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On the comparison principle for capillary surfaces

To Hiroshi Fujita on the occasion of his sixtieth birthday.

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Abstract: It is known that the capillary surface equation over a bounded domain Ω admits a comparison principle under significantly weaker restrictions than are needed for, e.g., harmonic functions. In the present work we show that in a uniform "positive" gravity field the same principle extends without change to arbitrary Ω, bounded or unbounded, and without growth conditions on the functions being compared. It is shown by example that the analogous extension in the absence of gravity would be false.

Let $\Omega\subset \boldsymbol{R}^n$, ν be exterior normal on $\partial\Omega$, let κ be a nonnegative constant and let γ be defined on $\partial\Omega$, $0\leqq\gamma\leqq\pi$.

The boundary data problem

$$\text{(1)} \qquad \begin{cases} \operatorname{div} Tu=\kappa u+\lambda \quad \text{in } \Omega,\ Tu=\dfrac{Du}{\sqrt{1+|Du|^2}} \\ \nu\cdot Tu=\cos\gamma \quad \text{on } \partial\Omega \end{cases}$$

determines the height $u(x)$ of a capillary surface interface S in a vertical cylinder Z over Ω, in a uniform gravity field directed downward across S, with heavier fluid below S. Here λ is a Lagrange parameter, determined by an eventual volume constraint. For background details, see, e.g. [1] Chapter 1.

In 1974, Concus and Finn [2] published a general comparison principle for solutions of (1) in bounded Ω, which holds under much weaker restrictions than are needed, e.g., for harmonic functions. The distinction has significant consequences for the behavior of the solutions (for some applications, see, e.g., Chapter 5 in [1]).

In 1980, the principle was extended by Siegel [3] to certain types of unbounded domains, and new uniqueness theorems were given. We show in

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the present note that in a positive gravity field ($\kappa>0$) the principle holds in general for arbitrary Ω, bounded or unbounded. We point out that no growth conditions are imposed, nor are conditions imposed on the form of Ω at infinity. Specifically, we shall prove:

THEOREM. *Let $\kappa>0$ and suppose $\partial\Omega$ admits a decomposition $\partial\Omega=\Sigma^0\cup\Sigma^\alpha\cup\Sigma^\beta$, where $\Sigma^\beta\in C^{(1)}$ and Σ^0 has $(n-1)$ dimensional Hausdorff measure zero. Let $u, v\in C^{(2)}(\Omega)$ with the properties*

(i) $\operatorname{div} Tu-\operatorname{div} Tv\geqq\kappa(u-v)$ *for every $x\in\Omega$ such that $u-v>0$.*
(ii) $\limsup(u-v)\leqq 0$ *for any approach to Σ^α from within Ω.*
(iii) $(Tu-Tv)\cdot\nu\leqq 0$ *almost everywhere on Σ^β as a limit from points of Ω.*

Then $u(x)\leqq v(x)$ in Ω; if (i) *holds throughout Ω and if $u(x)=v(x)$ at any $x\in\Omega$, then $u(x)\equiv v(x)$ in Ω.*

PROOF. If $u>v$ at some point of Ω, there would be positive constants m_1, m_2 and a set $\Omega_{12}\subset\Omega$ of positive measure, on which $0<m_1<u-v<m_2<\infty$. We set

$$w(x)=\begin{cases} 0, & u-v\leqq m_1 \\ u-v-m_1, & \text{in } \Omega_{12} \\ m_2-m_1, & u-v\geqq m_2 \end{cases}. \tag{2}$$

For any $R>0$, we set $B_R=\{x\in\boldsymbol{R}^n ; |x|<R\}$, $\Omega_R=\Omega\cap B_R$, $\Gamma_R=\Omega_R\cap\partial B_R$. We consider only those R sufficiently large that $|\Omega_R\cap\Omega_{12}|$ has positive measure.

Following the reasoning in [4, p. 193-195] or in [1, p. 110-113], we obtain

$$\int_{\Omega_R} w^{n-1}(\operatorname{div} Tu-\operatorname{div} Tv)dx\leqq\int_{\Gamma_R} w^{n-1}(Tu-Tv)\cdot\nu ds \\ -(n-1)\int_{\Omega_R} w^{n-2}(Tu-Tv)\cdot\nabla w dx \tag{3}$$

and thus

$$\kappa\int_{\Omega_R} w^n dx\leqq 2\int_{\Gamma_R} w^{n-1} dS \tag{4}$$

since $|Tu|<1$ for any function $u(x)$ and the integrand in the last term of (3) is non negative. There follows

$$Q(R)\equiv\int_{\Omega_R} w^n dx\leqq\frac{2}{\kappa}\left(\int_{\Gamma_R} w^n dS\right)^{(n-1)/n}\left(\int_{\Gamma_R} dS\right)^{1/n} \tag{5}$$

from which

$$J(R)\equiv\int_{R_1}^{R}\frac{1}{\rho}Q^{n/(n-1)}(\rho)d\rho\leqq C[Q(R)-Q(R_1)] \tag{6}$$

for a suitable constant C. If R_1 is sufficiently small then $J(R)>0$ and we conclude (for a possibly different C)

$$J^{-n/(n-1)}J'(R)\geqq CR^{-1} \tag{7}$$

for all sufficiently large R. Integration of (7) leads to a contradiction, and establishes that $u\leqq v$ in Ω. The remainder of the proof follows from the E. Hopf boundary point lemma as in [2] or in [1].

REMARK 1. The hypotheses could to some extent be weakened by allowing suitable decay of κ with increasing R.

REMARK 2. A particular consequence of the theorem is the uniqueness of solutions of (1) for arbitrary (bounded or unbounded) domains Ω, with no condition imposed at infinity. In the case of a bounded Ω, a uniqueness theorem holds also when $\kappa=0$, in the sense that the solution is determined up to an additive constant [4]. That theorem however does not extend to the unbounded case. In fact, we observe that for arbitrary α, β with $|\alpha|<\pi/2$, the (cylindrical) surface

$$u(x, y)=x\tan\alpha-\sqrt{1-y^2}\sec\alpha+\beta \tag{8}$$

satisfies $\operatorname{div} Tu=2$ in the strip $|y|<\cos\gamma$ and achieves the boundary angle γ.

We point out here that Tam [5] has proved that (8) yields the totality of solutions of the indicated boundary data problem for the infinite strip.

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