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Some remarks on group actions in symplectic geometry

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Introduction

In the previous paper [11], we proved the symplectic analogues of some results in Kähler geometry. In this note we discuss other phenomena, i.e. non-existence of non-trivial compact group actions on closed symplectic manifolds. We consider compact group actions mainly, because there are many non-trivial $\boldsymbol{R}$-actions on any symplectic manifolds. Our results are the following theorems.

THEOREM 1. *Let (M, ω) be a $2n$-dimensional closed symplectic manifold with $c_1(K_M)_{\boldsymbol{R}}=[\omega]$ where $K_M=\bigwedge_{\boldsymbol{C}}^{n} T^*M$ as defined in* § 1. *Then there are no non-trivial compact connected Lie group actions preserving the symplectic form ω.*

THEOREM 2. *Let (M, ω) be a $2n$-dimensional closed symplectic manifold with $c_1(K_M)_{\boldsymbol{R}}=0$. Then there are no non-trivial connected compact semi-simple Lie group actions preserving the symplectic form ω.*

THEOREM 3. *Let (M, ω) be a $2n$-dimensional closed symplectic manifold such that there is a symplectic immersion $f:(M, \omega)\longrightarrow(X, \Omega)$ where (X, Ω) is a compact quotient of bounded Hermitian symmetric domain. Then there are no non-trivial compact connected Lie group actions on M.*

These theorems are symplectic analogues of the following results in Kähler or complex geometry.

THEOREM (Kobayashi [5]). *Let M be a compact Kähler manifold with negative first Chern class. Then there are no non-trivial holomorphic vector fields. In particular, there are no non-trivial holomorphic actions of connected Lie groups on M.*

THEOREM (Lichnerowicz [8]). *Let M be a compact Kähler manifold with semi-negative first Chern class. Then every non-trivial holomorphic* *vector field is nowhere vanishing.*

Remark. From the above theorem we can deduce that there are no non-trivial holomorphic actions of connected semi-simple Lie groups on M. The proof roughly goes as follows. Let G be a connected semi-simple Lie group acting on M holomorphically and $\mathfrak{g}$ the Lie algebra of G. We may assume that G acts on M effectively. As $\mathfrak{g}$ is a semi-simple Lie algebra, there are two element X and Y of $\mathfrak{g}$ such that $[X, Y]\neq 0$. By direct calculation we can easily see that $i([X, Y])\omega$ is cohomologous to zero in $H^1(M, \mathcal{O})$. Applying a result of Carrell and Lieberman [1], the holomorphic vector field $[X, Y]$ has zeros. The above theorem of Lichnerowicz implies that $[X, Y]$ is everywhere vanishing vector field on M, and by the assumption that G acts on M effectively, $[X, Y]=0$ in $\mathfrak{g}$, which contradicts the assumption that $[X, Y]\neq 0$ in $\mathfrak{g}$.

Theorem (Kobayashi [6]). *Let M be a complex manifold and N a complex hyperbolic manifold in the sense of Kobayashi* [6]. *If there is a holomorphic immersion $f: M \longrightarrow N$, then there are no holomorphic actions of complex Lie groups on M.*

For the proof of Theorem 1 and Theorem 2 we use the existence theorem of moment maps and observe the weights of lifted actions on line bundles at fixed points. For the proof of Theorem 3 we use the Gromov norm which seems the counterpart of Kobayashi's pseudo-distance [6].

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§1. Proof of Theorem 1 and Theorem 2

For a symplectic manifold (M, ω) there exists an almost complex structure, unique up to homotopy, because $U(n)$ is a maximal compact subgroup of $Sp(n; \boldsymbol{R})$. More precisely there is a polar decomposition of the symplectic form as follows:

$\omega(x, y)=g(Jx, y)$ where g is a Riemannian metric and J is an almost complex structure.

If a compact Lie group G acts on (M, ω) symplectically, we choose a

G-invariant Riemannian metric g and a G-invariant almost complex structure J such that $\omega(x, y)=g(Jx, y)$. Especially the cotangent bundle T^*M becomes a complex vector bundle and we define the canonical line bundle of M as $K_M=\bigwedge_C^n T^*M$. We call a cohomology class c in $H^2(M, \boldsymbol{R})$ (semi-) positive, or (semi-)negative, if there is a 2-form σ representing c such that $\sigma(x, Jx)$ is positive (or zero) or negative (or zero) for any tangent vector x respectively.

Now we prove the following

PROPOSITION (1.1). *Let (M, ω) be a closed symplectic manifold and G a compact connected Lie group acting (M, ω) non-trivially and symplectically. If there is a moment map $\mu: M \longrightarrow \mathfrak{g}^*$, then the first Chern class of K_M is not semi-positive.*

PROOF. Without loss of generality, we can assume that $G=S^1$ a circle group. Since S^1 acts on M preserving the almost complex structure J, the action is lifted to K_M as automorphisms of complex line bundle. On the other hand there exists a moment map μ, so the fixed point set M^{S^1} is not empty. More precisely the moment map for the S^1-action is a real valued function μ such that $d\mu+i(v)\omega=0$ where v is the vector field determined by the S^1-action. The critical point set of μ is exactly same as the fixed point set M^{S^1}, so there are two connected components F_0 and F_∞ of M^{S^1} where μ attains its minimum and maximum respectively, especially the weights $m_{0,i}$ of the S^1-action on the normal bundle of F_0 is positive and the weights $m_{\infty,j}$ of the S^1-action on the normal bundle of F_∞ is negative.

For a point p of M, we define a subset $M(p)$ as $\bigcup_{t \in S^1} t(C_p)$ where C_p is the integral curve of the gradient flow of μ passing through p, and $\overline{M(p)}$ denotes the closure of $M(p)$ in M with the orientation defined by the almost complex structure J. For a generic point p, $\overline{M(p)}$ intersects F_0 and F_∞ at the points which we shall denote by q_0 and q_∞ respectively. We consider the S^1-complex line bundle K_M restricted to $\overline{M(p)}$, which is homeomorphic to S^2. Let the weights of the S^1 action on fibers over q_0 and q_∞ be m_0 and m_∞ respectively. As $K_M=\bigwedge_C^n T^*M$, $q_0 \in F_0$ and $q_\infty \in F_\infty$, $m_0=\sum_i -m_{0,i}$ is negative and $m_\infty=\sum_j -m_{\infty,j}$ is positive. On the other hand the following equality holds (see Lemma (1.2) below)

$m_0 - m_\infty = l \cdot c_1(K_M)[\overline{M(p)}]$, where l is the order of the isotropy subgroup at p.

Therefore $c_1(K_M)[\overline{M(p)}] < 0$. As the gradient vector field $\nabla \mu$ equals to JX where X is the vector field defined by the S^1-action, $c_1(K_M)$ is not represented by semi positive 2-forms with respect to J. //

The following lemma is well known.

LEMMA (1.2). *Let L be a complex line bundle on S^2 with a fixed orientation, and S^1 acts on S^2 by rotations fixing two points P and Q with rotation numbers m and $-m$ respectively. We denote by t_P and t_Q the weights of some lifted action on the fibers L_P and L_Q respectively. Then the following formula holds*

$$t_P - t_Q = m \cdot c_1(L)[S^2].$$

For the proof of Theorem 1 we show the following

LEMMA (1.3). *Let (M, ω) be a closed symplectic manifold, and G a compact Lie group acting (M, ω) preserving ω. If there exists a complex line bundle L such that*

1) $c_1(L) = -[\omega]$

and

2) *the G-action is lifted to L as automorphisms of a complex line bundle.*

Then there is a moment map $\mu : M \longrightarrow \mathfrak{g}^*$.

PROOF. By the equivariant projective imbedding theorem [11], M can be G-equivariantly imbedded to CP^N symplectically for sufficiently large N and a linear action on CP^N. It is well known that for a linear action on CP^N, there exists a moment map μ [4, 2.5 Lemma]. Let $\rho : G \longrightarrow U(N+1)$ be the representation which gives the G-action on CP^N above. Then the composition of $M \longrightarrow CP^N$, $CP^N \xrightarrow{\mu} \mathfrak{u}(N+1)^*$, and $\mathfrak{u}(N+1)^* \xrightarrow{d\rho^*} \mathfrak{g}^*$ gives a moment map for the G-action on (M, ω). //

Theorem 1 is a direct consequence of Proposition (1.1) and Lemma (1.3). Theorem 2 is derived from Proposition (1.1) and the existence theorem of moment maps for semi-simple Lie group actions [10]. We can also prove the following

PROPOSITION (1.4). *Let (M, ω) be a closed symplectic manifold with semi-positive $c_1(K_M)$. If (M, ω) satisfies the following condition*

$$(1.5) \qquad \wedge\omega^{n-1}: H^1(M; \boldsymbol{R}) \longrightarrow H^{2n-1}(M; \boldsymbol{R}) \quad \textit{is isomorphic}$$

then for any symplectic action on (M, ω) of a compact Lie group G, every vector field defined by the G-action has no zeros.

For the proof of this proposition we use the existence theorem for moment maps for compact Lie group actions [11]. For a general connected compact Lie group G, we can assume that G is a product of a compact semi-simple Lie group H and a toral group T by taking some covering group of G. If H is not a trivial group, then there exists a moment map for H-action [10]. Therefore it contradicts Proposition (1.1). If a vector field v_M corresponding to $v \in \mathfrak{t}$ has zeros, the existence theorem of moment maps [11] asserts that there exists a moment map for the closed subgroup of T generated by $v \in \mathfrak{t}$ under condition (1.5), which contradicts Proposition (1.1).

§ 2. Proof of Theorem 3

Gromov [3] and Yano [13] proved that the Gromov invariant of closed manifold is an obstruction to S^1-actions. Recently Domic and Toledo [2] extended the Kneser type inequality by using the Gromov norm of Kähler class. For the proof of Theorem 3 we review some properties of Gromov norms.

$\|\cdot\|_1$ and $\|\cdot\|_\infty$ denotes pseudo-norms on homology groups and cohomology groups respectively. They satisfy the following conditions.

$$(3.1) \qquad |\langle\alpha, c\rangle| \leqq \|\alpha\|_\infty \cdot \|c\|_1 \text{ for } \alpha \in H^*(M; \boldsymbol{R}) \text{ and } c \in H_*(M; \boldsymbol{R}).$$

$$(3.2) \qquad \|f_* c\|_1 \leqq \|c\|_1 \text{ for a continuous map } f: M \longrightarrow X.$$

$$(3.3) \qquad \|\alpha \cup \beta\|_\infty \leqq \|\alpha\|_\infty \cdot \|\beta\|_\infty \text{ for } \alpha, \beta \in H^*(M; \boldsymbol{R}).$$

(3.3) is easily seen from the definition of cup product.

PROOF OF THEOREM 3. Let (X, Ω) be as in Theorem 3. Then the Gromov norm of $[\Omega] \in H^*(M, \boldsymbol{R})$ is finite [2]. As (M, ω) is a closed symplectic manifold,

$$\begin{aligned} 0 &< \langle[\omega]^n, [M]\rangle \\ &= \langle[\Omega]^n, f_*[M]\rangle \end{aligned}$$

$$\leqq \|[\Omega]^n\|_\infty \cdot \|f_*[M]\|_1 \tag{3.1}$$
$$\leqq \|[\Omega]^n\|_\infty \cdot \|[M]\|_1. \tag{3.2}$$

From (3.3) and the fact $\|[\Omega]\|_\infty<\infty$, we see that $\|[M]\|_1\neq 0$. Therefore M does not admit non-trivial compact Lie group actions. //

§ 3. Conformal vector fields

For closed symplectic manifolds of dimension greater than 2, every conformal symplectic vector field is a symplectic vector field [7, Chap I. Theorem 6.5]. There is the Kähler analogue of this result.

PROPOSITION 4. *Let M be a compact Kähler manifold of* $\dim_C M\geqq 2$. *Then every conformal vector field is a Killing vector field.*

Proposition 4 is a direct consequence of the following facts.

FACT (Lichnerowicz [9]). *Let M and N be compact Kähler manifolds and $f: M\longrightarrow N$ a smooth map. Then we have $E'(f)-E''(f)$ depends only on the homotopy class of f where*

$$E'(f)=\int_M \|\partial f\|^2 dv_M$$
$$E''(f)=\int_M \|\bar{\partial} f\|^2 dv_M.$$

In particular, holomorphic maps are stable harmonic maps.

FACT (Smith [12]). *Let M be a closed Riemannian manifold of* $\dim M\geqq 3$. *Then we have*

$$\operatorname{index}(\mathrm{id.})\geqq \dim(c/i)$$

where c and i are the Lie algebras of conformal vector fields and Killing vector fields respectively, and index(id.) *is the index of the Hessian of the energy functional at the identity map. In particular, if* $\dim(c/i)\geqq 1$, *the identity map is an unstable harmonic map.*

References

[1] Carrell, J. B. and D. I. Lieberman, Holomorphic vector fields and Kaehler manifolds, Invent. Math. **21** (1973), 303-309.

[2] Domic, A. and D. Toledo, The Gromov norm of the Kähler class of symmetric domains, Math. Ann. **276** (1987), 425-432.

[3] Gromov, M., Volume and bounded cohomology, Publ. Math. IHES **56** (1982), 5-99.
[4] Kirwan, F. C., Cohomology of quotients in symplectic and algebraic geometry, Math. Notes vol. 31, Princeton Univ. Press, Princeton, 1984.
[5] Kobayashi, S., On the automorphism group of a certain class of algebraic manifolds, Tôhoku Math. J. **11** (1959), 184-190.
[6] Kobayashi, S., Hyperbolic Manifolds and Holomorphic Mappings, Marcel Dekker, New York, 1970.
[7] Kobayashi, S., Transformation groups in differential geometry, Ergeb. Math. Grenzgeb. Band 70, Springer-Verlag, Berline Heidelberg New York, 1972.
[8] Lichnerowicz, A., Variétés kählériennes et première classe de Chern, J. Differential Geom. **1** (1967), 195-224.
[9] Lichnerowicz, A., Applications harmoniques et variétés kählériennes, Symp. Math. III (Bologna) 1970, 341-402.
[10] Marsden, J. and A. Weinstein, Reduction of symplectic manifolds with symmetry, Rep. Math. Phys. **5** (1974), 121-130.
[11] Ono, K., Equivariant projective imbedding theorem for symplectic manifolds, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **35** (1988), 381-392.
[12] Smith, R. T., The second variation formula for harmonic maps, Proc. Amer. Math. Soc. **47** (1975), 229-236.
[13] Yano, K., Gromov invariant and S^1-actions, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **29** (1982), 493-501.

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