J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
35 (1988), 439-506.

On fixed and movable singularities of systems of rational differential equations of order n

By Yoshihiro MURATA

(Communicated by T. Kimura)

Contents

§0. Introduction.

Consider an ordinary differential equation in the complex domain. Singularities of solutions of the equation are divided into two categories: fixed singularities and movable singularities. The formers are those singularities which are situated on points determined by the equation and the latters are those singularities which appear at points depending on a particular solution. For example, the equation

$$y''=-\frac{(y')^2}{y}+\frac{y'}{x}-\frac{y}{2x^2}$$

has the general solution $y=\sqrt{x(A+B\log x)}$ $((A, B)\neq(0,0))$. $x=0$ is a fixed singularity and $x=\exp(-A/B)$ is a movable singularity.

As is well known, P. Painlevé [5] gave an exact definition of the set of fixed singularities of a rational differential equation of the first order, and obtained a fundamental theorem about singularities of solutions.

Consider an equation

$$(\mathrm{E}_1) \qquad \frac{dy}{dx}=\frac{P(x, y)}{Q(x, y)}$$

with relatively prime $P, Q\in\mathcal{O}_D[y]$, where $\mathcal{O}_D$ is the set of functions in x holomorphic in a domain $D\subset \boldsymbol{C}$ and $\mathcal{O}_D[y]$ is the polynomial ring in y with coefficients in $\mathcal{O}_D$. Suppose that by the change of the dependent variable $y=1/v$, (E_1) is transformed into the equation

$$(\mathrm{E}_1)_2 \qquad \frac{dv}{dx}=\frac{P_2(x, v)}{Q_2(x, v)}$$

with relatively prime $P_2, Q_2\in\mathcal{O}_D[v]$. Painlevé defined the set $\boldsymbol{\Theta}$ of fixed singularities as

$$\boldsymbol{\Theta}=\boldsymbol{\Theta}_1\cup\boldsymbol{\Theta}_2,$$

where

$$\begin{aligned}\boldsymbol{\Theta}_1&=\{\xi\in D|\, Q(\xi, y)\equiv 0\},\\ \boldsymbol{\Theta}_2&=\{\xi\in D-\boldsymbol{\Theta}_1|\ \text{for some } \eta\in \boldsymbol{C},\ P(\xi, \eta)=Q(\xi, \eta)=0\}\\ &\quad\cup\{\xi\in D-\boldsymbol{\Theta}_1|\, P_2(\xi, 0)=Q_2(\xi, 0)=0\},\end{aligned}$$

and obtained the following fundamental theorem.

THEOREM α. *Let Φ be a solution of* (E_1).

(1) *If Φ has a singularity $\boldsymbol{\omega}$ on $a\in D-\boldsymbol{\Theta}$, $\boldsymbol{\omega}$ is an algebraic branch point.*

(2) *If Φ has a transcendental singularity $\boldsymbol{\omega}$ on $\xi\in\boldsymbol{\Theta}_2$, $\boldsymbol{\omega}$ is an ordinary transcendental singularity.*

He also defined the set of fixed singularities of an algebraic differential equation

$$(\mathrm{A}_1) \qquad F\left(x, y, \frac{dy}{dx}\right)=0,$$

where $F \in \mathcal{O}_D[y, dy/dx]$ is irreducible, and proved the same result as in Theorem α (1).

In the cases of higher order differential equations or systems of differential equations, as examples show, movable transcendental singularities arise in general (See [1], [5].). For these equations, he tried to define the set of fixed singularities and to derive fundamental properties about singularities of solutions such as Theorem α (See [5].). He studied the rational differential equation of the second order:

$$(\mathrm{F}_2) \qquad \frac{d^2y}{dx^2} = \frac{P\left(x, y, \dfrac{dy}{dx}\right)}{Q\left(x, y, \dfrac{dy}{dx}\right)}$$

with relatively prime $P, Q \in \mathcal{O}_D[y, dy/dx]$, the algebraic differential equation of the second order:

$$(\mathrm{A}_2) \qquad F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0,$$

where $F \in \mathcal{O}_D[y, dy/dx, d^2y/dx^2]$ is irreducible, and the system of rational differential equations of order n:

$$(\mathrm{E}_n) \qquad \begin{cases} \dfrac{dy_1}{dx} = \dfrac{Y_1(x, y_1, \cdots, y_n)}{X(x, y_1, \cdots, y_n)} \\ \quad \cdots\cdots \\ \dfrac{dy_n}{dx} = \dfrac{Y_n(x, y_1, \cdots, y_n)}{X(x, y_1, \cdots, y_n)}, \end{cases}$$

where $X, Y_1, \cdots, Y_n \in \mathcal{O}_D[y_1, \cdots, y_n]$, especially (E_2). However, since his investigations of the set of fixed singularities were insufficient, he could not completely generalize Theorem α.

Later T. Kimura [1] investigated the set of fixed singularities of (F_2) and derived some properties of movable essential singularities of solutions (See § 4.).

The purposes of this paper are to give a general theory of fixed and movable singularities of rational differential equations and to study particular cases in detail.

(I) For the equation

$$(\mathrm{E}_n)\qquad \begin{cases} \dfrac{dy_1}{dx}=\dfrac{P_1(x, y_1, \cdots, y_n)}{Q_1(x, y_1, \cdots, y_n)} \\ \quad\cdots\cdots \\ \dfrac{dy_n}{dx}=\dfrac{P_n(x, y_1, \cdots, y_n)}{Q_n(x, y_1, \cdots, y_n)}, \end{cases}$$

where $P_l, Q_l \in \mathcal{O}_D[y_1, \cdots, y_n]$ are relatively prime ($l=1, \cdots, n$), we give an exact definition of a fixed singularity and that of a movable singularity, which are stated in Definition 1.12 and 1.13 in 1.4.

(II) Under these definitions, we prove two theorems (Theorem 2 and Theorem 3 in 1.5) which are generalizations of Theorem α.

(III) As a particular case, we study (E_2) in more detail in § 3. We prove Theorem 4, 5 and 6.

(IV) Using the results in (III), we study (F_2) and derive some results obtained by Kimura [1] (i.e. the definition of the set of fixed singularities (Proposition 4.1, 4.2 in § 4), the sufficient conditions for nonexistence of movable essential singularities (Theorem 7 in § 4)).

Before carrying out (I)~(IV), we make the following preparations to avoid confusions of concepts:

(P1) In 1.1, we clarify the concepts of values, singularities, cluster sets of n-tuple analytic functions.

(P2) We explain the prime factorization theorem for $\mathcal{O}_D[y_1, \cdots, y_n]$ (Theorem 1 in 1.2). In the Appendix, this theorem is proved in a more general situation.

§ 1. Fixed and movable singularities of the system (E_n) of rational differential equations of order n.

1.1. Singularities of n-tuple analytic functions.

An ordered n-tuple of analytic functions has some different properties from those of a single analytic function. So we survey the theory of n-tuple analytic functions in the following.

1° The definition of n-tuple analytic functions.

Let $D \subset C$ be a domain, and let $\boldsymbol{\phi}_a=(\phi_{a,1}, \cdots, \phi_{a,n})$ be an ordered n-tuple of convergent Puiseux series around $a \in D$ with finite principal parts. $\boldsymbol{\phi}_a=(\phi_{a,1}, \cdots, \phi_{a,n})$ and $\boldsymbol{\psi}_a=(\psi_{a,1}, \cdots, \psi_{a,n})$ are identified if and only if $\phi_{a,k}=\psi_{a,k}$ for $k=1, \cdots, n$. Let $\mathcal{A}_a^n$ be the set of all ordered n-tuples of convergent Puiseux series around $a\in D$ with finite principal parts. Let $\mathcal{A}^n=\bigcup_{a\in D}\mathcal{A}_a^n$, and let $\pi:\mathcal{A}^n\longrightarrow D$, $\boldsymbol{\phi}_a\longrightarrow a$ be the natural projection. In the usual way, $\mathcal{A}^n$ becomes a Hausdorff space and π is continuous. But, since π is not locally topological, $(\mathcal{A}^n, \pi, D)$ is not a sheaf.

Elements in $\mathcal{A}^n$ are classified as follows:

Let $\boldsymbol{\phi}_a=(\phi_{a,1},\cdots,\phi_{a,n})\in\mathcal{A}^n$.

1. If every $\phi_{a,i}$ $(i=1,\cdots,n)$ is holomorphic at $a\in D$, $\boldsymbol{\phi}_a$ is called *a holomorphic element.*
2. If some of $\phi_{a,i}$'s have poles at $a\in D$ and the others are holomorphic at $a\in D$, $\boldsymbol{\phi}_a$ is called *a polar element.*
3. If $\boldsymbol{\phi}_a$ is a holomorphic element or a polar element, $\boldsymbol{\phi}_a$ is called *a meromorphic element.*
4. If any one of $\phi_{a,i}$'s has an algebraic branch point at $a\in D$, $\boldsymbol{\phi}_a$ is called *a ramified element.*

Let $\mathcal{O}_a^n$ be the set of all holomorphic elements at $a\in D$ and let $\mathcal{O}^n=\bigcup_{a\in D}\mathcal{O}_a^n$. Then $\mathcal{O}^n$ is open dense in $\mathcal{A}^n$ and $(\mathcal{O}^n, \pi, D)$ is a sheaf.

DEFINITION 1.1. For $\boldsymbol{\phi}_a\in\mathcal{A}^n$, take a holomorphic element $\boldsymbol{\phi}_b$ in a connected open neighborhood $\mathcal{U}\subset\mathcal{A}^n$ of $\boldsymbol{\phi}_a$. And let $\boldsymbol{\Phi}_h$ be a connected component in $\mathcal{O}^n$ containing $\boldsymbol{\phi}_b$, and let $\boldsymbol{\Phi}=\overline{\boldsymbol{\Phi}_h}$ be the closure of $\boldsymbol{\Phi}_h$ in $\mathcal{A}^n$. $\boldsymbol{\Phi}$ is called *the n-tuple analytic function determined by* $\boldsymbol{\phi}_a$.

Note that $\boldsymbol{\Phi}$ contains $\boldsymbol{\phi}_a$, and that $\boldsymbol{\Phi}_h$ is independent of the choice of $\boldsymbol{\phi}_b$.

$\boldsymbol{\Phi}$ (resp. $\boldsymbol{\Phi}_h$) has a structure of Riemann surface. When we regard $\boldsymbol{\Phi}$ (resp. $\boldsymbol{\Phi}_h$) as a Riemann surface, we denote $\boldsymbol{\Phi}$ (resp. $\boldsymbol{\Phi}_h$) by $\mathcal{R}$ (resp. $\mathcal{R}_h$).

REMARK 1. Let $\mathcal{M}_a^n$ be the set of all meromorphic elements at $a\in D$ and let $\mathcal{M}^n=\bigcup_{a\in D}\mathcal{M}_a^n$. Then $\mathcal{M}^n$ is open dense in $\mathcal{A}^n$ and $(\mathcal{M}^n, \pi, D)$ is a sheaf. For $\boldsymbol{\phi}_a\in\mathcal{A}^n$, take a meromorphic element $\boldsymbol{\phi}_b$ in a connected open neighborhood $\mathcal{U}\subset\mathcal{A}^n$ of $\boldsymbol{\phi}_a$. Let $\boldsymbol{\Phi}_m$ denote a connected component in $\mathcal{M}^n$ containing $\boldsymbol{\phi}_b$. Then, $\boldsymbol{\Phi}_m$ is independent of the choice of $\boldsymbol{\phi}_b$, and $\boldsymbol{\Phi}_h\subset\boldsymbol{\Phi}_m\subset\boldsymbol{\Phi}$, $\overline{\boldsymbol{\Phi}_h}=\overline{\boldsymbol{\Phi}_m}=\boldsymbol{\Phi}$. $\boldsymbol{\Phi}_m$ has a structure of Riemann surface. If we regard $\boldsymbol{\Phi}_m$ as a Riemann surface, we denote $\boldsymbol{\Phi}_m$ by $\mathcal{R}_m$.

REMARK 2. An element $\boldsymbol{\phi}_a\in\mathcal{A}^n$ is often identified with an ordered

n-tuple of germs at a of analytic functions.

2° Values of n-tuple analytic functions.

DEFINITION 1.2. *A rational compactification of* $\boldsymbol{C}^n$ is an n-dimensional compact complex manifold M with the following properties:

1. M contains a nonempty nowhere dense closed analytic subset A such that $M-A$ is biholomorphic to $\boldsymbol{C}^n$.

2. M has an atlas $\{(U_i, \kappa_i)\}$ which satisfies the following conditions:

(1) $\{(U_i, \kappa_i)\}$ consists of a finite number of charts $(U_1, \kappa_1), \cdots, (U_m, \kappa_m)$.

(2) $U_1 = M - A \cong \boldsymbol{C}^n$, $\theta_1 = \mathrm{id} : U_1 = \boldsymbol{C}^n \longrightarrow \boldsymbol{C}^n$.

(3) $\kappa_j \circ \kappa_i^{-1} : (y_1^{(i)}, \cdots, y_n^{(i)}) \longrightarrow (y_1^{(j)}, \cdots, y_n^{(j)})$, where $y_k^{(j)} = R_k^{ji}(y_1^{(i)}, \cdots, y_n^{(i)}) \in \boldsymbol{C}(y_1^{(i)}, \cdots, y_n^{(i)})$, i.e. R_k^{ji} is rational in $y_1^{(i)}, \cdots, y_n^{(i)}$ over $\boldsymbol{C}$.

In what follows, we always take an atlas with the above properties whenever we consider a rational compactification of $\boldsymbol{C}^n$.

REMARK. An n-dimensional compact complex manifold which satisfies the condition 1 in Definition 1.2 is called a compactification of $\boldsymbol{C}^n$. In the cases $n=1, 2$, any compactification of $\boldsymbol{C}^n$ is a rational compactification. Strictly speaking, the compactification of $\boldsymbol{C}^1$ is nothing but $\boldsymbol{P}^1$ and any compactification of $\boldsymbol{C}^2$ is a rational surface (K. Kodaira [2], J. Morrow [3]). Especially, compactifications of $\boldsymbol{C}^2$ were studied by Morrow [3] in detail. Hirzebruch surfaces $\Sigma^{(k)}$ $(k=0, 1, 2, \cdots)$ as well as $\boldsymbol{P}^2$ are compactifications of $\boldsymbol{C}^2$.

Let $\mathfrak{R}$ be a Riemann surface determined by an n-tuple analytic function Φ $(=\overline{\Phi_h})$ on D, and let M be a rational compactification of $\boldsymbol{C}^n$. We define a mapping $\hat{\Phi} : \mathfrak{R} \longrightarrow M$ as follows:

Suppose $\boldsymbol{\phi}_a \in \mathfrak{R}$.

(A) If $\boldsymbol{\phi}_a$ is a holomorphic element, $\hat{\Phi}(\boldsymbol{\phi}_a) = \boldsymbol{\phi}_a(a) \in \boldsymbol{C}^n$.

(B) If $\boldsymbol{\phi}_a$ is a polar element or a ramified element, then for any small $r>0$, there exists a connected open neighborhood $\mathcal{U}_r \subset \pi^{-1}(\{x \mid |x-a| < r\}) \subset \mathfrak{R}$ such that any point in $\mathcal{U}_r - \{\boldsymbol{\phi}_a\}$ is a holomorphic element. Define $\hat{\Phi}(\boldsymbol{\phi}_a) = \bigcap_{r>0} \overline{\hat{\Phi}(\mathcal{U}_r - \{\boldsymbol{\phi}_a\})}$, where $-$ means the closure in M.

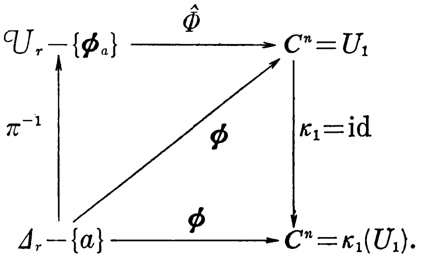
PROPOSITION 1.1. (1) *For any* $\boldsymbol{\phi}_a \in \mathfrak{R}$, $\hat{\Phi}(\boldsymbol{\phi}_a)$ *is a point in* M.

(2) $\hat{\Phi} : \mathfrak{R} \longrightarrow M$ *is a holomorphic mapping.*

(3) $\pi \times \hat{\Phi} : \mathfrak{R} \longrightarrow D \times M$, $\boldsymbol{\phi}_a \longrightarrow (a, \hat{\Phi}(\boldsymbol{\phi}_a))$ *is an immersion.*

PROOF. (1) It is sufficient to prove the proposition in the case that

$\boldsymbol{\phi}_a$ is a polar element or a ramified element. Suppose that $\boldsymbol{\phi}_a$ is a polar element, and let $(\boldsymbol{\phi}, \Delta_r)=((\phi_1, \cdots, \phi_n), \Delta_r)$ denote a representative of $\boldsymbol{\phi}_a$, where $r>0$ and $\Delta_r=\{x \mid |x-a|<r\}\subset D$. We may assume that $\boldsymbol{\phi}$ is holomorphic in $\Delta_r-\{a\}$ and $\pi: \mathcal{U}_r-\{\boldsymbol{\phi}_a\}\longrightarrow\Delta_r-\{a\}$ is biholomorphic, and that the following diagram is commutative:



Let $S=\bigcap_{r>0}\overline{\hat{\Phi}(\mathcal{U}_r-\{\boldsymbol{\phi}_a\})}$. It is clear that S is not empty, so we will prove that S consists of just one point. Suppose that S contains distinct two points p, q, and that p is in U_i, where (U_i, κ_i) is a chart of M. Let

$$\kappa_i\circ\kappa_1^{-1}: \kappa_1(U_1)\longrightarrow\kappa_i(U_i),\ (y_1^{(1)}, \cdots, y_n^{(1)})\longrightarrow(y_1^{(i)}, \cdots, y_n^{(i)}),$$

where $y_k^{(i)}=R_k^{i1}(y_1^{(1)}, \cdots, y_n^{(1)})\in \boldsymbol{C}(y_1^{(1)}, \cdots, y_n^{(1)})$, and let $E_k\subset\kappa_1(U_1)$ be the exceptional set of R_k^{i1} $(k=1, \cdots, n)$.

If there exists a k such that $\boldsymbol{\phi}(\Delta_r-\{a\})\subset E_k$, then $\hat{\Phi}(\mathcal{U}_r-\{\boldsymbol{\phi}_a\})=\boldsymbol{\phi}(\Delta_r-\{a\})\subset\kappa_1^{-1}(E_k)\subset M-U_i$. This means that $p\in S\subset\overline{\hat{\Phi}(\mathcal{U}_r-\{\boldsymbol{\phi}_a\})}\subset M-U_i$, which contradicts the assumption. Therefore, $\boldsymbol{\phi}(\Delta_r-\{a\})\not\subset E_k$ for any k and the function $\boldsymbol{\varphi}=(R_1^{i1}(\boldsymbol{\phi}), \cdots, R_n^{i1}(\boldsymbol{\phi}))$ defined on Δ_r is meromorphic at a. Since $p\in U_i$, $\boldsymbol{\varphi}: \Delta_r\longrightarrow C^n=\kappa_i(U_i)$ is holomorphic at a and $\boldsymbol{\varphi}(a)=\kappa_i(p)$. Then, for any neighborhood $V\subset M$ of p, we have $\overline{\hat{\Phi}(\mathcal{U}_r-\{\boldsymbol{\phi}_a\})}\subset V$ with a suitable small $r>0$.

By a similar consideration, we can conclude that for any neighborhood $W\subset M$ of q, we can choose a suitable $r>0$ such that $\overline{\hat{\Phi}(\mathcal{U}_r-\{\boldsymbol{\phi}_a\})}\subset W$. Hence, if we take V and W such that $V\cap W=\emptyset$, there exists a small $r>0$ such that $S\subset\overline{\hat{\Phi}(\mathcal{U}_r-\{\boldsymbol{\phi}_a\})}\subset V$ and $S\subset\overline{\hat{\Phi}(\mathcal{U}_r-\{\boldsymbol{\phi}_a\})}\subset W$. This is a contradiction. Therefore, S consists of one point.

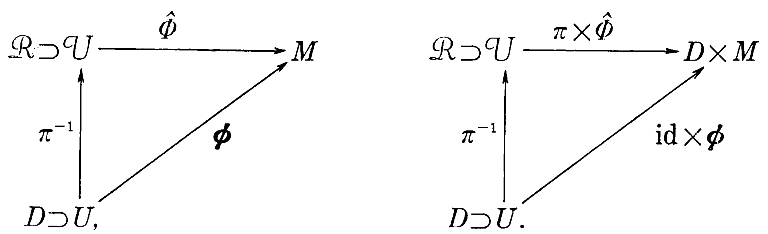
In the case that $\boldsymbol{\phi}_a$ is a ramified element, the proposition can be proved in a similar way.

(2) For any $\boldsymbol{\phi}_a\in\mathcal{R}$, there exist a neighborhood $\mathcal{U}$ of $\boldsymbol{\phi}_a$, an open disk $\Delta=\{t \mid |t|<\varepsilon\}\subset \boldsymbol{C}$ and a homeomorphism $\theta: \mathcal{U}\longrightarrow\Delta$ such that

$\kappa_1 \circ \hat{\Phi} \circ \theta^{-1}(t) = (f_1(t), \cdots, f_n(t))$, $f_k(t) = \sum_{\sigma=\sigma_k}^{\infty} C_{k,\sigma} t^{\sigma}$, $\sigma \in Z$, $(k=1, \cdots, n)$. Therefore, the same arguments as in (1) shows that $\hat{\Phi} : \mathfrak{R} \longrightarrow M$ is a holomorphic mapping.

(3) It is apparent from (2) and the definition of $\pi \times \hat{\Phi}$. q.e.d.

Suppose $\Phi = \overline{\Phi_m}$, and take any element $\boldsymbol{\phi}_a \in \Phi_m$. If $(\boldsymbol{\phi}, U)$ is a representative of $\boldsymbol{\phi}_a$ with an open neighborhood U of a, we can regard $\boldsymbol{\phi}$ as a holomorphic mapping $\boldsymbol{\phi} : U \longrightarrow M$ by the same arguments as the above. Then there is an open neighborhood $\mathcal{U} \subset \mathfrak{R}$ of $\boldsymbol{\phi}_a$ such that $\mathcal{U}$ and U are biholomorphic and the following diagrams are commutative:



3° Analytic continuations and singularities of n-tuple analytic functions.

Let I be a nonempty interval in $\boldsymbol{R}$ and let $l : I \longrightarrow D$ be a curve in D. Consider the Riemann surface $\mathfrak{R}$ determined by an n-tuple analytic function $\Phi (= \overline{\Phi_h} = \overline{\Phi_m})$. Let $\mathfrak{R}_h$, $\mathfrak{R}_m$ denote the Riemann surfaces determined by Φ_h, Φ_m respectively. A curve $\tilde{l} : I \longrightarrow \mathfrak{R}$ is said to be *an analytic continuation along* l if $\pi \circ \tilde{l} = l$. In particular, if $\tilde{l}(I) \subset \mathfrak{R}_h = \mathfrak{R} \cap \mathcal{O}^n$ (resp. $\mathfrak{R}_m = \mathfrak{R} \cap \mathcal{M}^n$), then $\tilde{l}$ is called *a holomorphic* (*resp. meromorphic*) *continuation*. When a curve $l : I \longrightarrow D$ starting from $a \in D$ and a point $\boldsymbol{\phi}_a \in \mathfrak{R}$ are given, $\boldsymbol{\phi}_a$ is said to be *analytically continuable along* l if there exists an analytic continuation along l starting from $\boldsymbol{\phi}_a$.

Suppose that $l : I = [\alpha, \beta) \longrightarrow D$ is a curve such that $\lim_{t \to \beta} l(t) = b \in D$, and that $\tilde{l}$ is a holomorphic continuation along l. Then we have the following four possibilities:

(1) $\tilde{l}$ is continuable to a holomorphic element on b, i.e., there is a curve $\hat{l} : [\alpha, \beta] \longrightarrow \mathcal{O}^n$ such that $\hat{l}|_{[\alpha, \beta)} = \tilde{l}$.

(2) $\tilde{l}$ is continuable to a polar element on b.

(3) $\tilde{l}$ is continuable to a ramified element on b.

(4) $\tilde{l}$ is not continuable to any element on b.

In each case of (1)$\sim$(3), $\tilde{l}$ reaches a certain point $\boldsymbol{\omega} \in \mathfrak{R}$ on b. In the case (4), $\tilde{l}$ does not reach any point in $\mathfrak{R}$ on b. But, including the case (4), $\tilde{l}$ is said to *determine a point* $\boldsymbol{\omega}$ *on* $b \in D$. Particularly, in the case (3), $\tilde{l}$ is said to determine *an algebraic singularity* (*or an algebraic branch point*) $\boldsymbol{\omega}$ on $b \in D$, and in the case (4), $\tilde{l}$ is said to determine *a transcendental singularity* $\boldsymbol{\omega}$ on $b \in D$. When any one of (1)～(3) occurs, $\tilde{l}$ is said to determine *at most an algebraic singularity* $\boldsymbol{\omega}$ on $b \in D$.

Using the above concepts, we find that the n-tuple analytic function $\Phi = \overline{\Phi_h}(=\mathfrak{R})$ is nothing but the set obtained by adding to $\Phi_h(=\mathfrak{R}_h)$ all polar elements and all ramified elements (i.e., all algebraic singularities) which are reached by holomorphic continuations on $\mathfrak{R}_h$.

DEFINITION 1.3. An n-tuple analytic function is called *an n-tuple algebroidal function* if it is of finite sheets and every singularity is algebraic.

We make a remark about transcendental singularities of $\Phi(=\mathfrak{R})$. Let l and l' be curves in D converging to $b \in D$. Suppose that there exist a holomorphic continuation $\tilde{l}$ along l which determines a transcendental singularity $\boldsymbol{\omega}$ on b and a holomorphic continuation $\tilde{l}'$ along l' which determines a transcendental singularity $\boldsymbol{\omega}'$ on b. Let $\mathcal{U}_r, \mathcal{U}'_r$ denote connected components in $\pi^{-1}\{x \mid 0<|x-b|<r\} \cap \mathfrak{R}_h$ which include terminal subarcs of $\tilde{l}, \tilde{l}'$ respectively. We define that $\boldsymbol{\omega}=\boldsymbol{\omega}'$ if and only if $\mathcal{U}_r=\mathcal{U}'_r$ for any $r>0$.

We make one more remark. Let l be a curve in D converging to $b \in D$, and let $\tilde{l}$ be a holomorphic continuation along l which determines a point $\boldsymbol{\omega}$ on b. A connected component in $\pi^{-1}\{x \mid 0<|x-b|<r\}$ (resp. $\pi^{-1}\{x \mid 0<|x-b|<r\} \cap \mathfrak{R}_h$, $\pi^{-1}\{x \mid 0<|x-b|<r\} \cap \mathfrak{R}_m$) which includes a terminal subarc of $\tilde{l}$ is called *an r-neighborhood* (*resp. a holomorphic r-neighborhood, a meromorphic r-neighborhood*) *of* $\boldsymbol{\omega}$.

4° Cluster sets and transcendental singularities.

Let l be a curve in D converging to $b \in D$, and let $\tilde{l}$ be a holomorphic continuation along l which determines a point $\boldsymbol{\omega}$ on b.

DEFINITION 1.4. Let $\mathcal{U}_r$ be the r-neighborhood of $\boldsymbol{\omega}$. The set $S_{\boldsymbol{\omega}} = \bigcap_{r>0} \overline{\hat{\Phi}(\mathcal{U}_r)} \subset M$ is called *the cluster set of* Φ *at* $\boldsymbol{\omega}$.

As usual, $S_{\boldsymbol{\omega}}$ has the following properties.

PROPOSITION 1.2. (1) *Let* ε *be any positive constant, n a positive* *integer, and* $\mathcal{U}_n$ *the holomorphic* ε/n*-neighborhood of* $\boldsymbol{\omega}$. *Then,*

$$S_{\omega}=\bigcap_{n=1}^{\infty}\overline{\hat{\Phi}(\mathcal{U}_n)}$$
$$=\left\{p\in M\mid \text{there exists a sequence } \{x_n\} \text{ in } \mathcal{R} \text{ such that } x_n\in\mathcal{U}_n,\ \lim_{n\to\infty}\hat{\Phi}(x_n)=p\right\}.$$

(2) S_{ω} *is closed and connected in* M. *Hence* S_{ω} *is either a set consisting of one point or a set containing infinitely many points.*

(3) *Suppose that* l' *is a curve in* D *converging to* $b\in D$, *and that the holomorphic continuation* $\tilde{l}'$ *along* l' *determines a point* $\boldsymbol{\omega}'$ *on* b. *If* $\boldsymbol{\omega}=\boldsymbol{\omega}'$, *then* $S_{\omega}=S_{\omega'}$.

(4) *If* $\boldsymbol{\omega}$ *is at most an algebraic singularity,* $S_{\omega}=\hat{\Phi}(\boldsymbol{\omega})$.

We omit the proof.

Before classifying transcendental singularities, we show an important example of the 2-tuple analytic function.

Example. Consider a 2-tuple analytic function $\boldsymbol{\Phi}=(1/\sqrt[k]{x},\ 1/\log x)$ defined on $\boldsymbol{C}$ with a transcendental singularity $\boldsymbol{\omega}$ on $x=0$, where k is a positive integer. Let $\mathcal{R}$ be the Riemann surface determined by $\boldsymbol{\Phi}$, M a rational compactification of $\boldsymbol{C}^2$ and $\hat{\boldsymbol{\Phi}}:\mathcal{R}\longrightarrow M$ the holomorphic mapping determined by $\boldsymbol{\Phi}$.

(1) In the case $M=\boldsymbol{P}\times\boldsymbol{P}$.

Take an atlas $\{(U_i,\kappa_i)\}_{i=1}^4$ on M as follows:

$$\kappa_1: U_1=\boldsymbol{C}\times\boldsymbol{C}\longrightarrow\boldsymbol{C}^2,\ P\longrightarrow\kappa_1(P)=(y,z),$$
$$\kappa_2: U_2=\boldsymbol{C}\times\boldsymbol{P}^*\longrightarrow\boldsymbol{C}^2,\ P\longrightarrow\kappa_2(P)=(y,s),$$
$$\kappa_3: U_3=\boldsymbol{P}^*\times\boldsymbol{C}\longrightarrow\boldsymbol{C}^2,\ P\longrightarrow\kappa_3(P)=(v,z),$$
$$\kappa_4: U_4=\boldsymbol{P}^*\times\boldsymbol{P}^*\longrightarrow\boldsymbol{C}^2,\ P\longrightarrow\kappa_4(P)=(v,s),$$

where $\boldsymbol{P}^*=\boldsymbol{P}-\{0\}$, $s=1/z$, $v=1/y$. Then from

$$\kappa_1(\hat{\Phi}(\pi^{-1}(x)))=(y,z)=(1/\sqrt[k]{x},1/\log x),$$
$$\kappa_2(\hat{\Phi}(\pi^{-1}(x)))=(y,s)=(1/\sqrt[k]{x},\log x),$$
$$\kappa_3(\hat{\Phi}(\pi^{-1}(x)))=(v,z)=(\sqrt[k]{x},1/\log x),$$
$$\kappa_4(\hat{\Phi}(\pi^{-1}(x)))=(v,s)=(\sqrt[k]{x},\log x),$$

it follows that $S_{\omega}=\{(\infty,0)\}$.

(2) In the case $M=\Sigma^{(k)}$ (Hirzebruch surface).

Take an atlas $\{(U_i,\kappa_i)\}_{i=1}^4$ on M as follows:

$$\kappa_1 : U_1 = \boldsymbol{C}^2 \longrightarrow \boldsymbol{C}^2,\ P \longrightarrow \kappa_1(P) = (y, z),$$
$$\kappa_2 : U_2 \longrightarrow \boldsymbol{C}^2,\ P \longrightarrow \kappa_2(P) = (y, s),$$
$$\kappa_3 : U_3 \longrightarrow \boldsymbol{C}^2,\ P \longrightarrow \kappa_3(P) = (v, w),$$
$$\kappa_4 : U_4 \longrightarrow \boldsymbol{C}^2,\ P \longrightarrow \kappa_4(P) = (v, t),$$

where $s = 1/z, v = 1/y, w = y^k z, t = 1/w$. Then from

$$\kappa_1(\hat{\Phi}(\pi^{-1}(x))) = (y, z) = (1/\sqrt[k]{x}, 1/\log x),$$
$$\kappa_2(\hat{\Phi}(\pi^{-1}(x))) = (y, s) = (1/\sqrt[k]{x}, \log x),$$
$$\kappa_3(\hat{\Phi}(\pi^{-1}(x))) = (v, w) = (\sqrt[k]{x}, 1/x \log x),$$
$$\kappa_4(\hat{\Phi}(\pi^{-1}(x))) = (v, t) = (\sqrt[k]{x}, x \log x),$$

it follows that $S_\omega = \{\infty\} \times \boldsymbol{P}^1$.

As this example shows, if $\boldsymbol{\omega}$ is a transcendental singularity, S_ω becomes either a single point set or an infinite set depending on the choice of M. Therefore, when $n \geqq 2$, the concept of ordinary (or essential) transcendental singularity has no a priori meaning. Giving attention to this fact, we classify transcendental singularities as follows.

DEFINITION 1.5. Let $\boldsymbol{\Phi}$ be an n-tuple analytic function defined on $D \subset \boldsymbol{C}$. Suppose that $\boldsymbol{\Phi}$ takes its values at M, where M is one of rational compactifications of $\boldsymbol{C}^n$, and that $\boldsymbol{\Phi}$ has a transcendental singularity $\boldsymbol{\omega}$ on $a \in D$. Let $S_\omega \subset M$ be the cluster set of $\boldsymbol{\Phi}$ at $\boldsymbol{\omega}$.

(1) If S_ω consists of one point, $\boldsymbol{\omega}$ is called *an ordinary transcendental singularity* (briefly, *ordinary singularity*) *with respect to M.*

(2) If S_ω contains infinitely many points, $\boldsymbol{\omega}$ is called *an essential transcendental singularity* (briefly, *essential singularity*) *with respect to M.*

According to this definition, the transcendental singularity $\boldsymbol{\omega}$ of $\boldsymbol{\Phi} = (1/\sqrt[k]{x}, 1/\log x)$ on $x = 0$ is an ordinary singularity with respect to $M = \boldsymbol{P} \times \boldsymbol{P}$, and an essential singularity with respect to $M = \Sigma^{(k)}$.

Next we give a definition on the figure of the cluster set S_ω.

DEFINITION 1.6. Suppose that $\boldsymbol{\Phi}$ and M are the same ones as in Definition 1.5. Let $\boldsymbol{\omega}$ be a singularity of $\boldsymbol{\Phi}$ on $a \in D$ and S_ω the cluster set of $\boldsymbol{\Phi}$ at $\boldsymbol{\omega}$. Take an atlas $\{(U_i, \kappa_i)\}_{i=1}^m$ on M, and let

$$p_k^{(i)} : \kappa(U_i) \longrightarrow \boldsymbol{C},\ (y_1^{(i)}, \cdots, y_n^{(i)}) \longrightarrow y_k^{(i)}.$$

(1) If there are $(p^{(i)}_{k_i})_{i\in I}$ $(I\in\{1,\cdots,m\})$ such that $p^{(i)}_{k_i}(\kappa_i(S_\omega\cap U_i))$ consists of one point, then S_ω (or $\boldsymbol{\omega}$) is said to be *at most ordinary in* $(k_i)_{i\in I}$*-direction.*

(2) If there are $(p^{(i)}_{k_i})_{i\in I}$ $(I\in\{1,\cdots,m\})$ such that $p^{(i)}_{k_i}(\kappa_i(S_\omega\cap U_i))$ contains infinitely many points, then S_ω (or $\boldsymbol{\omega}$) is said to be *essential in* $(k_i)_{i\in I}$*-direction.*

Note that when $\boldsymbol{\omega}$ is a transcendental singularity, $\boldsymbol{\omega}$ is ordinary if and only if S_ω is at most ordinary in every direction.

The cluster set S_ω of $\Phi=(1/\sqrt[k]{x}, 1/\log x)$ at the transcendental singularity $\boldsymbol{\omega}$ is at most ordinary in $(v^{(3)}, v^{(4)})$-direction whether $M=\boldsymbol{P}\times\boldsymbol{P}$ or $M=\Sigma^{(k)}$.

1.2. The prime factorization theorem for $\mathcal{O}_D[y_1,\cdots,y_n]$.

Let $D\subset\boldsymbol{C}$ be a domain, $\mathcal{O}_D$ the integral domain of all holomorphic functions on D, and $\mathcal{O}_D[y_1,\cdots,y_n]$ the polynomial ring in $y_1,\cdots,y_n$ over $\mathcal{O}_D$. A unit in $\mathcal{O}_D$ is a holomorphic function without zeros and a prime element in $\mathcal{O}_D$ is a holomorphic function with a simple zero. Then a holomorphic function with infinitely many zeros, the existence of which is guaranteed by Weierstrass' theorem, cannot be written as a product of finitely many primes. This means that $\mathcal{O}_D$ is not a Unique Factorization Domain (UFD). Therefore $\mathcal{O}_D[y_1,\cdots,y_n]$ is not a UFD. Nevertheless, the following prime factorization theorem holds.

THEOREM 1. (1) *Every irreducible element in* $\mathcal{O}_D[y_1,\cdots,y_n]$ *is a prime element.*

(2) *Any polynomial* $F\in\mathcal{O}_D[y_1,\cdots,y_n]$ *with* $\deg F\geqq 1$ *can be expressed as* $F=aF_1\cdots F_p$, *where* $a\in\mathcal{O}_D-\{0\}$ *and* F_j*'s* $(j=1,\cdots,p)$ *are irreducible polynomials in* $\mathcal{O}_D[y_1,\cdots,y_n]$ *with* $\deg F_j\geqq 1$.

(3) *If a polynomial* $F\in\mathcal{O}_D[y_1,\cdots,y_n]$ *with* $\deg F\geqq 1$ *is decomposed in two ways as* $F=aF_1\cdots F_p=bG_1\cdots G_q$ *in the sense of* (2), *then* a *and* b *are associates* (*i.e. there exists a unit* u *such that* $a=ub$), $p=q$, *and* F_j *and* G_j $(j=1,\cdots,p)$ *are associates by the proper rearrangement of indices.*

We will state a more generalized theorem than Theorem 1 in the Appendix. From now on, we apply the above prime factorization theorem to any polynomial in $\mathcal{O}_D[y_1,\cdots,y_n]$.

Let $F, G\in\mathcal{O}_D[y_1,\cdots,y_n]$. If F and G have no common divisor except units, F and G are said to be relatively prime and we write symbolically as $(F, G)=1$. On the resultant $R(F, G)$, see Theorem IV in the Appendix.

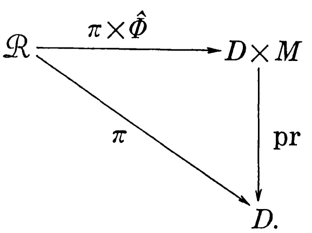
1.3. The extension E_n of (E_n) and the autonomous system A_n.

Let $D\subset C$ be a domain. Consider the following system of rational differential equations of order n:

$$(\mathrm{E}_n) \qquad \begin{cases} \dfrac{dy_1}{dx}=\dfrac{P_1(x, y_1, \cdots, y_n)}{Q_1(x, y_1, \cdots, y_n)} \\ \quad\cdots\cdots \\ \dfrac{dy_n}{dx}=\dfrac{P_n(x, y_1, \cdots, y_n)}{Q_n(x, y_1, \cdots, y_n)}, \end{cases}$$

where $P_l, Q_l\in \mathcal{O}_D[y_1, \cdots, y_n]$, $(P_l, Q_l)=1$ $(l=1, \cdots, n)$. We consider that (E_n) is defined on the total space $D\times C^n$ of a fiber space $\mathfrak{F}_0=(D\times C^n, \mathrm{pr}_0, D)$, where $\mathrm{pr}_0 : D\times C^n\longrightarrow D$ is a natural projection.

A solution of (E_n) which is guaranteed to exist by Cauchy's theorem is called *a local solution of* (E_n). A local solution $\boldsymbol{\phi}$ of (E_n) determines an n-tuple analytic function $\boldsymbol{\Phi}$ on D (See 1.1.). $\boldsymbol{\Phi}$ is called *a global solution of* (E_n). As in 1.1, let $\mathfrak{R}$ denote the Riemann surface determined by $\boldsymbol{\Phi}$. Let M be one of the rational compactifications of C^n, and let $\hat{\Phi} : \mathfrak{R}\longrightarrow M$ be the holomorphic mapping associated with $\boldsymbol{\Phi}$. Then, $\pi\times \hat{\Phi} : \mathfrak{R}\longrightarrow D\times M$, $\boldsymbol{\phi}_a\longrightarrow(a, \hat{\Phi}(\boldsymbol{\phi}_a))$ is an immersion and the following diagram is commutative (See 1.1.):



Therefore a study of the global solution $\boldsymbol{\Phi}$ consists of a study of properties of $\mathfrak{R}$ as a covering surface over D and a study of properties of the immersion $\pi\times\hat{\Phi}$ (in other words, properties of the motion $\pi\times\hat{\Phi}$: $\mathfrak{R}\longrightarrow D\times M$). To study the motion $\pi\times\hat{\Phi}$, we must extend (E_n) to the system on the total space $X=D\times M$ of a fiber space $\mathfrak{F}=(X, \mathrm{pr}, D)$.

Let $\{(U_i, \kappa_i)\}_{i=1}^m$ be an atlas on M which satisfies the conditions (1), (2), (3) in Definition 1.2. Then there exists an atlas $\{(\mathcal{U}_i, \theta_i)\}_{i=1}^m$ on X with the following properties:

A1. $\mathcal{U}_i = D\times U_i$, $\theta_i = \mathrm{id}\times\kappa_i$ $(i=1, \cdots, m)$,
i.e. $\theta_i : D\times U_i \longrightarrow D\times\kappa_i(U_i)$, $(a, b)\longrightarrow(a, \kappa_i(b))$.

In particular, $\theta_1 = \mathrm{id} : \mathcal{U}_1 = D\times C^n \longrightarrow D\times C^n$.

A2. $\theta_j\circ\theta_i^{-1} = \mathrm{id}\times(\kappa_j\circ\kappa_i^{-1}) : (x, y_1^{(i)}, \cdots, y_n^{(i)})\longrightarrow(x, y_1^{(j)}, \cdots, y_n^{(j)})$,
where $y_k^{(j)} = R_k^{ji}(y_1^{(i)}, \cdots, y_n^{(i)}) \in C(y_1^{(i)}, \cdots, y_n^{(i)})$.

From now on, we consider only the atlas which satisfies the above conditions. Note that if two atlases $\{(\mathcal{U}_i, \theta_i)\}_{i=1}^m$, $\{(\mathcal{V}_j, \tau_j)\}_{j=1}^{m'}$ on X are given, the coordinate transition between $\mathcal{U}_i$ and $\mathcal{V}_j$ (where $\mathcal{U}_i\cap\mathcal{V}_j\neq\emptyset$) is rational:

$$\tau_j\circ\theta_i^{-1} : (x, y_1^{(i)}, \cdots, y_n^{(i)})\longrightarrow(x, y_1'^{(j)}, \cdots, y_n'^{(j)}),$$

where $y_k'^{(j)} = Q_k^{ji}(y_1^{(i)}, \cdots, y_n^{(i)}) \in C(y_1^{(i)}, \cdots, y_n^{(i)})$.

Now take an atlas $\{(\mathcal{U}_i, \theta_i)\}_{i=1}^m$ on X. Then (E_n) is regarded as an equation on $\theta_1(\mathcal{U}_1) = D\times C^n$.

DEFINITION 1.7. Consider a system $(\mathrm{E}_n)_i$ of rational differential equations of order n defined on $\theta_i(\mathcal{U}_i)$ $(i=1, \cdots, m)$:

$$(\mathrm{E}_n)_i \qquad \begin{cases} \dfrac{dy_1^{(i)}}{dx} = \dfrac{P_1^{(i)}(x, y_1^{(i)}, \cdots, y_n^{(i)})}{Q_1^{(i)}(x, y_1^{(i)}, \cdots, y_n^{(i)})} \\ \cdots\cdots\cdots \\ \dfrac{dy_n^{(i)}}{dx} = \dfrac{P_n^{(i)}(x, y_1^{(i)}, \cdots, y_n^{(i)})}{Q_n^{(i)}(x, y_1^{(i)}, \cdots, y_n^{(i)})} . \end{cases}$$

The set $\boldsymbol{E}_n = \{(\mathrm{E}_n)_i\}_{i=1}^m$ is called *an extension of* (E_n) *onto* $X = D\times M$ if the following conditions are satisfied.

E1. $P_l^{(i)}, Q_l^{(i)} \in \mathcal{O}_D[y_1^{(i)}, \cdots, y_n^{(i)}]$, $(P_l^{(i)}, Q_l^{(i)}) = 1$ $(l=1, \cdots, n)$.
E2. $(\mathrm{E}_n)_1$ coincides with (E_n).
E3. If $\mathcal{U}_i\cap\mathcal{U}_j\neq\emptyset$, the change of dependent variables $y_k^{(j)} = R_k^{ji}(y_1^{(i)}, \cdots, y_n^{(i)})$ $(k=1, \cdots, n)$ transforms $(\mathrm{E}_n)_i$ into $(\mathrm{E}_n)_j$.

The extension $\boldsymbol{E}_n = \{(\mathrm{E}_n)_i\}_{i=1}^m$ of (E_n) is uniquely constructed. To show this, we prepare some notations.

Notations.

$$y^{(i)}=\begin{pmatrix} y_1^{(i)} \\ \vdots \\ y_n^{(i)} \end{pmatrix}.$$

$$y^{(j)}=\kappa_j\circ\kappa_i^{-1}(y^{(i)})=R^{ji}(y^{(i)})=\begin{pmatrix} R_1^{ji}(y_1^{(i)},\cdots,y_n^{(i)}) \\ \vdots \\ R_n^{ji}(y_1^{(i)},\cdots,y_n^{(i)}) \end{pmatrix}.$$

$$J(\kappa_j\circ\kappa_i^{-1})=J(R^{ji})=\begin{pmatrix} \dfrac{\partial R_1^{ji}}{\partial y_1^{(i)}},\cdots,\dfrac{\partial R_1^{ji}}{\partial y_n^{(i)}} \\ \cdots\cdot \\ \dfrac{\partial R_n^{ji}}{\partial y_1^{(i)}},\cdots,\dfrac{\partial R_n^{ji}}{\partial y_n^{(i)}} \end{pmatrix}.$$

$$\frac{P^{(i)}}{Q^{(i)}}=\begin{pmatrix} \dfrac{P_1^{(i)}(x,y_1^{(i)},\cdots,y_n^{(i)})}{Q_1^{(i)}(x,y_1^{(i)},\cdots,y_n^{(i)})} \\ \vdots \\ \dfrac{P_n^{(i)}(x,y_1^{(i)},\cdots,y_n^{(i)})}{Q_n^{(i)}(x,y_1^{(i)},\cdots,y_n^{(i)})} \end{pmatrix}.$$

If $A, B\in\mathcal{O}_D[y^{(j)}]$,

$$\frac{A}{B}*(\kappa_j\circ\kappa_i^{-1})(y^{(i)})=\frac{A}{B}*R^{ji}(y^{(i)})$$
$$=\frac{A(x,R_1^{ji}(y^{(i)}),\cdots,R_n^{ji}(y^{(i)}))}{B(x,R_1^{ji}(y^{(i)}),\cdots,R_n^{ji}(y^{(i)}))}$$

: the substitution of $R_k^{ji}(y^{(i)})$ for $y_k^{(j)}$ $(k=1,\cdots,n)$.

Since $\mathcal{U}_1=D\times U_1$ is open dense in $X=D\times M$, we see that $\mathcal{U}_1\cap\mathcal{U}_i\neq\emptyset$ for any i. By $y^{(i)}=R^{i1}(y^{(1)})$, we obtain

$$\frac{d}{dx}y^{(i)}=J(R^{i1})\frac{P^{(1)}}{Q^{(1)}}*R^{1i}(y^{(i)}). \tag{1.1}$$

Rewrite the right-hand member of (1.1) so that the condition E1 is satisfied. Then (1.1) becomes $(\mathrm{E}_n)_i$. Apparently $(\mathrm{E}_n)_1$ is (E_n). We will check the condition E3. If $\mathcal{U}_i\cap\mathcal{U}_j\neq\emptyset$, then $\boldsymbol{y}^{(j)}=\boldsymbol{R}^{ji}(\boldsymbol{y}^{(i)})$ transforms $(\mathrm{E}_n)_i$ into the equation

$$\frac{d}{dx}\boldsymbol{y}^{(j)}=J(\boldsymbol{R}^{ji})\frac{\boldsymbol{P}^{(i)}}{\boldsymbol{Q}^{(i)}}*\boldsymbol{R}^{ij}(\boldsymbol{y}^{(j)}). \tag{1.2}$$

Since $\mathcal{U}_1\cap\mathcal{U}_i\cap\mathcal{U}_j\neq\emptyset$, we can rewrite (1.2) as follows:

$$\begin{aligned}\frac{d}{dx}\boldsymbol{y}^{(j)}&=J(\kappa_j\circ\kappa_i^{-1})\frac{d}{dx}\boldsymbol{y}^{(i)}*(\kappa_i\circ\kappa_j^{-1})(\boldsymbol{y}^{(j)})\\&=J(\kappa_j\circ\kappa_i^{-1})J(\kappa_i\circ\kappa_1^{-1})\frac{\boldsymbol{P}^{(1)}}{\boldsymbol{Q}^{(1)}}*(\kappa_1\circ\kappa_i^{-1})\circ(\kappa_i\circ\kappa_j^{-1})(\boldsymbol{y}^{(j)})\\&=J(\kappa_j\circ\kappa_1^{-1})\frac{\boldsymbol{P}^{(1)}}{\boldsymbol{Q}^{(1)}}*(\kappa_1\circ\kappa_j^{-1})(\boldsymbol{y}^{(j)})\\&=J(R^{j1})\frac{\boldsymbol{P}^{(1)}}{\boldsymbol{Q}^{(1)}}*\boldsymbol{R}^{1j}(\boldsymbol{y}^{(j)}).\end{aligned}$$

Therefore, (1.2) coincides with $(\mathrm{E}_n)_j$.

REMARK. Take atlases $\{(\mathcal{U}_i,\theta_i)\}_{i=1}^m$, $\{(\mathcal{U}_j,\tau_j)\}_{j=1}^{m'}$ on X and let $\boldsymbol{E}_n=\{(\mathrm{E}_n)_i\}_{i=1}^m$, $\boldsymbol{E}'_n=\{(\mathrm{E}'_n)_j\}_{j=1}^{m'}$ be extensions of (E_n) onto respective atlases. If $\mathcal{U}_i\cap\mathcal{V}_j\neq\emptyset$, then the change of coordinates $y'^{(j)}_k=Q^{ji}_k(y^{(i)}_1,\cdots,y^{(i)}_n)$ $(k=1,\cdots,n)$ transforms $(\mathrm{E}_n)_i$ into $(\mathrm{E}'_n)_j$.

Next we introduce autonomous systems $(\mathrm{A}_n)_i$ $(i=1,\cdots,m)$ which are used to define the singularity set of (E_n). Rewrite $(\mathrm{E}_n)_i$ as

$$(\mathrm{E}_n)_i\qquad\left\{\begin{array}{l}\dfrac{dy^{(i)}_1}{dx}=\dfrac{P^{(i)}_1}{Q^{(i)}_1}=\dfrac{Y^{(i)}_1}{X^{(i)}}\\ \quad\cdots\cdots\\ \dfrac{dy^{(i)}_n}{dx}=\dfrac{P^{(i)}_n}{Q^{(i)}_n}=\dfrac{Y^{(i)}_n}{X^{(i)}}\end{array}\right.,$$

where $X^{(i)}, Y^{(i)}_1,\cdots,Y^{(i)}_n\in\mathcal{O}_D[\boldsymbol{y}^{(i)}]$, $(X^{(i)},Y^{(i)}_1,\cdots,Y^{(i)}_n)=1$, which is equivalent to $X^{(i)}=\mathrm{l.c.m.}(Q^{(i)}_1,\cdots,Q^{(i)}_n)$. From this, we obtain an autonomous system

$$\frac{dx}{X^{(i)}}=\frac{dy^{(i)}_1}{Y^{(i)}_1}=\cdots=\frac{dy^{(i)}_n}{Y^{(i)}_n}=dt^{(i)}$$

on $\theta_i(\mathcal{U}_i)$.

DEFINITION 1.8. Consider the autonomous system on $\theta_i(\mathcal{U}_i)$:

$$(\mathrm{A}_n)_i \qquad \begin{cases} \dfrac{dx}{dt^{(i)}} = X^{(i)} \\ \dfrac{dy_1^{(i)}}{dt^{(i)}} = Y_1^{(i)} \\ \quad \cdots\cdot \\ \dfrac{dy_n^{(i)}}{dt^{(i)}} = Y_n^{(i)}. \end{cases}$$

$A_n = \{(\mathrm{A}_n)_i\}_{i=1}^m$ is called *the autonomous system associated with* $\boldsymbol{E}_n$.

REMARK. Since $X^{(i)}, Y_1^{(i)}, \cdots, Y_n^{(i)}$ are determined only by the relation $(X^{(i)}, Y_1^{(i)}, \cdots, Y_n^{(i)}) = 1$, they have ambiguities of multiplications by units.

In the following, we study the relationship between $(\mathrm{A}_n)_i$ and $(\mathrm{A}_n)_j$. For that purpose, consider a vector field

$$\mathcal{X}_i = X^{(i)} \frac{\partial}{\partial x} + Y_1^{(i)} \frac{\partial}{\partial y_1^{(i)}} + \cdots + Y_n^{(i)} \frac{\partial}{\partial y_n^{(i)}}$$

defined on $\theta_i(\mathcal{U}_i)$.

PROPOSITION 1.3. *Suppose* $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$. *Let* $\mathcal{X}_{ji}$ *denote the meromorphic vector field on* $\theta_j(\mathcal{U}_j)$ *obtained from* $\mathcal{X}_i$ *by the coordinate transition* $\theta_i \circ \theta_j^{-1}$. *Then,*

$$\mathcal{X}_j = \frac{L_{ji}(x, \boldsymbol{y}^{(j)})}{M_{ji}(x, \boldsymbol{y}^{(j)})} \mathcal{X}_{ji},$$

where $L_{ji}, M_{ji} \in \mathcal{O}_D[\boldsymbol{y}^{(j)}]$ *and* L_{ji}, M_{ji} *have no zeros on* $\theta_j(\mathcal{U}_i \cap \mathcal{U}_j)$.

From this proposition, we obtain the following result.

PROPOSITION 1.4. *Suppose* $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$. *Let* $P \in \mathcal{U}_i \cap \mathcal{U}_j$, $\theta_i(P) = (a, \boldsymbol{b}) \in D \times \kappa_i(U_i)$ *and* $\theta_j(P) = (a, \boldsymbol{b}') \in D \times \kappa_j(U_j)$. *Let* $(x(s), \boldsymbol{\phi}(s))$ *denote the solution of* $(\mathrm{A}_n)_i$ *which is defined on an open neighborhood* $\Delta_i \subset C$ *of* $s=0$ *and satisfies the initial condition* $(x(0), \boldsymbol{\phi}(0)) = (a, \boldsymbol{b})$, *where* $s = t^{(i)}$. *Similarly, let* $(x'(t), \boldsymbol{\phi}'(t))$ *denote the solution of* $(\mathrm{A}_n)_j$ *which is defined on an* *open neighborhood* $\Delta_j \subset \boldsymbol{C}$ *of* $t=0$ *and satisfies the initial condition* $(x'(0), \boldsymbol{\phi}'(0))=(a, \boldsymbol{b}')$, *where* $t=t^{(j)}$. *Then there exist open neighborhoods* $V_i \subset \Delta_i$, $V_j \subset \Delta_j$ *and a biholomorphic mapping* $\xi: V_j \longrightarrow V_i$, $t \longrightarrow s=\xi(t)$ *such that* $\xi(0)=0$ *and* $\theta_j^{-1}(x'(t), \boldsymbol{\phi}'(t))=\theta_i^{-1}(x(\xi(t)), \boldsymbol{\phi}(\xi(t)))$ *for any* $t \in V_j$.

The proofs of these propositions are easy. So we omit them.

REMARK. Let $\boldsymbol{E}_n$, $\boldsymbol{E}_n'$ be the extensions of (E_n) onto atlases $\{(\mathcal{U}_i, \theta_i)\}_{i=1}^m$, $\{(\mathcal{V}_j, \tau_j)\}_{j=1}^{m'}$ on X respectively, and let $\boldsymbol{A}_n=\{(\mathrm{A}_n)_i\}_{i=1}^m$, $\boldsymbol{A}_n'=\{(\mathrm{A}_n')_j\}_{j=1}^{m'}$ be the autonomous systems associated with $\boldsymbol{E}_n$, $\boldsymbol{E}_n'$ respectively. Let $\mathcal{X}_i$ be the vector field on $\theta_i(\mathcal{U}_i)$ defined by $(\mathrm{A}_n)_i$, and let $\mathcal{X}_j'$ be the vector field on $\tau_j(\mathcal{V}_j)$ defined by $(\mathrm{A}_n')_j$. Then, when $\mathcal{U}_i \cap \mathcal{V}_j \neq \emptyset$, $\mathcal{X}_i$ and $\mathcal{X}_j'$ have the same relation as in Proposition 1.3, $(\mathrm{A}_n)_i$ and $(\mathrm{A}_n')_j$ have the same relation as in Proposition 1.4.

DEFINITION 1.9. The fiber space $\mathcal{F}=(X, \mathrm{pr}, D)$ is called *a definition space of* (E_n) when on X the extension $\boldsymbol{E}_n$ of (E_n) and the autonomous system $\boldsymbol{A}_n$ are defined. The total space X is called *a definition manifold of* (E_n).

Note that if $n \geqq 2$, then there exist various definition spaces of (E_n) according to rational compactifications of $\boldsymbol{C}^n$.

1.4. The singular initial set $\mathcal{S}$, the fixed singularity set $\boldsymbol{\Theta}$ and movable singularities of (E_n).

Let $\mathcal{F}=(X, \mathrm{pr}, D)$ be one of definition spaces of (E_n), $\{(\mathcal{U}_i, \theta_i)\}_{i=1}^m$ an atlas on $X=D \times M$, $\boldsymbol{E}_n$ the extension of (E_n), and $\boldsymbol{A}_n$ the autonomous system associated with $\boldsymbol{E}_n$. As is stated in the definition of atlases on X, $\mathcal{U}_i=D \times U_i$ and $\theta_i=\mathrm{id} \times \kappa_i$, where $\{(U_i, \kappa_i)\}_{i=1}^m$ is an atlas on M.

DEFINITION 1.10. Let

$$\mathrm{pr}_i: \theta_i(\mathcal{U}_i)=D \times \kappa_i(U_i) \longrightarrow D, \ (a, \boldsymbol{b}) \longrightarrow a.$$

We define $\mathcal{S}_i$ to be the set of points $(a, \boldsymbol{b}) \in \theta_i(\mathcal{U}_i)$ such that the solution $(x(t^{(i)}), \boldsymbol{\phi}(t^{(i)}))$ of $(\mathrm{A}_n)_i$ defined on a set $\Delta \subset \boldsymbol{C}$ and passing $(a, \boldsymbol{b})$ satisfies the condition $\mathrm{pr}_i(\{(x(t^{(i)}), \boldsymbol{\phi}(t^{(i)}))| \ t^{(i)} \in \Delta\})=\{a\}$.

It follows from Proposition 1.4 that $\theta_i^{-1}(\mathcal{S}_i)=\theta_j^{-1}(\mathcal{S}_j)$ on $\mathcal{U}_i \cap \mathcal{U}_j$ if $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$.

DEFINITION 1.11. We set

$$\mathcal{S}=\bigcup_{i=1}^{m}\theta_i^{-1}(\mathcal{S}_i).$$

$\mathcal{S}$ is called *the singular initial set of* (E_n) *in* X.

REMARK. The Remark after Proposition 1.4 tells us that $\mathcal{S}$ is uniquely determined for the pair $((\mathrm{E}_n), X)$ independent of the way of extending (E_n) (i.e. the choice of an atlas on X). However, $\mathcal{S}$ depends on the definition manifold X. Strictly speaking, $\mathcal{S}_i$'s depend on the rational compactification M of $\boldsymbol{C}^n$ except $\mathcal{S}_1$.

The sets $\mathcal{S}_i$ $(i=1, \cdots, m)$ and $\mathcal{S}$ have the following properties,

PROPOSITION 1.5. (1) *$\mathcal{S}_i$ is an analytic set in $\theta_i(\mathcal{U}_i)\subset D\times\boldsymbol{C}^n$.*
(2) *$\mathcal{S}$ is an analytic set in $X=D\times M$.*

PROOF. For the sake of simplicity, we omit the suffix (i) of the variables $t^{(i)}$, $y_l^{(i)}$ and the polynomials $X^{(i)}$, $Y_l^{(i)}$ of $(\mathrm{A}_n)_i$.

Let $(a, \boldsymbol{b})\in\theta_i(\mathcal{U}_i)\subset D\times\boldsymbol{C}^n$ and consider an equation

$$(1.3)\qquad \begin{cases} \dfrac{dy_1(t)}{dt}=Y_1(a, y_1(t), \cdots, y_n(t)) \\ \quad\cdots\cdots\cdots \\ \dfrac{dy_n(t)}{dt}=Y_n(a, y_1(t), \cdots, y_n(t)) . \end{cases}$$

Let $\boldsymbol{y}(t)=(y_1, \cdots, y_n)(t)$ be the solution of (1.3) which is defined on an open neighborhood $\varDelta\subset C$ of $t=0$ and satisfies the initial condition $\boldsymbol{y}(0)=(y_1, \cdots, y_n)(0)=\boldsymbol{b}$, and let $X(t)$ denote the function $X(a, y_1(t), \cdots, y_n(t))$ in t. From the definition of $\mathcal{S}_i$, we see that $(a, \boldsymbol{b})\in\mathcal{S}_i$ if and only if $X(t)\equiv 0$ on $\varDelta$. Apparently the condition "$X(t)\equiv 0$ on $\varDelta$" is equivalent to the condition:

$$(1.4)\qquad \left.\frac{d^lX(t)}{dt^l}\right|_{t=0}=0 \text{ for any nonnegative integer } l.$$

Moreover, (1.4) is equivalent to the following sequence of equalities:

$$(1.5)\qquad \begin{cases} X(a, \boldsymbol{b})=0, \\ (X_{y_1}Y_1+\cdots+X_{y_n}Y_n)(a, \boldsymbol{b})=0, \\ [(X_{y_1y_1}Y_1+\cdots+X_{y_1y_n}Y_n)Y_1+X_{y_1}((Y_1)_{y_1}Y_1+\cdots+(Y_1)_{y_n}Y_n)+\cdots \\ \quad +X_{y_n}((Y_n)_{y_1}Y_1+\cdots+(Y_n)_{y_n}Y_n)](a, \boldsymbol{b})=0, \\ \qquad\vdots \end{cases}$$

Therefore $\mathcal{S}_i$ is expressed as

$$\mathcal{S}_i=\{(a, b)\in\theta_i(\mathcal{U}_i)\mid X(a, b)=0,\ Z_l(a, b)=0\ \ (l=1, 2, 3, \cdots)\},$$

where $Z_l\in\mathcal{O}_D[y_1, \cdots, y_n]$ denotes the $(l+1)$-th polynomial of the sequence in (1.9). This means that $\mathcal{S}_i$ is an analytic set in $\theta_i(\mathcal{U}_i)\subset D\times C^n$ (Refer to [4]).

(2) For any i, $\theta_i(\mathcal{S}\cap\mathcal{U}_i)=\mathcal{S}_i$. Then, by (1), $\mathcal{S}$ is an analytic set in $X=D\times M$. q.e.d.

REMARK. When we study $\mathcal{S}$ more concretely, we divide $\mathcal{S}$ into the following two subsets S and T (See § 2, § 3.):

$$S_i=\{(a, b)\in\theta_i(\mathcal{U}_i)\mid X^{(i)}(a, b)=Y_1^{(i)}(a, b)=\cdots=Y_n^{(i)}(a, b)=0\},$$

$$T_i=\mathcal{S}_i-S_i,$$

$$S=\bigcup_{i=1}^{m}\theta_i^{-1}(S_i),$$

$$T=\bigcup_{i=1}^{m}\theta_i^{-1}(T_i)=\mathcal{S}-S.$$

$S_i\subset\theta_i(\mathcal{U}_i)$ is the singularity set of the vector field $\mathcal{X}_i$ (See 1.3.). From Proposition 1.4, we see that $\theta_i^{-1}(S_i)=\theta_j^{-1}(S_j)$, $\theta_i^{-1}(T_i)=\theta_j^{-1}(T_j)$ on $\mathcal{U}_i\cap\mathcal{U}_j$ if $\mathcal{U}_i\cap\mathcal{U}_j\neq\emptyset$. Furthermore, the Remark after Proposition 1.4 ensure that S and T are uniquely determined for the pair $((\mathrm{E}_n), X)$ independent of the extension $\boldsymbol{E}_n$ of (E_n).

Suppose that $\mathcal{S}$ is decomposed into irreducible components as

$$\mathcal{S}=\bigcup_{\sigma\in\Lambda_1}\mathcal{S}_\sigma^{(1)}\cup\cdots\cup\bigcup_{\sigma\in\Lambda_k}\mathcal{S}_\sigma^{(k)}\cup\cdots\cup\bigcup_{\sigma\in\Lambda_{n+1}}\mathcal{S}_\sigma^{(n+1)},$$

where some of Λ_k's $(k=1, \cdots, n+1)$ may be empty sets, $\mathrm{codim}_C\mathcal{S}_\sigma^{(k)}=k$, $\{\mathcal{S}_\sigma^{(k)}\}_{k,\sigma}$ is locally finite, and $\mathcal{S}_\sigma^{(k)}\not\subset\bigcup_{(l,\tau)\neq(k,\sigma)}\mathcal{S}_\tau^{(l)}$ for any $\mathcal{S}_\sigma^{(k)}$. Then we have the following

PROPOSITION 1.6. *For any* $\mathcal{S}_\sigma^{(k)}$, $\mathrm{pr}(\mathcal{S}_\sigma^{(k)})$ *is either a single point set or the domain* D.

PROOF. Let A denote the set $\mathrm{pr}(\mathcal{S}_\sigma^{(k)})$. Since M is compact, the projection $\mathrm{pr}: X=D\times M\longrightarrow D$ is a proper holomorphic mapping. Then, by a theorem of Remmert (See [4].), A is an irreducible analytic set in D. If $\dim_C A=0$, then A consists of one point. If $\dim_C A=1$, then $A=D$. q.e.d.

DEFINITION 1.12. (1) If $\mathrm{pr}(\mathcal{S}_\sigma^{(k)})$ consists of one point, $\mathcal{S}_\sigma^{(k)}$ is called *a vertical singularity set of the k-th kind of* (E_n). If $\mathrm{pr}(\mathcal{S}_\sigma^{(k)})=D$, $\mathcal{S}_\sigma^{(k)}$ is called *a covering singularity set of the k-th kind of* (E_n).

(2) In the base space D, take the following subsets:

$$\Theta_1=\{\xi\in D|\ \text{for some } \mathcal{S}_\sigma^{(1)},\ \mathrm{pr}(\mathcal{S}_\sigma^{(1)})=\{\xi\}\},$$
$$\Theta_k=\{\xi\in D-\Theta_1\cup\cdots\cup\Theta_{k-1}|\ \text{for some } \mathcal{S}_\sigma^{(k)},\ \mathrm{pr}(\mathcal{S}_\sigma^{(k)})=\{\xi\}\}\quad(k=2,\cdots,n+1),$$
$$\Theta=\Theta_1\cup\cdots\cup\Theta_{n+1}.$$

Θ_k is called *the fixed singularity set of the k-th kind of* (E_n), and Θ is called *the fixed singularity set of* (E_n). A point in Θ_k is called *a fixed singularity of the k-th kind of* (E_n), and a point in Θ is called *a fixed singularity of* (E_n).

REMARK 1. As is noted in the Remark after Definition 1.11, $\mathcal{S}$ is uniquely determined for the pair $((\mathrm{E}_n), X)$. Then vertical singularity sets, covering singularity sets, fixed singularity sets are uniquely determined for the pair $((\mathrm{E}_n), X)$.

REMARK 2. By the definition, vertical singularity sets and fixed singularity sets are classified into $n+1$ types respectively (from the first kind to the $(n+1)$-th kind), and covering singularity sets are classified into n types (from the first kind to the n-th kind).

REMARK 3. If (E_n) is an autonomous system, then $(\mathrm{E}_n)_i$'s $(i=1,\cdots,m)$ are autonomous systems. Therefore, in this case, $\mathcal{S}$ consists of the union of covering singularity sets, and $\Theta=\varnothing$.

REMARK 4. Suppose that (E_n) is a system of linear differential equations:

$$(\mathrm{L}_n)\qquad \begin{cases} \dfrac{dy_1}{dx}=\dfrac{1}{b_1(x)}\displaystyle\sum_{p=1}^{n}a_{1,p}(x)y_p \\ \cdots\cdots \\ \dfrac{dy_n}{dx}=\dfrac{1}{b_n(x)}\displaystyle\sum_{p=1}^{n}a_{n,p}(x)y_p\,, \end{cases}$$

where $a_{l,p}(x),\ b_l(x)\in\mathcal{O}_D,\ \left(\sum_{p=1}^{n}a_{l,p}(x)y_p,\ b_l(x)\right)=1$. In this case, it is sufficient to take the fiber space $\mathcal{F}_0=(D\times C^n, \mathrm{pr}_0, D)$ as a definition space of (E_n). Therefore,

$$\mathcal{S}=\mathcal{S}_1=\{\xi\in D|\ b(\xi)=0\}\times C^n,$$

where $b(x)=\prod_{l=1}^{n} b_l(x)$, and

$$\Theta=\Theta_1=\{\xi\in D|\ b(\xi)=0\}.$$

The fixed singularity set Θ and covering singularity sets have the following good properties.

PROPOSITION 1.7. *Θ is a discrete set in D.*

PROOF. Let $\{\mathcal{S}_\sigma\}$ denote all the vertical singularity sets. Since $\{\mathcal{S}_\sigma\}$ is locally finite, Θ is discrete in D. q.e.d.

PROPOSITION 1.8. *The number of covering singularity sets is finite.*

PROOF. Let $\{\mathcal{S}_\sigma\}_{\sigma\in\Lambda}$ denote all the covering singularity sets. Take a point $a\in D-\Theta$ and set $M_a=\{a\}\times M$. Then for any $\sigma\in\Lambda$, $M_a\cap\mathcal{S}_\sigma\neq\emptyset$. From the locally finiteness of $\{\mathcal{S}_\sigma\}_{\sigma\in\Lambda}$, for any $P\in M_a$, there exists an open neighborhood $\Delta_P\subset X$ of P such that $\{\sigma|\ \Delta_P\cap\mathcal{S}_\sigma\neq\emptyset\}$ is a finite set. $\{\Delta_P\}_{P\in M_a}$ makes an open covering of M_a. Since M_a is compact, M_a is covered by finite numbers of $\{\Delta_P\}_{P\in M_a}$. Therefore, $\Lambda=\{\sigma|\ M_a\cap\mathcal{S}_\sigma\neq\emptyset\}$ is a finite set. q.e.d.

DEFINITION 1.13. If a global solution Φ of (E_n) has a singularity on $a\in D-\Theta$, a is called *a movable singularity of Φ.*

1.5. Fundamental theorem.

We will study relationships among the singular initial set $\mathcal{S}$, the fixed singularity set Θ and singularities of solutions of (E_n).

DEFINITION 1.14. Let $\mathcal{F}=(X, \mathrm{pr}, D)$ be a definition space of (E_n) and $\mathcal{S}$ the singular initial set in X. If $P\in X-\mathcal{S}$, then P is called *an ordinary initial point* (briefly, *an ordinary point*) *in X of* (E_n).

The definition of $\mathcal{S}$ shows us that for any ordinary initial point P, there exists a unique local solution of E_n which passes P. Furthermore, we obtain

PROPOSITION 1.9. *Let $P=(a, \boldsymbol{b})\in X=D\times M$ be an ordinary initial* *point. Then there exist an open neighborhood $\Delta_a \subset D$ of a and an open neighborhood $\Delta_b \subset M$ of $\boldsymbol{b}$ which satisfies the following conditions*:

O1. *$\Delta_a \times \Delta_b \subset X - \mathcal{S}$ and $\Delta_a \times \Delta_b$ is contained in some chart $\mathcal{U}_i$.*

O2. *For any $(x_0, \boldsymbol{y}_0) \in \Delta_a \times \kappa_i(\Delta_b) \subset \theta_i(\mathcal{U}_i)$, let $\boldsymbol{y}(x) = \boldsymbol{\phi}(x; x_0, \boldsymbol{y}_0)$ be the solution of $(\mathrm{E}_n)_i$ which passes $(x_0, \boldsymbol{y}_0)$. Then $\boldsymbol{\phi}(x; x_0, \boldsymbol{y}_0)$ is an n-tuple algebroidal function on Δ_a.*

PROOF. Suppose $P \in \mathcal{U}_i - \mathcal{S}$. Since $\mathcal{S}_i = \theta_i(\mathcal{U}_i \cap \mathcal{S})$ is a closed set in $\theta_i(\mathcal{U}_i)$, there is a neighborhood $W_1 \times W_2 \subset \theta_i(\mathcal{U}_i) \subset D \times \boldsymbol{C}^n$ of $\theta_i(P) = (a, \boldsymbol{b}^{(i)})$ such that $W_1 \times W_2 \subset \theta_i(\mathcal{U}_i) - \mathcal{S}_i$, where $W_1 \subset D$ is an open neighborhood of a, $W_2 \subset \boldsymbol{C}^n$ is an open neighborhood of $\boldsymbol{b}^{(i)}$.

For any $(x_0, \boldsymbol{y}_0) \in W_1 \times W_2$, let $(x, \boldsymbol{y}) = \boldsymbol{\phi}(t; x_0, y_0) = (\phi_0, \phi_1, \cdots, \phi_n)(t; x_0, \boldsymbol{y}_0)$ be the solution of $(\mathrm{A}_n)_i$ which satisfies the initial condition $x(0) = x_0$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$. Then $\boldsymbol{\phi}$ is holomorphic in the domain $W \times W_1 \times W_2 \subset \boldsymbol{C}^{n+2}$, where $W = \{t \mid |t| < R\}$ with a positive constant R.

Now set $F(x, t, x_0, \boldsymbol{y}_0) = x - \phi_0(t; x_0, \boldsymbol{y}_0)$. Then F is holomorphic in $W_1 \times W \times W_1 \times W_2$ and $F(a, 0, a, \boldsymbol{b}^{(i)}) = 0$. Since $P = (a, \boldsymbol{b})$ is an ordinary point, we have $F(a, t, a, \boldsymbol{b}^{(i)}) = a - \phi_0(t; a, \boldsymbol{b}^{(i)}) \not\equiv 0$. Therefore, by the Weierstrass preparation theorem, F is written as

$$\begin{aligned} F(x, t, x_0, \boldsymbol{y}_0) &= x - \phi_0(t; x_0, \boldsymbol{y}_0) \\ &= G(x, t, x_0, \boldsymbol{y}_0) H(x, t, x_0, \boldsymbol{y}_0) \end{aligned}$$

in an open neighborhood $U = \Delta_1 \times \Delta \times \Delta_1 \times \Delta_2 \subset W_1 \times W \times W_1 \times W_2$ of $(a, 0, a, \boldsymbol{b}^{(i)})$. Here the following conditions are satisfied:

(1) $G(x, t, x_0, \boldsymbol{y}_0) = t^k + g_1(x, x_0, \boldsymbol{y}_0) t^{k-1} + \cdots + g_k(x, x_0, \boldsymbol{y}_0)$.

(2) k is the order of zero of $F(a, t, a, \boldsymbol{b}^{(i)})$ at $t = 0$.

(3) $g_1, \cdots, g_k$ are holomorphic in $U' = \Delta_1 \times \Delta_1 \times \Delta_2$.

(4) For any $(\bar{x}, \bar{x}_0, \overline{\boldsymbol{y}_0}) \in \Delta_1 \times \Delta_1 \times \Delta_2$, $G(\bar{x}, t, \overline{x_0}, \overline{\boldsymbol{y}_0})$ has k zeros in Δ with multiplicity.

(5) $H(x, t, x_0, \boldsymbol{y}_0)$ is holomorphic and has no zeros in U.

Take a point $(\bar{x}_0, \overline{\boldsymbol{y}_0}) \in \Delta_1 \times \Delta_2$ and consider the set of zeros in $\Delta_1 \times \Delta$ of $F(x, t, \overline{x_0}, \overline{\boldsymbol{y}_0})$. We can easily infer that $G(x, t, \overline{x_0}, \overline{\boldsymbol{y}_0}) = \{G_1(x, t)\}^q$, where $G_1(x, t) = t^p + g_{1,1}(x) t^{p-1} + \cdots + g_{1,p}(x)$ is an irreducible polynomial in $\mathcal{O}_{\Delta_1}[t]$, and p, q are positive integers with $pq = k$. In addition, we find that the inverse function of $x = \phi_0(t; \overline{x_0}, \overline{\boldsymbol{y}_0})$ is a p-valued algebroidal function $t = \phi_0(x)$ in Δ_1 which is determined by the equation $G_1(x, t) = 0$. Since $\phi_0(\Delta_1) \subset \Delta \subset W$ and $\boldsymbol{y} = (\phi_1, \cdots, \phi_n)(t; \overline{x_0}, \overline{\boldsymbol{y}_0})$ is defined in W, $\boldsymbol{y} = \boldsymbol{\phi}(x; \overline{x_0}, \overline{\boldsymbol{y}_0}) =$

$(\psi_1, \cdots, \psi_n)(\phi_0(x); \overline{x_0}, \overline{y_0})$ is an n-tuple algebroidal function in Δ_1 and is a local solution of $(\mathrm{E}_n)_i$ which passes $(\overline{x_0}, \overline{y_0})$. Therefore, if we set $\Delta_a = \Delta_1$, $\Delta_b = \kappa_i^{-1}(\Delta_2)$, then Δ_a and Δ_b are the desired neighborhoods. q.e.d.

REMARK. Let $P=(a, \boldsymbol{b}) \in \mathcal{U}_i$ be an ordinary initial point, and let $\boldsymbol{\phi}(x; P)$ be a local solution of $(\mathrm{E}_n)_i$ which is defined on a neighborhood Δ_a of a and passes $\theta_i(P)=(a, \boldsymbol{b}^{(i)})$. In the case $i=1$, $\boldsymbol{\phi}(x; P)$ is a solution of (E_n). Suppose $i \neq 1$. In this case, if $\{\theta_i^{-1}(x, \boldsymbol{\phi}(x; P)) \mid x \in \Delta_a\} \subset X - \mathcal{U}_1$, then $\boldsymbol{\phi}(x; P)$ cannot be continued to any solution of (E_n), and if $\{\theta_i^{-1}(x, \boldsymbol{\phi}(x; P)) \mid x \in \Delta_a\} \not\subset X - \mathcal{U}_1$, then $\boldsymbol{\phi}(x) = \boldsymbol{R}^{1i}(\boldsymbol{\phi}(x; P))$ is a solution of (E_n).

On an analytic continuation of a local solution of (E_n), we have the following result.

PROPOSITION 1.10. *Let $l : [\alpha, \beta) \longrightarrow D$ be a curve which starts from $l(\alpha) = x_0$ and converges to $a \in D$. Let $\boldsymbol{\phi}(x)$ be a local solution of* (E_n) *holomorphic in a neighborhood $\Delta \subset D$ of x_0. Suppose that $Q_k(x, \boldsymbol{\phi}(x))$'s $(k=1, \cdots, n)$ have no zeros in Δ, and that $\boldsymbol{\phi}(x)$ is holomorphically continuable along l and determines a point $\boldsymbol{\omega}$ on a, where $\boldsymbol{\omega}$ is either a singular point or a nonsingular point* (*See* 1.1, 3°.). *Then there exists a slightly deformed curve l' of l which satisfies the following conditions*:

C1. $l' : [\alpha, \beta) \longrightarrow D,\ l'(\alpha) = x_0,\ \lim_{s \to \beta} l'(s) = a.$

C2. *$\boldsymbol{\phi}$ is holomorphically continuable along l' and determines the same point $\boldsymbol{\omega}$ on a.*

C3. *Let $\{\boldsymbol{\phi}_{l'(s)}\}_{s \in [\alpha, \beta)}$ be the holomorphic continuation of $\boldsymbol{\phi}$ along l' and $(\boldsymbol{\phi}_s, \Delta_s)$ a representative of $\boldsymbol{\phi}_{l'(s)}$, where Δ_s is a sufficiently small open neighborhood of $l'(s)$. Then,* (1) *$\boldsymbol{\phi}_s$ is holomorphic in Δ_s and $Q_k(x, \boldsymbol{\phi}_s(x))$'s $(k=1, \cdots, n)$ have no zeros in Δ_s,* (2) *$\boldsymbol{\phi}_s$ is a local solution of* (E_n), (3) *for any $s \in [\alpha, \beta)$, $\{(x, \boldsymbol{\phi}_s(x)) \mid x \in \Delta_s\} \cap \mathcal{S} = \varnothing$.*

PROOF. Let $\{\boldsymbol{\phi}_{l(s)}\}_{s \in [\alpha, \beta)}$ be the holomorphic continuation of $\boldsymbol{\phi}$ along l and $(\boldsymbol{\phi}_s, D_s)$ a representative of $\boldsymbol{\phi}_{l(s)}$, where D_s is an open neighborhood of $l(s)$. From the assumption on $\boldsymbol{\phi}$, it follows that $Q_k(x, \boldsymbol{\phi}_s(x)) \neq 0$ $(k=1, \cdots, n)$ in D_s for any $s \in [\alpha, \beta)$. Then the set $Z = \{s \in [\alpha, \beta) \mid$ for some k, $Q_k(l(s), \boldsymbol{\phi}_s(l(s))) = 0\}$ is a discrete subset in $[\alpha, \beta)$. Therefore we can deform l into a curve $l' : [\alpha, \beta) \longrightarrow D$ on which there exists a holomorphic continuation of $\boldsymbol{\phi}$ satisfying the conditions C1, C2, C3. q.e.d.

From Proposition 1.9 and 1.10, we obtain the following theorem on relationships between the singular initial set $\mathcal{S}$ and a singularity of a global solution of (E_n).

THEOREM 2 (Fundamental theorem). *Let* $\boldsymbol{\Phi}$ *be a global solution of* (E_n), $\boldsymbol{\omega}$ *a singularity of* $\boldsymbol{\Phi}$ *on* $a\in D$ *and* $S_{\omega}\subset M$ *the cluster set of* $\boldsymbol{\Phi}$ *at* $\boldsymbol{\omega}$. *Set* $\mathcal{S}_a=\mathrm{pr}^{-1}(a)\cap\mathcal{S}$.

(1) *If* $\{a\}\times S_{\omega}$ *contains an ordinary initial point* $P\in\{a\}\times M$, *then* $\{a\}\times S_{\omega}=\{P\}$ *and* $\boldsymbol{\omega}$ *is an algebraic singularity.*

(2) *If* $\boldsymbol{\omega}$ *is a transcendental singularity,* $\{a\}\times S_{\omega}\subset\mathcal{S}_a$.

PROOF. (1) Let $\boldsymbol{\phi}$ be a local solution of (E_n) which determines $\boldsymbol{\Phi}$. We may assume that $\boldsymbol{\phi}$ is holomorphic in an open neighborhood Δ of a certain point $x_0\in D$, and that $Q_k(x,\boldsymbol{\phi}(x))$'s $(k=1,\cdots,n)$ have no zeros in Δ. Then, by Proposition 1.10, there exists a curve $l:[\alpha,\beta)\longrightarrow D$ which starts from $l(\alpha)=x_0\in D$ and converges to a and satisfies the conditions C2, C3 in Proposition 1.10.

If $\{a\}\times S_{\omega}$ contains an ordinary initial point $P=(a,\boldsymbol{b})\in U_i$, then there exists an open neighborhood $\Delta_a\times\Delta_b\subset D\times M$ of P which satisfies the conditions O1, O2 in Proposition 1.9, where $\Delta_a=\{x|\ |x-a|<r\}$ with $r>0$. Moreover since $P=(a,\boldsymbol{b})\in\{a\}\times S_{\omega}$, it follows from Proposition 1.2 (1) and Proposition 1.10 that in the holomorphic r-neighborhood $\mathcal{U}_r$ of $\boldsymbol{\omega}$ there exists a point $\boldsymbol{\phi}_c$ a representative $(\boldsymbol{\phi},\Delta_c)$ of which satisfies the following conditions:

(i) $\boldsymbol{\phi}_c$ is the terminal germ of a holomorphic continuation of some $\boldsymbol{\phi}_{l(s)}\in\mathcal{U}_r$ along a certain curve l' in Δ_a.

(ii) $\boldsymbol{\phi}$ is holomorphic in an open neighborhood Δ_c of $c\in\Delta_a$.

(iii) $\{(x,\boldsymbol{\phi}(x))|\ x\in\Delta_c\}\subset\Delta_a\times\Delta_b$.

(iv) $\boldsymbol{\phi}$ is a local solution of (E_n).

Since $\boldsymbol{R}^{i1}(\boldsymbol{\phi}(x))$ is a solution of $(\mathrm{E}_n)_i$ which passes $(c,\kappa_i(\boldsymbol{\phi}(c)))\in\Delta_a\times\kappa_i(\Delta_b)$, it follows from Proposition 1.9 that $\boldsymbol{R}^{i1}(\boldsymbol{\phi}(x))$ is an n-tuple algebroidal function in Δ_a. Then $\boldsymbol{\phi}(x)$ is also an n-tuple algebroidal function in Δ_a and $\boldsymbol{\omega}$ is an algebraic singularity. Therefore $\{a\}\times S_{\omega}=\{P\}$.

(2) It is obvious from (1). q.e.d.

REMARK. Theorem 2 (2) tells us that if a l-parameter family $\boldsymbol{\Phi}(x;C_1,\cdots,C_l)$ $(1\leqq l\leqq n)$ of solutions of (E_n) has a transcendental singularity $\boldsymbol{\omega}(C_1,\cdots,C_l)$ on $x=a(C_1,\cdots,C_l)\in D-\boldsymbol{\Theta}$ with the cluster set $S_{\omega}(C_1,\cdots,C_l)$ and $a(C_1,\cdots,C_l)$ moves depending on $C_1,\cdots,C_l$, then the set $\{a(C_1,\cdots,C_l)\}\times S_{\omega}(C_1,\cdots,C_l)$ moves in the covering singularity sets of (E_n).

Theorem 2 is the most fundamental theorem on singularities of solutions of (E_n). In fact, we find that Painlevé's theorem for (E_1) (Theorem α in § 0), the next theorem 3 for (E_n) and Kimura's theorem for (F_2) (Theorem 7 in § 4) are corollaries of Theorem 2.

THEOREM 3. *Let Φ be a global solution of* (E_n).

(1) *Suppose that* (E_n) *has no covering singularity sets. If Φ has a singularity $\boldsymbol{\omega}$ on $a\in D-\boldsymbol{\Theta}$, then $\boldsymbol{\omega}$ is an algebraic singularity. If Φ has a singularity $\boldsymbol{\omega}$ on $\xi\in\boldsymbol{\Theta}_{n+1}$, then $\boldsymbol{\omega}$ is either an algebraic singularity or an ordinary transcendental singularity.* (*Hence, if Φ has an essential singularity $\boldsymbol{\omega}$ on $\xi\in\boldsymbol{\Theta}$, then $\xi\in\boldsymbol{\Theta}_1\cup\cdots\cup\boldsymbol{\Theta}_n$.*)

(2) *Suppose that* (E_n) *has no covering singularity sets from the first kind to the* $(n-1)$*-th kind. If Φ has a singularity $\boldsymbol{\omega}$ on $a\in D-\boldsymbol{\Theta}$ or on $\xi\in\boldsymbol{\Theta}_{n+1}$, then $\boldsymbol{\omega}$ is either an algebraic singularity or an ordinary transcendental singularity.* (*Hence, if Φ has an essential singularity $\boldsymbol{\omega}$ on $\xi\in\boldsymbol{\Theta}$, then $\xi\in\boldsymbol{\Theta}_1\cup\cdots\cup\boldsymbol{\Theta}_n$.*)

PROOF. (1) Let $S_{\boldsymbol{\omega}}$ denote the cluster set of Φ at $\boldsymbol{\omega}$. Since (E_n) has no covering singularity sets, $\mathcal{S}\subset\boldsymbol{\Theta}\times M$. Hence, if $a\in D-\boldsymbol{\Theta}$, then $\{a\}\times S_{\boldsymbol{\omega}}$ contains an ordinary initial point. By Theorem 1 (1), $\boldsymbol{\omega}$ on a is an algebraic singularity. If $\xi\in\boldsymbol{\Theta}_{n+1}$, then $\mathcal{S}_\xi=\mathrm{pr}^{-1}(\xi)\cap\mathcal{S}=\bigcup_{\sigma\in\Lambda'}\{P_\sigma^{(n+1)}\}$, where $\{P_\sigma^{(n+1)}\}$ is a vertical singularity set of the $(n+1)$-th kind such that $\mathrm{pr}(P_\sigma^{(n+1)})=\{\xi\}$, and Λ' is a finite set. Therefore, if $\boldsymbol{\omega}$ on ξ is not an algebraic singularity, then there exists a point $P_\sigma^{(n+1)}\in\mathcal{S}_\xi$ such that $\{\xi\}\times S_{\boldsymbol{\omega}}=\{P_\sigma^{(n+1)}\}$. Hence $\boldsymbol{\omega}$ is an ordinary transcendental singularity.

(2) By the assumption, the covering singularity sets of (E_n), if exist, are of the n-th kind. Let them be $\{\mathcal{S}_\sigma\}_{\sigma\in\Lambda}$, where Λ is a finite set. Suppose that a is a point in $D-\boldsymbol{\Theta}$. If we set $M_a=\mathrm{pr}^{-1}(a)$ $(=\{a\}\times M)$, then $M_a=\{(x,y)\mid x-a=0\}$ is an irreducible analytic set and $M_a\not\supset\mathcal{S}_\sigma$ for any $\sigma\in\Lambda$. Hence $M_a\cap\mathcal{S}_\sigma=\bigcup_{\tau\in\Gamma_\sigma}\{P_{\sigma,\tau}\}$, where $P_{\sigma,\tau}$ is a point in M_a and Γ_σ is a finite set. Therefore $\mathcal{S}_a=M_a\cap\mathcal{S}=\bigcup_{\sigma\in\Lambda}\bigcup_{\tau\in\Gamma_\sigma}\{P_{\sigma,\tau}\}$ is a finite set. If a singularity $\boldsymbol{\omega}$ of Φ on a is not an algebraic singularity, then there exists a point $P_{\sigma,\tau}\in\mathcal{S}_a$ such that $\{a\}\times S_{\boldsymbol{\omega}}=\{P_{\sigma,\tau}\}$. Therefore $\boldsymbol{\omega}$ is an ordinary transcendental singularity. Let ξ be a point in $\boldsymbol{\Theta}_{n+1}$ and $M_\xi=\mathrm{pr}^{-1}(\xi)$ $(=\{\xi\}\times M)$, then $M_\xi\cap\mathcal{S}_\sigma=\bigcup_{\tau\in\Delta_\sigma}\{Q_{\sigma,\tau}\}$, where $Q_{\sigma,\tau}$ is a point in M_ξ and Δ_σ is a finite set. Then $\mathcal{S}_\xi=M_\xi\cap\mathcal{S}=\bigcup_{\sigma\in\Lambda}\bigcup_{\tau\in\Gamma_\sigma}\{Q_{\sigma,\tau}\}\cup\bigcup_{\sigma\in\Lambda'}\{P_\sigma^{(n+1)}\}$ is a finite set, where $\{P_\sigma^{(n+1)}\}$ is a vertical singularity set of the $(n+1)$-th

kind such that $\mathrm{pr}(P_\sigma^{(n+1)})=\{\xi\}$. Therefore, if a singularity $\boldsymbol{\omega}$ on ξ is not an algebraic singularity, $\{\xi\}\times S_\omega=\{\text{a point}\}\subset\mathcal{S}_\xi$ and $\boldsymbol{\omega}$ is an ordinary transcendental singularity. q.e.d.

REMARK. As will be shown in Proposition 2.1 (1) in § 2, (E_1) has no covering singularity sets. On the other hand, if $n\geqq 2$, then (E_n) has covering singularity sets in general. Moreover, the larger n becomes, the more various kinds of covering singularity sets (E_n) has. So, by the Remark after Theorem 2, the larger n becomes, the more possibility there is that (E_n) has movable transcendental singularities.

§ 2. Fixed and movable singularities of (E_1).

For the equation (E_1), we have two ways to define the fixed singularity set $\boldsymbol{\Theta}$. One is the classical definition by Painlevé (See § 0.) and the other is the new definition stated in Definition 1.12. In this section, we shall show that these two definitions coincide and that Painlevé's theorem (Theorem α in § 0) is a corollary of Theorem 3.

Let $D\subset\boldsymbol{C}$ be a domain and consider the equation

$$(\mathrm{E}_1) \qquad \frac{dy}{dx}=\frac{P(x, y)}{Q(x, y)}$$

with relatively prime $P, Q\in\mathcal{O}_D[y]$. The equation (E_1) is defined on the total space $D\times\boldsymbol{C}$ of a fiber space $\mathcal{F}_0=(D\times\boldsymbol{C}, \mathrm{pr}_0, D)$. As is noticed in the Remark after Definition 1.2, the rational compactification of $\boldsymbol{C}$ is the projective space $\boldsymbol{P}$. Then we extend (E_1) to the system $\boldsymbol{E}_1$ on the manifold $X=D\times\boldsymbol{P}$. A fiber space $\mathcal{F}=(X, \mathrm{pr}, D)$ is the definition space of (E_1).

As an atlas on X, take $\{(\mathcal{U}_1, \theta_1), (\mathcal{U}_2, \theta_2)\}$:

$$\theta_1 : \mathcal{U}_1=D\times\boldsymbol{C}\longrightarrow D\times\boldsymbol{C},\ (x, y)\longrightarrow(x, y),$$
$$\theta_2 : \mathcal{U}_2=D\times\boldsymbol{P}^*\longrightarrow D\times\boldsymbol{C},\ (x, y)\longrightarrow(x, v)=(x, 1/y),$$

where $\boldsymbol{P}^*=\boldsymbol{P}-\{0\}$. The system $\boldsymbol{E}_1$ is constructed as follows:
We set

$$(\mathrm{E}_1)_1 \qquad \frac{dy}{dx}=\frac{P(x, y)}{Q(x, y)}.$$

Since

$$\frac{dv}{dx}=-v^2\frac{P(x, 1/v)}{Q(x, 1/v)}=\frac{-v^{2+q}\mathcal{P}(x, v)}{v^p\mathcal{Q}(x, v)}=\frac{P_2(x, v)}{Q_2(x, v)},$$

where $p=\deg_v P$, $q=\deg_v Q$, $P(x, 1/v)=(1/v^p)\mathcal{P}(x, v)$, $Q(x, 1/v)=(1/v^q)\mathcal{Q}(x, v)$ and $P_2, Q_2\in\mathcal{O}_D[v]$ with $(P_2, Q_2)=1$, we set

$$(\mathrm{E}_1)_2 \qquad \frac{dv}{dx}=\frac{P_2(x, v)}{Q_2(x, v)}.$$

$\boldsymbol{E}_1=\{(\mathrm{E}_1)_1, (\mathrm{E}_1)_2\}$ is the extension of (E_1).

PROPOSITION 2.1. (1) *The singular initial set* $\mathcal{S}$ *of* (E_1) *in the definition manifold* $X=D\times\boldsymbol{P}$ *is written as*

$$\begin{aligned}\mathcal{S}=&\{a\in D|\ Q(a, y)\equiv 0\}\times\{y|\ y\in\boldsymbol{P}\}\\ &\cup\{(a, b)\in D\times\boldsymbol{C}|\ P(a, b)=Q(a, b)=0\}\\ &\cup\{(a, \infty)|\ P_2(a, 0)=Q_2(a, 0)=0\}.\end{aligned}$$

Therefore (E_1) *has vertical singularity sets of the first and the second kinds, but does not have covering singularity sets.*

(2) *The fixed singularity sets* $\boldsymbol{\Theta}_1$, $\boldsymbol{\Theta}_2$ *and* $\boldsymbol{\Theta}$ *of* (E_1) *defined by Definition* 1.12 *coincide with those defined by Painlevé.*

PROOF. (1) The autonomous system associated with $\boldsymbol{E}_1$ is $\boldsymbol{A}_1=\{(\mathrm{A}_1)_1, (\mathrm{A}_1)_2\}$, where

$$(\mathrm{A}_1)_1\begin{cases}\dfrac{dx}{dt}=Q(x, y)\\[2mm] \dfrac{dy}{dt}=P(x, y)\end{cases}, \qquad (\mathrm{A}_1)_2\begin{cases}\dfrac{dx}{ds}=Q_2(x, v)\\[2mm] \dfrac{dv}{ds}=P_2(x, v)\end{cases}.$$

Then,

$$\begin{aligned}S=&\theta_1^{-1}(S_1)\cup\theta_2^{-1}(S_2)\\ =&\{(a, b)\in D\times\boldsymbol{C}|\ P(a, b)=Q(a, b)=0\}\\ &\cup\{(a, b)\in D\times\boldsymbol{P}^*|\ P_2(a, 1/b)=Q_2(a, 1/b)=0\}\\ =&\{(a, b)\in D\times\boldsymbol{C}|\ P(a, b)=Q(a, b)=0\}\\ &\cup\{(a, \infty)|\ P_2(a, 0)=Q_2(a, 0)=0\}\end{aligned}\tag{2.1}$$

(See the Remark after Proposition 1.5.).

Suppose that $(a, b)\in\mathcal{U}_1-S$. If a local solution of $(\mathrm{A}_1)_1$ passing $\theta_1(a, b)=(a, b)$ is given by $x\equiv a$, $y=y(t)$, then $Q(a, y(t))\equiv 0$ and $y(t)$ is not a constant function. Therefore $Q(a, y) \equiv 0$ as a polynomial in y. Conversely, suppose that $Q(a, y) \equiv 0$, and take any $b \in \boldsymbol{C}$. If $y(t)$ is the solution of $dy/dt = P(a, y)$ which satisfies the initial condition $y(0) = b$, then $x \equiv a$, $y = y(t)$ is a solution of $(\mathrm{A}_1)_1$ passing (a, b). Hence we see that

$$T_1 = \{a \in D \mid Q(a, y) \equiv 0\} \times \{y \mid y \in \boldsymbol{C}\} - S_1.$$

By similar arguments, we obtain

$$T_2 = \{a \in D \mid Q_2(a, v) \equiv 0\} \times \{v \mid v \in \boldsymbol{C}\} - S_2.$$

Since $\theta_1^{-1}(T_1) = \theta_2^{-1}(T_2)$ on $\mathcal{U}_1 \cap \mathcal{U}_2$, we have

$$\begin{aligned} T &= \theta_1^{-1}(T_1) \cup \theta_2^{-1}(T_2) \\ &= \{a \in D \mid Q(a, y) \equiv 0\} \times \{y \mid y \in \boldsymbol{P}\} - S. \end{aligned} \tag{2.2}$$

By (2.1) and (2.2), $\mathcal{S}$ is written as the desired form (See the Remark after Proposition 1.5.).

For any $a \in D$ which satisfies $Q(a, y) \equiv 0$, the set $\{a\} \times \{y \mid y \in \boldsymbol{P}\}$ is a vertical singularity set of the first kind. The set $\{(a, b)\}$ which satisfies $P(a, b) = Q(a, b) = 0$ and the set $\{(a, \infty)\}$ which satisfies $P_2(a, 0) = Q_2(a, 0) = 0$ are vertical singularity sets of the second kind. Consequently (E_1) has vertical singularity sets of the first and the second kinds, but does not have covering singularity sets.

(2) From (1) and Definition 1.12, it follows that

$$\begin{aligned} &\boldsymbol{\Theta}_1 = \{\xi \in D \mid Q(\xi, y) \equiv 0\}, \\ &\boldsymbol{\Theta}_2 = \{\xi \in D - \boldsymbol{\Theta}_1 \mid \text{ for some } \eta \in \boldsymbol{C},\ P(\xi, \eta) = Q(\xi, \eta) = 0\} \\ &\qquad \cup \{\xi \in D - \boldsymbol{\Theta}_1 \mid P_2(\xi, 0) = Q_2(\xi, 0) = 0\}, \\ &\boldsymbol{\Theta} = \boldsymbol{\Theta}_1 \cup \boldsymbol{\Theta}_2. \end{aligned}$$

Therefore $\boldsymbol{\Theta}_1$, $\boldsymbol{\Theta}_2$, $\boldsymbol{\Theta}$ are the same sets that are defined by Painlevé (See § 0.). q.e.d.

Lastly we note that Painlevé's theorem (Theorem α in § 0) is derived from Theorem 3 (1), because (E_1) has no covering singularity sets.

§ 3. Fixed and movable singularities of (E_2).

In this section, we study the singular initial set $\mathcal{S}$ and the fixed singularity set $\boldsymbol{\Theta}$ of (E_2) in detail. Preliminaries and results are stated in 3.1 and proofs are given in 3.2.

3.1. The singular initial set $\mathcal{S}$ and the fixed singularity set Θ of (E_2).

Let $D\subset C$ be a domain and consider a system of rational differential equations of order two defined on $D\times C^2$:

$$(\mathrm{E}_2)\qquad \begin{cases} \dfrac{dy}{dx}=\dfrac{P(x, y, z)}{F(x, y, z)Q(x, y, z)} \\ \dfrac{dz}{dx}=\dfrac{V(x, y, z)}{F(x, y, z)W(x, y, z)} \end{cases},$$

where $P, Q, V, W, F\in\mathcal{O}_D[y, z]$, $(P, FQ)=1$, $(V, FW)=1$ and $(Q, W)=1$. By Kodaira [2] and Morrow [3], any compactification of C^2 is a rational surface. Hence it is always a rational compactification. Therefore let us take a rational surface M which is a compactification of C^2 and extend (E_2) to the system $\boldsymbol{E}_2$ on the manifold $X=D\times M$. A fiber space $\mathcal{F}=(X, \mathrm{pr}, D)$ is a definition space of (E_2).

Let $\{(U_i, \kappa_i)\}_{i=1}^m$ be an atlas on M which satisfies the conditions (1), (2), (3) in Definition 1.2. We can take the atlas so that the condition $\kappa_i(U_i)=C^2$ is satisfied for any i, because M is obtained from one of Hirzebruch surfaces $\Sigma^{(k)}$ $(k=0, 2, 3, \cdots)$ or P^2 by blowing up points. Let $\{(\mathcal{U}_i, \theta_i)\}_{i=1}^m$ be the atlas on X which satisfies the following conditions:

A1. $\mathcal{U}_i=D\times U_i$, $\theta_i=\mathrm{id}\times\kappa_i$ $(i=1, \cdots, m)$,

i.e. $\theta_i : D\times U_i\longrightarrow D\times\kappa_i(U_i)=D\times C^2$,
$(a, b)\longrightarrow(a, \kappa_i(b))=(a, b_i, c_i)$.

Particularly, $\mathcal{U}_1=D\times C^2$, $\theta_1=\mathrm{id} : D\times C^2\longrightarrow D\times C^2$.

A2. $\theta_j\circ\theta_i^{-1}=\mathrm{id}\times(\kappa_j\circ\kappa_i^{-1}) : (x, y_i, z_i)\longrightarrow(x, y_j, z_j)$,
where $y_j=A_{ji}(y_i, z_i)$ and $z_j=B_{ji}(y_i, z_i)$ with $A_{ji}, B_{ji}\in C(y_i, z_i)$.

Then (E_2) is extended to the system $\boldsymbol{E}_2=\{(\mathrm{E}_2)_i\}_{i=1}^m$ given by

$$(\mathrm{E}_2)_i\qquad \begin{cases} \dfrac{dy_i}{dx}=\dfrac{P_i(x, y_i, z_i)}{F_i(x, y_i, z_i)Q_i(x, y_i, z_i)} \\ \dfrac{dz_i}{dx}=\dfrac{V_i(x, y_i, z_i)}{F_i(x, y_i, z_i)W_i(x, y_i, z_i)} \end{cases}.$$

The equation $(\mathrm{E}_2)_i$ is defined on $\theta_i(\mathcal{U}_i)=D\times C^2$ and satisfies the following properties: P_i, Q_i, V_i, W_i, $F_i\in\mathcal{O}_D[y_i, z_i]$, $(P_i, F_iQ_i)=1$, $(V_i, F_iW_i)=1$ and

$(Q_i, W_i)=1$.

In order to study the irreducible decomposition of the singular initial set $\mathcal{S}$ of (E_2), we consider the irreducible decomposition of $\theta_i(\mathcal{S}\cap\mathcal{U}_i)=\mathcal{S}_i=S_i\cup T_i$ in $\theta_i(\mathcal{U}_i)=D\times \boldsymbol{C}^2$. First, we obtain the following

PROPOSITION 3.1. *Let F_i be decomposed into irreducible polynomials as follows*:

$$F_i=e_i(x)\prod_{\alpha=1}^{k} f_i^{(\alpha)}(x, y_i)\prod_{\beta=1}^{k'} g_i^{(\beta)}(x, z_i)\prod_{\gamma=1}^{k''} h_i^{(\gamma)}(x, y_i, z_i).$$

Then,

$$\begin{aligned} S_i=&\{\boldsymbol{x}\mid P_i(\boldsymbol{x})=Q_i(\boldsymbol{x})=0\}\\ &\cup\{\boldsymbol{x}\mid V_i(\boldsymbol{x})=W_i(\boldsymbol{x})=0\}\\ &\cup\{\boldsymbol{x}\mid Q_i(\boldsymbol{x})=W_i(\boldsymbol{x})=0\}\\ &\cup\{\boldsymbol{x}\mid F_i(\boldsymbol{x})=P_i(\boldsymbol{x})=V_i(\boldsymbol{x})=0\}, \end{aligned}$$

where $\boldsymbol{x}=(a, b_i, c_i)\in\theta_i(\mathcal{U}_i)=D\times\boldsymbol{C}^2$, *and*

$$\begin{aligned} T_i=[&\{(a, c_i)\mid Q_i(a, y_i, c_i)\equiv 0\}\times\{y_i\mid y_i\in\boldsymbol{C}\}\\ &\cup\{(a, b_i)\mid W_i(a, b_i, z_i)\equiv 0\}\times\{z_i\mid z_i\in\boldsymbol{C}\}\\ &\cup\{a\mid e_i(a)=0\}\times\{(y_i, z_i)\mid (y_i, z_i)\in\boldsymbol{C}^2\}\\ &\cup\bigcup_{\alpha=1}^{k}\{(a, b_i)\mid f_i^{(\alpha)}(a, b_i)=0,\ P_i(a, b_i, z_i)\equiv 0\}\times\{z_i\mid z_i\in\boldsymbol{C}\}\\ &\cup\bigcup_{\beta=1}^{k'}\{(a, c_i)\mid g_i^{(\beta)}(a, c_i)=0,\ V_i(a, y_i, c_i)\equiv 0\}\times\{y_i\mid y_i\in\boldsymbol{C}\}\\ &\cup\bigcup_{\gamma=1}^{k''} H_i^{(\gamma)}]-S_i, \end{aligned}$$

where $H_i^{(\gamma)}$ is the set of points $(a, b_i, c_i)\in\theta_i(\mathcal{U}_i)$ such that there exist holomorphic functions $y_i(t)$, $z_i(t)$ defined on a neighborhood of $t=0$ and satisfying the following conditions:

(1) $y_i(0)=b_i,\ z_i(0)=c_i$.

(2) $h_i^{(\gamma)}(a, y_i(t), z_i(t))\equiv 0$.

(3) $$\begin{cases} \dfrac{dy_i(t)}{dt}=P_i W_i(a, y_i(t), z_i(t)) \\[2ex] \dfrac{dz_i(t)}{dt}=Q_i V_i(a, y_i(t), z_i(t)). \end{cases}$$

All of the sets which constitute S_i are analytic sets in $D\times\boldsymbol{C}^2$. By the same arguments as in Proposition 1.5 (1), we find that $H_i^{(\gamma)}$'s are analytic sets in $D\times C^2$. Then all of the sets which constitute T_i are also analytic sets in $D\times C^2$. To study the irreducible decompositions of these analytic sets, we introduce some terminologies.

DEFINITION 3.1. (1) Let $a\in D$. The analytic set $\{a\}\times C^2\subset D\times C^2$, which is irreducible and of codim.1, is called *an analytic set at a of the type* V1.

(2) (i) Let $a\in D$, and let $A(y_i, z_i)$ be an irreducible polynomial in

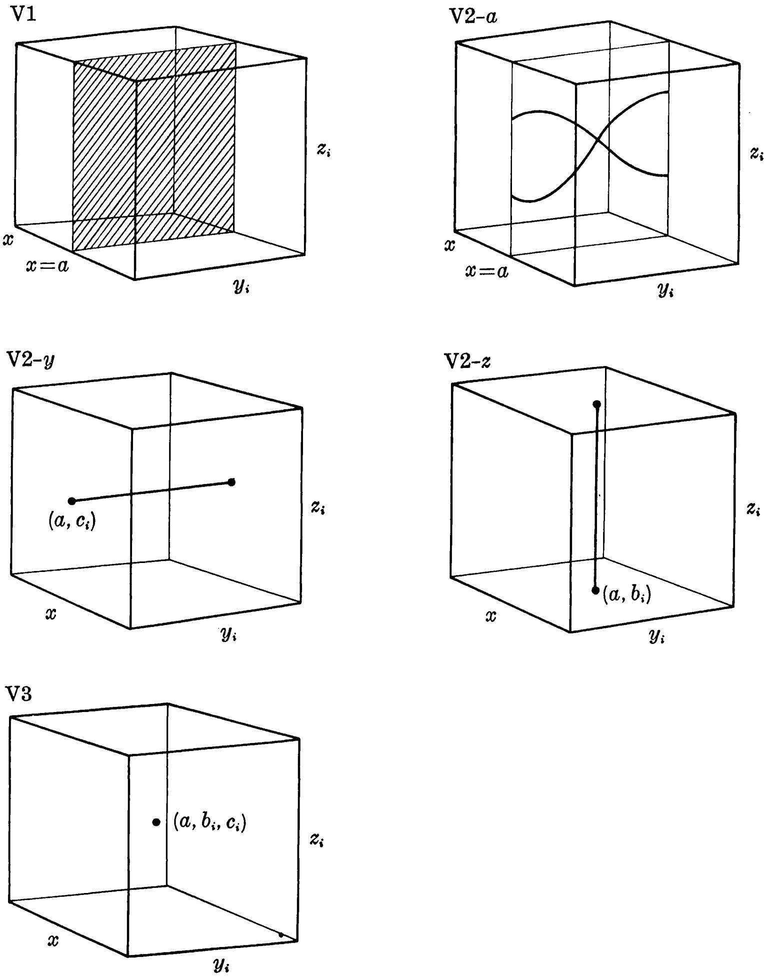


Figure 1.

$C[y_i, z_i]$. The analytic set $\{a\}\times\{(b_i, c_i)|\, A(b_i, c_i)=0\}\subset D\times C^2$, which is irreducible and of codim.2, is called *an analytic set at a of the type* V2-a.

(ii) Let $(a, c_i)\in D\times C$. The analytic set $\{(a, c_i)\}\times\{y_i|\, y_i\in C\}\subset D\times C^2$, which is irreducible and of codim.2, is called *an analytic set at a of the type* V2-y.

(iii) Let $(a, b_i)\in D\times C$. The analytic set $\{(a, b_i)\}\times\{z_i|\, z_i\in C\}\subset D\times C^2$, which is irreducible and of codim.2, is called *an analytic set at a of the type* V2-z.

Analytic sets at a of the type V2-a, V2-y, V2-z are called *analytic sets at a of the type* V2.

(3) Let $a\in D$. The analytic set $\{(a, b_i, c_i)\}\subset D\times C^2$, which is irreducible and of codim.3, is called *an analytic set at a of the type* V3.

DEFINITION 3.2. (1) Let $A(x, y_i, z_i)$, $Y(x, z_i)$ and $Z(x, y_i)$ be irreducible polynomials in $\mathcal{O}_D[y_i, z_i]$, where $\deg_{y_i}A\geqq 1$, $\deg_{z_i}A\geqq 1$, $\deg_{y_i}Y=0$, $\deg_{z_i}Y\geqq 1$, $\deg_{y_i}Z\geqq 1$ and $\deg_{z_i}Z=0$. The analytic sets in $D\times C^2$:

$$\{(a, b_i, c_i)|\, A(a, b_i, c_i)=0\},$$

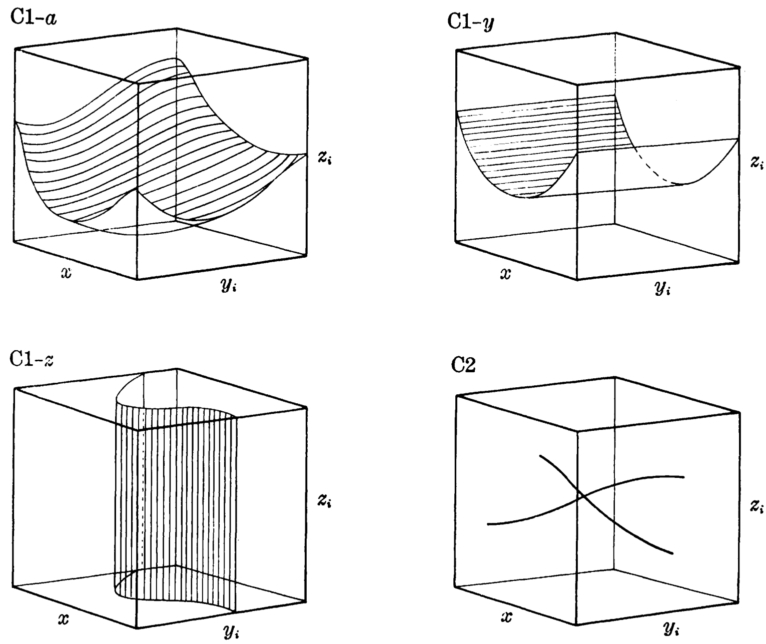


Figure 2.

$\{(a, c_i) | Y(a, c_i)=0\} \times \{y_i | y_i \in \boldsymbol{C}\}$,
$\{(a, b_i) | Z(a, b_i)=0\} \times \{z_i | z_i \in \boldsymbol{C}\}$,

which are irreducible and of codim.1, are called *analytic sets of the type* C1-a, C1-y, C1-z respectively. In general, these analytic sets are called *analytic sets of the type* C1.

(2) Let $(y_i, z_i)=(\phi(x), \psi(x))$ be a 2-tuple algebroidal function on D (See Definition 1.3.). The analytic set $\{(a, \phi(a), \psi(a)) | a \in D\} \subset D \times \boldsymbol{C}^2$, which is irreducible and of codim.2, is called *an analytic set of the type* C2.

Using terminologies in Definition 3.1 and 3.2, we obtain the following

THEOREM 4. *Irreducible components of the analytic sets which constitute* S_i, T_i *are analytic sets of the type* V *or the type* C *and as follows*:

Analytic sets	*Types of irreducible components*				
$\{P_i=Q_i=0\}$, $\{V_i=W_i=0\}$, $\{Q_i=W_i=0\}$		V2-a, V2-y, V2-z			C2
$\{F_i=P_i=V_i=0\}$		V2-a, V2-y, V2-z	V3		C2
$\{(a, c_i) \| Q_i(a, y_i, c_i)\equiv 0\} \times \boldsymbol{C}$	V1	V2-y		C1-y	
$\{(a, b_i) \| W_i(a, b_i, z_i)\equiv 0\} \times \boldsymbol{C}$	V1	V2-z		C1-z	
$\{a \| e_i(a)=0\} \times \boldsymbol{C}^2$	V1				
$\{(a, b_i) \| f_i^{(\alpha)}(a, b_i)=0,$ $P_i(a, b_i, z_i)\equiv 0\} \times \boldsymbol{C}$		V2-z			
$\{(a, c_i) \| g_i^{(\beta)}(a, c_i)=0,$ $V_i(a, y_i, c_i)\equiv 0\} \times \boldsymbol{C}$		V2-y			
$H_i^{(\gamma)}$		V2-a, V2-y, V2-z	V3 (*)	C1-a	C2 (**)

REMARK. Irreducible components of the type V3 of $H_i^{(\gamma)}$ (the above (*)) are either the same ones as in $\{F_i=P_i=V_i=0\}$ or the sets contained in the irreducible components of the types V2, C2 of $\{P_i=Q_i=0\}$, $\{V_i=W_i=0\}$, $\{Q_i=W_i=0\}$ and $\{F_i=P_i=V_i=0\}$. Irreducible components of the type C2 of $H_i^{(\gamma)}$ (the above (**)) are the same ones as in $\{P_i=Q_i=0\}$, $\{V_i=W_i=0\}$, $\{Q_i=W_i=0\}$ and $\{F_i=P_i=V_i=0\}$. More detailed results on irreducible decompositions will be stated in Proposition 3.6~Proposition

3.11 in 3.2.

Suppose that $\mathcal{S}$ is decomposed into irreducible components as

$$\mathcal{S}=\bigcup_{\sigma\in\Lambda_1}\mathcal{S}_\sigma^{(1)}\cup\bigcup_{\sigma\in\Lambda_2}\mathcal{S}_\sigma^{(2)}\cup\bigcup_{\sigma\in\Lambda_3}\mathcal{S}_\sigma^{(3)},$$

where $\operatorname{codim}_{\mathcal{C}}\mathcal{S}_\sigma^{(k)}=k$ $(k=1, 2, 3)$. Then we obtain the following result for $\mathcal{S}_\sigma^{(k)}$'s.

PROPOSITION 3.2. (1) *The following three conditions are equivalent*:

(i) $\mathcal{S}_\sigma^{(k)}$ *is a vertical singularity set of the* k*-th kind with* $\mathrm{pr}(\mathcal{S}_\sigma^{(k)})=\{a\}$.

(ii) *For any* i, *every component of* $\theta_i(\mathcal{S}_\sigma^{(k)}\cap\mathcal{U}_i)$ *is an analytic set at* a *of the type* Vk.

(iii) *For an* i, *a component of* $\theta_i(\mathcal{S}_\sigma^{(k)}\cap\mathcal{U}_i)$ *is an analytic set at* a *of the type* Vk.

(2) *The following conditions are equivalent*:

(i) $\mathcal{S}_\sigma^{(k)}$ *is a covering singularity set of the* k*-th kind.*

(ii) *For any* i, *every component of* $\theta_i(\mathcal{S}_\sigma^{(k)}\cap\mathcal{U}_i)$ *is an analytic set of the type* Ck.

(iii) *For an* i, *a component of* $\theta_i(\mathcal{S}_\sigma^{(k)}\cap\mathcal{U}_i)$ *is an analytic set of the type* Ck.

We can easily check this proposition. So we omit the proof. This proposition means that $\mathcal{S}_\sigma^{(k)}$ is obtained by gluing analytic sets of the same type. So a vertical (resp. covering) singularity set of the k-th kind is called *a vertical* (*resp. covering*) *singularity set of the type* Vk (*resp.* Ck).

Next we define some subsets in D to describe the fixed singularity set Θ of (E_2).

DEFINITION 3.3. Suppose that Q_i, W_i are decomposed into irreducible components as

$$Q_i=q_i(x)\prod_{\alpha=1}^{l}q_i^{(\alpha)}(x, y_i)\prod_{\beta=1}^{l'}q_i'^{(\beta)}(x, z_i)\prod_{\gamma=1}^{l''}q_i''^{(\gamma)}(x, y_i, z_i),$$
$$W_i=w_i(x)\prod_{\alpha=1}^{r}w_i^{(\alpha)}(x, y_i)\prod_{\beta=1}^{r'}w_i'^{(\beta)}(x, z_i)\prod_{\gamma=1}^{r''}w_i''^{(\gamma)}(x, y_i, z_i).$$

We define the subsets $\Delta_i^{(j)}$ $(j=1\sim4)$ and Δ_i as follows:

$\Delta_i^{(1)}=\{a|\ P_i(a, y),\ Q_i(a, y)$ have a common divisor with deg.$\geqq1\}$

$\cup\{a|\ V_i(a, y),\ W_i(a, y)$ have a common divisor with deg. $\geqq 1\}$
$\cup\{a|\ Q_i(a, y),\ W_i(a, y)$ have a common divisor with deg. $\geqq 1\}$
$\cup\{a|\ F_i(a, y),\ P_i(a, y),\ V_i(a, y)$ have a common divisor with deg. $\geqq 1\}$,

where $y=(y_i, z_i)$.

$$\Delta_i^{(2)}=\bigcup_{\gamma=1}^{l''}\{a|\ \text{for some } c_i\in C,\ q''^{(\gamma)}_i(a, y_i, c_i)\equiv 0\}$$
$$\cup\bigcup_{\gamma=1}^{r''}\{a|\ \text{for some } b_i\in C,\ w''^{(\gamma)}_i(a, b_i, z_i)\equiv 0\}.$$

$$\Delta_i^{(3)}=\bigcup_{\alpha=1}^{k}\{a|\ \text{for some } b_i\in C,\ f_i^{(\alpha)}(a, b_i)=0,\ P_i(a, b_i, z_i)\equiv 0\}$$
$$\cup\bigcup_{\beta=1}^{k'}\{a|\ \text{for some } c_i\in C,\ g_i^{(\beta)}(a, c_i)=0,\ V_i(a, y_i, c_i)\equiv 0\}.$$

$$\Delta_i^{(4)}=\bigcup_{\gamma=1}^{k''}\{a|\ \text{there exists an irreducible component } A \text{ of the type V2 of a } H_i^{(\gamma)} \text{ such that } \mathrm{pr}_i(A)=\{a\}\},$$

where $\mathrm{pr}_i: \theta_i(\mathcal{U}_i)=D\times C^2\longrightarrow D,\ (a, b)\longrightarrow a$.

$\Delta_i=\{a|$ for some $(b_i, c_i)\in C^2$, (a, b_i, c_i) is an isolated point of the set of common zeros of F_i, P_i and $V_i\}$.

Using Definition 3.3, we obtain the following

THEOREM 5. *The fixed singularity set* Θ *of* (E_2) *is written as*

$$\Theta=\Theta_1\cup\Theta_2\cup\Theta_3,$$

where

$\Theta_1=\{a|\ e_1(a)=0$ *or* $q_1(a)=0$ *or* $w_1(a)=0\}$,

$\Theta_2=\{a\in\bigcup_{i=1}^{m}(\Delta_i^{(1)}\cup\Delta_i^{(2)}\cup\Delta_i^{(3)}\cup\Delta_i^{(4)})-\Theta_1|$ *there exist vertical singularity sets at* a *of the type* V2 *which are not contained in any covering singularity sets of the type* C1$\}$,

$\Theta_3=\{a\in\bigcup_{i=1}^{m}\Delta_i-\Theta_1\cup\Theta_2|$ *there exist vertical singularity sets at* a *of the type* V3 *which are not contained in any covering singularity sets*$\}$.

From Theorem 3, Theorem 4, Proposition 3.2 and Definition 1.6, we obtain the following theorem on singularities of solutions of (E_2).

THEOREM 6. *Let Φ be a global solution of* (E_2).

(1) *Suppose that* (E_2) *has no covering singularity sets. If Φ has a singularity $\boldsymbol{\omega}$ on $a\in D-\Theta$, then $\boldsymbol{\omega}$ is an algebraic singularity. If Φ has a singularity $\boldsymbol{\omega}$ on $\xi\in\Theta_3$, then $\boldsymbol{\omega}$ is either an algebraic singularity or an ordinary transcendental singularity. (Hence, if Φ has an essential singularity $\boldsymbol{\omega}$ on $\xi\in\Theta$, then $\xi\in\Theta_1\cup\Theta_2$.)*

(2) *Suppose that* (E_2) *has no covering singularity sets of the type* C1. *If Φ has a singularity $\boldsymbol{\omega}$ on $a\in D-\Theta$ or on $\xi\in\Theta_3$, then $\boldsymbol{\omega}$ is either an algebraic singularity or an ordinary transcendental singularity. (Hence, if Φ has an essential singularity $\boldsymbol{\omega}$ on $\xi\in\Theta$, then $\xi\in\Theta_1\cup\Theta_2$.)*

(3) *Suppose that* (E_2) *has no covering singularity sets of the type* C1-*a and* C1-*y (resp.* C1-*z). If Φ has a transcendental singularity $\boldsymbol{\omega}$ on $a\in D-\Theta_1\cup\Theta_2$, then $\boldsymbol{\omega}$ is at most ordinary in $(y_1,\cdots,y_m)$-direction (resp. $(z_1,\cdots,z_m)$-direction). In addition, if all of the vertical singularity sets of the type* V2 *are of the type* V2-*z (resp.* V2-*y), then transcendental singularities on $\xi\in\Theta_2$ are at most ordinary in $(y_1,\cdots,y_m)$-direction (resp. $(z_1,\cdots,z_m)$-direction).*

3.2. Proofs of Propositions and Theorems.

1° Proof of Proposition 3.1.

PROOF. For the sake of simplicity, we omit the suffix i of the variables y_i, z_i, t_i and the polynomials P_i, Q_i, V_i, W_i, F_i.

Since

$$(A_2)_i\qquad \begin{cases} \dfrac{dx}{dt}=FQW(x,y,z)\\[2mm] \dfrac{dy}{dt}=PW(x,y,z)\\[2mm] \dfrac{dz}{dt}=QV(x,y,z)\,, \end{cases}$$

we have $S_i=\{\boldsymbol{x}\mid FQW(\boldsymbol{x})=PW(\boldsymbol{x})=QV(\boldsymbol{x})=0\}$, where $\boldsymbol{x}=(a,b,c)$ (See the Remark after Proposition 1.5.). Hence, apparently,

$$S_i=\{P=Q=0\}\cup\{V=W=0\}\cup\{Q=W=0\}\cup\{F=P=V=0\}.$$

We also find that

$T_i=\Big\{(a,b,c)\mid$ there exist holomorphic functions $y(t)$, $z(t)$ defined on a neighborhood $\Delta\subset C$ of $t=0$ and satisfying the following conditions:

at least one of $y(t)$, $z(t)$ is not a constant function,

(3.1) $y(0)=b,\ z(0)=c,$

(3.2) $FQW(a, y(t),\ z(t))\equiv 0$

(3.3) $\dfrac{dy(t)}{dt}=PW(a, y(t),\ z(t))$

(3.4) $\left.\dfrac{dz(t)}{dt}=QV(a, y(t),\ z(t))\right\}.$

We will study T_i in the following. By (3.2), we see that $Q(a, y(t), z(t))\equiv 0$ or $W(a, y(t), z(t))\equiv 0$ or $F(a, y(t), z(t))\equiv 0$.

(i) If $Q(a, y(t), z(t))\equiv 0$, then $z\equiv c$ by (3.4). Since $y(t)\not\equiv$const., $Q(a, y, c)\equiv 0$ as a polynomial in y. Conversely, suppose $Q(a, y, c)\equiv 0$ and let $y(t)$ be a solution of the equation $dy(t)/dt=PW(a, y(t), c)$ satisfying the initial condition $y(0)=b$. If $y(t)$ is not a constant, then $(a, b, c)\in T_i$, and if $y(t)\equiv b$, then $(a, b, c)\in S_i$.

(ii) Similarly, we obtain $W(a, b, z)\equiv 0$ from $W(a, y(t), z(t))\equiv 0$.

(iii) Suppose that

$$F(a, y(t), z(t))=e(a)\prod_{\alpha=1}^{k} f^{(\alpha)}(a, y(t))\prod_{\beta=1}^{k'} g^{(\beta)}(a, z(t))\prod_{\gamma=1}^{k''} h^{(\gamma)}(a, y(t), z(t)).$$

(A) Suppose $e(a)=0$. Let $y(t)$, $z(t)$ be solutions of (3.3), (3.4) satisfying the condition (3.1). If at least one of $y(t)$, $z(t)$ is not a constant, then $(a, b, c)\in T_i$, and if $y(t)\equiv b$, $z(t)\equiv c$, then $(a, b, c)\in S_i$.

(B) Suppose $f^{(\alpha)}(a, y(t))\equiv 0$. If $y(t)\not\equiv$const., then $f^{(\alpha)}(a, y)\equiv 0$ as a polynomial in y. This implies that $f^{(\alpha)}$ is decomposed as $f^{(\alpha)}(x, y)=d(x)f_1(x, y)$ with a non-unit $d(x)$, which contradicts the assumption that $f^{(\alpha)}$ is irreducible. Hence $y(t)\equiv b$ and $f^{(\alpha)}(a, b)=0$. From (3.3), $W(a, b, z(t))\equiv 0$ or $P(a, b, z(t))\equiv 0$. The case $W(a, b, z(t))\equiv 0$ is reduced to the case (ii). Since $z(t)\not\equiv$const., it follows from $P(a, b, z(t))\equiv 0$ that $P(a, b, z)\equiv 0$ as a polynomial in z. Conversely assume that (a, b) satisfies $f^{(\alpha)}(a, b)=0$, $P(a, b, z)\equiv 0$, and let $z(t)$ be a solution of the equation $dz(t)/dt=QV(a, b, z(t))$ which satisfies the initial condition $z(0)=c$. If $z(t)$ is not a constant, then $(a, b, c)\in T_i$, if $z(t)\equiv c$, then $(a, b, c)\in S_i$.

(C) Suppose $g^{(\beta)}(a, z(t))\equiv 0$. By the same arguments as in (B), we find that $z(t)\equiv c$ and $V(a, y, c)\equiv 0$ as a polynomial in y. Conversely assume that (a, c) satisfies $g^{(\beta)}(a, c)=0$, $V(a, y, c)\equiv 0$, and let $y(t)$ be a solution of the equation $dy(t)/dt=PW(a, y(t), c)$ which satisfies $y(0)=b$.

If $y(t)$ is not a constant, then $(a, b, c) \in T_i$, and if $y(t) \equiv b$, then $(a, b, c) \in S_i$.

(D) If $h^{(r)}(a, y(t), z(t)) \equiv 0$, then we define the set $H_i^{(r)}$ as in Proposition 3.1.

From (i), (ii), (iii), T_i is the desired one. q.e.d.

2° Preparatory propositions for Theorem 4.

In the following, we prepare some propositions for the proof of Theorem 4.

PROPOSITION 3.3. *Let*

$$P(x, y, z) = p_0(x, z)y^l + \cdots + p_l(x, z) \in \mathcal{O}_D[y, z]$$

be an irreducible polynomial with $\deg_y P = l \geqq 1$, $\deg_z P \geqq 1$.

(1) *The following* (i) *and* (ii) *are equivalent:*

(i) *For a point* $a \in D$, *there is* $c \in C$ *such that* $P(a, y, c) \equiv 0$.

(ii) *For a point* $a \in D$, *either* (A) *or* (B) *occurs.*

(A) *For more than one of* $p_j(a, z)$*'s, the condition* $\deg_z p_j(a, z) \geqq 1$ *holds. Those have a common divisor with* $\deg. \geqq 1$ *in* $C[z]$. *The others are identically zero.*

(B) *For a unique one of* $p_j(a, z)$*'s, the condition* $\deg_z p_j(a, z) \geqq 1$ *holds. The others are identically zero.*

(2) *For any* $a \in D$, $\{c \in C \mid P(a, y, c) \equiv 0\}$ *is a finite set.*

(3) $\{a \mid$ *for some* $c \in C$, $P(a, y, c) \equiv 0\}$ *is a discrete subset in* D.

PROOF. (1) Since $P(x, y, z)$ is irreducible, $P(a, y, z) \not\equiv 0$ for any $a \in D$. Note that $p_l(x, z) \not\equiv 0$, and that some of $p_j(x, z)$'s are of $\deg_z \geqq 1$.

We set $\Delta = \{a \in D \mid$ for some $c \in C$, $P(a, y, c) \equiv 0\}$. Let a be a point in D. For $p_j(a, z)$'s, we have the following four possibilities:

1. More than one of them are of $\deg_z \geqq 1$.
2. A unique one of them is of $\deg_z \geqq 1$. Among the others, there are nonzero constants.
3. A unique one of them is of $\deg_z \geqq 1$. The others are identically zero.
4. All of them are constants. Then, among them, there are nonzero constants.

In the case 1, if all the constants among $p_j(a, z)$'s are zero and all the polynomials among them have a common divisor with $\deg. \geqq 1$, then $a \in \Delta$, if not, then $a \notin \Delta$. Apparently, in the case 2 or 4, $a \notin \Delta$, and in the case 3, $a \in \Delta$. From these, we obtain the desired result.

(2) It is apparent from (1).

(3) As in (1), we set $\Delta=\{a\in D|$ for some $c\in C,\ P(a,y,c)\equiv 0\}$, and set $Z=\{a\in D|\ a$ is a zero of a certain coefficient of some $p_j(x,z)\}$.

Suppose $a\in\Delta$. If a satisfies the condition (B) in (1), then obviously $a\in Z$. So we consider the case where a satisfies the condition (A) in (1). In this case, $a\in Z$ or $a\notin Z$. Let us consider the case where $a\notin Z$. If $a\notin Z$, then all of $p_j(a,z)$'s are of $\deg_z\geqq 1$ and have a common divisor. Let $p_j(x,z)=p_{j,1}(x,z)\cdots p_{j,k_j}(x,z)$ $(j=0,\cdots,l)$ be the irreducible decomposition in $\mathcal{O}_D[z]$. We note that there exist $p_{0,h_0}(x,z),\cdots,p_{l,h_l}(x,z)$ such that $p_{j,h_j}(x,z)$ is an irreducible divisor of $p_j(x,z)$ for $j=0,\cdots,l$ and $p_{j,h_j}(a,z)$'s have a common root c. Since $P(x,y,z)$ is irreducible, we can choose a pair, say, $p(x,z)$ and $p'(x,z)$, out of $p_{j,h_j}(x,z)$'s so that they are not associates. Here we define the set Z_1 as $Z_1=\{a\in D|\ a$ is a zero of a certain coefficient of some $p_{j,k}(x,z)\}$. Then, $a\in Z_1$ or $a\notin Z_1$. We study the case where $a\notin Z_1$. Let $R(x)$ be the resultant of $p(x,y)$ and $p'(x,y)$. Then, the holomorphic function $R(x)$ on D is not a constant and $R(a)=0$. Hence,

$$a\in Z_2=\bigcup_{i,j,\alpha,\beta}\{a|\ a \text{ is a zero of the resultant } R_{i,j,\alpha,\beta}(x)=R(p_{i,\alpha}(x,z),\ p_{j,\beta}(x,z))\},$$

where (i,j,α,β) is an ordered 4-tuple for which $\deg_z p_{i,\alpha}\geqq 1$, $\deg_z p_{j,\beta}\geqq 1$, and $p_{i,\alpha}(x,z)$ and $p_{j,\beta}(x,z)$ are not associates. From the above consideration, we find that $\Delta\subset Z\cup Z_1\cup Z_2\subset D$. Therefore Δ is discrete in D.

q.e.d.

PROPOSITION 3.4. *Let* $B(x,z)$, $P(x,y,z)$ *be irreducible polynomials in* $\mathcal{O}_D[y,z]$ *such that*

$$B(x,z)=b_0(x)z^k+\cdots+b_k(x),$$
$$P(x,y,z)=p_0(x,z)y^l+\cdots+p_l(x,z),$$

where $k\geqq 1$, $l=\deg_y P\geqq 1$, $\deg_z P\geqq 1$. *Let* $z=\psi(x)$ *be the algebroidal function defined by the equation* $B(x,z)=0$, *and set*

$$Z_2=\{(a,c)\in D\times C|\ B(a,c)=0,\ P(a,y,c)\equiv 0\}.$$

Then,

$$\begin{aligned}&\{(a,b,c)\in D\times C^2|\ B(a,c)=0,\ P(a,b,c)=0\}\\ &=Z_2\times\{y|\ y\in C\} \qquad (\textit{the type } \mathrm{V2}\text{-}y)\\ &\cup\bigcup_{\sigma\in\Lambda}\{(a,\phi_\sigma(a),\ \psi(a))|\ a\in D\}, \qquad (\textit{the type } \mathrm{C2})\end{aligned}$$

where Λ *is a finite set and* $(\phi_\sigma(x),\ \psi(x))$ *is a 2-tuple algebroidal function* *which satisfies the condition* $P(x, \phi_\sigma(x), \psi(x))\equiv 0$.

PROOF. If $p_0(x, \psi(x))=\cdots=p_l(x, \psi(x))\equiv 0$, then $B|p_0, \cdots, B|p_l$. Hence B divides P. This contradicts the assumption. Therefore there exists an integer s such that $0\leqq s\leqq l$, $p_0(x, \psi(x))=\cdots=p_{s-1}(x, \psi(x))\equiv 0$, $p_s(x, \psi(x))\not\equiv 0$.

We set

$$P^*(x, y, z)=p_s(x, z)y^{l-s}+\cdots+p_l(x, z).$$

Suppose that P^* is decomposed into irreducible polynomials as

$$P^*(x, y, z)=e(x)\prod_{\alpha=1}^{t} f_\alpha(x, y)\prod_{\beta=1}^{t'} g_\beta(x, z)\prod_{\gamma=1}^{t''} h_\gamma(x, y, z).$$

Then,

$$\begin{aligned}&\{(a, b, c)\in D\times \boldsymbol{C}^2|\ B(a, c)=0,\ P(a, b, c)=0\}\\ &=\{(a, b, \psi(a))\in D\times \boldsymbol{C}^2|\ P^*(a, b, \psi(a))=0\}\\ &=\{(a, \psi(a))\in D\times \boldsymbol{C}|\ e(a)=0\}\times\{y|\ y\in \boldsymbol{C}\}\\ &\quad\cup\bigcup_{\alpha=1}^{t}\bigcup_{a\in D}\{a\}\times\{\phi_\alpha(a)\}\times\{\psi(a)\}\\ &\quad\cup\bigcup_{\beta=1}^{t'}\{(a, \psi(a))\in D\times \boldsymbol{C}|\ g_\beta(a, \psi(a))=0\}\times\{y|\ y\in \boldsymbol{C}\}\\ &\quad\cup\bigcup_{\gamma=1}^{t''}\{(a, b, \psi(a))\in D\times \boldsymbol{C}^2|\ h_\gamma(a, b, \psi(a))=0\},\end{aligned}$$

where $y=\phi_\alpha(x)$ is the algebroidal function defined by the equation $f_\alpha(x, y)=0$. Let us study these sets in detail.

(i) Obviously, $\{(a, \psi(a))|\ e(a)=0\}\subset Z_2$.

(ii) Fix a point $a\in D$, and take an ordered 2-tuple (ϕ_a, ψ_a) of germs ϕ_a, ψ_a which are germs of ϕ_α, ψ at $a\in D$ respectively. Continuing (ϕ_a, ψ_a) analytically in D as far as possible, we obtain the following result:

$$\bigcup_{a\in D}\{a\}\times\{\phi_\alpha(a)\}\times\{\psi(a)\}=\bigcup_{\sigma\in\Lambda_1}\{(a, \phi_\sigma(a), \psi(a))|\ a\in D\},$$

where Λ_1 is a finite set, $(\phi_\sigma(x), \psi(x))$ is a 2-tuple algebroidal function.

(iii) Since $g_\beta(x, \psi(x))\not\equiv 0$, $\{(a, \psi(a))|\ g_\beta(a, \psi(a))=0\}$ is a discrete set in $D\times \boldsymbol{C}$ and apparently it is contained in Z_2.

(iv) Let $\Delta(B)(x)=R(B, B_z)$ be the discriminant of B. Since B is irreducible, $\Delta(B)(x)$ is not identically zero. Take any point $\bar{a}\in D-\{a\in D|$ $\Delta(B)(a)=0\}$. On a small neighborhood U of $\bar{a}$, $z=\psi(x)$ has distinct holomorphic k branches $\psi_1(x), \cdots, \psi_k(x)$. Suppose that h_γ is written as

$$h_{\gamma}(x, y, z)=H_0(x, z)y^r+\cdots+H_r(x, z).$$

Let us consider the equation

$$\begin{aligned} h_{\gamma}(x, y, \psi_l(x)) &= H_0(x, \psi_l(x))y^r+\cdots+H_r(x, \psi_l(x)) \\ &= A(x)[\hat{H}_0(x)y^r+\cdots+\hat{H}_r(x)]=0 \end{aligned}$$

for $l=1, \cdots, k$, where $A(x)\in \mathcal{O}_U$ is a G.C.D. of $H_0(x, \psi_l(x)), \cdots, H_r(x, \psi_l(x))$. Then we see that

$$\begin{aligned} &\{(a, b, \psi_l(a))\in U\times C^2 \mid h_{\gamma}(a, b, \psi_l(a))=0\} \\ =&\{(a, \psi_l(a))\in U\times C \mid A(a)=0\}\times\{y \mid y\in C\} \\ &\cup \bigcup_{\sigma=1}^{w} \{(a, \phi_\sigma(a), \psi_l(a)) \mid a\in U\} \end{aligned}$$

where $(\phi_\sigma(x), \psi_l(x))$ is a 2-tuple algebroidal function on $U\subset D$ and $1\leqq w\leqq r$. Note that $(\phi_\sigma(x), \psi_l(x))$ is analytically continuable on D and determines a 2-tuple algebroidal function $(\phi_\sigma(x), \psi(x))$ on D. Set

$$Z_{2,\gamma}=\{(a, \psi(a))\in D\times C \mid h_{\gamma}(a, y, \psi(a))\equiv 0\}.$$

Then, obviously $Z_{2,\gamma}\subset Z_2$. By the above arguments, we have

$$\begin{aligned} &\{(a, b, \psi(a))\in D\times C^2 \mid h_{\gamma}(a, b, \psi(a))=0\} \\ =&Z_{2,\gamma}\times\{y \mid y\in C\} \\ &\cup \bigcup_{\sigma\in\Lambda_2} \{(a, \phi_\sigma(a), \psi(a)) \mid a\in D\}, \end{aligned}$$

where Λ_2 is a finite set, $(\phi_\sigma(x), \psi(x))$ is a 2-tuple algebroidal function which satisfies the condition $h_{\gamma}(x, \phi_\sigma(x), \psi_\sigma(x))\equiv 0$.

From (i)$\sim$(iv), we obtain the desired result. q.e.d.

REMARK. We obtain a similar proposition for the set of common zeros of irreducible polynomials

$$\begin{aligned} &C(x, y)=c_0(x)y^h+\cdots+c_h(x), \\ &P(x, y, z)=p_0'(x, y)z^r+\cdots+p_r'(x, y) \end{aligned}$$

in $\mathcal{O}_D[y, z]$, where $h\geqq 1$, $\deg_y P\geqq 1$, $r=\deg_z P\geqq 1$.

PROPOSITION 3.5. *Let $P(x, y, z)$, $Q(x, y, z)$ be irreducible polynomials in $\mathcal{O}_D[y, z]$ which satisfy the conditions $(P, Q)=1$, $\deg_y P\geqq 1$, $\deg_z P\geqq 1$, $\deg_y Q\geqq 1$ and $\deg_z Q\geqq 1$. Then,*

$$\begin{aligned}&\{(a, b, c)\in D\times \boldsymbol{C}^2 |\ P(a, b, c)=Q(a, b, c)=0\}\\ &=\bigcup_{a\in\Delta_2} (\textit{analytic sets at } a \textit{ of the type } \mathrm{V2}) \qquad (\textit{the type } \mathrm{V2})\\ &\quad\cup\bigcup_{\sigma\in\Lambda}\{(a, \phi_\sigma(a), \psi_\sigma(a)) |\ a\in D\}, \qquad (\textit{the type } \mathrm{C2})\end{aligned}$$

where $\Delta_2=\{a\in D|\ P(a, y, z),\ Q(a, y, z)$ *have a common divisor with* deg.$\geqq 1\}$, Λ *is a finite set and* $(\phi_\sigma(x), \psi_\sigma(x))$ *is a* 2-*tuple algebroidal function which satisfies the conditions* $P(x, \phi_\sigma(x), \psi_\sigma(x))\equiv 0,\ Q(x, \phi_\sigma(x), \psi_\sigma(x))\equiv 0.$

PROOF. Suppose that P, Q are written as

$$\begin{aligned}P(x, y, z)&=p_0(x, z)y^l+p_1(x, z)y^{l-1}+\cdots+p_l(x, z),\\ Q(x, y, z)&=q_0(x, z)y^m+q_1(x, z)y^{m-1}+\cdots+q_m(x, z).\end{aligned}$$

Assume that p_0, q_0 are decomposed into irreducible polynomials as

$$\begin{aligned}p_0(x, z)&=\alpha(x)\prod_{j=1}^{s}\beta_j(x, z),\\ q_0(x, z)&=\gamma(x)\prod_{j=1}^{t}\delta_j(x, z),\end{aligned}$$

and that the resultant $\Delta(x, z)=R(P, Q)(x, z)\ (\not\equiv 0)$ is decomposed into irreducible polynomials as

$$\Delta(x, z)=\lambda(x)\prod_{j=1}^{u}\mu_j(x, z).$$

Apparently,

$$\begin{aligned}&\{(a, b, c) |\ P(a, b, c)=Q(a, b, c)=0\}\\ &=\{(a, b, c) |\ p_0(a, c)=P(a, b, c)=Q(a, b, c)=0\}\\ &\quad\cup\{(a, b, c) |\ q_0(a, c)=P(a, b, c)=Q(a, b, c)=0\}\\ &\quad\cup\{(a, b, c) |\ \Delta(a, c)=P(a, b, c)=Q(a, b, c)=0\}.\end{aligned}$$

Let us study the set

$$\begin{aligned}&\{(a, b, c) |\ p_0(a, c)=P(a, b, c)=Q(a, b, c)=0\}\\ &=\{(a, b, c) |\ \alpha(a)=P(a, b, c)=Q(a, b, c)=0\}\\ &\quad\cup\bigcup_{j=1}^{s}\{(a, b, c) |\ \beta_j(a, c)=P(a, b, c)=Q(a, b, c)=0\}.\end{aligned}$$

From the irreducible decompositions of $P(a, y, z)$ and $Q(a, y, z)$, we see that

$$\begin{aligned}&\{(a, b, c) |\ \alpha(a)=P(a, b, c)=Q(a, b, c)=0\}\\ &=\bigcup_{a\in S_{2,\alpha}}(\text{analytic sets at } a \text{ of the type V2 or the type V3}),\end{aligned}$$

where $S_{2,\alpha}$ is a subset of the set $\{a\in D|\ \alpha(a)=0\}$. From Proposition 3.4, we have

$$\begin{aligned}&\{(a,b,c)|\ \beta_j(a,c)=P(a,b,c)=0\}\\&=Z'_{2,\beta_j}\times\{y|\ y\in C\}\\&\quad\cup\bigcup_{\sigma\in\Lambda'_{\beta_j}}\{(a,\phi_\sigma(a),\psi(a))|\ a\in D\},\end{aligned}$$

where $Z'_{2,\beta_j}=\{(a,c)\in D\times C|\ \beta_j(a,c)=0,\ P(a,y,c)\equiv 0\}$, Λ'_{β_j} is a finite set and $(\phi_\sigma(x),\psi(x))$ is a 2-tuple algebroidal function which satisfies the condition $\beta_j(x,\psi(x))\equiv 0$, $P(x,\phi_\sigma(x),\psi(x))\equiv 0$. Therefore,

$$\begin{aligned}&\{(a,b,c)|\ \beta_j(a,c)=P(a,b,c)=Q(a,b,c)=0\}\\&=Z_{2,\beta_j}\times\{y|\ y\in C\}\\&\quad\cup\bigcup_{\sigma\in\Lambda_{\beta_j}}\{(a,\phi_\sigma(a),\psi(a))|\ a\in D\}\\&\quad\cup(\text{analytic sets of the type V3}),\end{aligned}$$

where $Z_{2,\beta_j}=\{(a,c)\in D\times C|\ \beta_j(a,c)=0,\ P(a,y,c)\equiv 0,\ Q(a,y,c)\equiv 0\}$, Λ_{β_j} is a finite set and $(\phi_\sigma(x),\psi(x))$ is a 2-tuple algebroidal function which satisfies the conditions $\beta_j(x,\psi(x))\equiv 0$, $P(x,\phi_\sigma(x),\psi(x))\equiv 0$, $Q(x,\phi_\sigma(x),\psi(x))\equiv 0$.

By similar arguments for the sets $\{q_0=P=Q=0\}$, $\{\Delta=P=Q=0\}$ and the fact that $\operatorname{codim}_C\{(a,b,c)|\ P(a,b,c)=Q(a,b,c)=0\}\leqq 2$, it turns out that

$$\begin{aligned}&\{(a,b,c)|\ P(a,b,c)=Q(a,b,c)=0\}\\&=\bigcup_{a\in S_2}(\text{analytic sets of the type V2 at } a)\\&\quad\cup Z_2\times\{y|\ y\in C\}\\&\quad\cup\bigcup_{\sigma\in\Lambda}\{(a,\phi_\sigma(a),\psi_\sigma(a))|\ a\in D\},\end{aligned}$$

where S_2 is a subset of the set $\{a\in D|\ \alpha(a)=0 \text{ or } \gamma(a)=0 \text{ or } \lambda(a)=0\}$, $Z_2=\left\{(a,c)\,\middle|\,\prod_{j=1}^{s}\beta_j(a,c)\prod_{j=1}^{t}\delta_j(a,c)\prod_{j=1}^{u}\mu_j(a,c)=0,\ P(a,y,c)\equiv 0,\ Q(a,y,c)\equiv 0\right\}$, Λ is a finite set and $(\phi_\sigma(x),\psi_\sigma(x))$ is a 2-tuple algebroidal function which satisfies the conditions $\prod_{j=1}^{s}\beta_j(x,\psi_\sigma(x))\prod_{j=1}^{t}\delta_j(x,\psi_\sigma(x))\prod_{j=1}^{u}\mu_j(x,\psi_\sigma(x))\equiv 0$, $P(x,\phi_\sigma(x),\psi_\sigma(x))\equiv 0$, $Q(x,\phi_\sigma(x),\psi_\sigma(x))\equiv 0$.

Since $\Delta_2=S_2\cup\{a\in D|\text{ for some } c\in C,\ (a,c)\in Z_2\}$, the desired result is derived. q.e.d.

3° Proof of Theorem 4.

Theorem 4 is reduced to the following Proposition 3.6～Proposition 3.11.

PROPOSITION 3.6. *Putting*

$$\Delta_{2,P_i,Q_i}=\{a\in D|\ P_i(a, y_i, z_i) \text{ and } Q_i(a, y_i, z_i) \text{ have a common divisor with } \deg.\geqq 1\},$$

we have

$$\begin{aligned}&\{(a, b_i, c_i)\in D\times C^2|\ P_i(a, b_i, c_i)=Q_i(a, b_i, c_i)=0\}\\ &=\bigcup_{a\in\Delta_{2,P_i,Q_i}} (\text{analytic sets at } a \text{ of the type V2}) \qquad (\text{the type V2})\\ &\cup\bigcup_{\sigma\in\Lambda}\{(a, \phi_\sigma(a), \psi_\sigma(a))\,|\ a\in D\}, \qquad (\text{the type C2})\end{aligned}$$

where Λ is a finite set and $(\phi_\sigma(x), \psi_\sigma(x))$ is a 2-tuple algebroidal function which satisfies the conditions $P_i(x, \phi_\sigma(x), \psi_\sigma(x))\equiv 0$, $Q_i(x, \phi_\sigma(x), \psi_\sigma(x))\equiv 0$.

PROOF. Suppose that P_i and Q_i are decomposed into irreducible components as

$$P_i=p_i(x)\prod_{\alpha=1}^{h} p_i^{(\alpha)}(x, y_i)\prod_{\beta=1}^{h'} p_i'^{(\beta)}(x, z_i)\prod_{\gamma=1}^{h''} p_i''^{(\gamma)}(x, y_i, z_i),$$
$$Q_i=q_i(x)\prod_{\alpha=1}^{l} q_i^{(\alpha)}(x, y_i)\prod_{\beta=1}^{l'} q_i'^{(\beta)}(x, z_i)\prod_{\gamma=1}^{l''} q_i''^{(\gamma)}(x, y_i, z_i).$$

Then,

$$\begin{aligned}&\{P_i=Q_i=0\}\\ &=\{p_i=Q_i=0\}\cup\{q_i=P_i=0\}\\ &\quad\cup\bigcup_{\alpha,\alpha'}\{p_i^{(\alpha)}=q_i^{(\alpha')}=0\}\cup\bigcup_{\alpha,\beta}\{p_i^{(\alpha)}=q_i'^{(\beta)}=0\}\\ &\quad\cup\bigcup_{\alpha,\gamma}\{p_i^{(\alpha)}=q_i''^{(\gamma)}=0\}\cup\bigcup_{\beta,\alpha}\{p_i'^{(\beta)}=q_i^{(\alpha)}=0\}\\ &\quad\cup\bigcup_{\beta,\beta'}\{p_i'^{(\beta)}=q_i'^{(\beta')}=0\}\cup\bigcup_{\beta,\gamma}\{p_i'^{(\beta)}=q_i''^{(\gamma)}=0\}\\ &\quad\cup\bigcup_{\gamma,\alpha}\{p_i''^{(\gamma)}=q_i^{(\alpha)}=0\}\cup\bigcup_{\gamma,\beta}\{p_i''^{(\gamma)}=q_i'^{(\beta)}=0\}\\ &\quad\cup\bigcup_{\gamma,\gamma'}\{p_i''^{(\gamma)}=q_i''^{(\gamma')}=0\}.\end{aligned}$$

Set

$$\begin{aligned}&\Theta_{2,P_i,Q_i}^{(1)}=\{a\in D|\ p_i(a)=0 \text{ or } q_i(a)=0\},\\ &\Theta_{2,P_i,Q_i}^{(2)}=\bigcup_{\gamma,\gamma'}\{a\in D|\ p_i''^{(\gamma)}(a, y_i, z_i) \text{ and } q_i''^{(\gamma')}(a, y_i, z_i) \text{ has a common divisor with } \deg.\geqq 1\}.\\ &Y_{2,P_i,Q_i}^{(1)}=\bigcup_{\alpha,\alpha'}\{(a, b_i)\in D\times C|\ p_i^{(\alpha)}(a, b_i)=q_i^{(\alpha')}(a, b_i)=0\},\\ &Y_{2,P_i,Q_i}^{(2)}=\bigcup_{\alpha,\gamma}\{(a, b_i)\in D\times C|\ p_i^{(\alpha)}(a, b_i)=0,\ q_i''^{(\gamma)}(a, b_i, z_i)\equiv 0\}\\ &\qquad\cup\bigcup_{\alpha,\gamma}\{(a, b_i)\in D\times C|\ q_i^{(\alpha)}(a, b_i)=0,\ p_i''^{(\gamma)}(a, b_i, z_i)\equiv 0\},\end{aligned}$$

$$Z^{(1)}_{2,P_i,Q_i}=\bigcup_{\beta,\beta'}\{(a,c_i)\in D\times C|\ p'^{(\beta)}_i(a,c_i)=q'^{(\beta')}_i(a,c_i)=0\},$$
$$Z^{(2)}_{2,P_i,Q_i}=\bigcup_{\beta,\gamma}\{(a,c_i)\in D\times C|\ p'^{(\beta)}_i(a,c_i)=0,\ q''^{(\gamma)}_i(a,y_i,c_i)\equiv 0\}$$
$$\cup\bigcup_{\beta,\gamma}\{(a,c_i)\in D\times C|\ q'^{(\beta)}_i(a,c_i)=0,\ p''^{(\gamma)}_i(a,y_i,c_i)\equiv 0\},$$
$$\Theta_{2,P_i,Q_i}=\Theta^{(1)}_{2,P_i,Q_i}\cup\Theta^{(2)}_{2,P_i,Q_i},$$
$$Y_{2,P_i,Q_i}=Y^{(1)}_{2,P_i,Q_i}\cup Y^{(2)}_{2,P_i,Q_i},$$
$$Z_{2,P_i,Q_i}=Z^{(1)}_{2,P_i,Q_i}\cup Z^{(2)}_{2,P_i,Q_i}.$$

By Proposition 3.4 and 3.5, we obtain the following result:

$$\{(a,b_i,c_i)|\ P_i(a,b_i,c_i)=Q_i(a,b_i,c_i)=0\}$$
$$=\bigcup_{a\in\Theta_{2,P_i,Q_i}}(\text{analytic sets at } a \text{ of the type V2}) \qquad (\text{the type V2})$$
$$\cup\, Y_{2,P_i,Q_i}\times\{z|\ z\in C\} \qquad (\text{the type V2-}z)$$
$$\cup\, Z_{2,P_i,Q_i}\times\{y|\ y\in C\} \qquad (\text{the type V2-}y)$$
$$\cup\bigcup_{\sigma\in\Lambda}\{(a,\phi_\sigma(a),\psi_\sigma(a)|\ a\in D\}, \qquad (\text{the type C2})$$

where Λ is a finite set and $(\phi_\sigma(x),\psi_\sigma(x))$ is a 2-tuple algebroidal function which satisfies the conditions $P_i(x,\phi_\sigma(x),\psi_\sigma(x))\equiv 0$, $Q_i(x,\phi_\sigma(x),\psi_\sigma(x))\equiv 0$.

Since

$$\Delta_{2,P_i,Q_i}=\Theta_{2,P_i,Q_i}$$
$$\cup\{a\in D|\ \text{for some } b_i\in C,\ (a,b_i)\in Y_{2,P_i,Q_i}\}$$
$$\cup\{a\in D|\ \text{for some } c_i\in C,\ (a,c_i)\in Z_{2,P_i,Q_i}\},$$

we obtain the desired result. q.e.d.

We obtain similar propositions for the sets

$$\{(a,b_i,c_i)|\ V_i(a,b_i,c_i)=W_i(a,b_i,c_i)=0\},$$
$$\{(a,b_i,c_i)|\ Q_i(a,b_i,c_i)=W_i(a,b_i,c_i)=0\}.$$

PROPOSITION 3.7. *Putting*

$$\Delta_{2,F_i}=\{a\in D|\ F_i(a,y_i,z_i),\ P_i(a,y_i,z_i) \text{ and } V_i(a,y_i,z_i) \text{ have a common divisor with deg.}\geqq 1\},$$

we have

$$\{(a,b_i,c_i)|\ F_i(a,b_i,c_i)=P_i(a,b_i,c_i)=V_i(a,b_i,c_i)=0\}$$
$$=\bigcup_{a\in\Delta_{2,F_i}}(\textit{analytic sets at } a \textit{ of the type V2}) \qquad (\textit{the type V2})$$

$\cup$ (*analytic sets of the type* V3) (*the type* V3)
$\cup \bigcup_{\sigma \in \Lambda} \{(a, \phi_\sigma(a), \psi_\sigma(a)) \mid a \in D\}$, (*the type* C2)

where Λ is a finite set and $(\phi_\sigma(x), \psi_\sigma(x))$ is a 2-tuple algebroidal function which satisfies the conditions $F_i(x, \phi_\sigma(x), \psi_\sigma(x)) \equiv 0$, $P_i(x, \phi_\sigma(x), \psi_\sigma(x)) \equiv 0$ and $V_i(x, \phi_\sigma(x), \psi_\sigma(x)) \equiv 0$.

PROOF. Suppose that P_i and V_i are expressed as

$$P_i = R_i P_i', \quad V_i = R_i V_i', \quad (P_i', V_i') = 1,$$

where $R_i, P_i', V_i' \in \mathcal{O}_D[y_i, z_i]$. Then,

$$\{F_i = P_i = V_i = 0\} = \{F_i = R_i = 0\} \cup \{F_i = P_i' = V_i' = 0\}.$$

Since $(F_i, R_i) = 1$, it follows from the same arguments as in Proposition 3.6 that

$$\begin{aligned} &\{F_i = R_i = 0\} \\ &= \bigcup_{a \in \Delta_{2, F_i, R_i}} (\text{analytic sets of the type V2 at } a) \\ &\quad \cup \bigcup_{\sigma \in \Lambda_1} \{(a, \phi_\sigma(a), \psi_\sigma(a)) \mid a \in D\}, \end{aligned}$$

where $\Delta_{2, F_i, R_i} = \{a \in D \mid F_i(a, y_i, z_i)$ and $R(a, y_i, z_i)$ have a common divisor with deg.$\geqq 1\}$, Λ_1 is a finite set and $(\phi_\sigma(x), \psi_\sigma(x))$ is a 2-tuple algebroidal function which satisfies the conditions $F_i(x, \phi_\sigma(x), \psi_\sigma(x)) \equiv 0$, $R_i(x, \phi_\sigma(x), \psi_\sigma(x)) \equiv 0$.

By similar consideration, we find that

$$\begin{aligned} &\{F_i = P_i' = V_i' = 0\} \\ &= \{P_i' = V_i' = 0\} \cap \{F_i = 0\} \\ &= \bigcup_{a \in \Delta'_{2, F_i}} (\text{analytic sets at } a \text{ of the type V2}) \\ &\quad \cup (\text{analytic sets of the type V3}) \\ &\quad \cup \bigcup_{\sigma \in \Lambda_2} \{(a, \phi_\sigma(a), \psi_\sigma(a)) \mid a \in D\}, \end{aligned}$$

where $\Delta'_{2, F_i} = \{a \in D \mid F_i(a, y_i, z_i)$, $P_i'(a, y_i, z_i)$ and $V_i'(a, y_i, z_i)$ have a common divisor with deg.$\geqq 1\}$, Λ_2 is a finite set and $(\phi_\sigma(x), \psi_\sigma(x))$ is a 2-tuple algebroidal function which satisfies the conditions $F_i(x, \phi_\sigma(x), \psi_\sigma(x)) \equiv 0$, $P_i'(x, \phi_\sigma(x), \psi_\sigma(x)) \equiv 0$ and $V_i'(x, \phi_\sigma(x), \psi_\sigma(x)) \equiv 0$.

From these facts, we obtain the desired result. q.e.d.

PROPOSITION 3.8. *Suppose that Q_i is decomposed into irreducible components as*

$$Q_i = q_i(x) \prod_{\alpha=1}^{l} q_i^{(\alpha)}(x, y_i) \prod_{\beta=1}^{l'} q'^{(\beta)}_i(x, z_i) \prod_{\gamma=1}^{l''} q''^{(\gamma)}_i(x, y_i, z_i).$$

Set

$$\Theta_{1,Q_i} = \{a \in D \mid q_i(a) = 0\},$$
$$Z_{2,Q_i} = \bigcup_{\gamma=1}^{l''} \{(a, c_i) \in D \times C \mid q''^{(\gamma)}_i(a, y_i, c_i) \equiv 0\},$$

and let $z_i = \phi^{(\beta)}_i(x)$ *be the algebroidal function defined by the equation* $q'^{(\beta)}_i(x, z_i) = 0$. *Then, we have*

$$\begin{aligned} &\{(a, c_i) \mid Q_i(a, y_i, c_i) \equiv 0\} \times \{y_i \mid y_i \in C\} \\ &= \Theta_{1,Q_i} \times \{(y_i, z_i) \mid (y_i, z_i) \in C^2\} && \text{(the type V1)} \\ &\cup Z_{2,Q_i} \times \{y_i \mid y_i \in C\} && \text{(the type V2-}y\text{)} \\ &\cup \bigcup_{\beta=1}^{l'} \{(a, \phi^{(\beta)}_i(a)) \mid a \in D\} \times \{y_i \mid y_i \in C\}. && \text{(the type C1-}y\text{)} \end{aligned}$$

PROOF. From

$$Q_i(a, y_i, c_i) = q_i(a) \prod_{\alpha=1}^{l} q_i^{(\alpha)}(a, y_i) \prod_{\beta=1}^{l'} q'^{(\beta)}_i(a, c_i) \prod_{\gamma=1}^{l''} q''^{(\gamma)}_i(a, y_i, c_i) \equiv 0,$$

it turns out that $q_i(a) = 0$ or $q'^{(\beta)}_i(a, c_i) = 0$ or $q''^{(\gamma)}_i(a, y_i, c_i) \equiv 0$, because $q_i^{(\alpha)}(x, y_i)$ is irreducible ($\alpha = 1, \cdots, l$). Then $\{(a, c_i) \mid Q_i(a, y_i, c_i) \equiv 0\} \times \{y_i \mid y_i \in C\}$ is written as the desired form. q.e.d.

In a similar way, we obtain the following

PROPOSITION 3.9. *Suppose that* W_i *is decomposed into irreducible components as*

$$W_i = w_i(x) \prod_{\alpha=1}^{r} w_i^{(\alpha)}(x, y_i) \prod_{\beta=1}^{r'} w'^{(\beta)}_i(x, z_i) \prod_{\gamma=1}^{r''} w''^{(\gamma)}_i(x, y_i, z_i).$$

Set

$$\Theta_{1,W_i} = \{a \in D \mid w_i(a) = 0\},$$
$$Y_{2,W_i} = \bigcup_{\gamma=1}^{r''} \{(a, b_i) \in D \times C \mid w''^{(\gamma)}_i(a, b_i, z_i) \equiv 0\},$$

and let $y_i = \phi^{(\alpha)}_i(x)$ *be the algebroidal function defined by the equation* $w^{(\alpha)}_i(x, y_i) = 0$. *Then, we have*

$$\begin{aligned} &\{(a, b_i) \mid W_i(a, b_i, z_i) \equiv 0\} \times \{z_i \mid z_i \in C\} \\ &= \Theta_{1,W_i} \times \{(y_i, z_i) \mid (y_i, z_i) \in C^2\} && \text{(the type V1)} \end{aligned}$$

$$\cup Y_{2,w_i}\times\{z_i|\ z_i\in C\} \qquad \text{(the type V2-}z)$$
$$\cup \bigcup_{\alpha=1}^{r}\{(a, \phi_i^{(\alpha)}(a))|\ a\in D\}\times\{z_i|\ z_i\in C\}. \qquad \text{(the type C1-}z)$$

Furthermore, we obtain the following two propositions.

PROPOSITION 3.10. (1) *The sets*

$$\{(a, b_i)|\ f_i^{(\alpha)}(a, b_i)=0,\ P_i(a, b_i, z_i)\equiv 0\},$$
$$\{(a, c_i)|\ g_i^{(\beta)}(a, c_i)=0,\ V_i(a, y_i, c_i)\equiv 0\}$$

are discrete subsets in $D\times C$.

(2) *The set* $\{(a, b_i)|\ f_i^{(\alpha)}(a, b_i)=0,\ P_i(a, b_i, z_i)\equiv 0\}\times\{z_i|\ z_i\in C\}$ *is a union of analytic sets of the type* V2-z.

Similarly the set $\{(a, c_i)|\ g_i^{(\beta)}(a, c_i)=0,\ V_i(a, y_i, c_i)\equiv 0\}\times\{y_i|\ y_i\in C\}$ *is a union of analytic sets of the type* V2-y.

PROOF. (1) We prove only the case of $\{(a, b_i)|\ f_i^{(\alpha)}(a, b_i)=0,\ P_i(a, b_i, z_i)\equiv 0\}$. Let us recall the prime decomposition of P_i supposed in the proof of Proposition 3.6:

$$P_i=p_i(x)\prod_{\alpha'=1}^{h}p_i^{(\alpha')}(x, y_i)\prod_{\beta=1}^{h'}p_i'^{(\beta)}(x, z_i)\prod_{\gamma=1}^{h''}p_i''^{(\gamma)}(x, y_i, z_i).$$

Noting that $p_i'^{(\beta)}(x, z_i)$ is irreducible, we see that

$$\begin{aligned}
&\{(a, b_i)|\ f_i^{(\alpha)}(a, b_i)=0,\ P_i(a, b_i, z_i)\equiv 0\}\\
=&\{(a, b_i)|\ f_i^{(\alpha)}(a, b_i)=p_i(a)=0\}\\
&\cup\bigcup_{\alpha'=1}^{h}\{(a, b_i)|\ f_i^{(\alpha)}(a, b_i)=p_i^{(\alpha')}(a, b_i)=0\}\\
&\cup\bigcup_{\gamma=1}^{h''}\{(a, b_i)|\ f_i^{(\alpha)}(a, b_i)=0,\ p_i''^{(\gamma)}(a, b_i, z_i)\equiv 0\}.
\end{aligned}$$

Obviously $\{(a, b_i)|\ f_i^{(\alpha)}(a, b_i)=p_i(a)=0\}$ is discrete in $D\times C$. Since $(f_i^{(\alpha)}, p_i^{(\alpha')})=1$, $\{(a, b_i)|\ f_i^{(\alpha)}(a, b_i)=p_i^{(\alpha')}(a, b_i)=0\}$ is discrete in $D\times C$. Moreover, from Proposition 3.3 (2), (3), it follows that $\{(a, b_i)|\ p_i''^{(\gamma)}(a, b_i, z_i)\equiv 0\}$ is discrete in $D\times C$, and so $\{(a, b_i)|\ f_i^{(\alpha)}(a, b_i)=0,\ p_i''^{(\gamma)}(a, b_i, z_i)\equiv 0\}$ is also discrete in $D\times C$. Therefore we have finished the proof.

(2) It is apparent from (1). q.e.d.

PROPOSITION 3.11. (1) *The set* $H_i^{(\gamma)}$ *is the hypersurface* $\{(a, b_i, c_i)\in D\times C^2|\ h_i^{(\gamma)}(a, b_i, c_i)=0\}$ *or decomposed into irreducible components of the types* V2, V3 *and* C2.

(2) *Irreducible components of the type* V3 *of* $H_i^{(\gamma)}$ *are either the same sets that are obtained by the decompositions of* $\{F_i=P_i=Q_i=0\}$ *or the sets contained in irreducible components of the types* V2, C2 *of* $\{P_i=Q_i=0\}$, $\{V_i=W_i=0\}$, $\{Q_i=W_i=0\}$ *and* $\{F_i=P_i=V_i=0\}$.

(3) *Irreducible components of the type* C2 *of* $H_i^{(\gamma)}$ *are the same sets that are obtained by the decompositions of* $\{P_i=Q_i=0\}$, $\{V_i=W_i=0\}$, $\{Q_i=W_i=0\}$ *and* $\{F_i=P_i=V_i=0\}$.

PROOF. (1) By the same arguments as in Proposition 1.5 (1), we find that $H_i^{(\gamma)}$ is written as

$$H_i^{(\gamma)}=\{(a,b_i,c_i)\in D\times C^2 \mid h_i^{(\gamma)}(a,b_i,c_i)=0,\ K_l(a,b_i,c_i)=0 \quad (l=1,2,3,\cdots)\}$$

with infinitely many polynomials K_l's in $\mathcal{O}_D[y_i,z_i]$. If $h_i^{(\gamma)}|K_l$ $(l=1,2,3,\cdots)$, then

$$H_i^{(\gamma)}=\{(a,b_i,c_i)\mid h_i^{(\gamma)}(a,b_i,c_i)=0\}.$$

Suppose that there exists a K_{l_1} such that $h_i^{(\gamma)}\nmid K_{l_1}$. Since $(h_i^{(\gamma)},K_{l_1})=1$, it follows from the same arguments as in Proposition 3.6 that

$$\{(a,b_i,c_i)\mid h_i^{(\gamma)}(a,b_i,c_i)=K_{l_1}(a,b_i,c_i)=0\}=\bigcup_{\sigma\in\Lambda_1}V_\sigma^{(2)}\cup\bigcup_{\sigma\in\Lambda_2}C_\sigma^{(2)},$$

where $V_\sigma^{(2)}$ is an analytic set of the type V2, $C_\sigma^{(2)}$ is an analytic set of the type C2 and Λ_2 is a finite set. Hence,

$$H_i^{(\gamma)}=\bigcup_{\sigma\in\Lambda_1}[V_\sigma^{(2)}\cap\{x\mid K_l(x)=0,\ l\neq l_1\}]\cup\bigcup_{\sigma\in\Lambda_2}[C_\sigma^{(2)}\cap\{x\mid K_l(x)=0,\ l\neq l_1\}],$$

where $x=(a,b_i,c_i)$. If $K_l|_{V_\sigma^{(2)}}\equiv 0$ for any $l\neq l_1$, then $V_\sigma^{(2)}\cap\{x\mid K_l(x)=0,\ l\neq l_1\}=V_\sigma^{(2)}$, if not, then $V_\sigma^{(2)}\cap\{x\mid K_l(x)=0,\ l\neq l_1\}$ is a union of analytic sets of the type V3 or an empty set. Similarly, $C_\sigma^{(2)}\cap\{x\mid K_l(x)=0,\ l\neq l_1\}$ is $C_\sigma^{(2)}$ or a union of analytic sets of the type V3 or an empty set. Therefore we have finished the proof.

(2) Suppose that $\{(a,b_i,c_i)\}$ is an irreducible component of the type V3 of $H_i^{(\gamma)}$. By the definition of $H_i^{(\gamma)}$, there exist holomorphic functions $y_i(t)$, $z_i(t)$ defined on a neighborhood of $t=0$ which satisfy the conditions:

$$y_i(0)=b_i,\ z_i(0)=c_i,$$
$$h_i^{(\gamma)}(a,y_i(t),z_i(t))\equiv 0,$$

$$\begin{cases} \dfrac{dy_i(t)}{dt} = P_i W_i(a, y_i(t), z_i(t)) \\ \dfrac{dz_i(t)}{dt} = Q_i V_i(a, y_i(t), z_i(t)) . \end{cases}$$

If at least one of $y_i(t)$ and $z_i(t)$ is not a constant, (a, b_i, c_i) is contained in a certain irreducible component of $H_i^{(7)}$ of higher dimension. Hence, $y_i(t) = b_i$, $z_i(t) = c_i$. Therefore $h_i^{(7)}(a, b_i, c_i) = 0$, $P_i W_i(a, b_i, c_i) = 0$ and $Q_i V_i(a, b_i, c_i) = 0$, which implies the desired result.

(3) The proof is given by the same arguments as in (2). q.e.d.

4° Proof of Theorem 5.

PROOF. From Proposition 3.2, Theorem 4 and Proposition 3.6$\sim$ Proposition 3.11, we find that

$$\begin{aligned} \Theta_1 &= \bigcup_{i=1}^{m} (\Theta_{1,e_i} \cup \Theta_{1,Q_i} \cup \Theta_{1,W_i}) \\ &= \bigcup_{i=1}^{m} \{a \in D \mid e_i(a) = 0 \text{ or } q_i(a) = 0 \text{ or } w_i(a) = 0\} \\ &= \{a \in D \mid e_1(a) = 0 \text{ or } q_1(a) = 0 \text{ or } w_1(a) = 0\}, \end{aligned}$$

where $\Theta_{1,e_i} = \{a \in D \mid e_i(a) = 0\}$,

$$\begin{aligned} \Theta_2 = \Big\{ a \in \bigcup_{i=1}^{m} (\Delta_{2,P_i,Q_i} \cup \Delta_{2,V_i,W_i} \cup \Delta_{2,Q_i,W_i} \cup \Delta_{2,F_i} \cup \Delta_i^{(2)} \cup \Delta_i^{(3)} \cup \Delta_i^{(4)}) - \Theta_1 \,\Big|\, &\text{there exist vertical singularity sets at } a \text{ of the type V2 which} \\ &\text{are not contained in any covering singularity sets of the type C1} \Big\}, \\ = \Big\{ a \in \bigcup_{i=1}^{m} (\Delta_i^{(1)} \cup \Delta_i^{(2)} \cup \Delta_i^{(3)} \cup \Delta_i^{(4)}) - \Theta_1 \,\Big|\, &\text{there exist vertical singularity sets at } a \text{ of the type V2 which are not contained in any} \\ &\text{covering singularity sets of the type C1} \Big\}, \end{aligned}$$

where $\Delta_i^{(1)} = \Delta_{2,P_i,Q_i} \cup \Delta_{2,V_i,W_i} \cup \Delta_{2,Q_i,W_i} \cup \Delta_{2,F_i}$, and

$$\Theta_3 = \Big\{ a \in \bigcup_{i=1}^{m} \Delta_i - \Theta_1 \cup \Theta_2 \,\Big|\, \text{there exist vertical singularity sets at } a \text{ of the type V3 which do not contained in any covering singularity sets} \Big\}.$$

q.e.d.

§ 4. Fixed and Movable Singularities of (F_2).

In this section, using the results in § 3, we study the rational differential equation of order two:

$$(\mathrm{F}_2)\qquad \frac{d^2y}{dx^2}=\frac{P\left(x, y, \dfrac{dy}{dx}\right)}{Q\left(x, y, \dfrac{dy}{dx}\right)},$$

where $P, Q\in \mathcal{O}_D[y, dy/dx]$, $(P, Q)=1$.

According to particular equations, there exist various ways of rewriting (F_2) into a system of rational differential equations. Here we rewrite (F_2) into the system

$$(\mathrm{E}_2')\qquad \begin{cases} \dfrac{dy}{dx}=z \\ \dfrac{dz}{dx}=\dfrac{P(x, y, z)}{Q(x, y, z)}. \end{cases}$$

In the case of (E_2'), it is better to take the Hirzebruch surface $\Sigma^{(2)}$ as a rational compactification of C^2, and the fiber space $\mathcal{F}=(X, \mathrm{pr}, D)$ as a definition space of (E_2'), where $X=D\times\Sigma^{(2)}$. Note that $\Sigma^{(2)}$ is a manifold obtained by gluing four copies U_i $(i=1\sim4)$ of C^2 by the following change of coordinates:

$$(y, z)\in U_1,\ (y, s)\in U_2,\ (v, w)\in U_3,\ (v, t)\in U_4,$$
$$s=\frac{1}{z},\ v=\frac{1}{y},\ w=-\frac{z}{y^2},\ t=\frac{1}{w}.$$

The extension $\boldsymbol{E}_2'=\{(\mathrm{E}_2')_i\}_{i=1}^4$ of (E_2') onto the definition manifold $X=D\times\Sigma^{(2)}$ is constructed as follows:

$$(\mathrm{E}_2')_1\qquad \begin{cases} \dfrac{dy}{dx}=z \\ \dfrac{dz}{dx}=\dfrac{P(x, y, z)}{Q(x, y, z)}. \end{cases}$$

$$(\mathrm{E}_2')_2\qquad \begin{cases} \dfrac{dy}{dx}=\dfrac{1}{s} \\ \dfrac{ds}{dx}=-s^2\dfrac{P\left(x, y, \dfrac{1}{s}\right)}{Q\left(x, y, \dfrac{1}{s}\right)}=\dfrac{-s^{q+2}\mathcal{P}_2(x, y, s)}{s^pQ_2(x, y, s)}, \end{cases}$$

where $P(x, y, 1/s)=(1/s^p)\mathcal{P}_2(x, y, s)$, $Q(x, y, 1/s)=(1/s^q)Q_2(x, y, s)$, $p=\deg_z P$. $q=\deg_z Q$.

$$(\mathrm{E}_2')_3 \qquad \begin{cases} \dfrac{dv}{dx}=w \\[2ex] \dfrac{dw}{dx}=\dfrac{2w^2}{v}-v^2\dfrac{P\left(x, \dfrac{1}{v}, -\dfrac{w}{v^2}\right)}{Q\left(x, \dfrac{1}{v}, -\dfrac{w}{v^2}\right)} \\[2ex] \quad =\dfrac{2w^2}{v}-\dfrac{v^{\tau+2}\mathcal{P}_3(x, v, w)}{v^\sigma Q_3(x, v, w)}=\dfrac{P_3(x, v, w)}{Q_3(x, v, w)}, \end{cases}$$

where $P(x, 1/v, -w/v^2)=(1/v^\sigma)\mathcal{P}_3(x, v, w)$, $Q(x, 1/v, -w/v^2)=(1/v^\tau)Q_3(x, v, w)$, $\mathcal{P}_3$ and Q_3 do not have v as a divisor, $2p\leqq\sigma\leqq 2p+\deg_y P$, $2q\leqq\tau\leqq 2q+\deg_y Q$, and P_3, $Q_3\in\mathcal{O}_D[v, w]$ with $(P_3, Q_3)=1$.

$$(\mathrm{E}_2')_4 \qquad \begin{cases} \dfrac{dv}{dx}=\dfrac{1}{t} \\[2ex] \dfrac{dt}{dx}=-t^2\dfrac{P_3\left(x, v, \dfrac{1}{t}\right)}{Q_3\left(x, v, \dfrac{1}{t}\right)}=\dfrac{-t^{q+2}\mathcal{P}_4(x, v, t)}{t^r Q_4(x, v, t)}, \end{cases}$$

where $P_3(x, v, 1/t)=(1/t^r)\mathcal{P}_4(x, v, t)$, $Q_3(x, v, 1/t)=(1/t^q)Q_4(x, v, t)$,

$$r=\deg_w P_3=\begin{cases}\max(p, q+2) & (\text{if } p\neq q+2) \\ \leqq q+2 & (\text{if } p=q+2)\end{cases}, \quad q=\deg_w Q_3=\deg_z Q.$$

The reason why we take $\Sigma^{(2)}$ as a rational compactification of $\boldsymbol{C}^2$ is that the change of coordinates: $v=1/y$, $w=-z/y^2$ transforms the derivative $z=dy/dx$ to the derivative $w=dv/dx$.

Let $\mathcal{S}$ be the singular initial set of (F_2) (i.e. (E_2')). Applying Proposition 3.1 to the system E_2', we find that $\mathcal{S}$ is written as

$$\mathcal{S}=\mathcal{S}'\cup\mathcal{S}''.$$

Here,

$$\begin{aligned}\mathcal{S}'=&\{(a, b, c)\mid P(a, b, c)=Q(a, b, c)=0\} \\ &\cup\{(a, b, d)\mid \mathcal{P}_2(a, b, d)=Q_2(a, b, d)=0\} \\ &\cup\{(a, \beta, \gamma)\mid P_3(a, \beta, \gamma)=Q_3(a, \beta, \gamma)=0\}\end{aligned}$$

$$\cup\{(a,\beta,\delta)\mid \mathcal{P}_4(a,\beta,\delta)=Q_4(a,\beta,\delta)=0\}$$
$$\cup\{(a,b)\mid Q(a,b,z)\equiv 0\}\times\{z\mid z\in \boldsymbol{P}\}$$
$$\cup\{(a,\beta)\mid Q_3(a,\beta,w)\equiv 0\}\times\{w\mid w\in \boldsymbol{P}\},$$

and

$$\mathcal{S}''=\begin{cases}\{(a,b,\infty)\mid \mathcal{P}_2(a,b,0)=0\}\cup\{(a,\beta,\infty)\mid \mathcal{P}_4(a,\beta,0)=0\} & (\text{if } p>q+3)\\ \{(a,\infty)\mid \mathcal{P}_2(a,y,0)\equiv 0\}\times\{y\mid y\in \boldsymbol{P}\} & (\text{if } p=q+3)\\ \text{the surface } \{s=t=0\} & (\text{if } p<q+3).\end{cases}$$

Therefore, from Proposition 3.6 and 3.9, immediately we obtain the following

PROPOSITION 4.1. *Suppose that Q and Q_3 are decomposed into irreducible components as*

$$Q=q(x)\prod_{\alpha=1}^{l}q^{(\alpha)}(x,y)\prod_{\beta=1}^{l'}q'^{(\beta)}(x,z)\prod_{\gamma=1}^{l''}q''^{(\gamma)}(x,y,z),$$
$$Q_3=q_3(x)\prod_{\alpha=1}^{r}q_3^{(\alpha)}(x,v)\prod_{\beta=1}^{r'}q_3'^{(\beta)}(x,w)\prod_{\gamma=1}^{r''}q_3''^{(\gamma)}(x,v,w).$$

Set

$\boldsymbol{\Theta}_1=\{a\in D\mid q(a)=0\},$

$\boldsymbol{\Theta}_2'=\{a\in D\mid P(a,y,z),\ Q(a,y,z)$ *have a common divisor with* $\deg.\geqq 1\}$
$\cup\{a\in D\mid \mathcal{P}_2(a,y,s),\ Q_2(a,y,s)$ *have a common divisor with* $\deg.\geqq 1\}$
$\cup\{a\in D\mid P_3(a,v,w),\ Q_3(a,v,w)$ *have a common divisor with* $\deg.\geqq 1\}$
$\cup\{a\in D\mid \mathcal{P}_4(a,v,t),\ Q_4(a,v,t)$ *have a common divisor with* $\deg.\geqq 1\},$

$$Y_2=\bigcup_{\gamma=1}^{l''}\{(a,b)\in D\times \boldsymbol{C}\mid q''^{(\gamma)}(a,b,z)\equiv 0\},$$
$$Y_2'=\bigcup_{\gamma=1}^{r''}\left\{(a,b)\in D\times \boldsymbol{P}^*\mid q_3''^{(\gamma)}\left(a,\frac{1}{b},w\right)\equiv 0\right\},$$
$$Z_2=\{(a,\infty)\mid a\in D,\ \mathcal{P}_2(a,y,0)\equiv 0\},$$

and let $y=G^{(\alpha)}(x)$, $v=G_3^{(\alpha)}(x)$ be the algebroidal functions obtained by the equations $q^{(\alpha)}(x,y)=0$, $q_3^{(\alpha)}(x,v)=0$ respectively. Then the singular initial set $\mathcal{S}$ of (F_2) *is written as*

$$\mathcal{S}=\mathcal{S}'\cup\mathcal{S}''.$$

Here

$\mathcal{S}'=\boldsymbol{\Theta}_1\times\Sigma^{(2)}$ (*the type* V1)
$\cup\bigcup_{a\in\boldsymbol{\Theta}_2'}$ (*vertical singularity sets at a of the type* V2) (*the type* V2)
$\cup\, Y_2\times\{z\mid z\in\boldsymbol{P}\}$ (*the type* V2-z)

$$\cup Y_2'\times\{w|\ w\in \boldsymbol{P}\} \qquad (\textit{the type } \mathrm{V2}\text{-}w)$$
$$\cup \bigcup_{\alpha=1}^{l}\{(a,\ G^{(\alpha)}(a))\in D\times \boldsymbol{P}|\ a\in D\}\times\{z|\ z\in \boldsymbol{P}\} \qquad (\textit{the type } \mathrm{C1}\text{-}z)$$
$$\cup \bigcup_{\alpha=1}^{r}\{(a,\ G_3^{(\alpha)}(a))\in D\times \boldsymbol{P}|\ a\in D\}\times\{w|\ w\in \boldsymbol{P}\} \qquad (\textit{the type } \mathrm{C1}\text{-}w)$$
$$\cup \bigcup_{i=1}^{4}\bigcup_{\sigma\in\Lambda_i'}\{(a,\ \phi_i^{(\sigma)}(a),\ \psi_i^{(\sigma)}(a))\in D\times\kappa_i(U_i)=D\times \boldsymbol{C}^2|\ a\in D\}, \qquad (\textit{the type } \mathrm{C2})$$

Λ_i' is a finite set $(i=1\sim 4)$, $(\phi_i^{(\sigma)}(x),\ \psi_i^{(\sigma)}(x))$ is a 2-tuple algebroidal function which satisfies the following conditions:

$$P_i(x,\phi_i^{(\sigma)}(x),\ \psi_i^{(\sigma)}(x))\equiv 0,\ Q_i(x,\phi_i^{(\sigma)}(x),\ \psi_i^{(\sigma)}(x))\equiv 0, \qquad (\textit{if } i=1 \textit{ or } 3)$$
$$\mathcal{P}_i(x,\phi_i^{(\sigma)}(x),\ \psi_i^{(\sigma)}(x))\equiv 0,\ \mathcal{Q}_i(x,\phi_i^{(\sigma)}(x),\ \psi_i^{(\sigma)}(x))\equiv 0 \qquad (\textit{if } i=2 \textit{ or } 4).$$

Furthermore, the set S'' is given in the following way:

(i) *If $p>q+3$, then*

$$S''=Z_2\times\{y|\ y\in \boldsymbol{P}\} \qquad (\textit{the type } \mathrm{V2}\text{-}y)$$
$$\cup\bigcup_{i=2,4}\bigcup_{\sigma\in\Lambda_i''}\{(a,\ \phi_i^{(\sigma)}(a),\ 0)\in D\times\kappa_i(U_i)|\ a\in D\}, \qquad (\textit{the type } \mathrm{C2})$$

Λ_i'' is a finite set $(i=2,4)$ and $y=\phi_2^{(\sigma)}(x)$ (resp. $v=\phi_4^{(\sigma)}(x)$) is an algebroidal function which satisfies the condition:

$$\mathcal{P}_2(x,\phi_2^{(\sigma)}(x),0)\equiv 0 \ \ (\textit{resp. } \mathcal{P}_4(x,\phi_4^{(\sigma)}(x),0)\equiv 0).$$

(ii) *If $p=q+3$, then*

$$S''=Z_2\times\{y|\ y\in \boldsymbol{P}\}. \qquad (\textit{the type } \mathrm{V2}\text{-}y)$$

(iii) *If $p<q+3$, then*

$$S''=\textit{the surface } \{s=t=0\}. \qquad (\textit{the type } \mathrm{C1}\text{-}y)$$

From this proposition, we obtain the next proposition.

PROPOSITION 4.2. *The fixed singularity set $\boldsymbol{\Theta}$ of* (F_2) *is written as*

$$\boldsymbol{\Theta}=\boldsymbol{\Theta}_1\cup\boldsymbol{\Theta}_2,$$

where $\boldsymbol{\Theta}_1$ is the same set as in Proposition 4.1 *and*

$\{a\in D|\ a\in\boldsymbol{\Theta}_2'$ *or for some* $b\in \boldsymbol{P}$, $(a,b)\in Y_2\cup Y_2'$, *and there exist vertical singularity sets at a of the type* V2 *which are not contained in any covering singularity sets of the*

$$\boldsymbol{\Theta}_2=\begin{cases} \textit{type } \mathrm{C1}\} \cup \{a \in D \mid (a, \infty) \in Z_2\} & (\textit{if } p \geqq q+3), \\ \{a \in D \mid a \in \boldsymbol{\Theta}_2' \textit{ or for some } b \in \boldsymbol{P},\ (a, b) \in Y_2 \cup Y_2', \textit{ and there exist vertical singularity sets at } a \textit{ of the type } \mathrm{V2} \textit{ which are not contained in any covering singularity sets of the type } \mathrm{C1}\} & (\textit{if } p<q+3). \end{cases}$$

In particular, (E_2) *does not have the fixed singularity set of the third kind.*

From the above propositions and Theorem 6, we can derive the following

THEOREM 7. *Suppose that a global solution* $\boldsymbol{\Phi}$ *of* (F_2) *has a singularity* $\boldsymbol{\omega}$ *on* $a \in D$. *Let* $S_{\boldsymbol{\omega}}$ *be the cluster set of* $\boldsymbol{\Phi}$ *at* $\boldsymbol{\omega}$.

(1) *Suppose* $a \in D-\boldsymbol{\Theta}$. *If* ($F_2$) *has no covering singularity sets, then* $\boldsymbol{\omega}$ *is an algebraic singularity.*

(2) *Suppose* $p \geqq q+3$ *and* $a \in D-\boldsymbol{\Theta}$. *If* $\boldsymbol{\omega}$ *is an essential singularity, then,*

$$\{a\} \times S_{\boldsymbol{\omega}} \subset \bigcup_{\alpha=1}^{l} \{(a, G^{(\alpha)}(a)) \in D \times \boldsymbol{P}\} \times \{z \mid z \in \boldsymbol{P}\} \cup \bigcup_{\alpha=1}^{r} \{(a, G_3^{(\alpha)}(a)) \in D \times \boldsymbol{P}\} \times \{w \mid w \in \boldsymbol{P}\}.$$

Hence, any singularity on a is at most ordinary in (y, y, v, v)*-direction. If there do not exist* $G^{(\alpha)}(x)$*'s and* $G_3^{(\alpha)}(x)$*'s, then any singularity on a is either an algebraic singularity or an ordinary singularity.*

(3) *Suppose* $p<q+3$ *and* $a \in D-\boldsymbol{\Theta}$. *If* $\boldsymbol{\omega}$ *is an essential singularity, then*

$$\{a\} \times S_{\boldsymbol{\omega}} \subset \bigcup_{\alpha=1}^{l} \{(a, G^{(\alpha)}(a)) \in D \times \boldsymbol{P}\} \times \{z \mid z \in \boldsymbol{P}\} \cup \bigcup_{\alpha=1}^{r} \{(a, G_3^{(\alpha)}(a)) \in D \times \boldsymbol{P}\} \times \{w \mid w \in \boldsymbol{P}\} \cup \{(a, y, \infty) \mid y \in \boldsymbol{P}\}.$$

Hence, if there do not exist $G^{(\alpha)}(x)$*'s and* $G_3^{(\alpha)}(x)$*'s, then any singularity on a is at most ordinary in* (z, s, w, t)*-direction. If there do not exist* $G^{(\alpha)}(x)$*'s,* $G_3^{(\alpha)}(x)$*'s and the surface* $\{s=t=0\}$*, then any singularity on a is either an algebraic singularity or an ordinary singularity.*

(4) *Suppose* $a \in \boldsymbol{\Theta}_2$. *If there do not exist* $G^{(\alpha)}(x)$*'s,* $G_3^{(\alpha)}(x)$*'s and vertical singularity sets of the type* V2-*a*, V2-*z*, *then* $\boldsymbol{\omega}$ *is at most*

ordinary in (z, s, w, t)*-direction. If there do not exist the surface* $\{s=t=0\}$ *(in the case* $p<q+3$*) and the vertical singularity sets of the type* V2-a, V2-y, *then* $\boldsymbol{\omega}$ *is at most ordinary in* (y, y, v, v)*-direction.*

(2) and (3) in the above theorem are obtained by Kimura [1].

Appendix: The prime factorization theorem for $\mathcal{O}_M[X_1, \cdots, X_n]$**.**

As is stated in Theorem 1, a prime factorization theorem holds in $\mathcal{O}_D[y_1, \cdots, y_n]$. We will prove this theorem in a more general situation.

Let $\mathcal{O}_M$ be the integral domain of holomorphic functions on a complex manifold M, and let $\mathcal{O}_M[X_1, \cdots, X_n]$ be a polynomial ring over $\mathcal{O}_M$. We prove that if the Cousin II problem is solvable on M, then a prime factorization theorem very similar to that in a UFD holds in $\mathcal{O}_M[X_1, \cdots, X_n]$, though $\mathcal{O}_M[X_1, \cdots, X_n]$ is not a UFD.

In order to prove this fact, we define a ring called w-UFD (weak UFD) in A1. We show that a prime factorization theorem holds in any polynomial ring over any w-UFD. Next, in A2, we show that $\mathcal{O}_M$ is a w-UFD if the Cousin II problem is solvable on M. From these results, the prime factorization theorem for $\mathcal{O}_M[X_1, \cdots, X_n]$ is obvious. In A2, we study the relationship between the prime factorization in $\mathcal{O}_M[X_1, \cdots, X_n]$ and that in $\mathcal{M}_M[X_1, \cdots, X_n]$, where $\mathcal{M}_M$ is the ring of meromorphic functions on M. We also study the relationship between the prime factorization in $\boldsymbol{C}[X_1, \cdots, X_n]$ and that in $\mathcal{O}_C[X_1, \cdots, X_n]$. Lastly, in A3, we prove the theorems in A1 and A2.

A1. Polynomial rings over w-UFD's.

First, let us recall some fundamental definitions.

Let R be an integral domain.

If a and b are elements in R and there exists $c \in R$ such that $a=bc$, then it is said that b divides a (symbolically, $b|a$) and that b is a divisor of a. Let 1 be the identity of R. An element $u \in R$ is called a unit if it is a divisor of 1. When $a=bu$ and u is a unit, a and b are called associated elements, or simply accociates, and we symbolically write as $a \sim b$.

An element $a \in R$ is said to be irreducible if it is neither zero nor a unit and every divisor of a is a unit or an associated element of a. An element $a \in R$ is said to be prime if it is neither zero nor a unit and the ideal (a) generated by a is a prime ideal. In an integral domain, a prime element is irreducible, but an irreducible element is not prime in general.

Let $a_1, \cdots, a_m$ be in R. An element $d \in R$ is called a greatest common divisor (GCD) of $a_1, \cdots, a_m$ when it satisfies the following conditions:

(1) d is a common divisor of $a_1, \cdots, a_m$.

(2) If c is a common divisor of $a_1, \cdots, a_m$, then $c|d$.

A GCD of $a_1, \cdots, a_m$ does not always exist, but if both d and d' are GCD's, then $d \sim d'$. Elements $a_1, \cdots, a_m$ are said to be relatively prime if any common divisor of a_j's is a unit.

DEFINITION I. An integral domain R is called *a weak* UFD (w-UFD) if it satisfies the following three conditions:

W1. Every irreducible element in R is a prime element.

W2. Any element in R which is neither zero nor a unit is divisible by an irreducible element.

W3. If $a_1, \cdots, a_m$ are in R and any one of them is not zero, then there exists a GCD of $a_1, \cdots, a_m$.

REMARK. Let R be an integral domain. R is a UFD if and only if it satisfies the following two conditions:

U1. Every irreducible element in R is a prime element.

U2. The ascending chain condition for principal ideals holds in R.

From U2, we can immediately derive the following property.

U3. Any element in R which is neither zero nor a unit is divisible by an irreducible element.

From U1, U2 and U3, we obtain the prime factorization theorem for R. From this theorem, the next property is deduced.

U4. If $a_1, \cdots, a_m$ are in R and any one of them is not zero, then there exists a GCD of $a_1, \cdots, a_m$.

Therefore a w-UFD is nothing but an integral domain in which the finiteness (any element which is neither zero nor a unit can be expressed as a product of finitely many primes) required by U2 is not assumed, but U1, U3, U4 are assumed.

THEOREM I. (1) *A* UFD *is a* w-UFD.

(2) *A* w-UFD *R is a* UFD *if and only if it satisfies the following*

condition: Any element in R which is neither zero nor a unit can be expressed as a product of finitely many irreducible elements.

(3) *A* w-UFD *is a normal ring.*

From the above Remark and Theorem I, we see that a w-UFD is an integral domain with weaker axioms than those of a UFD, and that w-UFD's are intermediate rings of UFD's and normal rings. However, a polynomial ring over a w-UFD have the following properties very similar to those of a polynomial ring over a UFD.

THEOREM II. *Let R be a* w-UFD, *and let $R[X_1, \cdots, X_n]$ be a polynomial ring in n indeterminates over R.*

(1) *Any polynomial $f \in R[X_1, \cdots, X_n]$ with* $\deg f \geqq 1$ *can be expressed as $f = af_1 \cdots f_p$, where $a \in R - \{0\}$ and f_j's $(j=1, \cdots, p)$ are irreducible polynomials in $R[X_1, \cdots, X_n]$ with* $\deg f_j \geqq 1$.

(2) *If a polynomial $f \in R[X_1, \cdots, X_n]$ with* $\deg f \geqq 1$ *is decomposed in two ways as $f = af_1 \cdots f_p = bg_1 \cdots g_q$ in the sense of* (1), *then a and b are associates, $p=q$, and f_j and g_j $(j=1, \cdots, p)$ are associates by the proper rearrangement of indices.*

(3) *$R[X_1, \cdots, X_n]$ is a* w-UFD.

(4) *If R is not a* UFD, *then $R[X_1, \cdots, X_n]$ is not a* UFD.

DEFINITION II. Let f be a polynomial in $R[X_1, \cdots, X_n]$ with $\deg f \geqq 1$. f is said to be *primitive* if and only if coefficients of f are relatively prime. In the case $f = aX_1^{k_1} \cdots X_n^{k_n}$, f is *primitive* if and only if a is a unit.

THEOREM III. *Let R be a* w-UFD, *Q a quotient field of R, and let f be a polynomial in $R[X_1, \cdots, X_n]$ with* $\deg f \geqq 1$.

(1) *f is irreducible in $R[X_1, \cdots, X_n]$ if and only if it is primitive in $R[X_1, \cdots, X_n]$ and irreducible in $Q[X_1, \cdots, X_n]$.*

(2) *Let $f = af_1 \cdots f_p$ be the prime factorization in $R[X_1, \cdots, X_n]$ in the sense of Theorem* II. *Then $f = (af_1) \cdots f_p$ is the prime factorization in $Q[X_1, \cdots, X_n]$ in the sense of* UFD. *Conversely, if f is decomposed into irreducible polynomials in $Q[X_1, \cdots, X_n]$ as $f = F_1 \cdots F_p$, then f can be decomposed into irreducible polynomials in $R[X_1, \cdots, X_n]$ as $f = af_1 \cdots f_p$, where $a \in R - \{0\}$, F_j and f_j $(j=1, \cdots, p)$ are associates in $Q[X_1, \cdots, X_n]$.*

On the resultant of two polynomials f, $g \in R[X_1, \cdots, X_n]$, we have the following

THEOREM IV. *Let R be a* w-UFD, *and let*

$$f=a_0X^m+a_1X^{m-1}+\cdots+a_m,$$
$$g=b_0X^n+b_1X^{n-1}+\cdots+b_n$$

be polynomials in $R[X]$, where $m\geqq 1$, $n\geqq 1$, $a_0\neq 0$ and $b_0\neq 0$. Then, f and g have a common divisor $h\in R[X]$ with $\deg h\geqq 1$ *if and only if $R(f,g)=0$, where*

$$R(f,g)=\left|\begin{matrix} a_0\cdots\cdots\cdots a_m & & 0 \\ \ddots & & \ddots \\ 0 & a_0\cdots\cdots\cdots & a_m \\ b_0\cdots\cdots\cdots b_n & & 0 \\ \ddots & & \ddots \\ 0 & b_0\cdots\cdots\cdots & b_n \end{matrix}\right| \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} n \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} m \end{matrix}.$$

$R(f,g)$ is called *the resultant of f and g.*

A2. A w-UFD $\mathcal{O}_M$.

Let M be a complex manifold, and let $\mathcal{O}_M$ be the ring of holomorphic functions on M. If $\mathcal{O}_M=\boldsymbol{C}$, then, of course, $\mathcal{O}_M$ is a UFD. Therefore we restricted ourself to the case $\mathcal{O}_M\supsetneqq\boldsymbol{C}$ in the following.

Now let $\mathcal{O}_M^*=\mathcal{O}_M-\{0\}$, $\mathrm{Div}(M)=\{$divisors on $M\}$, $\mathrm{Div}^+(M)=\{D\in\mathrm{Div}(M)\mid D\geqq 0\}$, and let ϕ denote the mapping $\mathcal{O}_M^*\longrightarrow\mathrm{Div}^+(M)$, $f\longrightarrow\phi(f)=\sum_\nu n_\nu D_\nu$, where $\sum_\nu n_\nu D_\nu$ is the divisor of zeros of f. Then we obtain the following theorems.

THEOREM V. *Suppose that f is in $\mathcal{O}_M^*$ and that ϕ is surjective.*

(1) *f is a unit in $\mathcal{O}_M$ if and only if $\phi(f)=0$.*

(2) *f is irreducible in $\mathcal{O}_M$ if and only if $\phi(f)=D$ with an irreducible hypersurface D.*

THEOREM VI. *Suppose that ϕ is surjective.*

(1) *$\mathcal{O}_M$ is a* w-UFD.

(2) *If there exists a divisor $\sum_\nu n_\nu D_\nu\in\mathrm{Div}^+(M)$ with infinitely many nonzero n_ν's, then $\mathcal{O}_M$ is not a* UFD.

REMARK I. Let $\mathcal{M}_M$ denote the field of meromorphic functions on M, and let $\mathcal{M}_M^* = \mathcal{M}_M - \{0\}$. Then, $\phi : \mathcal{O}_M^* \longrightarrow \mathrm{Div}^+(M)$ is surjective if and only if $\phi : \mathcal{M}_M^* \longrightarrow \mathrm{Div}(M)$, $f \longrightarrow \phi(f) = (f)_0 - (f)_\infty$ is surjective. Namely, surjectivity of ϕ is equivalent to the solvability of the Cousin II problem on M.

REMARK II. Let D be a domain in C. Then $\phi : \mathcal{O}_D^* \longrightarrow \mathrm{Div}^+(D)$ is surjective (i.e. Weierstrass' theorem) and there exists a divisor $\sum_\nu n_\nu D_\nu$ with infinitely many nonzero n_ν's. Therefore $\mathcal{O}_D$ is a w-UFD, but not a UFD. Similarly, it is well known that on a cylindrical domain $\Delta = \prod_{j=1}^n D_j \subset C^n$ whose components are simply connected, the Cousin II problem is solvable, i.e., $\phi : \mathcal{O}_\Delta^* \longrightarrow \mathrm{Div}^+(\Delta)$ is surjective and that there exists a divisor $\sum_\nu n_\nu D_\nu$ with infinitely many nonzero n_ν's. Therefore $\mathcal{O}_\Delta$ is also a w-UFD, but not a UFD.

From Theorem II and VI, it is obvious that if the Cousin II problem is solvable on M, a prime factorization theorem holds in $\mathcal{O}_M[X_1, \cdots, X_n]$. The relationship between the prime factorization in the w-UFD $\mathcal{O}_M[X_1, \cdots, X_n]$ and that in the UFD $\mathcal{M}_M[X_1, \cdots, X_n]$ is given by Theorem III. Suppose that f is a polynomial in $C[X_1, \cdots, X_m]$. The following theorem shows the relationship between the prime factorization of f in the UFD $C[X_1, \cdots, X_n]$ and that in the w-UFD $\mathcal{O}_C[X_1, \cdots, X_n]$.

THEOREM VII. *Let f be a polynomial in $C[X_1, \cdots, X_n]$ with* $\deg f \geqq 1$.

(1) *f is irreducible in $C[X_1, \cdots, X_n]$ if and only if f is irreducible in $\mathcal{O}_C[X_1, \cdots, X_n]$.*

(2) *Let $f = f_1 \cdots f_p$ be the prime factorization in $C[X_1, \cdots, X_n]$ in the sense of* UFD. *Then it is also the prime factorization of f in $\mathcal{O}_C[X_1, \cdots, X_n]$ in the sense of Theorem* II. *Conversely, if f is decomposed into irreducible polynomials in $\mathcal{O}_C[X_1, \cdots, X_n]$ as $f = aF_1 \cdots F_p$, then f can be decomposed into irreducible polynomials in $C[X_1, \cdots, X_n]$ as $f = f_1 \cdots f_p$, where f_j $(j = 1, \cdots, p)$ are irreducible polynomials in $C[X_1, \cdots, X_n]$, and $a \sim 1$, $F_j \sim f_j$ $(j = 1, \cdots, p)$ in $\mathcal{O}_C[X_1, \cdots, X_n]$.*

A3. Proofs of Theorems.

We will prove Theorem II, VI and VII in the following.

1° Preparatory Propositions for Theorem II.

First we prove Theorem II in the case $n=1$, and as a corollary, next we prove Theorem II in the case $n>1$.

Hereafter, we assume that R is a w-UFD.

PROPOSITION I. *Let f be a polynomial in $R[X]$ with* $\deg f \geqq 1$. *If $f=ag=bh$ with $a, b \in R$ and primitive polynomials $g, h \in R[X]$, then $a \sim b$, $g \sim h$ in $R[X]$.*

PROPOSITION II. *If $f, g \in R[X]$ are primitive, then fg is also primitive.*

PROPOSITION III. *Let Q be a quotient field of R, and let f be a primitive polynomial in $R[X]$.*

(1) *If* $\deg f=1$, *then f is irreducible in $R[X]$.*

(2) *Suppose* $\deg f \geqq 2$. *If f is reducible in $Q[X]$, then f is reducible in $R[X]$. Explicitly speaking, if $f=GH$, where G, H are polynomials in $Q[X]$ with* $\deg G \geqq 1$, $\deg H \geqq 1$, *then there exist primitive polynomials g, h in $R[X]$ and elements $\bar{a}$, $\bar{b}$ in Q such that $f=gh$, $g=\bar{a}G$, $h=\bar{b}H$.*

PROPOSITION IV. *Let Q be a quotient field of R. A polynomial $f \in R[X]$ is irreducible in $R[X]$ if and only if*

(1) $\deg f=0$, *and f is irreducible in R*

or

(2) $\deg f \geqq 1$, *and f is primitive in $R[X]$ and irreducible in $Q[X]$.*

PROPOSITION V (The Case $n=1$ in Theorem II).

(1) *Any polynomial $f \in R[X]$ with* $\deg f \geqq 1$ *can be expressed as $f=af_1 \cdots f_p$, where $a \in R-\{0\}$ and f_j's $(j=1, \cdots, p)$ are irreducible polynomials in $R[X]$ with* $\deg f_j \geqq 1$.

(2) *If a polynomial $f \in R[X]$ with* $\deg f \geqq 1$ *is decomposed in two ways as $f=af_1 \cdots f_p=bg_1 \cdots g_q$ in the sense of* (1), *then a and b are associates, $p=q$, and f_j and g_j $(j=1, \cdots, p)$ are associates by the proper rearrangement of indices.*

(3) $R[X]$ *is a* w-UFD.

PROOF. (1) We denote by Q a quotient field of R. Note that f is expressible as $f=af_0$ with $a \in R-\{0\}$ and a primitive polynomial $f_0 \in R[X]$.

(i) In the case $\deg f=1$.

Since $\deg f=\deg f_0=1$, it follows from Proposition III (1) that f_0 is irreducible in $R[X]$.

(ii) In the case $\deg f \geqq 2$.

We decompose f_0 in $Q[X]$ as $f_0 = F_1 \cdots F_p$, where F_j's are irreducible polynomials in $Q[X]$. If $p=1$, then, by Proposition IV, f_0 itself is irreducible in $R[X]$. If $p \geqq 2$, then, applying Proposition III (2) repeatedly, we obtain $f_0 = f_1 \cdots f_p$, where f_j $(j=1, \cdots, p)$ is primitive in $R[X]$ and $f_j = a_j F_j$ with $a_j \in Q$. $f_j = a_j F_j$ implies that f_j is irreducible in $Q[X]$. Therefore, by Proposition IV, f_j is irreducible in $R[X]$.

From the above, we obtain the desired result.

(2) By Proposition IV, f_j and g_j are primitive in $R[X]$ and irreducible in $Q[X]$. Since $f_1 \cdots f_p$, $g_1 \cdots g_q$ are primitive (by Proposition II), $a \sim b$ and $f_1 \cdots f_p \sim g_1 \cdots g_q$ in $R[X]$ (by Proposition I). Then, $f_1 \cdots f_p = (ug_1) \cdots g_q$ with a unit $u \in R$. Since we can regard this decomposition as that in $Q[X]$, $p=q$ and $f_j \sim g_j$ $(j=1, \cdots, p)$ in $Q[X]$ by the proper rearrangement of indices. If f_j is written as $f_j = (a_j/b_j) g_j$ with $a_j, b_j \in R$, then $b_j f_j = a_j g_j$. From Proposition I, we see that $f_j \sim g_j$ in $R[X]$ $(j=1, \cdots, p)$. Thus we have finished the proof.

(3) We will check the conditions W1, W2, W3 of w-UFD in this order.

(W1) Let f be an irreducible element in $R[X]$, and $(f)_R$ the ideal generated by f in $R[X]$.

(i) In the case $\deg f = 0$.

f is a prime element in R because f is irreducible in R. We denote by (f) the ideal generated by f in R. Now suppose $g, h \in R[X]$ and $gh \in (f)_R$. Then $gh = fk$ with $k \in R[X]$. If we express g, h, k as $g = ag_0$, $h = bh_0$, $k = ck_0$, where $a, b, c \in R$ and g_0, h_0, k_0 are primitive polynomials in $R[X]$, then we obtain $(ab) g_0 h_0 = (cf) k_0$ (When $\deg g = 0$, we set $g_0 = 1$. Similarly, in the cases $\deg h = 0$, $\deg k = 0$, we set $h_0 = 1$, $k_0 = 1$ respectively). By Proposition I, $ab = cfu \in (f)$ with a unit $u \in R$. Since f is prime in R, one of a and b belongs to (f). We may assume that $a \in (f)$, and that $a = fd$ with $d \in R$. Then, $g = ag_0 = (fd) g_0 \in (f)_R$. Thus $(f)_R$ is a prime ideal.

(ii) In the case $\deg f \geqq 1$.

By Proposition IV, f is primitive in $R[X]$, and irreducible in $Q[X]$. Let $(f)_Q$ be the ideal generated by f in $Q[X]$. Suppose that $g, h \in R[X]$ and $gh \in (f)_R$. Then $gh \in (f)_Q$. Since $Q[X]$ is a UFD, $(f)_Q$ is a prime ideal and one of g and h belongs to $(f)_Q$. So we may assume that $g = fL$ with $L \in Q[X]$. If we set $g = ag_0$, $L = (b/c) l_0$, where $a, b, c \in R$, g_0 and l_0 are primitive polynomials in $R[X]$, then we obtain $ag_0 = f(b/c) l_0$. From

$(ac)g_0=bfl_0$, Proposition I and II, it follows that $g_0=ufl_0$ with a unit $u\in R$. Then, $g=ag_0=(au)fl_0\in(f)_R$. Therefore $(f)_R$ is a prime ideal.

We have finished the check of W1.

(W2) Let f be an element in $R[X]$ which is neither zero nor a unit.

(i) In the case $\deg f=0$.

Since R is a w-UFD, there exists an irreducible element p in R such that $p|f$. p is also irreducible in $R[X]$.

(ii) In the case $\deg f\geqq 1$.

As is shown in (1) in the present proposition, f is written as $f=af_1\cdots f_p$ with nonzero $a\in R$ and irreducible polynomials $f_1,\cdots,f_p$ in $R[X]$. Therefore, in particular, $f_1|f$.

Thus we have checked W2.

(W3) Take elements $f_1,\cdots,f_p\in R[X]-\{0\}$. If there is a unit u among f_j's, then, since a divisor of a unit is a unit, u is a GCD of f_j's. Therefore, we may assume that every one of f_j's is neither zero nor a unit.

By (1) in the present proposition, we can decompose f_j's into primes as follows:

$$f_1=a_1f_{1,1}\cdots f_{1,k_1},$$
$$\cdots\cdots\cdots\cdots$$
$$f_p=a_pf_{p,1}\cdots f_{p,k_p},$$

where $a_j\in R-\{0\}$ for any j, and $f_{i,j}$ is an irreducible polynomial in $R[X]$ with $\deg f_{i,j}\geqq 1$ for any i and any j (When $\deg f_j=0$, we consider $f_j=a_j$.). If $\{f_{1,1},\cdots,f_{1,k_1}\},\cdots,\{f_{p,1},\cdots,f_{p,k_p}\}$ include common irreducible polynomials, then we pick up all of them and rename them $g_1,\cdots,g_l$, where ambiguities of multiplications by units are pressed into a_j's. Moreover, we let a be a GCD of a_j's. Then $g=ag_1\cdots g_l$ is clearly a common divisor of f_j's. Therefore f_j is written as $f_j=gh_j$ with $h_j\in R[X]$ $(j=1,\cdots,p)$.

Suppose that $s\in R[X]$ is a common divisor of f_j's, i.e., $f_j=st_j$ with $t_j\in R[X]$ $(j=1,\cdots,p)$. Then we may assume that

$$\begin{aligned}f_j&=gh_j=(ag_1\cdots g_l)(b_jf_{j,l+1}\cdots f_{j,k_j})\\&=st_j=(cs_1\cdots s_m)(d_jt_{j,m+1}\cdots t_{j,k_j}),\end{aligned}$$

where b_j, c, $d_j\in R$, $ab_j=a_j$, $g_1\cdots g_lf_{j,l+1}\cdots f_{j,k_j}=f_{j,1}\cdots f_{j,k_j}$ and s_j, $t_{j,i}$ are irreducible polynomials in $R[X]$ with $\deg s_j\geqq 1$, $\deg t_{j,i}\geqq 1$.

(i) In the case $m=0$.

By (2) in the present proposition, we see that $a_j=ab_j=cd_ju_j$ with a unit $u_j \in R$ for $j=1, \cdots, p$. Then c is a common divisor of a_j's. Since a is a GCD of a_j's, $c(=s)$ divides a. Therefore s divides g.

(ii) In the case $m \geqq 1$.

If s_1 is not an associate of anyone of g_j's, then it must be included in $\{f_{1,l+1}, \cdots, f_{1,k_1}\}, \cdots, \{f_{p,l+1}, \cdots, f_{p,k_p}\}$ in common. Therefore, by the definition of g_j's, s_1 must be a member of g_j's. But this contradicts the assumption. Thus s_1 is an associate of a certain g_j. In the same manner, we see that each one of s_j's is an associate of some one of g_j's. Hence $m \leqq l$ and $s_1 \cdots s_m | g_1 \cdots g_l$. On the other hand, $a_j = ab_j = cd_j u_j$ with a unit $u_j \in R$ $(j=1, \cdots, p)$. Then c is a common divisor of a_j's. Since a is a GCD of a_j's, c divides a. From these, we find that $cs_1 \cdots s_m | ag_1 \cdots g_l$, i.e., $s | g$.

Therefore we have proved that g is a GCD of f_j's. q.e.d.

2° Proof of Theorem II.

PROOF. (1), (2), (3). We prove (1), (2) and (3) by induction. We have proved the case of $n=1$ in Proposition V. Suppose that (1), (2), (3) in Theorem II are true in the case $n=k$ $(\geqq 1)$. We will check (1), (2), (3) in Theorem II in the case $n=k+1$.

Since $R[X_1, \cdots, X_{k+1}] \cong R[X_1, \cdots, X_k][X_{k+1}]$ and $R[X_1, \cdots, X_k]$ is a w-UFD, it follows from Proposition V that $R[X_1, \cdots, X_k][X_{k+1}]$, i.e., $R[X_1, \cdots, X_{k+1}]$ is also a w-UFD. We have proved (3).

Next we will show (1). For the sake of simplicity, we set $R[X_1, \cdots, X_k]=S$. By Proposition V, any polynomial $F \in S[X_{k+1}]$ with $\deg_{X_{k+1}} F \geqq 1$ can be expressed as $F = fF_1 \cdots F_p$, where $f \in S - \{0\}$ and F_j's $(j=1, \cdots, p)$ are irreducible polynomials in $S[X_{k+1}]$ with $\deg_{X_{k+1}} F_j \geqq 1$. By the assumption, f can be written as $f = af_1 \cdots f_q$, where $a \in R - \{0\}$ and f_j's $(j=1, \cdots, q)$ are irreducible polynomials in $R[X_1, \cdots, X_k]$ with $\deg f_j \geqq 1$. Therefore F is expressed as $F = af_1 \cdots f_q F_1 \cdots F_p$, where f_j's and F_j's are irreducible in $R[X_1, \cdots, X_{k+1}]$ and $\deg f_j \geqq 1$, $\deg F_j \geqq 1$ in $X_1, \cdots, X_{k+1}$. Next suppose that F is a polynomial in $S[X_{k+1}]$ with $\deg_{X_{k+1}} F = 0$ and $\deg F \geqq 1$ in $X_1, \cdots, X_k$. Then F belongs to $S = R[X_1, \cdots, X_k]$ and is expressed as $F = af_1 \cdots f_p$, where $a \in R - \{0\}$ and f_j's $(j=1, \cdots, p)$ are irreducible polynomials in $R[X_1, \cdots, X_k]$ with $\deg f_j \geqq 1$. f_j's are also irreducible in $R[X_1, \cdots, X_{k+1}]$ and $\deg f_j \geqq 1$ in $X_1, \cdots, X_{k+1}$. Thus we have proved (1).

Lastly, we prove (2). Suppose that F is a polynomial in $R[X_1, \cdots, X_{k+1}]$ with $\deg F \geqq 1$, and that it is expressed in two ways as $F=$

$a\bar{F}_1\cdots\bar{F}_p=b\bar{G}_1\cdots\bar{G}_q$, where a, $b\in R-\{0\}$, $\bar{F}_j$'s and G_j's are irreducible polynomials in $R[X_1, \cdots, X_{k+1}]$ with $\deg\bar{F}_j\geqq 1$, $\deg\bar{G}_j\geqq 1$. Classifying $\bar{F}_j$'s and $\bar{G}_j$'s, we may assume that $F=af_1\cdots f_lF_1\cdots F_m=bg_1\cdots g_sG_1\cdots G_t$, where a, $b\in R-\{0\}$, $l+m=p$, $s+t=q$, f_j's and g_j's are irreducible polynomials in $S=R[X_1, \cdots, X_k]$ with $\deg f_j\geqq 1$, $\deg g_j\geqq 1$, and F_j's and G_j's are irreducible polynomials in $S[X_{k+1}]$ with $\deg_{X_{k+1}}F_j\geqq 1$, $\deg_{X_{k+1}}G_j\geqq 1$. Then, by the assumption and by Proposition V, it follows that $af_1\cdots f_l\sim bg_1\cdots g_s$, $m=t$, $F_j\sim G_j$ $(j=1, \cdots, m)$ in $S[X_{k+1}]$ by the proper rearrangement of indices. Since $af_1\cdots f_l=(bu)g_1\cdots g_s$ with a unit $u\in R$, by the assumption, we see that $a\sim b$, $l=s$, $f_j\sim g_j$ $(j=1, \cdots, l)$ in $S=R[X_1, \cdots, X_k]$ by the proper rearrangement of indices. From these, we can conclude that $a\sim b$, $p=l+m=s+t=q$, $\bar{F}_j\sim\bar{G}_j$ $(j=1, \cdots, p)$ in $S[X_{k+1}]=R[X_1, \cdots, X_{k+1}]$ by the proper rearrangement of indices. Thus we have finished the proof of (2).

From the above arguments, (1), (2) and (3) are true for any positive integer n.

(4) If R is not a UFD, there exists an element $a\in R$ which is reducible but cannot be decomposed into finitely many primes. Since a cannot be decomposed into finitely many primes in $R[X_1, \cdots, X_n]$, $R[X_1, \cdots, X_n]$ is not a UFD. q.e.d.

3° Proof of Theorem VI.

PROOF. (1) We will show that $\mathcal{O}_M$ satisfies the conditions W1, W2, W3 in Definition I.

(W1) Let f be an irreducible element in $\mathcal{O}_M$ and let $\phi(f)=D$. If g, $h\in\mathcal{O}_M^*$ and $gh\in(f)$, then $gh=fk$ with $k\in\mathcal{O}_M^*$. Therefore $\phi(g)+\phi(h)=\phi(f)+\phi(k)=D+\sum_\nu n_\nu D_\nu$, where $\phi(k)=\sum_\nu n_\nu D_\nu$. From this, we may assume that $\phi(g)=D+\sum_\nu m_\nu D_\nu$, where $\sum_\nu m_\nu D_\nu\geqq 0$. If we set $l=g/f$, then $l\in\mathcal{O}_M^*$ and $g=fl$. Since $g\in(f)$, (f) is a prime ideal. Hence f is a prime element in $\mathcal{O}_M$.

(W2) Let $f\in\mathcal{O}_M$ be an element which is neither zero nor a unit, and let $\phi(f)=\sum_\nu n_\nu D_\nu$. Putting $D_\nu=D$ for a suitable nonzero ν, we may set $\sum_\nu n_\nu D_\nu=D+\sum_\nu n'_\nu D_\nu$, where $\sum_\nu n'_\nu D_\nu\geqq 0$. Since ϕ is surjective, there exists an irreducible element $g\in\mathcal{O}_M^*$ such that $\phi(g)=D$ and $f/g\in\mathcal{O}_M^*$. Thus we have checked W2.

(W3) Suppose that $f_1, \cdots, f_m$ are nonzero elements in $\mathcal{O}_M$ and that $\phi(f_j)=\sum_\nu n_\nu^{(j)}D_\nu$ $(j=1, \cdots, m)$. Let $n_\nu=\min_j n_\nu^{(j)}$ for any ν. If we take a

$g\in\mathcal{O}_M^*$ which satisfies $\phi(g)=\sum_\nu n_\nu D_\nu$, then $f_j/g\in\mathcal{O}_M^*$ $(j=1,\cdots,m)$. On the other hand, if $h\in\mathcal{O}_M^*$ satisfies $f_j/h\in\mathcal{O}_M^*$ for any j, then $f_j=hl_j$ with $l_j\in\mathcal{O}_M^*$ and $\phi(f_j)=\phi(h)+\phi(l_j)$ $(j=1,\cdots,m)$. If we set $\phi(h)=\sum_\nu n'_\nu D_\nu$, then it follows from $\phi(h)\geqq 0$ and $\phi(l_j)\geqq 0$ that $n'_\nu\leqq \min_j n_\nu^{(j)}=n_\nu$ for any ν. Therefore, $g/h\in\mathcal{O}_M^*$ and g is a GCD of $f_1,\cdots,f_m$. We have finished the check of W3.

(2) Suppose that there exists a divisor $\sum_\nu n_\nu D_\nu\in \mathrm{Div}^+(M)$ with infinitely many nonzero n_ν's. Then, since ϕ is surjective, there exists a $f\in\mathcal{O}_M^*$ which satisfies $\phi(f)=\sum_\nu n_\nu D_\nu$. If f can be expressed as $f=g_1\cdots g_p$ with irreducible elements g_j's in $\mathcal{O}_M^*$, then $\phi(f)=\sum_{j=1}^p\phi(g_j)$, i.e., $\sum_\nu n_\nu D_\nu=\sum_{j=1}^p D_j$, where $\phi(g_j)=D_j$. This is a contradiction. Therefore $\mathcal{O}_M$ cannot be a UFD. q.e.d.

4° Proof of Theorem VII.

PROOF. We first note that $C[X_1,\cdots,X_n]$ is a subring of $\mathcal{O}_C[X_1,\cdots,X_n]$ and that $\mathcal{O}_C[X_1,\cdots,X_n]$ is a subring of $\mathcal{O}_{C^{n+1}}$:

$$C[X_1,\cdots,X_n]\subset\mathcal{O}_C[X_1,\cdots,X_n]\subset\mathcal{O}_{C^{n+1}}.$$

(1) Let f be irreducible in $C[X_1,\cdots,X_n]$. If we put $\Phi:\mathcal{O}^*_{C^{n+1}}\longrightarrow \mathrm{Div}^+(C^{n+1})$, then $\Phi(f)$ is an analytically irreducible hypersurface in C^{n+1}. Therefore, by Theorem V (2), f is irreducible in $\mathcal{O}_{C^{n+1}}$. Now suppose that f is written as $f=GH$ with $G, H\in\mathcal{O}_C[X_1,\cdots,X_n]$. Since we can regard this decomposition as that in $\mathcal{O}_{C^{n+1}}$, one of G and H, for example G, is a unit in $\mathcal{O}_{C^{n+1}}$, i.e., a holomorphic function on C^{n+1} without zeros. Then G must be of degree zero in $X_1,\cdots,X_n$ and a holomorphic function on C without zeros. Thus G is a unit in $\mathcal{O}_C$. Hence f is irreducible in $\mathcal{O}_C[X_1,\cdots,X_n]$.

Conversely, suppose that f is irreducible in $\mathcal{O}_C[X_1,\cdots,X_n]$ and that f is written as $f=gh$ with $g,h\in C[X_1,\cdots,X_n]$. Since we can regard this decomposition as that in $\mathcal{O}_C[X_1,\cdots,X_n]$, one of g and h, for example g, is a unit in $\mathcal{O}_C[X_1,\cdots,X_n]$, i.e., a unit in $\mathcal{O}_C$. Therefore g is an element in $C-\{0\}$, i.e., a unit in $C[X_1,\cdots,X_n]$. Then f is irreducible in $C[X_1,\cdots,X_n]$.

(2) The first part is apparent from (1). We will show the latter part. Suppose that f is decomposed into primes in $\mathcal{O}_C[X_1,\cdots,X_n]$ as

$f=aF_1\cdots F_p$, and that f is decomposed into primes in $C[X_1, \cdots, X_n]$ as $f=f_1\cdots f_q$. Since we can regard both expressions as prime factorizations in $\mathcal{O}_C[X_1, \cdots, X_n]$, by Theorem II (2), we see that $a\sim 1$, $p=q$, $F_j\sim f_j$ $(j=1, \cdots, p)$ in $\mathcal{O}_C[X_1, \cdots, X_n]$ by the proper rearrangement of indices.

q.e.d.

Acknowledgements.

The author would like to express his gratitude to Professor Tosihusa Kimura, Professor Masahiro Iwano, Professor Kyoji Saito and Professor Kazuo Okamoto. This study started from the author's wish to comprehend [1] clearly and was constantly supported by Professor T. Kimura's valuable advice. Without his advice and encouragement, this work could not have been accomplished. Professor M. Iwano listened this work many times in his seminar and gave the author many valuable suggestions and encouragement. Fruitful discussions with Professor K. Saito and Professor K. Okamoto improved this work.

References

[1] Kimura. T., Sur les points singuliers essentiels mobiles des équations différentielles du second order, Comment. Math. Univ. St. Paul. **5** (1956), 81-94.

[2] Kodaira, K., Holomorphic mappings of polydiscs into compact complex manifolds, J. Differential Geom. **6** (1971), 33-46.

[3] Morrow, J. A., Minimal normal compactifications of C^2, Rice Univ. Stud. **59** (1973), 97-112.

[4] Narasimhan, R., Introduction to the theory of analytic spaces, Lecture Notes in Math. vol. 25, Springer, Berlin-Heidelberg-New York, 1966.

[5] Painlevé, P., Leçons sur la théorie analytique des équations différentielles, professées à Stockholm, Œuvres de Paul Painlevé, t. III, Centre national de la recherche scientifique, France, 1975.

(Received January 8, 1988)

Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan