.14}$$

For this equation we have a priori estimates:

$$|\nabla^2\partial^\alpha U|_r\leqq C|\partial^\alpha F|_r, \tag{2.15}$$

with C independent of F, where $|f|_r$ denotes the norm of f in $L^r(\boldsymbol{R}^n)$. Indeed, for $\phi\in C_0^\infty(\boldsymbol{R}^n)$, we see

$$\langle\partial_i\partial_j U, \Delta\phi\rangle=\langle\Delta U, \partial_i\partial_j\phi\rangle=\langle F, \partial_i\partial_j\phi\rangle .$$

Using the Hölder and the Calderón-Zygmund inequality [23, § 2.2] yields

$$|\langle F, \partial_i\partial_j\phi\rangle|\leqq|F|_r|\partial_i\partial_j\phi|_{r'}\leqq C|F|_r|\Delta\phi|_{r'}$$

with $1/r+1/r'=1$. Since $\{\Delta\phi\,;\,\phi\in C_0^\infty(\boldsymbol{R}^n)\}$ is dense in $L^{r'}(\boldsymbol{R}^n)$, we obtain (2.15) for $|\alpha|=0$. Differentiating (2.14) now yields (2.15) for general α. By (2.13) we have

$$|\nabla^{k-1}F|_r\leqq C_7(\|\nabla^k f\|_r+\|f\|_q) .$$

Applying (2.15) yields

$$\|\nabla^{k+1}(\varphi p)\|_r\leqq C_8(\|\nabla^k f\|_r+\|f\|_q) .$$

The estimate (2.9) follows from this and (2.13). □

COROLLARY 2.4. *Let Ω be an exterior domain in $\boldsymbol{R}^n$ $(n\geqq 3)$ with $C^{2+\mu}$ boundary $(0<\mu<1)$. We have*

$$\|\nabla Pf\|_r\leqq C\|\nabla f\|_r, \qquad 1<r<n \tag{2.16}$$

$$\|\nabla^2 Pf\|_r\leqq C\|\nabla^2 f\|_r, \qquad 1<r<n/2 \tag{2.17}$$

for all $f\in L^r(\Omega)^n$ with C independent of f.

PROOF. If $\|\nabla f\|_r<\infty$ and $f\in L^r(\Omega)^n$, it follows from (2.4) that $\|f\|_s\leqq C\|\nabla f\|_r$ with $1/n+1/s=1/r$. Applying (2.9) with $k=1$ and $q=s$ yields (2.16). The proof of (2.17) is similar to that of (2.16), so is omitted. □

§ 3. A priori estimates for the Stokes equations

We consider the Stokes equations

$$\lambda u-\Delta u+\nabla p=f, \quad \operatorname{div} u=g \quad \text{in } \Omega , \tag{3.1}$$

$$u|_{\partial\Omega}=0\,, \tag{3.2}$$

where Ω is an exterior domain in $\boldsymbol{R}^n$ with $C^{2+\mu}$ boundary $(0<\mu<1)$ and λ is a complex number. When $g=0$, we have

THEOREM 3.1 ([6]). *Assume that* $n\geqq 3$, $1<r<n/2$ *and* $0<\varepsilon<\pi/2$. *Suppose that* $u\in H^{2,r}(\Omega)^n$ *and* $p\in L^r_{\mathrm{loc}}(\bar{\Omega})$ *with* $\nabla p\in L^r(\Omega)^n$ *solve* (3.1) *and* (3.2) *with* $g=0$. *Then*

$$|\lambda|\,\|u\|_r+\|\nabla^2 u\|_r+\|\nabla p\|_r\leqq C\|f\|_r \tag{3.3}$$

for $|\arg\lambda|\leqq\pi-\varepsilon$ *with* C *depending only on* n, r, ε, *and* Ω. (*We may take* $\lambda=0$.)

This section gives a proof which is a simplified version of that in [6]. As is pointed out in [6], one can extend Theorem 3.1 for some $g\neq 0$. This is useful in the next section.

THEOREM 3.2. *Assume that* $n\geqq 3$, $1<r<n/2$ *and* $0<\varepsilon<\pi/2$. *Suppose that* $u\in H^{2,r}(\Omega)^n$ *and* $p\in L^r_{\mathrm{loc}}(\bar{\Omega})$ *with* $\nabla p\in L^r(\Omega)^n$ *solve* (3.1) *and* (3.2) *with*

$$\int_\Omega g(x)dx=0\,,\quad \operatorname{supp} g\subset B \textit{ and } g|_{\partial\Omega}=0\,, \tag{3.4}$$

where B *is an open ball in* $\boldsymbol{R}^n$ *with* $B\supset\partial\Omega$. *Then*

$$|\lambda|\,\|u\|_r+\|\nabla^2 u\|_r+\|\nabla p\|_r\leqq C(\|f\|_r+\|\nabla g\|_r+|\lambda|\,\|g\|_r) \tag{3.5}$$

for $|\arg\lambda|\leqq\pi-\varepsilon$ *with* C *depending only on* n, r, ε, B *and* Ω.

Proof that Theorem 3.1 implies Theorem 3.2. We may assume that $\|\nabla g\|_r<\infty$. By (3.4) we can apply Lemma 2.1 (ii) with $D=B\cap\Omega$ and get a function $v\in H^{2,r}_0(D)$ which solves $\operatorname{div} v=g$ and satisfies

$$\|\nabla^2 v\|_{rD}\leqq C\|\nabla g\|_{rD}\,,\qquad \|v\|_{rD}\leqq C'\|\nabla v\|_{rD}\leqq CC'\|g\|_{rD} \tag{3.6}$$

with C and C' independent of g. Defining $v=0$ outside D we extend v to function $v\in H^{2,r}_0(\Omega)$. The difference $w=u-v$ solves

$$(\lambda-\Delta)w+\nabla p=F=f-(\lambda-\Delta)v\,,\qquad \operatorname{div} w=0 \quad \text{in } \Omega,\ w|_{\partial\Omega}=0\,.$$

The estimates (3.3) and (3.6) yields (3.5) since $\|v\|_{rD}=\|v\|_r$ and $\|\nabla^2 v\|_r=\|\nabla^2 v\|_{rD}$. □

It is well known that (3.3) holds for large λ (cf. [10, 21]) for all $1<r<\infty$ and $n\geqq 2$ so the crucial step is to prove (3.3) for small λ. This problem is related to the behavior of solutions near space infinity. Since (3.1) and (3.2) is an elliptic system ([2, II]), a general a priori estimate yields (3.3) with adding extra terms

$$C(\|\nabla u\|_r+\|u\|_r+\|p\|_r)$$

in the right hand side. As is pointed out in [2, I, pp. 668-669], when Ω is bounded one can delete these extra terms if solution of (3.1) and (3.2) is unique for given f and g. Indeed, one can prove (3.3) for small λ when Ω is bounded. Since Ω is unbounded, this idea does not apply directly to our problem, although we still reduce our problem to uniqueness of solutions of (3.1) and (3.2).

Below we shall prove (3.5) when $\Omega=\boldsymbol{R}^n$. The assumption (3.4) is necessary because (3.5) does not hold even if $f=0$ and $\Omega=\boldsymbol{R}^n$ without (3.4). We next prove a priori estimate near space infinity. Applying this estimate, we prove that solution of (3.1) and (3.2) is unique even if $\lambda=0$. This and a priori estimates near space infinity and for bounded domain yield Theorem 3.1. We pay attention on decay assumptions for u so that we get a good uniqueness result.

Let us recall the Stokes operator $A_r=-P_r\Delta$ with domain

$$D(A_r)=\{w\in(H^{2,r}(\Omega))^n\cap L_\sigma^r(\Omega)\,;\ w|_{\partial\Omega}=0\}$$

(dense in $L_\sigma^r(\Omega)$), where P_r is the continuous projection from $L^r(\Omega)^n$ to $L_\sigma^r(\Omega)$ associated with the Helmholtz decomposition (2.7); we assume $n\geqq 3$ for exterior Ω. The Stokes equations (3.1), (3.2) with $g=0$ and

$$u\in H^{2,r}(\Omega),\quad p\in L_{\mathrm{loc}}^r(\bar{\Omega})\quad\text{and}\quad\nabla p\in L^r(\Omega) \tag{3.7}$$

are equivalent to

$$(\lambda+A_r)u=P_rf,\qquad u\in D(A_r)$$

where the operator of multiplication with constant λ is simply denoted by λ. We say $(u,p)\in E^r(\Omega)$ (resp. $E_{\mathrm{loc}}^r(\bar{\Omega})$) if u and p satisfy (3.7) (with replacing Ω by $\Omega\cap B$ where B is an arbitrary open ball in $\boldsymbol{R}^n$).

When $\Omega=\boldsymbol{R}^n$, we denote A_r and P_r by $\tilde{A}_r$ and $\tilde{P}_r$, respectively. These operators are expressed explicitly:

$$\tilde{P}=\tilde{P}_r=F^{-1}hF,\quad h(\xi)=(\delta^{ij}-\xi_i\xi_j/|\xi|^2)_{1\leq i,j\leq n},\quad \tilde{A}_r=-\tilde{P}_r\Delta \tag{3.8}$$

where F is the Fourier transformation defined by

$$(Fw)(\xi)=\int_{\boldsymbol{R}^n}e^{-ix\cdot\xi}w(x)dx,\qquad \xi\in\boldsymbol{R}^n.$$

For λ, $|\arg\lambda|\neq\pi$ and $\lambda\neq 0$ it is well-known (cf. [10, 11]) that $\lambda+\tilde{A}_r$ has the bounded inverse and

$$(3.9)\qquad (\lambda+\tilde{A}_r)^{-1}=(\lambda-\Delta)^{-1}\tilde{P}$$

since Δ commutes with $\tilde{P}$. Let us rewrite estimates for the resolvent $(\lambda+\tilde{A}_r)^{-1}$ in a form slightly different from [10].

LEMMA 3.3. *Assume that* $n\geqq 2$, $1<r<\infty$ *and* $0<\varepsilon<\pi/2$. *Suppose that* $u\in L^s(\boldsymbol{R}^n)^n$, $\nabla p\in L^q(\boldsymbol{R}^n)^n$ *with* $(u, p)\in E^q_{\mathrm{loc}}(\boldsymbol{R}^n)$ *for some* $1<q\leqq s<\infty$ *and that* (u, p) *solves* (3.1) *with* $g=0$ *in* $\boldsymbol{R}^n$. *Then*

$$(3.10)\qquad |\lambda||u|_r+|\nabla^2 u|_r+|\nabla p|_r\leqq C|f|_r$$

for $|\arg\lambda|\leqq\pi-\varepsilon$ *with* C *depending only on* n, r *and* ε.

PROOF. We first assume that $(u, p)\in E^s(\boldsymbol{R}^n)$. This implies that $u\in D(\tilde{A}_s)$ and solves

$$(\lambda+\tilde{A}_s)u=\tilde{P}_s f,$$

which yields

$$(\lambda-\Delta)u=\tilde{P}_s f.$$

Applying a multiplier theorem [23, § 2.2] to $\nabla^l(\lambda-\Delta)^{-1}$ $(l=0, 2)$ yields

$$|\lambda||u|_r+|\nabla^2 u|_r\leqq C|f|_r \qquad \text{(cf. [10])}.$$

provided that $\lambda\neq 0$; even if $\lambda=0$, this holds because of (2.15). This yields (3.10) for $(u, p)\in E^s(\boldsymbol{R}^n)$ since

$$\nabla p=f-(\lambda-\Delta)u\ .$$

For general (u, p) we consider mollified functions $u_\delta=u*\varphi_\delta$ and $p_\delta=p*\varphi_\delta$ where $\varphi_\delta\in C_0^\infty(\boldsymbol{R}^n)$ is a mollifier $\varphi_\delta(x)=\varphi(x/\delta)\delta^{-n}\geqq 0$ with $\int\varphi dx=1$. By estimating the convolution, we see $(u_\delta, p_\delta)\in E^s(\boldsymbol{R}^n)$, $\delta>0$ since $s\geqq q$. Since (u_δ, p_δ) solves (3.1) with $f=f_\delta$, $g=0$, (3.10) is valid for (u_δ, p_δ). Letting δ tend to zero leads to (3.10) for general (u, p). □

LEMMA 3.4. *Assume that* $n\geqq 2$, $1<r<\infty$ *and* $0<\varepsilon<\pi/2$. *Suppose that* $u\in L^s(\boldsymbol{R}^n)$, $\nabla p\in L^q(\boldsymbol{R}^n)^n$ *with* $(u, p)\in E^q_{\mathrm{loc}}(\boldsymbol{R}^n)$ *for some* $1<q\leqq s<\infty$ *and that* (u, p) *solves* (3.1) *in* $\boldsymbol{R}^n$ *with* (3.4). *Then*

$$(3.11)\qquad |\lambda||u|_r+|\nabla^2 u|_r+|\nabla p|_r\leqq C(|f|_r+|\nabla g|_r+|\lambda||g|_r)$$

for $|\arg\lambda|\leqq\pi-\varepsilon$ *with* C *depending only on* n, r, B *in* (3.4) *and* ε.

PROOF. The estimate (3.11) is reduced to (3.10). The proof is almost the same as the proof that (3.3) implies (3.5), so is omitted. □

REMARK. When $(u,p)\in E^r(\boldsymbol{R}^n)$, (3.11) is proved by [6]. We assume that (u,p) belongs to a function class so that u decays at space infinity. We may weaken this assumption in Lemmas 3.3 and 3.4. For example, we only need to assume that $u\in L^s(\boldsymbol{R}^n)$ and $\nabla p\in L^q(\boldsymbol{R}^n)^n$ with $1<q\leqq s<\infty$.

LEMMA 3.5 (Estimates near space infinity). *Assume that* $n\geqq 2$, $1<r<\infty$ *and* $0<\varepsilon<\pi/2$. *Let* Ω *be an exterior domain in* $\boldsymbol{R}^n$ *such that* Ω *contains* $\Omega'=\boldsymbol{R}^n-B$, *where* B *is an open ball. Suppose that* $u\in L^s(\Omega)^n$, $\nabla p\in L^q(\Omega)^n$ *with* $(u,p)\in E^q_{\text{loc}}(\bar{\Omega})$ *for some* $1<q\leqq s<\infty$ *and that* (u,p) *solves* (3.1) *in* Ω *with* (3.4). *Then*

$$
\begin{aligned}
(3.12)\qquad & |\lambda|\,\|u\|_{r\Omega'}+\|\nabla^2 u\|_{r\Omega'}+\|\nabla p\|_{r\Omega'} \\
& \leqq C(\|f\|_r+\|\nabla g\|_r+|\lambda|\,\|g\|_r+\|u\|_{rD}+\|\nabla u\|_{rD}+\|p\|_{rD}),\quad D=\Omega\cap B
\end{aligned}
$$

for $|\arg\lambda|\leqq\pi-\varepsilon$ *with* C *depending only on* n, r, ε, B *and* Ω.

PROOF. We just cut off space infinity. Multiplying $\varphi\in C^\infty(\boldsymbol{R}^n)$ with f and g in (3.1) yields

$$
(3.13)\qquad
\begin{aligned}
&(\lambda-\Delta)(\varphi u)+\nabla(\varphi p)=\varphi f-2\nabla u\cdot\nabla\varphi-u\Delta\varphi+p\nabla\varphi \\
&\operatorname{div}\varphi u=u\nabla\varphi+\varphi g\,.
\end{aligned}
$$

If $\operatorname{supp}\varphi\subset\Omega$, (3.13) is regarded as equations in $\boldsymbol{R}^n$. We take φ such that $\varphi=1$ on Ω' and $\varphi\geqq 0$ on Ω. Since $\operatorname{div}(\varphi u)$ satisfies (3.4), applying (3.11) to (3.13) now yields (3.12). □

COROLLARY 3.6. *Assume that* $n\geqq 2$ *and* $1<r$, $q<\infty$ *and that* λ *is not negative real. Assume that* Ω *is an exterior domain in* $\boldsymbol{R}^n$ *with* $C^{2+\mu}$ $(0<\mu<1)$ *boundary or* $\boldsymbol{R}^n$ *itself.*

(i) (Uniqueness) *Suppose that* $u\in L^s(\Omega)^n$ *and* $\nabla p\in L^q(\Omega)^n$ *with* $(u,p)\in E^q_{\text{loc}}(\bar{\Omega})$ *for some* $1<q\leqq s<\infty$ *and that* (u,p) *solves* (3.1), (3.2) *on* Ω *with* $f=g=0$. *Then* $u=0$. *In particular,* $(\lambda+A_q)u=0$ *implies* $u=0$ $(n\geqq 3)$.

(ii) *The range* $R(A_r)$ *is dense in* $L^r_\sigma(\Omega)$ *when* $n\geqq 3$.

PROOF. (i) By elliptic regularity theory for (3.1) and (3.2) we see u is at least C^2 on $\bar{\Omega}$. (cf. [2, II]). We now apply (3.12) with $\Omega'=\boldsymbol{R}^n-B(R')$ for large R' and conclude that

$$
|\lambda|\,\|u\|_r+\|\nabla^2 u\|_r+\|\nabla p\|_r<\infty
$$

for all $r, 1<r<\infty$. (even if $\lambda=0$). Applying (2.4) with $u\in L^s(\Omega)^n$ yields

$$\|\nabla u\|_q+\|p+c\|_q<\infty$$

for all q, $1/q<1-1/n$ with some constant c. Since $u\in L^s(\Omega)^n$, we thus obtain

$$\|u\cdot\nabla u\|_{n'}+\|u(p+c)\|_{n'}<\infty \tag{3.14}$$

with $1/n+1/n'=1$. This justifies integration by parts

$$\int_\Omega \bar{u}\cdot\Delta u dx=-\|\nabla u\|_2^2, \qquad \int_\Omega \bar{u}\cdot\nabla p dx=0\,, \tag{3.15}$$

where $\bar{u}$ denotes the complex conjugate. Indeed, there are cut-off functions $\varphi_R\in C_0^\infty(\boldsymbol{R}^n)$ such that $\operatorname{supp}\varphi_R\subset B(2R)$ and $\varphi_R=1$ on $B(R)$ and that

$$|\nabla\varphi_R(x)|\leqq C/R, \qquad 0\leqq\varphi_R\leqq 1$$

with C independent of R, x. Integrating by parts yields

$$\int_\Omega \varphi_R\bar{u}\cdot\Delta u dx=-\int_\Omega \varphi_R|\nabla u|^2 dx-\int_\Omega \nabla\varphi_R\bar{u}\cdot\nabla u dx$$

since $u=0$ on $\partial\Omega$. By the Hölder inequality we have

$$\left|\int_\Omega \nabla\varphi_R\bar{u}\cdot\nabla u dx\right|\leqq|\nabla\varphi_R|_n\left(\int_{R^n-B(R)}|u\cdot\nabla u|^{n'}dx\right)^{1/n'}\longrightarrow 0 \quad \text{as } R\longrightarrow\infty$$

since $|\nabla\varphi_R|_n$ is bounded and (3.14) holds. This leads to the first identity in (3.15); the proof of the second one is the same.

We now multiply $\bar{u}$ with (3.1), and integrate over Ω and apply (3.15) to get

$$\lambda\|u\|_2^2+\|\nabla u\|_2^2=0\,.$$

Since λ is not negative real, this implies $\nabla u=0$ so $u=0$ since $u\in L^s(\Omega)^n$.

(ii) Since the dual operator $A_r^*=A_{r'}$ with $1/r+1/r'=1$ [9, 19], (i) implies that $R(A_r)$ is dense in $L_\sigma^r(\Omega)$. □

PROOF OF THEOREM 3.1. We first observe that for $M>0$

$$|\lambda|\|u\|_r+\|\nabla^2 u\|_r+\|\nabla p\|_r\leqq C(\|f\|_r+\|u\|_{rD}+\|\nabla u\|_{rD}+\|p\|_{rD}) \tag{3.16}$$

is valid for $|\lambda|\leqq M$ and $D=\Omega\cap B$, where B is an open ball. Indeed, we take $\varphi\in C_0^\infty(\boldsymbol{R}^n)$ with $\varphi=1$ on B' and $\operatorname{supp}\varphi\subset B$, when $\bar{B}'\subset B$, where B' is another ball with $\partial\Omega\subset B'$. Then φu solves (3.13) in $D'=\Omega\cap B'$ with $\varphi u|_{\partial D'}=0$. We know (3.3) holds for *bounded* domain D with $n\geqq 2$ and

$1<r<\infty$ (cf. [10, 21]), so does (3.5) by applying a result of Bogovski (Lemma 2.1 (ii)) (cf. [6]). Applying (3.5) to (3.13) in D' and (3.12) with B' instead of B yields (3.16).

It suffices to prove (3.3) for $|\lambda|\leqq M$ since (3.3) holds for large λ [10, 21]. Suppose that (3.3) for $|\lambda|\leqq M$ were false. Then there would exist a sequence (λ_m, u_m, p_m) such that

$$\begin{aligned} &|\lambda_m|\|u_m\|_r+\|\nabla^2 u_r\|_r+\|\nabla p_m\|=1, \quad |\lambda_m|\leqq M, \quad |\arg\lambda_m|\leqq\pi-\varepsilon \\ &\|f_m\|_r\longrightarrow 0, \quad f_m=(\lambda-\Delta)u_m+\nabla p_m\,. \end{aligned} \tag{3.17}$$

Since $1<r<n/2$, applying the Sobolev inequality (2.5) in Ω now yields

$$\|u_m\|_s\leqq C_1\|\nabla u_m\|_q\leqq C_2\|\nabla^2 u_m\|_r\leqq C_2$$

with $2/n+1/s=1/r$, $1/n+1/s=1/q$. It follows from (2.4)

$$\|p_m+K_m\|_q\leqq C_3\|\nabla p_m\|_r\leqq C_3$$

with some constant K_m; we may assume that $K_m=0$ since p_m are determined up to constants. We may assume that there are λ_0, $u\in L^s(\Omega)^n$, $p\in L^q(\Omega)$ with $\lambda_m\to\lambda_0$ such that

(a) $u_m\rightharpoonup u$ in $L^s(\Omega)^n$

(b) $\partial_j u_m\rightharpoonup\partial_j u$, $p_m\rightharpoonup p$ in $L^q(\Omega)^n$ resp. $L^q(\Omega)$

(c) $\partial_i\partial_j u_m\rightharpoonup\partial_i\partial_j u$, $\partial_j p_m\rightharpoonup\partial_j p$ in $L^r(\Omega)^n$, $1\leqq i$, $j\leqq n$

weakly by taking a subsequence, where λ_0 is not negative real. Since D is bounded, by compactness strong convergence replaces weak one in (a) and (b) provided that Ω is replaced by D. Since (3.16) and (3.17) imply

$$1\leqq C(\|u_m\|_{rD}+\|\nabla u_m\|_{rD}+\|\nabla p_m\|_{rD})$$

and since $L^r(D)\supset L^q(D)\supset L^s(D)$, this strong convergence yields

$$1\leqq C(\|u\|_{rD}+\|\nabla u\|_{rD}+\|p\|_{rD}) \tag{3.18}$$

with $C>0$. From (a), (b), (c) it follows that $u\in L^s(\Omega)^n$, $\nabla p\in L^r(\Omega)^n$ with $(u, p)\in E^r_{\mathrm{loc}}(\bar{\Omega})$ and that (u, p) solves (3.1) and (3.2) with $f=g=0$ and $\lambda=\lambda_0$. Corollary 3.6 (i) now implies $u=0$ so we obtain $p=0$. This contradicts to (3.18) so the proof is now complete. □

REMARK. We may weaken the assumption $(u, p)\in E^r(\Omega)$ in Theorems 3.1 and 3.2. Indeed, we only need to assume $u\in L^s(\Omega)$ and $\nabla p\in L^q(\Omega)$ with $(u, p)\in E^r_{\mathrm{loc}}(\bar{\Omega})$ for some $1<q\leqq s<\infty$ since (3.16) implies $(u, p)\in E^r(\Omega)$.

§4. Asymptotic behavior of the resolvent $(\lambda+A_r)^{-1}$ near $\lambda=0$

We showed in Theorem 3.1 that $\lambda+A_r$ has the bounded inverse unless λ is negative real or zero, and that for $0<\varepsilon<\pi/2$

$$|\lambda|\,\|(\lambda+A_r)^{-1}f\|_r\leqq C\|f\|_r,\quad |\arg\lambda|\leqq\pi-\varepsilon,\quad \lambda\neq 0 \tag{4.1}$$

with C independent of f and λ provided that $1<r<n/2$. A simple duality and interpolation argument [6] shows that (4.1) holds for every $1<r<\infty$. Here and hereafter we assume that Ω is an exterior domain in $\boldsymbol{R}^n$ with $C^{2+\mu}$ boundary $(0<\mu<1)$ and $n\geqq 3$, unless otherwise claimed. The estimate (4.1) gives a behavior of $(\lambda+A_r)^{-1}$ near $\lambda=0$ in a coarse term. We shall prove that $(\lambda+A_r)^{-1}$ behaves like $P_r\ \mathrm{res}(\lambda+\tilde{A}_r)^{-1}\,\mathrm{ext}$ near $\lambda=0$, where $\tilde{A}_r$ is the Stokes operator on $\boldsymbol{R}^n$. Here res denotes the restriction of functions on Ω and ext denotes the operator of extension of functions in Ω by defining zero outside Ω. In what follows we suppress both res and ext.

THEOREM 4.1. *Assume that* $1<r<\infty$ *and* $0<\varepsilon<\pi/2$, *and set*

$$W_\lambda=P_r(\lambda+\tilde{A}_r)^{-1}-(\lambda+A_r)^{-1}. \tag{4.2}$$

Then there are positive constants α *and* C *with* $\alpha<1$ *such that*

$$\|W_\lambda f\|_r\leqq C|\lambda|^{\alpha-1}\|f\|_r \tag{4.3}$$

holds for all $f\in L^r_\sigma(\Omega)$, $|\arg\lambda|\leqq\pi-\varepsilon$, $\lambda\neq 0$.

PROOF. We may assume that $0<|\lambda|\leqq M$ for some $M>0$ since (3.3) and (3.10) imply $\|W_\lambda f\|_r\leqq C|\lambda|^{-1}\|f\|_r$ for $|\lambda|\geqq M$. We may also assume that $f\in C^\infty_{0,\sigma}(\Omega)$ since $C^\infty_{0,\sigma}(\Omega)$ is dense in $L^r_\sigma(\Omega)$. We set $\tilde{u}_\lambda=(\lambda+\tilde{A}_r)^{-1}f$ and see $\tilde{u}_\lambda$ solves (3.1) in Ω with $p=\tilde{p}_\lambda$ and $g=0$ and that $(\tilde{u}_\lambda,\tilde{p}_\lambda)\in E^r(\Omega)$. So for $u_\lambda=P_\lambda\tilde{u}_\lambda$ we see u_λ solves (3.1) in Ω with $g=0$, where $p=q_\lambda$ is given by

$$\nabla q_\lambda=\nabla\tilde{p}_\lambda+(\lambda-\Delta)(1-P_r)\tilde{u}_\lambda.$$

Lemma 2.3 implies $(u_\lambda,q_\lambda)\in E^r(\Omega)$. The function $w_\lambda=W_\lambda f$ now solves (3.1) with $g=0$ for some $p=p_\lambda$, where $(w_\lambda,p_\lambda)\in E^r(\Omega)$. Since $u\in L^r_\sigma(\Omega)$ implies $u\cdot N=0$ on $\partial\Omega$ [19], we have

$$w_\lambda\cdot N=u_\lambda\cdot N=0\quad\text{on }\partial\Omega \tag{4.4}$$

although w_λ may not be identically zero on $\partial\Omega$. We set $v_\lambda=w_\lambda-\varphi u_\lambda$ where $\varphi\in C^\infty_0(\boldsymbol{R}^n)$ with $\varphi=1$ in a neighborhood of $\partial\Omega$ so that $v_\lambda=0$ on $\partial\Omega$. A direct calculation shows that v_λ solves

$$(\lambda-\Delta)v_\lambda+\nabla p_\lambda=-\lambda\varphi u_\lambda+\Delta(\varphi u_\lambda) \quad \text{in } \Omega \tag{4.5}$$

$$\operatorname{div} v_\lambda=-u_\lambda\cdot\nabla\varphi=g \quad \text{in } \Omega \tag{4.6}$$

$$v_\lambda|_{\partial\Omega}=0\,. \tag{4.7}$$

We first assume that $1<r<n/2$, and choose α and ρ such that $0<\alpha<n(1-1/r)/2$, $1<\rho<r$ and $1/\rho=1/r+2\alpha/n$. We may apply (3.5) to (4.5)-(4.7) with ρ instead of r. Indeed, we take ball B such that $\operatorname{supp}\nabla\varphi\subset B$. Since $\varphi=1$ in a neighborhood of $\partial\Omega$, we see $g|_{\partial\Omega}=0$. The property $\int_\Omega g(x)dx=0$ follows from (4.4) by integration by parts, so (3.4) holds for $g=-u_\lambda\cdot\nabla\varphi$. The estimate (3.5) with (4.5)-(4.7) yields

$$\begin{aligned}&|\lambda|\,\|w_\lambda\|_\rho+\|\nabla^2 w_\lambda\|_\rho\\ &\leqq C_1(|\lambda|\,\|\varphi u_\lambda\|_\rho+\|\nabla^2(\varphi u_\lambda)\|_\rho+\|\nabla(u_\lambda\cdot\nabla\varphi)\|_\rho+|\lambda|\,\|u_\lambda\cdot\nabla\varphi\|_\rho)\\ &\leqq C_2(|\lambda|\,\|u_\lambda\|_r+\|u_\lambda\|_s+\|\nabla u_\lambda\|_q+\|\nabla^2 u_\lambda\|_r)\end{aligned}$$

where $s, q\geqq r$ since $\operatorname{supp}\varphi$ is bounded; C_j $(j=1, 2, \cdots)$ represents a positive constant depending only on $r, \varepsilon, n, s, q, \varphi, \alpha$ and Ω. We now apply the Sobolev inequality (2.5) and obtain

$$|\lambda|\,\|w_\lambda\|_\rho+\|\nabla^2 w_\lambda\|_\rho\leqq C_3(|\lambda|\,\|u_\lambda\|_r+\|\nabla^2 u_\lambda\|_r)$$

by taking $2/n+1/s=1/r$, $1/n+1/s=1/q$, where the constant C_3 depends only on r, n, ε and Ω. Since P_r is bounded in $L^r_\sigma(\Omega)$ and satisfies (2.17), applying Lemma 3.3 to $\tilde{u}_\lambda$ now yields

$$|\lambda|\,\|w_\lambda\|_\rho+\|\nabla^2 w_\lambda\|_\rho\leqq C_4\|f\|_r\,.$$

Applying (2.6) we see

$$\begin{aligned}|\lambda|^{1-\alpha}\|w_\lambda\|_r&\leqq C_5|\lambda|^{1-\alpha}\|w_\lambda\|_\rho^{1-\alpha}\|\nabla^2 w_\lambda\|_\rho^\alpha\\ &\leqq C_6(|\lambda|\,\|w_\lambda\|_\rho+\|\nabla^2 w_\lambda\|_\rho)\leqq C_7\|f\|_r\end{aligned}$$

where C_7 is independent of f and λ. This proves (4.3) when $1<r<n/2$.

We write W_λ by $W_{\lambda r}$ when W_λ acts on $L^r_\sigma(\Omega)$. Taking dual of $W_{\lambda r}$ yields

$$W^*_{\lambda r}=W_{\lambda r'},\quad 1/r+1/r'=1$$

since $A^*_r=A_{r'}$ and $P^*_r=P_{r'}$ [9, 19]. We thus see (4.3) for $1<r<n/2$ implies (4.3) for $n/(n-2)<r<\infty$. Interpolating these two results by the Riesz-Thorin theorem [20, p. 40], we obtain (4.3) for arbitrary r, $1<r<\infty$. This completes the proof. □

§5. The complex powers A_r^z of the Stokes operator A_r

Our main goal in this section is to prove Theorem A, *i.e.*, a local bound on pure imaginary powers A_r^{iy} of the Stokes operator A_r in the space of bounded linear operators from $L_\sigma^r(\Omega)$ into itself, where Ω is an exterior domain in $\boldsymbol{R}^n$ $(n \geqq 3)$.

To get a bound on A_r^{iy} we shall use an integral representation of fractional powers A_r^z for $\operatorname{Re} z<0$. Such a formula is found in standard textbooks [17, 22] if the operator has the bounded inverse. Since the Stokes operator on an exterior domain does not have the bounded inverse, a more general theory is necessary. A unified theory of fractional powers was established by Komatsu [16]. We first recall some of his general theory. Let A be a closed linear operator in a Banach space X equipped with norm $\|\cdot\|$. Suppose that the resolvent set of A contains the negative real ray $(-\infty, 0)$ and that

$$|\lambda|\|(\lambda+A)^{-1}f\| \leqq M\|f\|, \quad f \in X, \quad \lambda>0 \tag{5.1}$$

with M independent of λ and f. Komatsu [16] gave a definition of fractional powers A^z for all complex numbers z by integral formulas and showed that his definition is consistent with other definitions. For example, the operator A^z for $z=x+iy$, $-1<x=\operatorname{Re} z<1$ is a smallest closed extension of $A_\#^z$ defined by

$$\begin{aligned} A_\#^z f = -\frac{\sin \pi z}{\pi}\Big(\int_0^m \lambda^z(-\lambda(\lambda+A)^{-1}g)d\lambda + \frac{m^{z+1}}{z+1}g \\ -\frac{m^z}{z}f + \frac{m^{z-1}}{z-1}Af + \int_m^\infty \lambda^{z-2}A(\lambda+A)^{-1}Af d\lambda\Big) \end{aligned} \tag{5.2}$$

for $f \in D(A) \cap R(A)$ with $Ag=f$, where m is an arbitrary fixed positive number (cf. [16, p. 305]). This is well-defined since (5.1) guarantees that all integrals in (5.2) converge absolutely in X, *i.e.*, the norm of integrands is integrable. The value $A_\#^z f$ does not depend on m and g. This definition looks slightly different from (4.11) of [16] since the class of f is smaller than his. However, by a density theorem [16, Theorem 3.6] and relation among $D(A)$, $R(A)$ and his spaces, we see that above definition agrees with his. We shall simply write $A_\#^z$ by A^z.

For later applications we shall always assume that $D(A)$ and $R(A)$ is dense in X (cf. [16, Lemma 3.5, Theorem 3.6], Lemma 6.2 in §6). By (5.2) we now observe that A^z is a densely defined closed linear operator in X at least for z, $-1<\operatorname{Re} z<1$; this property holds for arbitrary $z \in \boldsymbol{C}$

according to Komatsu [16]. For $-1<\operatorname{Re} z<0$ or $0<\operatorname{Re} z<1$ the expression of A^z becomes simpler than (5.2) and agrees with standard definitions [17, 22].

PROPOSITION 5.1. (i) *For* $f\in D(A)\cap R(A)$, $z\mapsto A^z f$ *is analytic on the strip* $-1<\operatorname{Re} z<1$ *with value in* X.

(ii) *For* $f\in R(A)$ *and* $-1<\operatorname{Re} z<0$, *we have*

$$A^z f=-\frac{\sin\pi z}{\pi}\int_0^\infty \lambda^z(\lambda+A)^{-1}f\,d\lambda\,. \tag{5.3}$$

(iii) *For* $f\in D(A)$ *and* $0<\operatorname{Re} z<1$, *we have*

$$A^z f=\frac{\sin\pi z}{\pi}\int_0^\infty \lambda^{z-1}(\lambda+A)^{-1}Af\,d\lambda\,. \tag{5.4}$$

(*Both integrals in* (5.3) *and* (5.4) *converge absolutely in* X.)

PROOF. (i) As is pointed out in [16, p. 305], this is clear from the expression (5.2).

(ii) Since

$$(\lambda+A)^{-1}F=\lambda^{-1}f-\lambda^{-2}f+\lambda^{-2}A(\lambda+A)^{-1}Af,$$

the sum of last three terms of (5.2) equals

$$-\frac{\sin\pi z}{\pi}\int_m^\infty \lambda^z(\lambda+A)^{-1}f\,d\lambda\,;$$

this integral converges absolutely in X even for $f\in X$ since $\operatorname{Re} z<0$ and (5.1) holds. Since

$$(\lambda+A)^{-1}f=(\lambda+A)^{-1}Ag=g-\lambda(\lambda+A)^{-1}g \quad \text{with } f=Ag, \tag{5.5}$$

(5.1) implies that the integral

$$-\frac{\sin\pi z}{\pi}\int_0^m \lambda^z(\lambda+A)^{-1}f\,d\lambda$$

converges absolutely in X provided that $-1<\operatorname{Re} z$ and $f\in R(A)$. Then a direct calculation shows that this integral equals the sum of first two terms of (5.2), so (5.3) equals (5.2).

(iii) The proof is similar to (ii) and is in [16], so is omitted. See Proposition 4.12 and (4.2) in [16]. □

If the resolvent set of $-A$ contains some sector and there (5.1) holds, one can move the line of integration in (5.3) into the complex plane.

PROPOSITION 5.2. *Suppose that the resolvent set of* $-A$ *includes a sector* $\{\lambda\in\boldsymbol{C};\ |\arg\lambda|\leqq\pi-\varepsilon,\ \lambda\neq 0\}$ *and that*

$$|\lambda|\,\|(\lambda+A)^{-1}f\|\leqq M\|f\|,\quad |\arg\lambda|\leqq\pi-\varepsilon \tag{5.6}$$

with M *depending only on* ε, $0<\varepsilon<\pi$. *Let* $\Gamma=\Gamma_\varepsilon$ *denote the path which consists of two rays from* $\infty e^{i(\varepsilon-\pi)}$ *to zero and zero to* $\infty e^{i(\pi-\varepsilon)}$ *in* $\boldsymbol{C}$. *Then for* $f\in R(A)$ *and* $-1<\mathrm{Re}\,z<0$, *it holds*

$$A^z f=\frac{1}{2\pi i}\int_\Gamma(-\lambda)^z(\lambda+A)^{-1}f\,d\lambda \tag{5.7}$$

where $(-\lambda)^z$ *means the principal branch.*

PROOF. By (5.5) and (5.6) the integrals

$$\int_{\Gamma_{\delta m}}(-\lambda)^z(\lambda+A)^{-1}f\,d\lambda \quad\text{and}\quad \int_{\Gamma'_{\delta m}}(-\lambda)^z(\lambda+A)^{-1}f\,d\lambda,\quad \varepsilon\leqq\delta<\pi$$

converge absolutely for $f\in R(A)$, $-1<\mathrm{Re}\,z$ and for $f\in X$, $\mathrm{Re}\,z<0$, where $\Gamma_{\delta m}=\Gamma_\delta\cap\{\lambda\in\boldsymbol{C};\ |\lambda|\geqq m\}$ and $\Gamma_\delta-\Gamma_{\delta m}=\Gamma'_{\delta m}$, respectively. We now apply Cauchy's integral theorem with (5.6) and modify the path of integration to obtain (5.7) from (5.3). □

We now go back to the Stokes operators. By (3.10), (4.1) and Corollary 3.6 (ii), we see $\tilde{A}_r$ and A_r satisfy all assumptions on A of Propositions 5.1 and 5.2 with $X=L^r_\sigma(\boldsymbol{R}^n)$ and $L^r_\sigma(\Omega)$ respectively. We first prove that assertions in Theorem A hold if $\Omega=\boldsymbol{R}^n$.

LEMMA 5.3. *Assume that* $n\geqq 2$. *Then for every* $\gamma>0$, *there is a constant* C *such that*

$$|\tilde{A}^{iy}f|_r\leqq Ce^{\gamma|y|}|f|_r \tag{5.8}$$

holds for $f\in L^r_\sigma(\boldsymbol{R}^n)$ *and* $y\in\boldsymbol{R}$.

PROOF. By (3.8) and (3.9) a direct calculation of (5.2) shows that

$$\tilde{A}_r^z f=F^{-1}|\zeta|^{2z}F\tilde{P}_r f,\qquad f\in D(\tilde{A}_r)\cap R(\tilde{A}_r)$$

at least for $-1<\mathrm{Re}\,z<1$. Since $D(\tilde{A}_r)\cap R(\tilde{A}_r)$ is dense in $L^r_\sigma(\boldsymbol{R}^n)$ and

$$|\partial_\zeta^\alpha|\zeta|^{2z}|\leqq a_\alpha(y)|\zeta|^{2\mathrm{Re}\,z-|\alpha|}$$

with a polynomial a_α of degree $|\alpha|$, applying a multiplier theorem [23, § 2.2] yields (5.8). □

For technical reasons we need to show that $R(A_q)$ contains "good functions" when q is large.

LEMMA 5.4. *Assume that* $n \geqq 3$ *and* $q > n/(n-2)$. *Assume that* Ω *is an exterior domain in* $\boldsymbol{R}^n$ *or* $\boldsymbol{R}^n$ *itself. Then* $R(A_q)$ *contains* $L^q_\sigma(\Omega) \cap L^r(\Omega)^n$ *for* $1/r = 2/n + 1/q$. *In particular,* $R(A_q)$ *contains* $C^\infty_{0,\sigma}(\Omega)$.

PROOF. We observe that for $1 < s < \infty$

$$\lim_{\lambda \downarrow 0} \|\lambda(\lambda + A_s)^{-1} f\|_s = 0 \qquad \text{for } f \in L^s_\sigma(\Omega)\,. \tag{5.9}$$

This follows from (4.1) (or (5.1)) and $\overline{R(A_s)} = L^s_\sigma(\Omega)$ (Corollary 3.6 (ii)) by a standard argument (cf. [16]). Indeed for $f \in R(A_s)$ we see by resolvent equations that $\lambda(\lambda + A_s)^{-1} f \to 0$ in $L^s_\sigma(\Omega)$ as $\lambda \downarrow 0$. For general $f \in L^s_\sigma(\Omega)$ we approximate by elements of $R(A_s)$. The uniform estimate (4.1) now yields (5.9).

Suppose that $f \in L^r(\Omega) \cap L^q_\sigma(\Omega)$. We set $u_\lambda = (\lambda + A_r)^{-1} f$. We observe that $\{u_\lambda\}$ is Cauchy in $L^q_\sigma(\Omega)$ and that $A_q u_\lambda$ converges to f in $L^q_\sigma(\Omega)$ as $\lambda \downarrow 0$. Indeed for $0 < \mu < \lambda$, applying (2.5) and (3.3) yields,

$$\begin{aligned}\|u_\lambda - u_\mu\|_q &= \|(\mu - \lambda)(\mu + A_r)^{-1}(\lambda + A_r)^{-1} f\|_q \\ &\leqq |\lambda|\,\|\nabla^2(\mu + A_r)^{-1}(\lambda + A_r)^{-1} f\|_r \\ &\leqq |\lambda|\,\|(\lambda + A_r)^{-1} f\|_r\,.\end{aligned}$$

By (5.9) we see $\{u_\lambda\}$ is Cauchy in $L^q_\sigma(\Omega)$. Since

$$\|A_q u_\lambda - f\|_q = \|\lambda(\lambda + A_q)^{-1} f\|_q$$

applying (5.9) yields that $A_q u_\lambda \to f$ as $\lambda \downarrow 0$ in $L^q_\sigma(\Omega)$. Since A_q is closed there is $u \in D(A_q)$ such that $A_q u = f$ so $f \in R(A_q)$. This completes the proof. □

PROOF OF THEOREM A. We approximate A^z_r by $\tilde{A}^z_r$. We first prove that for every $\gamma > 0$

$$\|A^z_r f - P_r \tilde{A}^z_r f\|_r \leqq C e^{\gamma|y|} \|f\|_r, \qquad f \in R(A_r), \quad -a < \operatorname{Re} z < 0 \tag{5.10}$$

with C and a, $0 < a < 1$ independent of f and $y = \operatorname{Im} z$. By (4.2) and (5.7) we have

$$A^z_r f = \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} (-\lambda)^z P_r (\lambda + \tilde{A}_r)^{-1} f \, d\lambda - \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} (-\lambda)^z W_\lambda f \, d\lambda$$

$$=P_r\tilde{A}_r^z f-\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}(-\lambda)^z W_\lambda f d\lambda, \qquad f\in R(A_r).$$

For large λ, $|\lambda|\geqq m>0$, applying Theorem 4 in [11] yields that

$$\left\|\int_{\Gamma_{\varepsilon m}}(-\lambda)^z W_\lambda f d\lambda\right\|_r \leqq Ce^{\gamma|y|}\|f\|, \quad -1<-a<\operatorname{Re} z<0 \tag{5.11}$$

with C depending only on $r, n, \Omega, \varepsilon, m$ and a. In [11] boundedness of Ω is assumed. However, this is not necessary to get (5.11). In [11, p. 264], the boundedness is only needed to conclude that W_λ is uniformly bounded for $|\lambda|\leqq m$. Even when Ω is an exterior domain, as shown in Theorem 4.1 the estimate (4.3) for W_λ is available. From (4.3) and (5.11) it follows that

$$\left\|\int_{\Gamma_\varepsilon}(-\lambda)^z W_\lambda f d\lambda\right\|_r \leqq Ce^{\gamma|y|}\|f\|_r, \quad -a<\operatorname{Re} z<0$$

with $a=\alpha$ in (4.3). This leads to (5.10).

We next prove (5.10) for $z=iy$ and $f\in C_{0,\sigma}^\infty(\Omega)$. For technical reasons we restrict $r>n/(n-2)$. Lemma 5.4 shows that $C_{0,\sigma}^\infty(\Omega)$ is contained both in $R(A_r)$ and $R(\tilde{A}_r)$. Applying Proposition 5.1 (i) now yields that $z\mapsto A_r^z f$, $z\mapsto\tilde{A}_r^z f$ is analytic in $-1<\operatorname{Re} z<1$ provided $f\in C_{0,\sigma}^\infty(\Omega)$. Letting $\operatorname{Re} z\uparrow 0$ in (5.10), we have

$$\|A_r^{iy}f-P_r\tilde{A}_r^{iy}f\|_r\leqq Ce^{\gamma|y|}\|f\|_r, \quad f\in C_{0,\sigma}^\infty(\Omega)$$

where $r>n/(n-2)$.

This and Lemma 5.3 yield

$$\|A_r^{iy}f\|_r\leqq Ce^{\gamma|y|}\|f\|_r, \qquad f\in L_\sigma^r(\Omega) \tag{5.12}$$

for $r>n/(n-2)$, since $C_{0,\sigma}^\infty(\Omega)$ is dense in $L_\sigma^r(\Omega)$. Estimates for other r follows from a duality and interpolation argument. Indeed, from Theorem 13.4 of [16] it follows that $(A_r^{iy})^*=(A_r^*)^{iy}$ holds for the dual (transposed) operator A_r^*. Using $A_r^*=A_{r'}$ in [19] with $1/r+1/r'=1$ we see $(A_r^{iy})^*=A_{r'}^{iy}$ and therefore we have

$$\|A_{r'}^{iy}\|_{r'}=\|(A_r^{iy})^*\|_{r'}=\|A_r^{iy}\|_r\leqq Ce^{\gamma|y|}$$

for all r' such that $r>n/(n-2)$, where $\|\ \|_r$ denotes the operator norm in L^r. Thus we have (5.12) for $1<r<n/2$. By the Riesz-Thorin interpolation theorem [20, p. 40], we see (5.12) holds for all $1<r<\infty$. This completes the proof of Theorem 1. $\square$

§ 6. Applications

This section, as an application of Theorem A, proves Theorem B and compares $\|A_r^\alpha u\|_r$ with various norms when $0 \leqq \alpha \leqq 1$, where A_r^α are the fractional powers of the Stokes operator A_r on the exterior domain Ω in $\boldsymbol{R}^n$ with $n \geqq 3$ and $1 < r < \infty$. We shall compare $\|A_r^{1/2}u\|_r$ with $\|\nabla u\|_r$.

We begin with an abstract interpolation theorem. Let A be a densely defined closed linear operator in a Banach space X equipped with $\| \ \|$. Suppose that A is injective and that the range $R(A)$ is dense in X. We assume that A satisfies (5.1). Let D_A^α denote the completion of $D(A)$ in the norm $\|A^\alpha u\|$, where $0 \leqq \alpha \leqq 1$. This is well-defined. Indeed (5.4) shows that $D(A^\alpha)$ contains $D(A)$ and $\|A^\alpha u\|_r$ is a norm on $D(A)$ since $A^\alpha u = 0$ implies that $Au = 0$ so $u = 0$. Although $D_A^0 = X$ may not include D_A^1, we see two norms $\|u\|$ and $\|Au\|$ are consistent ([20, p. 35]) on $D(A)$ since A is closed, so we may define the complex interpolation spaces $[D_A^0, D_A^1]_\alpha$.

PROPOSITION 6.1. *Suppose that*

$$\|A^{iy}f\| \leqq Ce^{\gamma|y|}\|f\|, \quad y \in \boldsymbol{R}, \quad f \in X \tag{6.1}$$

holds with some C, $\gamma > 0$ *independent of* y *and* f. *Assume that* $0 \leqq \sigma \leqq \tau \leqq 1$ *and* $0 \leqq \alpha \leqq 1$. *Then* $D_A^\beta = [D_A^\sigma, D_A^\tau]_\alpha$ *as Banach spaces with* $\beta = (1-\alpha)\sigma + \alpha\tau$. *In particular* $D_A^\alpha = [D_A^0, D_A^1]_\alpha$.

The space D_A^α includes $D(A^\alpha)$ since $D(A^\alpha)$ is the completion of $D(A)$ in the norm $\|u\| + \|Au\|$. Although D_A^α may be larger than $D(A^\alpha)$, we still have a density result.

LEMMA 6.2. (i) $\|\lambda(\lambda + A)^{-1}f\| \to 0$ *as* $\lambda \downarrow 0$ *for* $f \in X$.
(ii) $D(A) \cap R(A)$ *is dense in* D_A^α.

PROOF. (i) The proof is the same as that of (5.9).
(ii) For $f \in D(A)$ we take $f_\lambda = (\lambda + A)^{-1}Af$, $\lambda > 0$. Since $\|A^\alpha(f_\lambda - f)\| = \|\lambda(\lambda + A)^{-1}A^\alpha f\|$, applying (i) yields $f_\lambda \to f$ in D_A^α as $\lambda \downarrow 0$. Since $f_\lambda \in D(A) \cap R(A)$ and $D(A)$ is dense in D_A^α, this completes the proof. □

PROOF OF PROPOSITION 6.1. We may assume $0 < \alpha < 1$, and $a = \tau - \sigma > 0$. For $f \in D(A) \cap R(A)$ we consider

$$F(z) = e^{(z-\alpha)^2} A^{a(\alpha - z)} f$$

for $z \in S = \{z \in \boldsymbol{C}\,;\, 0 \leqq \mathrm{Re}\, z \leqq 1\}$. Since $e^{(z-\alpha)^2} e^{\gamma|y|}$ is bounded on S for $\gamma > 0$, Re $z=y$ and since $A^{-az}=A^{-iay}A^{a(\alpha-x)}$, Re $z=x$, applying (6.1) yields

$$\|A^{\sigma}F(z)\| \leqq C\|A^{\beta-ax}f\|, \quad z\in S, \quad 0\leqq x\leqq 1, \tag{6.2}$$

$$\|A^{\tau}F(z)\| \leqq C\|A^{\beta+a(1-x)}f\|, \quad z\in S, \quad 1-(1-\beta)/a<x\leqq 1 \tag{6.3}$$

with C independent of f and z. By (6.2) and Proposition 5.1 (i) we see F is analytic in S^0, the interior of S, and is continuous on S with value in D_A^{σ}. Let $[f]_{\beta}$ denote the norm in $[D_A^{\sigma}, D_A^{\tau}]_{\alpha}$. Since $F(\alpha)=f$ the estimate (6.2), (6.3) yields

$$[f]_{\beta} \leqq \max\left(\sup_y\|A^{\sigma}F(iy)\|,\ \sup_y\|A^{\tau}F(1+iy)\|\right) \leqq C\|A^{\beta}f\|\,.$$

By Lemma 6.2, we see this holds for all $f\in D_A^{\beta}$.

It remains to prove the converse inequality. We consider an arbitrary function $G(z)$ expressed as a finite linear combination of functions of the form $\exp(\delta z^2+\sigma z)b$, $\delta>0$, $\sigma\in\boldsymbol{R}$ and $b\in D(A)$ with $G(\alpha)=f\in D(A)\cap R(A)$. By Proposition 5.1 (i) and (6.1), $A^{(1-z)\sigma+z\tau}G(z)$ is analytic in S^0 and bounded continuous in S with value in D_A^{σ}. The three-line theorem yields

$$\|A^{\beta}f\| = \|A^{\beta}G(\alpha)\| \leqq C\max\left(\sup_y\|A^{\sigma}G(iy)\|,\ \sup_y\|A^{\tau}G(1+iy)\|\right).$$

Since $D(A)$ is dense in D_A^{σ} and D_A^{τ}, it follows from the Theorem in [23, § 1.9.1] that $\|A^{\beta}f\|\leqq C[f]_{\beta}$ for $f\in[D_A^{\sigma}, D_A^{\tau}]_{\alpha}$ so the proof is complete. □

We take $A=A_r$ with $X=L_{\sigma}^r(\Omega)$, when A_r is the Stokes operator and Ω is an exterior domain in $\boldsymbol{R}^n$ with $n\geqq 3$. By Corollary 3.6 and (4.1) one can define $D_{A_r}^{\alpha}$ which simply denoted by D_r^{α}, the completion of $D(A_r)$ in the norm $\|A^{\alpha}u\|_r$.

Theorem A and Proposition 6.1 yield a stronger result than Theorem B.

THEOREM 6.3. *For every* $0\leqq\alpha\leqq 1$ *and* $1<r<\infty$, $D_r^{\beta}=[D_r^{\sigma}, D_r^{\sigma}]_{\alpha}$ *as Banach spaces with* $\beta=(1-\alpha)\sigma+\alpha\tau$, $0\leqq\sigma<\tau\leqq 1$. *In particular* $D_r^{\alpha}=[D_r^0, D_r^1]_{\alpha}$, $D_r^{\alpha/2}=[D_r^0, D_r^{1/2}]_{\alpha}$, *where* $D_r^0=L_{\sigma}^r(\Omega)$.

REMARK. When Ω is bounded this is known by [11, Theorem 2] with $D_r^{\alpha}=D(A_r^{\alpha})$. For exterior domains in $\boldsymbol{R}^3$ when $r=2$ this is essentially known by Miyakawa [19, Theorem 2.4]. When $r=2$, Theorem A is trivial so he deduced the characterization following an argument in [23, p. 103-104]. Our proof of Proposition 6.1 follows their argument; when A has the bounded inverse, Proposition 6.1 is found in [23, p. 103-104].

Using Proposition 6.1 we shall study $\hat{H}^{k,r}(\boldsymbol{R}^n)$, the completion of $C_0^{\infty}(\boldsymbol{R}^n)$ in the norm $|\nabla^k u|_r$. We set $A=-\Delta$ with domain $D(-\Delta)=\{f\in L^r(\boldsymbol{R}^n)$;

$|\nabla^2 f|_r<\infty\}$ in $X=L^r(\boldsymbol{R}^n)$. Since A is injective, a duality argument shows that $R(-\Delta)$ is dense in $L^r(\boldsymbol{R}^n)$. Since

$$(\lambda-\Delta)^{-1}f=F^{-1}(\lambda+|\xi|^2)^{-1}Ff, \quad (-\Delta)^z f=F^{-1}|\xi|^{2z}Ff$$

applying a multiplier theorem [23, § 2.2], we see $A=-\Delta$ satisfies all assumptions of Proposition 6.1 including (6.1). Let $\hat{H}^{2\alpha,r}(\boldsymbol{R}^n)$ be the completion of $C_0^\infty(\boldsymbol{R}^n)$ in the norm $|(-\Delta)^\alpha u|_r$, $\alpha\geqq 0$. This definition is consistent with previous definition of $\hat{H}^{k,r}(\boldsymbol{R}^n)$, $k=0,1,2,\cdots$ since two norms $|\nabla^k u|_r$ and $|(-\Delta)^{k/2}u|_r$ are equivalent on $C_0^\infty(\boldsymbol{R}^n)$ by a multiplier theorem [23, § 2.2] (cf. [18, p. 9]).

LEMMA 6.4. (i) *For every r and α with $1<r<\infty$, $0\leqq\alpha\leqq 1$,*

$$\hat{H}^{2\alpha,r}(\boldsymbol{R}^n)=[L^r(\boldsymbol{R}^n),\hat{H}^{2,r}(\boldsymbol{R}^n)]_\alpha \quad \textit{as Banach spaces.}$$

(ii) *Assume that $1<r<n/k$. Then*

$$\hat{H}^{k,r}(\boldsymbol{R}^n)=\{u\in L^q(\boldsymbol{R}^n), |\nabla^k v|_r<\infty\}, \quad k=0,1,2,\cdots$$

with $k/n+1/q=1/r$.

PROOF. (i) Proposition 6.1 implies that two norms $|(-\Delta)^\alpha u|_r$ and $[u]_\alpha$ *i.e.* the norm in $[D^0_{-\Delta}, D^1_{-\Delta}]_\alpha$, are equivalent on $D(-\Delta)$. Since $C_0^\infty(\boldsymbol{R}^n)$ is dense in $D(-\Delta)$ in the norm $|\nabla^2 u|_r$ we see $\hat{H}^{2,r}(\boldsymbol{R}^n)=D^1_{-\Delta}$ and $L^r(\boldsymbol{R}^n)=D^0_{-\Delta}$ so $D^\alpha_{-\Delta}$ agrees with the completion of $C_0^\infty(\boldsymbol{R}^n)$ in the norm $[u]_\alpha$ or $|(-\Delta)^{\alpha/2}u|_r$. This completes the proof of (i).

(ii) This follows from the Sobolev inequality in $\boldsymbol{R}^n$ ([8, p. 24]):

$$|v|_q\leqq C|\nabla^k v|_r, \quad v\in C_0^\infty(\boldsymbol{R}^n), \quad k/n+1/q=1/r. \qquad \square$$

The rest of this section is devoted to comparison of two norms $\|A_r^{1/2}u\|_r$ and $\|\nabla u\|_r$.

THEOREM 6.5. *Assume that $1<r<\max(n/2,2)$ or $r=2$. Then*

$$\|\nabla u\|_r\leqq C\|A_r^{1/2}u\|_r \tag{6.4}$$

holds for $u\in D_r^{1/2}$ $(\supset D(A_r^{1/2}))$ with C independent of u.

To prove this theorem we need some properties of extension operators which will be proved in the last part of this section.

LEMMA 6.6. *There is a bounded linear operator $E: H^{2,r}(\Omega)\to H^{2,r}(\boldsymbol{R}^n)$ such that $Eu=u$ in Ω and that for $k=0,1,2$ the estimate*

(6.5) $$|\nabla^k Eu|_r \leqq C\|\nabla^k u\|_r \qquad \text{for } 1<r<n/k$$

holds with C *independent of* u.

PROOF OF THEOREM 6.5. Since (3.3) implies that

$$\|\nabla^2 Eu\|_r \leqq C_1\|Au\|_r, \qquad u\in D(A), \quad 1<r<n/2,$$

Lemma 6.6 yields

$$|\nabla^2 Eu|_r \leqq C_2\|Au\|_r, \quad |Eu|_r \leqq C_3\|u\|_r, \qquad u\in D(A_r)$$

with C_j ($j=1,2,\cdots$) independent of u.

By the Calderón-Lions interpolation theorem [20, p. 37], we see E is bounded linear from $[D_r^0, D_r^1]_\alpha$ to $[L^r(\boldsymbol{R}^n)^n, \hat{H}^{2,r}(\boldsymbol{R}^n)^n]_\alpha$. Theorem 6.3 and Lemma 6.4 yield

$$\|\nabla u\|_r \leqq |\nabla Eu|_r \leqq C_4\|A^{1/2}u\|_r \qquad \text{for } u\in D_r^{1/2}$$

provided that $1<r<n/2$.

It remains to prove (6.4) for $1<r\leqq 2$. We consider $\partial_j A_r^{-1/2}$ on $R(A_r)$, where $1\leqq j\leqq n$. When $r=2$, it holds $\|\nabla u\|_r=\|A_r^{1/2}u\|_r$, so $\partial_j A_r^{-1/2}$ extends to a bounded linear operator form $L_\sigma^r(\Omega)$ to $L^r(\Omega)^n$ with $r=2$. The estimate (6.4) shows that $\partial_j A_r^{-1/2}$ is bounded when $1<r<n/2$. By the Riesz-Thorin interpolation theorem we see

$$\|\nabla A_r^{-1/2}f\|_r \leqq C\|f\|_r, \qquad f\in L_\sigma^r(\Omega), \quad 1<r\leqq 2$$

with C independent of f. This leads to (6.4) when $1<r\leqq 2$ by setting $f=A_r^{-1/2}u$, since we may assume $u\in D(A_r)$ in (6.4).

COROLLARY 6.7. *Assume that* $1<r<n/2$ *and* $0\leqq\alpha\leqq 1$ *or that* $1<r\leqq 2$, $0\leqq\alpha\leqq 1/2$. *Then*

(6.6) $$\|u\|_\rho \leqq C\|A_r^\alpha u\|_r, \qquad 1/\rho=1/r-2\alpha/n$$

holds for $u\in D_r^\alpha$ *with* C *independent of* u.

PROOF. The estimates (2.5) and (3.3) yield

$$\|u\|_s \leqq C_1\|\nabla^2 u\|_r \leqq C_2\|A_r u\|_r, \qquad u\in D(A_r)$$

with $1/s=1/r-2n$. Applying the Calderón-Lions interpolation theorem, we see the inclusion $I: L_\sigma^r(\Omega)\to L^r(\Omega)^n$ is bounded linear from $[D_r^0, D_r^1]_\alpha$ to $[L^r(\Omega)^n, L^s(\Omega)^n]_\alpha$. Since $[L^r(\Omega), L^s(\Omega)]_\alpha=L^\rho(\Omega)$, $1/\rho=\alpha/s+(1-\alpha)/r$, [20, p. 38]. Theorem 6.3 yields (6.6) for $1<r<n/2$ and $0\leqq\alpha\leqq 1$.

For $1<r\leqq 2$, (2.4) and (6.4) yields

$$\|u\|_q\leqq C_3\|\nabla u\|_r\leqq C_4\|A^{1/2}u\|_r, \quad u\in D(A_r)$$

with $1/q=1/r-1/n$. As above, the inclusion I is bounded linear from $[D_r^0, D_r^{1/2}]_{2\alpha}$ to $[L^r(\Omega)^n, L^q(\Omega)^n]_{2\alpha}$. Theorem 6.3 now yields (6.6) for $1<r\leqq 2$. □

REMARK. For the entire space and the half space, (6.4) holds for all r with $1<r<\infty$ (cf. [5]). Recently, Borchers and Miyakawa [4] proves (6.4) for $1<r<n$ when Ω is an exterior domain. In the half space (6.6) is still valid for all r, α with $1/r>2\alpha/n$, $1<r<\infty$, $0\leqq\alpha\leqq 1$. (cf. [5]).

PROOF OF LEMMA 6.6. We use the following well-known extension operator [1]. Let B be an open ball with $\boldsymbol{R}^n-\Omega\subset B$. Then there is a bounded linear operator $E: H^{2,r}(\Omega)^n\rightarrow H^{2,r}(\boldsymbol{R}^n)^n$ with $Eu=u$ in Ω and

$$|\nabla^k Eu|_r\leqq C(\|\nabla^k u\|_r+\|u\|_{rD}), \quad D=B\cap\Omega\,.$$

If $1<r<n/k$, then from (2.4) and (2.5) it follows that

$$\|u\|_{rD}\leqq C\|u\|_q\leqq C'\|\nabla^k u\|_r, \quad 1/q=1/r-k/n\,.$$

We thus obtain (6.5) and the proof is now complete. □

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