J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
35 (1988), 507-518.

# *A mathematical study of the charge simulation method I*

By Masashi KATSURADA and Hisashi OKAMOTO*)

**Abstract.** The charge simulation method, which we abbreviate to CSM, is one of numerical techniques to solve the boundary value problem for the Laplace operator $\Delta=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_n^2}$, and has been used effectively in some fields of engineering, e.g. in electrical engineering. In spite of its effectiveness and popularity very little mathematical study has been done for CSM. Actually even the convergence of CSM has not yet been established yet. Here in this paper, we confine our analysis to a very special case where the domain is a two dimensional disk, and establish the following facts: if the boundary data is analytic, the error is of the order of $a^N$, where $a$ is a certain positive constant less than one and $N$ is the number of the collocation points. This exponential rate of convergence is rather remarkable, because usual methods such as the finite difference method or the finite element method can offer solutions with the error of the order of $N^{-2}$ or $N^{-3}$.

## § 1. Introduction.

In this paper we discuss the mathematical aspect of the charge simulation method (CSM). In applications it is experienced that CSM is very effective in solving the boundary value problems for the Laplace operator. Let us consider the following problem:

(1.1) $$\Delta u=0 \qquad \text{in } \Omega,$$

(1.2) $$u=f \qquad \text{on } \Gamma,$$

where $\Omega$ is a bounded domain in $\boldsymbol{R}^m$ with a smooth boundary. $\Gamma$ is the boundary of $\Omega$. $f$ is a given function defined on $\Gamma$. We introduce the fundamental solution $E$ of the Laplace operator:

$$E(x, y)=-\frac{1}{2\pi}\log|x-y| \qquad (x, y \in \boldsymbol{R}^2)$$

or

*) Partially supported by the Fūjukai.

$$E(x, y)=\frac{1}{4\pi}\cdot\frac{1}{|x-y|} \qquad (x, y \in \boldsymbol{R}^3).$$

We also take $N$ points $y_1, y_2, \cdots, y_N$ outside $\Omega$. In CSM we intend to construct an approximation $u^{(N)}$ to $u$ in the following form:

$$u^{(N)}(x)=\sum_{j=1}^{N} Q_j E(x, y_j) \qquad (x \in \Omega).$$

Here $Q_j$'s are constants which are to be chosen so that $u^{(N)}$ satisfies the boundary condition (1.2) approximately. To be more specific, we take $N$ points $x_1, x_2, \cdots, x_N$ on the boundary $\Gamma$, and determine the coefficients $Q_j$'s by the equations below:

$$u^{(N)}(x_j)=f(x_j) \qquad (j=1, 2, \cdots, N). \tag{1.3}$$

This is a description of CSM. In terms of the electric field, the required static field is approximated by the field generated by point-charges located at $y_1, \cdots, y_N$. Note that the approximate solution $u^{(N)}$ satisfies the equation (1.1) exactly. In principle, this rather primitive method is applicable to any domain of any dimension. In spite of its simple procedure, this method sometimes yields very good simulation results (see, e.g., [1], [2]). But the mathematical proof of convergence of CSM is not so easy as might be expected from the simplicity of its formulation.

In order to obtain a clear-cut result we consider in this paper the following case with nice symmetry. Namely the domain $\Omega$ is assumed to be a two dimensional disk. The charge points $y_1, y_2, \cdots, y_N$ are distributed evenly on some circle which is concentric to $\Gamma$. The collocation points $x_1, x_2, \cdots, x_N$ are distributed evenly on $\Gamma$, too. In this situation we prove that $u^{(N)}$ exists uniquely and converges exponentially to the exact solution $u$ under a certain condition on the radii of the circles (see § 2).

We think this is a rather surprising result because other popular methods such as finite difference method or finite element method can approximate the solution with the error of order $N^{-s}$ ($s$ is some positive number). Although our proof covers only a very special case, there are practical evidences of numerical results which show that CSM is very effective even for general domains (see the final comments in § 4).

This paper is composed of 4 sections. In § 2 we state the theorems which assert the solvability and convergence of CSM and we prove them in § 3. In § 4 we give several remarks and comments.

## § 2. Exponential convergence.

In this section we state the exponential convergence of $u^{(N)}$ to $u$ in the case where the domain is a two dimensional disk. From now on we assume that the domain $\Omega$ is a disk with the center at the origin and with the radius $\rho$. We denote by $\Gamma$ the circle which is the boundary of $\Omega$. We fix a number $R>\rho$. Using the polar coordinates $(r, \theta)$, we define the charge points $y_j$ as the points in $\boldsymbol{R}^2$ which have the coordinates $r=R$, $\theta=2\pi(j-1)/N$ $(j=1, 2, \cdots, N)$. Finally we define the collocation points $x_j$ as the points which have the coordinates $r=\rho$, $\theta=2\pi(j-1)/N$ $(j=1, 2, \cdots, N)$.

Now we can write (1.3) in terms of $Q_j$'s as follows:

$$G\vec{Q}=\vec{f}, \tag{2.1}$$

where $\vec{Q}=(Q_1, Q_2, \cdots, Q_N)$, $\vec{f}=(f(x_1), f(x_2), \cdots, f(x_N))$ and $G$ is an $N\times N$ matrix with the entries $g_{ij}=E(x_i, y_j)$ $(1\leqq i,\ j\leqq N)$. We are now in a position to state the theorems:

THEOREM 1. *Suppose that* $R^N-\rho^N\neq 1$. *Then we can determine* $u^{(N)}$ *by* (1.3) *uniquely. In other words, the matrix* $G$ *is nonsingular.*

THEOREM 2. *In addition to the hypothesis of Theorem* 1, *we assume that* $R\neq 1$ *and the boundary data* $f$ *is real analytic. In this case the exact solution* $u$ *admits of a harmonic extension to some neighborhood of* $\bar{\Omega}$. *Hence we may assume that* $u$ *is harmonic in* $0\leqq r\leqq r_0$ *with* $r_0>\rho$. *Then there are constants* $A>0$ *and* $a$ *with* $0<a<1$ *which are independent of* $N$ *and* $u$ *such that*

$$\sup_{x\in\Omega}|u(x)-u^{(N)}(x)|\leqq Aa^N\cdot\sup_{|x|\leqq r_0}|u(x)|. \tag{2.2}$$

REMARK. If $R=1$, the approximate solution does not converge to the exact solution. This would be clearly understood by observing the following example (this example was communicated to the authors by Prof. S. Yotsutani):

Let $\Omega$ be a domain which is not necessarily a disk such that its closure is included in the interior of the unit circle. Then the approximate function $u^{(N)}(x)=-\dfrac{1}{2\pi}\sum_{j=1}^{N}Q_j\log|x-y_j|$ satisfies that $u^{(N)}(0)=-\dfrac{1}{2\pi}\sum_{j=1}^{N}Q_j\log|y_j|=0$, no matter what the choice of $\{y_j\}$ and $\{Q_j\}$ may be. Hence $u^{(N)}$ cannot approximate a harmonic function which does not vanish at the origin.

## § 3. Proof of the theorems.

First we define the following symbols:

$$\omega=\omega^{(N)}=\exp\left(\frac{2\pi i}{N}\right),$$

$$L_k=E(\rho, R\omega^k) \qquad (k\in \boldsymbol{Z}),$$

$$\varphi_p^{(N)}(z)=\sum_{k=0}^{N-1}\omega^{pk}E(z, R\omega^k) \qquad (z\in \boldsymbol{C},\ p\in \boldsymbol{Z}).$$

In this section, we identify $\boldsymbol{R}^2$ with $\boldsymbol{C}$. The function $\varphi_p^{(N)}(z)$ plays a fundamental role in the following analysis. We first prove the following:

LEMMA 1.

$$\varphi_p^{(N)}(z)=\begin{cases} -\dfrac{1}{2\pi}\log|z^N-R^N| & (p\equiv 0 \pmod N) \\ \dfrac{N}{4\pi}\displaystyle\sum_{\substack{m\equiv p \pmod N \\ m\in \boldsymbol{Z}}}\frac{1}{|m|}\left(\frac{r}{R}\right)^{|m|}e^{im\theta} & (\textit{otherwise}) \end{cases}$$

*for all* $z=re^{i\theta}$ *such that* $r<R$.

PROOF OF LEMMA 1. First we note that if $p\equiv q \pmod N$, it holds that

$$\varphi_p^{(N)}=\varphi_q^{(N)}.$$

*Case* 1. $(p\equiv 0 \pmod N)$

In this case $\omega^p=1$. Hence we can compute $\varphi_p^{(N)}$ as follows:

$$\begin{aligned}\varphi_p^{(N)}(z)&=\sum_{k=0}^{N-1}E(z, R\omega^k)\\ &=-\frac{1}{2\pi}\sum_{k=0}^{N-1}\log|z-R\omega^k|\\ &=-\frac{1}{2\pi}\log\prod_{k=0}^{N-1}|z-R\omega^k|\\ &=-\frac{1}{2\pi}\log|z^N-R^N|.\end{aligned}$$

*Case* 2. $(p\not\equiv 0 \pmod N)$

First, we expand $E(z, R\omega^k)$ as

$$E(z, R\omega^k)=-\frac{1}{2\pi}\log|z-R\omega^k|$$

$$=-\frac{1}{2\pi}\left\{\log R+\log\left|1-\frac{\omega^{-k}}{R}z\right|\right\}$$

$$=-\frac{1}{2\pi}\left\{\log R+\operatorname{Re}\log\left(1-\frac{\omega^{-k}}{R}z\right)\right\}$$

$$=-\frac{1}{2\pi}\left\{\log R-\operatorname{Re}\sum_{n=0}^{\infty}\frac{1}{n}\left(\frac{z\omega^{-k}}{R}\right)^{n}\right\}$$

$$=-\frac{1}{2\pi}\left\{\log R-\sum_{n=0}^{\infty}\frac{1}{2n}\left(\frac{r}{R}\right)^{n}(e^{in\theta}\omega^{-nk}+e^{-in\theta}\omega^{nk})\right\}.$$

Using the fact that

$$\sum_{k=0}^{N-1}\omega^{lk}=\begin{cases}N & (\text{if } l\equiv 0 \pmod N)\\ 0 & (\text{otherwise}),\end{cases}$$

we have

$$\varphi_p^{(N)}(z)=\sum_{k=0}^{N-1}\omega^{pk}E(z, R\omega^k)$$

$$=\frac{-1}{2\pi}\sum_{k=0}^{N-1}\omega^{pk}\left\{\log R-\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{r}{R}\right)^{n}(e^{in\theta}\omega^{-nk}+e^{-in\theta}\omega^{nk})\right\}$$

$$=\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{r}{R}\right)^{n}\sum_{k=0}^{N-1}\{e^{in\theta}\omega^{(p-n)k}+e^{-in\theta}\omega^{(p+n)k}\}$$

$$=\frac{N}{4\pi}\left\{\sum_{m=1}^{\infty}\frac{1}{mN-p}\left(\frac{r}{R}\right)^{mN-p}e^{-i(mN-p)\theta}+\sum_{m=0}^{\infty}\frac{1}{mN+p}\left(\frac{r}{R}\right)^{mN+p}e^{i(mN+p)\theta}\right\}$$

$$=\frac{N}{4\pi}\sum_{\substack{m\equiv p\\ m\in\mathbf{Z}}}\frac{1}{|m|}\left(\frac{r}{R}\right)^{|m|}e^{im\theta}.$$

Q.E.D.

PROOF OF THEOREM 1. We see

$$g_{ij}=E(\rho, R\omega^{j-i})=L_{j-i}\qquad(1\leqq i, j\leqq N).$$

Hence the matrix $G$ is cyclic:

$$G=\begin{pmatrix} L_0 & L_1 & \cdots & L_{N-1}\\ L_{N-1} & L_0 & \cdots & L_{N-2}\\ & \cdots\cdots & & \\ L_1 & L_2 & \cdots & L_0\end{pmatrix}.$$

Therefore we can compute the determinant as

$$\det G = \prod_{p=0}^{N-1}\left(\sum_{k=0}^{N-1}\omega^{pk}L_k\right) = \prod_{p=0}^{N-1}\varphi_p^{(N)}(\rho).$$

By Lemma 1, we know that if $1 \leqq p \leqq N-1$ and $\rho>0$, then it holds that $\varphi_p^{(N)}(\rho)>0$. Hence

$$\det G \neq 0 \Longleftrightarrow \varphi_0^{(N)}(\rho) \neq 0 \Longleftrightarrow R^N - \rho^N \neq 1.$$

Q.E.D.

Before proving Theorem 2, we introduce norms $\|\cdot\|_{\infty,r}$ by

$$\|v\|_{\infty,r} = \sup_{|x| \leqq r} |v(x)|.$$

PROOF OF THEOREM 2. From Theorem 1, we know that if $R^N - \rho^N \neq 1$, then there exists the inverse $G^{-1}$. In fact, we have

$$\text{the } (i, j)\text{-element of } G^{-1} = \frac{1}{N}\sum_{p=0}^{N-1}\frac{\omega^{p(i-j)}}{\varphi_p^{(N)}(\rho)} \equiv e_{ij}.$$

To prove this formula, we verify that the matrix product $G\cdot(e_{ij})$ is the identity matrix. Namely

$$\begin{aligned}\text{the } (i, j)\text{-element of } G\cdot(e_{ij}) &= \sum_{k=1}^{N} g_{ik}e_{kj} \\ &= \sum_{k=1}^{N} L_{k-i}\cdot\frac{1}{N}\sum_{p=0}^{N-1}\frac{\omega^{p(k-j)}}{\varphi_p^{(N)}(\rho)} \\ &= \frac{1}{N}\sum_{p=0}^{N-1}\frac{\omega^{p(i-j)}}{\varphi_p^{(N)}(\rho)}\sum_{k=1}^{N}\omega^{p(k-i)}L_{k-i} \\ &= \frac{1}{N}\sum_{p=0}^{N-1}\frac{\omega^{p(i-j)}}{\varphi_p^{(N)}(\rho)}\cdot\varphi_p^{(N)}(\rho) \\ &= \frac{1}{N}\sum_{p=0}^{N-1}\omega^{p(i-j)} \\ &= \delta_{ij} \qquad (\delta_{ij} \text{ is the Kronecker's delta}).\end{aligned}$$

Thus we obtain the following expression of our CSM solution:

$$u^{(N)}(x) = \frac{1}{N}\sum_{j,p=0}^{N-1} u(\rho\omega^j)\omega^{-pj}\frac{\varphi_p^{(N)}(x)}{\varphi_p^{(N)}(\rho)} \quad (x \in C,\ |x| \leqq \rho). \tag{3.1}$$

Let $f_n$ be the Fourier coefficients of $f$, then the solution of Dirichlet problem (1.1)-(1.2) can be written as follows:

$$u(re^{i\theta}) = \sum_{n\in Z} f_n\left(\frac{r}{\rho}\right)^{|n|} e^{in\theta}. \tag{3.2}$$

Evaluating $u(\rho\omega^j)$ by (3.2) and substituting them into (3.1), we have

$$\begin{aligned} u^{(N)}(x) &= \frac{1}{N}\sum_{j,p=0}^{N-1}\left(\sum_{n\in Z} f_n\omega^{nj}\right)\omega^{-pj}\frac{\varphi_p^{(N)}(x)}{\varphi_p^{(N)}(\rho)} \\ &= \frac{1}{N}\sum_{p=0}^{N-1}\frac{\varphi_p^{(N)}(x)}{\varphi_p^{(N)}(\rho)}\sum_{n\in Z} f_n\cdot\sum_{j=0}^{N-1}\omega^{(n-p)j} \\ &= \sum_{p=0}^{N-1}\frac{\varphi_p^{(N)}(x)}{\varphi_p^{(N)}(\rho)}\sum_{\substack{n\in Z\\ n\equiv p}} f_n \\ &= \sum_{p=0}^{N-1}\sum_{\substack{n\in Z\\ n\equiv p}} f_n\frac{\varphi_n^{(N)}(x)}{\varphi_n^{(N)}(\rho)} \\ &= \sum_{n\in Z} f_n\frac{\varphi_n^{(N)}(x)}{\varphi_n^{(N)}(\rho)}. \end{aligned} \tag{3.3}$$

Consequently we can write the error function $e^{(N)}(x) = u(x) - u^{(N)}(x)$ as

$$e^{(N)}(re^{i\theta}) = \sum_{n\in Z} f_n\left\{\left(\frac{r}{\rho}\right)^{|n|} e^{in\theta} - \frac{\varphi_n^{(N)}(re^{i\theta})}{\varphi_n^{(N)}(\rho)}\right\}. \tag{3.4}$$

Since $e^{(N)}$ is a harmonic function, we obtain by the maximum principle

$$\sup_{r\leqq\rho} |e^{(N)}(re^{i\theta})| \leqq \sum_{n\in Z} |f_n| g_{n,\rho}^{(N)},$$

where

$$g_{n,\rho}^{(N)} = \sup_{0\leqq\theta\leqq 2\pi}\left| e^{in\theta} - \frac{\varphi_n^{(N)}(\rho e^{i\theta})}{\varphi_n^{(N)}(\rho)}\right|.$$

On the other hand, by the last assumption of Theorem 2, we have

$$|f_n| \leqq \left(\frac{\rho}{r_0}\right)^{|n|} \|u\|_{\infty, r_0} \qquad (n\in Z), \tag{3.5}$$

for the Fourier coefficients. At this stage we claim the following Lemma 2 concerning $g_{n,\rho}^{(N)}$.

LEMMA 2. *Put*

$$C(R,\rho) = \max\left\{1, \left|\frac{\log(R^N+\rho^N)}{\log(R^N-\rho^N)}\right|\right\},$$

*then we have*

$$g_{-n,\rho}^{(N)}=g_{n,\rho}^{(N)} \qquad (n\in \boldsymbol{Z}), \tag{3.6}$$

$$g_{n,\rho}^{(N)}\leqq 1+C(R,\rho) \qquad (n\in \boldsymbol{Z}), \tag{3.7}$$

*and for sufficiently large $N$,*

$$g_{0,\rho}^{(N)}\leqq\frac{8}{N|\log R|}\left(\frac{\rho}{R}\right)^{N}, \tag{3.8}$$

$$g_{n,\rho}^{(N)}\leqq\frac{8n}{N-n}\left(\frac{\rho}{R}\right)^{N-2n} \qquad \left(1\leqq n\leqq\frac{N}{2}\right). \tag{3.9}$$

The proof of this lemma will be given later.

Now we are ready to derive the error estimate in the theorem. Namely we have

$$\begin{aligned}\sup_{|x|\leqq\rho}|e^{(N)}(x)|&\leqq\sum_{n\in \boldsymbol{Z}}|f_n|g_{n,\rho}^{(N)}\\ &=|f_0|g_{0,\rho}^{(N)}+\sum_{n=1}^{m}(|f_n|+|f_{-n}|)g_{n,\rho}^{(N)}+\sum_{n=m+1}^{\infty}(|f_n|+|f_{-n}|)g_{n,\rho}^{(N)}\\ &\qquad\left(\text{where } m=\text{integral part of } \frac{N}{2}\right).\end{aligned}$$

We are going to estimate each term of the right hand side. First, by (3.5) and (3.8), we have

$$|f_0|g_{0,\rho}^{(N)}\leqq\frac{8}{N|\log R|}\left(\frac{\rho}{R}\right)^{N}\|u\|_{\infty,r_0}. \tag{3.10}$$

And by (3.5) and (3.9), we estimate the second term as follows:

$$\begin{aligned}\sum_{n=1}^{m}(|f_n|+|f_{-n}|)g_{n,\rho}^{(N)}&\leqq\sum_{n=1}^{m}2\left(\frac{\rho}{r_0}\right)^{n}\|u\|_{\infty,r_0}\cdot\frac{8n}{N-n}\left(\frac{\rho}{R}\right)^{N-2n}\\ &\leqq 16\left(\frac{\rho}{R}\right)^{N}\|u\|_{\infty,r_0}\sum_{n=1}^{m}\left(\frac{R^2}{\rho r_0}\right)^{n}.\end{aligned}$$

And using

$$\sum_{n=1}^{m}\tau^n\leqq\begin{cases}\tau^m/(\tau-1) & (\text{if } \tau>1)\\ m & (\text{if } \tau=1)\\ 1/(1-\tau) & (\text{if } 0<\tau<1)\end{cases}$$

for $\tau = R^2/(\rho r_0)$, we have

$$\sum_{n=1}^{m}(|f_n|+|f_{-n}|)g_{n,\rho}^{(N)} \leqq \begin{cases} \dfrac{16}{(R^2/\rho r_0)-1}\left(\dfrac{\rho}{r_0}\right)^{N/2}\|u\|_{\infty,r_0} & (R^2/\rho r_0>1) \\ 8N\left(\dfrac{\rho}{R}\right)^{N}\|u\|_{\infty,r_0} & (R^2/\rho r_0=1) \\ \dfrac{16}{1-(R^2/\rho r_0)}\left(\dfrac{\rho}{R}\right)^{N}\|u\|_{\infty,r_0} & (R^2/\rho r_0<1). \end{cases} \tag{3.11}$$

Finally for the last term, we have

$$\sum_{n=m+1}^{\infty}(|f_n|+|f_{-n}|)g_{n,\rho}^{(N)} \leqq \frac{2(1+C(R,\rho))}{1-(\rho/r_0)}\left(\frac{\rho}{r_0}\right)^{m+1}\|u\|_{\infty,r_0} \leqq \frac{2(1+C(R,\rho))}{1-(\rho/r_0)}\left(\frac{\rho}{r_0}\right)^{(N+1)/2}\|u\|_{\infty,r_0}. \tag{3.12}$$

By (3.10)-(3.12), we obtain the conclusion of Theorem 2. Q.E.D.

PROOF OF LEMMA 2. (3.6) is easily seen from

$$\varphi_{-n}^{(N)}(z)=\overline{\varphi_n^{(N)}(z)} \qquad (n\in Z).$$

By Lemma 1, we see that

$$|\varphi_n^{(N)}(z)|\leqq\varphi_n^{(N)}(|z|) \qquad (n\notin NZ=\{lN;\, l\in Z\}),$$

and this leads to $g_{n,\rho}^{(N)}\leqq 2$ if $n\notin NZ$. And for $n\in NZ$ we see directly

$$\sup_{|z|\leqq\rho}|\varphi_n^{(N)}(z)|\leqq\frac{1}{2\pi}\max\{|\log(R^N-\rho^N)|,\ |\log(R^N+\rho^N)|\}$$

and this leads to $g_{n,\rho}^{(N)}\leqq 1+C(R,\rho)$. Hence we have (3.7).

To prove (3.9), we write $g_{n,\rho}^{(N)}$ as

$$g_{n,\rho}^{(N)}\leqq\frac{\sup_{\theta\in[0,2\pi]}|e^{in\theta}\varphi_n^{(N)}(\rho)-\varphi_n^{(N)}(\rho e^{i\theta})|}{|\varphi_n^{(N)}(\rho)|}.$$

As for the denominator we can see

$$\varphi_n^{(N)}(\rho)\geqq\frac{N}{4n\pi}\cdot\left(\frac{\rho}{R}\right)^n \qquad \left(1\leqq n\leqq\frac{N}{2}\right),$$

by Lemma 1. On the other hand, if $N$ is so large that $(\rho/R)^N \leqq 1/2$, then we have for the numerator

$$\sup_{0\leqq\theta\leqq 2\pi} |e^{in\theta}\varphi_n^{(N)}(\rho)-\varphi_n^{(N)}(\rho e^{i\theta})| \leqq \frac{2N}{\pi(N-n)}\left(\frac{\rho}{R}\right)^{N-n}.$$

In fact,

$$\begin{aligned}
&|e^{in\theta}\varphi_n^{(N)}(\rho)-\varphi_n^{(N)}(\rho e^{i\theta})| \\
&=\left|\frac{N}{4\pi}\sum_{m\equiv n}\frac{1}{|m|}\left(\frac{\rho}{R}\right)^{|m|}(e^{in\theta}-e^{im\theta})\right| \\
&=\frac{N}{4\pi}\left|\sum_{l=1}^{\infty}\left\{\frac{1}{lN+n}\left(\frac{\rho}{R}\right)^{lN+n}(e^{in\theta}-e^{i(lN+n)\theta})+\frac{1}{lN-n}\left(\frac{\rho}{R}\right)^{lN-n}(e^{in\theta}-e^{-i(lN-n)\theta})\right\}\right| \\
&\leqq\frac{N}{2\pi}\sum_{l=1}^{\infty}\left\{\frac{1}{lN+n}\left(\frac{\rho}{R}\right)^{lN+n}+\frac{1}{lN-n}\left(\frac{\rho}{R}\right)^{lN-n}\right\} \\
&=\frac{N}{2\pi(N-n)}\left(\frac{\rho}{R}\right)^{N-n}\sum_{l=1}^{\infty}\left\{\frac{N-n}{lN+n}\left(\frac{\rho}{R}\right)^{(l-1)N+2n}+\frac{N-n}{lN-n}\left(\frac{\rho}{R}\right)^{(l-1)N}\right\} \\
&\leqq\frac{N}{2\pi(N-n)}\left(\frac{\rho}{R}\right)^{N-n}\left\{1+\left(\frac{\rho}{R}\right)^{2n}\right\}\sum_{l=1}^{\infty}\left(\frac{\rho}{R}\right)^{(l-1)N} \\
&=\frac{N}{2\pi(N-n)}\left(\frac{\rho}{R}\right)^{N-n}\frac{1+(\rho/R)^{2n}}{1-(\rho/R)^N} \\
&\leqq\frac{2N}{\pi(N-n)}\left(\frac{\rho}{R}\right)^{N-n}.
\end{aligned}$$

Hence (3.9) is true for large $N$.

(3.8) can be proved similarly. Q.E.D.

## § 4. Remarks.

Although we assumed the boundary data $f$ is real analytic in this paper, we shall show in a forthcoming paper that the convergence takes place under a weaker regularity condition that the Fourier series of $f$ is absolutely convergent. Moreover it is easy to show that if $f_n=O(|n|^{-\alpha})$ (as $|n|\to\infty$) for some $\alpha>1$, then it holds that $\|u-u^{(N)}\|_\infty=O(N^{-\alpha+1})$ (as $N\to\infty$). We shall also show that the CSM works in such domains as an exterior of a circle or an annular domain. We shall also consider Neumann problem. In the CSM, the approximate solution $u^{(N)}$ is represented by a linear combination of concrete functions, so we can compute its derivative grad $u^{(N)}$ directly. And when we take grad $u^{(N)}$

as an approximation of grad $u$, we can easily get an error estimate for the first derivatives. We carried out numerical experiments to see all these results. These will be given in the forthcoming paper.

**Remark on the case of general domains:** if $\Omega$ is a general domain, our proof does not work. However, in this connection, the following idea, which is used by Takahasi and Mori [3] in the error analysis of numerical quadrature, is worthy of notice. Let us regard the solution $u$ as a real part of an function $g(\zeta)$, which is analytic in a domain $D$. We assume that $\Omega$ is compactly contained in $D$, i.e., the closure of $\Omega$ is contained in $D$. Let $C$ be a contour which lies in $D$ but contains $\Omega$ in its inside. For each $q=1, 2, \cdots, N$, we define $Q_{j,q}$ by

$$\sum_{j=1}^{N} \log|x_p - y_j| \cdot Q_{j,q} = \delta_{pq} \qquad (1 \leqq p \leqq N).$$

Note that $Q_{j,q}$ are determined only by the geometrical data. We denote by $g(z)$ an analytic function whose real part is $u(z)$. Then we have

$$g(z) = \frac{1}{2\pi i} \int_C \frac{g(\zeta)}{\zeta - z} d\zeta$$

and

$$u^{(N)}(z) = \mathrm{Re}\left( \sum_{j,q=1}^{N} Q_{j,q} \log|z - y_j| \cdot \frac{1}{2\pi i} \int_C \frac{g(\zeta)}{\zeta - x_q} d\zeta \right).$$

Therefore we can represent the error function $e^{(N)}(z)$ as follows:

$$e^{(N)}(z) = \mathrm{Re} \frac{1}{2\pi i} \int_C \Phi^{(N)}(z, \zeta) g(\zeta) d\zeta,$$

where

$$\Phi^{(N)}(z, \zeta) = \frac{1}{\zeta - z} - \sum_{j,q=1}^{N} Q_{j,q} \log|z - y_j| \cdot \frac{1}{\zeta - x_q}.$$

This formula is useful because we have only to estimate $\Phi^{(N)}(z, \zeta)$, which is independent of particular solution $u(\zeta)$. We can obtain an error bound by estimating $\Phi^{(N)}$. The proof of Theorem 2 was first given in this way and later rearranged as in Section 3, for when we can apply Fourier series, it is a little simpler to use them in the proof. We consider that in general cases that the domain does not have a radial symmetry, although its analysis might be difficult, the method described above is of some use.

## References

[1] Shōji, M., An application of the charge simulation method to a free boundary problem, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **33** (1986), 523-539.
[2] Murashima, S., Charge Simulation Method and Its Applications, Morikita Syuppan, Tokyo, 1983 (in Japanese).
[3] Takahasi, H. and M. Mori, Error estimation in the numerical integration of analytic functions, Rep. Compt. Centre, Univ. of Tokyo **3** (1970), 41-108.

(Received February 15, 1988)

Masashi Katsurada
Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan

Hisashi Okamoto
Department of Pure and Applied Sciences
College of Arts and Sciences
University of Tokyo
Meguro-ku, Tokyo
153 Japan