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Remarks on positive eigenvalues of the Schrödinger-type operator on spherically symmetric Riemannian manifolds

Dedicated to Professor Hiroshi Fujita for his 60th birthday

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There is an abundant literature about the non-existence of positive eigenvalues of the Schrödinger operator or more general second order operators. However, not so many papers were found which dealt with the same problems considered on non-Euclidean manifolds (e.g. [4], [5], [6]). Recently, T. Tayoshi [11] investigated the spectrum of second order elliptic differential operators on noncompact Riemannian manifolds. He laid very general conditions on the coefficients and the metric, and, as one of notable results, he showed the non-existence of eigenvalues in a half line $(\exists E, \infty)$ by giving the growth order of non-zero eigenfunctions.

The present article treats the special case that the metric is spherically symmetric and the operator is $-\Delta+q(x)$. In this case however, we can offer another simple approach making the conditions slightly relaxed and concrete.

The main part of the proofs are described in an abstract manner. A part of the methods used there originates mostly from S. Agmon's works, especially from [1]. Furthermore, K. Masuda's work [7] [8] gives a great suggestion on developing the abstract theory.

The author would like to thank Professor H. Fujita for kind and valuable advice.

§1. Assumptions and theorems.

Let M be an n-dimensional Riemannian manifold $(n\geqq 2)$ of the structure

$$M=(r_0, \infty)\times S^{n-1}=\{(r, \omega)\mid r\in(r_0, \infty), \omega\in S^{n-1}\}$$

with the metric

$$ds^2=dr^2+\rho(r)^2 d\tilde{s}^2$$

where $d\tilde{s}$ is the ordinary line element of $n-1$ dimensional unit sphere S^{n-1} and $\rho(r)$ is a non-negative twice continuously differentiable function[1]. Then, the Laplace-Beltrami operator on M is expressed as

$$\Delta=\frac{1}{\rho^{n-1}}\frac{\partial}{\partial r}\left(\rho^{n-1}\frac{\partial}{\partial r}\right)+\frac{1}{\rho^2}\Lambda$$

where Λ is that on S^{n-1}.

We use the following notations throughout this paper: x denotes a point of M, $q(x)$ is a measurable real-valued function[2] defined on M, λ is a constant and $f(x)$ is the real-valued solution[2] of the Schrödinger-type equation

(1.1) $\qquad -\Delta f+qf=\lambda f \qquad$ on M (in the sense of distribution)

which belongs[3] to $H^2_{\mathrm{loc}}(M)$ and does not vanish identically in any neighbourhood of infinity. Moreover, we denote by a topside or superior dot the (ordinary or partial) derivative with respect to r and by a superior -1 the reciprocal number. Further, the expression "$(r\to\infty$, unif.)" should be read as "uniformly on S^{n-1} as $r\to\infty$".

ASSUMPTION$_{(\rho,0)}$. $\rho\in C^2((r_0,\infty))$ while $\rho(r)$ is monotone increasing and diverging.

ASSUMPTION$_{(\rho,1)}$. $\rho^{-1}\dot{\rho}=o(1), \qquad \dot{\rho}^{-1}\ddot{\rho}=o(1) \qquad (r\to\infty)$.

For economy of description, we shall refer to some properties of the potential functions by certain names.

DEFINITION OF PROPERTY$_{(q,a)}$. We say a function q has *Property*$_{(q,a)}$ if for any $\varepsilon>0$ there exists a positive constant C_ε depending only on q, ε, ρ and the demension n, such that for every $v\in H^1_{\mathrm{loc}}(M)$ and for any r', r'' $(r_0+1\leqq r'<r'')$ we have

1) The variable r corresponds to the length along the meridian and $\rho(r)$ to the radius of rotation if M is a surface of revolution.
It should be noted that we consider a local problem in the neighbourhood of infinity. Therefore it is of no significance that the boundary of M has the very specific shape.

2) With little modification, we can treat complex-valued ones.

3) $H^k_{\mathrm{loc}}(M)$ is the set of functions $\in L^2_{\mathrm{loc}}(M)$ whose derivatives in the sense of distribution up to k-th order with respect to the local coordinates also belong to $L^2_{\mathrm{loc}}(M)$.

(1.2)
$$\int_{r'<r<r''}|q(x)||v(x)|^2dx$$
$$\leqq\varepsilon\int_{r'-1<r<r''+1}|\nabla v(x)|^2dx+C_\varepsilon\int_{r'-1<r<r''+1}|v(x)|^2dx.$$

REMARK ON $\partial q/\partial r$. Let us agree that if we find, say, $\dot{q}$ in the sequel, $q(r,\omega)$ is assumed to be absolutely continuous with respect to $r>r_0$ for almost every fixed $\omega\in S^{n-1}$.

DEFINITION OF PROPERTY$_{(q,b)}$. q is said to possess *Property*$_{(q,b)}$ if $\dot{q}|v|^2\in L^1_{\mathrm{loc}}(M)$ for every $v\in H^2_{\mathrm{loc}}(M)$.

ASSUMPTION$_{(q,1)}$. q is decomposed into the sum of two real-valued functions q_1, q_2 such that

(i) q_1 possesses Properties$_{(q,a)(q,b)}$;

(ii) There exist constants γ $(0<\gamma\leqq 2)$ and E and a nonnegative function $e(r)$ such that

$$e(r)=o(1)\qquad (r\to\infty),$$
$$\gamma^{-1}\rho\dot{\rho}^{-1}\dot{q}_1+q_1\leqq E+e(r)\qquad \text{almost everywhere;}$$

(iii) $q_2=o(\rho^{-1}\dot{\rho})\qquad (r\to\infty,\ \text{unif.}).$

THEOREM 1. *Let Assumptions*$_{(\rho,0)(\rho,1)(q,1)}$ *be satisfied and let* $\lambda>E$. *For an arbitrary* ε $(0<\varepsilon<\min(\gamma,\lambda-E))$, *there exist a positive constant* C, *a constant* $\tilde{C}$ *and an* r_1 $(\geqq r_0)$ *such that*

$$\int_{r_0<r<R}|f(x)|^2dx\geqq C\int_{r_0}^{R}\rho(r)^{-\gamma/2-\varepsilon}dr+\tilde{C}\qquad (R\geqq r_1)$$

holds. $(dx=\rho(r)^{n-1}drd\omega$ *is the volume element of* M *and* $r_0<r<R$ *indicates the range of* x *whose* r*-coordinates fulfil the inequality.*)

REMARK. If we apply Tayoshi's result [11; Theorem 1.0.1] to this special case, we are led to the following statement. Assumptions: $\exists C_1\geqq\exists C_2>0$, $C_2r^{-1}\leqq\rho(r)^{-1}\dot{\rho}(r)\leqq C_1r^{-1}$ for large r; q_1 satisfies Properties$_{(q,a)(q,b)}$; $\exists\gamma$ $(0<\gamma<C_1)$ such that $\gamma^{-1}r\dot{q}_1+q_1\leqq E+e(r)$; $q_2=o(r^{-1})$. Conclusion: $\int_{r_0<r<R}|f(x)|^2dx\geqq CR^{1-\gamma/2-\varepsilon}$. His theorem treats of very general circumstances in comparison to our Theorem 1. But we have a little sharper results. Note that we need not exclude the case $\rho^{-1}\dot{\rho}\neq o(r^{-1})$ for example $\rho(r)=\log r$.

Properties$_{(q,a)(q,b)}$ are rather abstract. If M is a surface of bounded curvature which is imbedded isometrically in $\boldsymbol{R}^{n+1}$, we can find more concrete conditions. First, let us introduce the "quasi-Stummel space" $S^{\delta}(M)$ in a similar manner to that in the ordinary Euclidean spaces (cf. [3], p. 124).

DEFINITION OF $S^{\delta}(M)$. We denote by $S^{\delta}(M)$ (δ being a constant $0<\delta<2$) the set of functions $q(x)$ such that

$$\int_{\mathscr{B}(x)} \frac{|q(y)|}{\mathrm{dis}(x, y)^{n-2+\delta}} dy \tag{1.3}$$

is bounded on M, where $\mathrm{dis}(x, y)$ denotes the geodesic distance between x and y, and $\mathscr{B}(x)=\{y \in M \mid \mathrm{dis}(x, y)<1\}$. We denote further by $S_{\mathrm{loc}}^{\delta}(M)$ the set of $q(x)$ such that (1.3) is locally bounded.

DEFINITION OF PROPERTY$_{(q,c)}$. We say $q(x)$ has *Property*$_{(q,c)}$ if it satisfies at least *one of* the following conditions (i) and (ii).
(i) q is expressed as a sum of two functions $q^{(a)}$ and $q^{(b)}$ such that $q^{(i)} \in S_{\mathrm{loc}}^{\delta}(M)$ $(i=a, b)$ for some δ and $\dot{q}^{(a)} \geqq 0$, $\dot{q}^{(b)} \leqq 0$ almost everywhere;
(ii) $\dot{q} \in S_{\mathrm{loc}}^{\delta}(M)$ for some δ.

ASSUMPTION$_{(\rho,2)}$. $\dot{\rho}(r) \leqq 1$ (therefore, M is a surface of revolution in $\boldsymbol{R}^{n+1}$) and the normal curvature of the meridian does not exceed a constant in magnitude, i.e.

$$|\ddot{\rho}|/\sqrt{1-\dot{\rho}^2} \leqq c$$

for some constant $c>0$.

LEMMA 1. *If Assumption*$_{(\rho,2)}$ *is satisfied and if* $q \in S^{\delta}(M)$, *then* q *possesses Property*$_{(q,a)}$. *If* $q \in S_{\mathrm{loc}}^{\delta}(M)$ *then* q *satisfies the same inequality with* C_{ε} *depending also on* r' *and* r''.

REMARK. Thus, in particular, $q \in S_{\mathrm{loc}}^{\delta}(M)$ and $v \in H_{\mathrm{loc}}^{1}(M)$ imply $q|v|^2 \in L_{\mathrm{loc}}^{1}(\tilde{M})$ where $\tilde{M}=\{x \in M \mid r>r_0+1\}$.

LEMMA 2. *If Assumption*$_{(\rho,2)}$ *is satisfied, Property*$_{(q,c)}$ *implies Property*$_{(q,b)}$.

The proofs of these lemmas will be given later. Assumption$_{(\rho,2)}$ is needed only in proving Lemmas 1 and 2. Therefore, if q is a bounded C^1-function, we can dispense with this assumption because q has Properties$_{(q,a)(q,b)}$

automatically.

If M is an exterior domain of a Euclidean space, the ε in Theorem 1 can be taken away. Indeed, many works (e.g. [10] [13]) dealt with this case. Especially, Uchiyama [12], Mochizuki and Uchiyama [9] and Eastham and Kalf [3] gave sharpest forms. As an application of our method, we would like to offer another proof. We repeat the theorem in a rather rough form, though the conditions on $\dot{q}$ is slightly relaxed.

ASSUMPTION$_{(\rho,3)}$. $n\geqq 3$ and $\rho(r)=cr$ $(0<c\leqq 1)$, i.e. M is the exterior domain $\{|x|>r_0\}$ in $\boldsymbol{R}^n$ or the part $\{r>r_0\}$ of the surface of a cone.

ASSUMPTION$_{(q,2)}$.
(i) $q\in S^{\delta}(M)$ for some δ and possesses Property$_{(q,c)}$;
(ii) There exist constants γ $(0<\gamma\leqq 2)$ and E and a function $e(r)$ such that

$$e(r)=o(1) \qquad (r\to\infty),$$
$$\gamma^{-1}r\dot{q}+q\leqq E+e(r) \qquad \text{almost everywhere.}$$

THEOREM 2. *If Assumptions$_{(\rho,3)(q,2)}$ are satisfied and if $\lambda>E$, we can find a positive constant C and an r_1 $(\geqq r_0)$ so that*
(a) *if $0<\gamma<2$, we have*

$$\int_{r_0<|x|<R}|f(x)|^2dx\geqq CR^{1-\gamma/2} \qquad (R\geqq r_1),$$

(b) *and if $\gamma=2$,*

$$\int_{r_0<|x|<R}|f(x)|^2dx\geqq C\log R \qquad (R\geqq r_1).$$

The case "$\gamma=0$" corresponds to the requirement "q is repulsive". Theorem 2 fails in this case. But we can obtain two theorems for repulsive potentials by strengthening the conditions. The first one is, so to speak, on positive sub-homogeneous potentials such as the repulsive Coulomb potentials.

ASSUMPTION$_{(q,2')}$.
(i) $q\in S^{\delta}(M)$ for some δ;
(ii) $q\geqq 0$;
(iii) There exists a constant γ $(0<\gamma\leqq 2)$ such that

$$\gamma^{-1}r\dot{q}+q\leqq 0 \qquad \text{almost everywhere.}$$

THEOREM 2′. *If Assumptions*$_{(\rho,3)(q,2')}$ *are satisfied and if* $\lambda>0$, *we can find a positive constant* C *and an* r_1 ($\geqq r_0$) *such that*

$$\int_{r_0<|x|<R}|f(x)|^2dx\geqq CR \qquad (R\geqq r_1).$$

The second one relates to simple repulsive potentials. In exchange for weakening the restriction on q we assume that $M=\boldsymbol{R}^n$ or the entire surface of a cone. J. Weidmann gave a result in this connection ([14]; Corollary 1. *Cf.* also [3]; Example 5.3.6). We make it more precise under a little weaker conditions.

ASSUMPTION$_{(q,3)}$.
(i) $q\in S^\delta(M)$ for some δ.
(ii) $\dot{q}\leqq 0$ almost everywhere.

THEOREM 3. *If* $M=\boldsymbol{R}^n$ ($n\geqq 3$) *or the whole surface of a* (*half*) *cone, and if Assumption*$_{(q,3)}$ *is satisfied, then, for the nonzero solution* $f(x)$ *of* (1.1) *such that* $f(x)$ *and* $\nabla f(x)$ *are bounded near* $r=0$, *there exist a positive constant* C *and an* r_1 ($\geqq 0$) *for which*

$$\int_{|x|<R}|f(x)|^2dx\geqq CR \qquad (R\geqq r_1)$$

holds. (*This statement is valid in the case* $-\infty<\lambda<\infty$.)

In a very particular case, we can present one more theorem where the smallness of $\ddot{\rho}$ is not required.

ASSUMPTION$_{(\rho,4)}$. The dimension $n=2$.

ASSUMPTION$_{(q,4)}$.
(i) $q\in S^\delta(M)$ for some δ and possesses Property$_{(q,c)}$;
(ii) There exist a constant E and a nonnegative function $e(r)$ such that

$$e(r)=o(1) \qquad (r\to\infty),$$
$$\frac{1}{2}\rho\left(\frac{d\rho}{dr}\right)^{-1}\frac{\partial q}{\partial r}+q\leqq E+e(r) \qquad \text{almost everywhere.}$$

THEOREM 4. *If Assumptions*$_{(\rho,0)(\rho,2)(\rho,4)(q,4)}$ *are satisfied and if* $\lambda>E$, *then we can find positive constants* C *and* r_1 ($\geqq r_0$) *such that*

$$\int_{r_0<r<R}|f(x)|^2dx\geqq C\int_{r_0}^{R}\frac{dr}{\rho(r)} \qquad (R\geqq r_1).$$

REMARK. The right-hand member goes to ∞ as $R\to\infty$ because $\rho(r)^{-1}\geqq\rho(r)^{-1}\dot\rho(r)$ (this fact is essential in the proof). As was mentioned, if in addition $q\in C^1$ and is bounded, then Assumption$_{(\rho,2)}$ is not necessary though $\int_{r_0}^{\infty}\rho(r)^{-1}dr=\infty$ is required instead.

It is of some interest to describe this theorem in terms of surfaces of revolution. We prefer to assume q is bounded and C^1 in order to emphasize unnecessariness of restrictions on the curvature of M.

COROLLARY. *Let M be the two dimensional surface obtained by rotating the graph of a C^2-function $t=t(\rho)$, $0\leqq\rho_0\leqq\rho<\infty$, around the t-axis in $\boldsymbol{R}^3$. If q is a bounded C^1-function and if there exist a number E and a nonnegative function $e(\rho)$ such that*

$$e(\rho)=o(1) \qquad (\rho\to\infty),$$

$$\frac{1}{2}\rho\frac{\partial q}{\partial\rho}+q\leqq E+e(\rho) \qquad \textit{almost everywhere,}$$

then, for $\lambda>E$ there exist positive constants C and ρ_1 such that

$$\int_{\rho_0<\rho<P}|f(\rho,\omega)|^2d\sigma\geqq C\int_{\rho_0}^{P}\rho^{-1}\sqrt{1+t'(\rho)^2}d\rho \qquad (P\geqq\rho_1)$$

holds. Here $d\sigma$ means the surface element of M.

§ 2. Abstract differential equation.

In order to carry out the proofs systematically, we prefer to describe the process of estimation in an abstract manner. That is, we interpret (1.1) as an equation for a vector-valued function and closely examine a subsidiary function to obtain the estimate of the solution.

Let $\mathfrak{H}$ be a Hilbert space and $\mathfrak{D}_0$ its linear subset. $A_0=A_0(r)$ and $A_1=A_1(r)$ are assumed to be linear operators defined for each r $(>r_0)$. Further, we assume that for each value of r the domains of $A_0(r)$ and $A_1(r)$ contain $\mathfrak{D}_0$. We denote in the sequel by $(\ ,\ )$ and $\|\ \|$ the inner product and the norm of $\mathfrak{H}$ respectively, and by a dot the derivative with respect to r. Moreover, instead of writing as "for almost every $r>r_0$", we say simply "$(r>r_0)$".

We shall consider a differential equation in $\mathfrak{H}$ and want to get estimates for the norm of the solution. To this end, we begin with several conditions and definitions.

Condition 0.
(i) $(A_0(r)v, w)=(v, A_0(r)w)$ for every $v, w \in \mathfrak{D}_0$;
(ii) For each $v \in \mathfrak{D}_0$, $(A_0(r)v, v)$ is differentiable at almost every $r>r_0$ and its derivative $(\dot{A}_0(r)v, v) \in L^1_{\mathrm{loc}}((r_0, \infty))$.

DEFINITION OF $B(v, w)$. Let $p(r), a(r)$ and $\varphi(r)>0$ be real-valued C^2-functions. For every $v \in \mathfrak{D}_0$ and $w \in \mathfrak{H}$, we set

$$B(v, w)=(\{(\varphi A_0)^{\cdot}+p\varphi(A_0+A_1)+(a\varphi)^{\cdot}\}v, v) \\ +(\dot{\varphi}-p\varphi)\|w\|^2+(\{2\varphi(a-A_1)-(p\varphi)^{\cdot}\}v, w).$$

Condition 1. There exist a nonnegative function ψ and a number r_2 $(\geqq r_0)$ such that

(i) $\int_{r_0}^{\infty}\psi(r)dr=\infty$;
(ii) $B(v, w)\geqq\psi\|v\|^2 \qquad (r>r_2)$

for every $v \in \mathfrak{D}_0$, $w \in \mathfrak{H}$.

Condition 2. We can find a nonnegative function $b(r)$ and a number r_2 $(\geqq r_0)$[4] such that for an arbitrary $v \in \mathfrak{D}_0$,

(i) $(\{a(r)-A_1(r)\}v, v)\geqq -b(r)\|v\|^2 \qquad (r>r_2)$;
(ii) $\int_{r_0}^{\infty}e^{P(r)}b(r)dr<\infty$, where $P(r)=\int p(r)dr$.

Condition 3. Let K be an arbitrary positive constant and put

$$\zeta(R)=\int_{r_0}^{R}\frac{e^{P(r)}}{\varphi(r)}\int_{r}^{\infty}\psi(s)\exp\left\{-K\int_{r}^{s}e^{-P(t)}dt\right\}dsdr.$$

Then we have

$$\lim_{R\to\infty}\zeta(R)=\infty, \qquad \lim_{R\to\infty}e^{-P(R)}\zeta(R)=\infty.$$

Condition 4. There exist a number r_2 and a function $\eta(r) \in L^1_{\mathrm{loc}}((r_2, \infty))$ such that for every $v \in \mathfrak{D}_0$ and $w \in \mathfrak{H}$,

$$B(v, w)\geqq\eta(r)\dot{\varphi}\{\|w\|^2+(A_0v, v)-p(v, w)+a\|v\|^2\} \qquad (r>r_2)$$

holds.

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4) We can choose the same value of r_2 as in Condition 1.

Condition 4′. There exist a number r_2 and a function $\chi(r)\in L^1((r_2,\infty))$ such that for any $v\in\mathfrak{D}_0$ and $w\in\mathfrak{H}$,

$$B(v,w)\geqq(\dot{\varphi}-\varphi\chi)\{\|w\|^2+(A_0v,v)-p(v,w)+a\|v\|^2\}\qquad(r>r_2)$$

holds.

Now we consider the following second order differential equation in $\mathfrak{H}$:

$$\ddot{u}+A_0(r)u+A_1(r)u=0, \tag{2.1}$$

and study the solution $u(r)$ which does not vanish identically in any neighbourhood of infinity. We mean by "u is the solution" that (i) $u(r)\in\mathfrak{D}_0$ a.e.; (ii) $\ddot{u}(r)$ exists a.e. in the strong sense belonging to $L^2_{\mathrm{loc}}((r_0,\infty);\ \mathfrak{H})$ and enjoying (2.1); (iii) $u, \dot{u}$ are the indefinite integrals of $\dot{u}$, $\ddot{u}$ respectively in the strong sense; (iv) $(A_0(r)u,u)$ is absolutely continuous and satisfies $\dfrac{d}{dr}(A_0u,u)=(\dot{A}_0u,u)+2\,\mathrm{Re}(A_0u,\dot{u})$ a.e.

Definition of $F(r)$. For the solution u, we set

$$F(r)=\|\dot{u}\|^2+(A_0u,u)-p(\dot{u},u)+a\|u\|^2,$$

where $p(r)$ and $a(r)$ are the functions appearing in the definition of $B(v,w)$. (Note that $F(r)$ is an absolutely continuous function.)

We are now in a position to describe several estimates for $F(r)$.

Lemma 3. *If Conditions* 0 *and* 1 *are satisfied, we have*

$$(\varphi F)^{\cdot}=B(u,\dot{u})\geqq\dot{\varphi}\|u\|^2\qquad(r>r_2).$$

Proof. Differentiating φF by r and considering (2.1), we have

$$\begin{aligned}(\varphi F)^{\cdot}&=\varphi\{2(\ddot{u}+A_0u,\dot{u})+(\dot{A}_0u,u)\}+\dot{\varphi}\{\|\dot{u}\|^2+(A_0u,u)\}\\&\quad-(p\varphi)^{\cdot}(u,\dot{u})-p\varphi\|\dot{u}\|^2-p\varphi(\ddot{u},u)+2a\varphi(u,\dot{u})+(a\varphi)^{\cdot}\|u\|^2\\&=-2\varphi(A_1u,\dot{u})+(\{(\varphi A_0)^{\cdot}+(a\varphi)^{\cdot}\}u,u)+(\dot{\varphi}-p\varphi)\|\dot{u}\|^2\\&\quad+\{2a\varphi-(p\varphi)^{\cdot}\}(u,\dot{u})+p\varphi((A_0+A_1)u,u)\\&=(\{(\varphi A_0)^{\cdot}+p\varphi(A_0+A_1)+(a\varphi)^{\cdot}\}u,u)\\&\quad+(\dot{\varphi}-p\varphi)\|\dot{u}\|^2+(\{2a\varphi-2\varphi A_1-(p\varphi)^{\cdot}\}u,\dot{u})\\&=B(u,\dot{u}).\end{aligned}$$

Therefore, by Condition 1, we obtain

$$(\varphi F)^{\cdot} \geqq \phi \|u\|^2.$$

LEMMA 4. *Under Conditions* 0, 1, 2 *and* 3, *we can find an* r_3 $(\geqq r_2)$ *such that*

$$F(r_3) > 0.$$

PROOF. Suppose to the contrary that $F(r) \leqq 0$ for almost every r $(> r_2)$. Then, since F and $\dot{F}$ belong to $L^1_{\mathrm{loc}}((r_0, \infty))$, it follows from Lemma 3 that

$$\begin{aligned}-\varphi(r)F(r) &= -\varphi(t)F(t) + \int_r^t (\varphi(s)F(s))^{\cdot} ds \\ &\geqq \int_r^t \phi(s)\|u(s)\|^2 ds.\end{aligned}$$

The last member is an increasing function of t, while the first one does not depend on t. Hence, letting $t \to \infty$, we obtain

$$-\varphi(r)F(r) \geqq \int_r^\infty \phi(s)\|u(s)\|^2 ds \tag{2.2}$$

together with the finiteness of the right member.

Now, let I be an interval in which $u(r)$ does not vanish, and for $r \in I$, set

$$g(r) = \log \|u(r)\|^2.$$

Then,

$$\begin{aligned}&\dot{g} = 2(\dot{u}, u)/\|u\|^2, \\ &\ddot{g} = \{2(\ddot{u}, u) + 2\|\dot{u}\|^2\}/\|u\|^2 - 4(\dot{u}, u)^2/\|u\|^4.\end{aligned}$$

And the Schwarz inequality shows

$$\begin{aligned}\ddot{g} &\geqq \{2(\ddot{u}, u) - 2\|\dot{u}\|^2\}/\|u\|^2 \\ &= -2e^{-g}\{F + p(\dot{u}, u) - a\|u\|^2 + (A_1 u, u)\} \\ &= -2e^{-g}F - p\dot{g} + 2a - 2e^{-g}(A_1 u, u).\end{aligned}$$

Hence,

$$\begin{aligned}(e^P \dot{g})^{\cdot} &= e^P(\ddot{g} + p\dot{g}) \\ &\geqq -2e^{P-g}F + 2e^P\{a - (A_1 u, u)/\|u\|^2\}.\end{aligned} \tag{2.3}$$

Therefore, from Condition 2 and the assumption $F \leqq 0$, we see

$$(e^P\dot{g})^{\cdot} \geqq -2e^P b,$$

and hence, there exists a positive constant K such that

$$\begin{aligned} e^{P(t)}\dot{g}(t) &\geqq e^{P(r_2)}\dot{g}(r_2) - 2\int_{r_1}^{t} e^{P(r)} b(r) dr \\ &\geqq -K. \end{aligned}$$

Consequently, we obtain

$$g(s) - g(r) \geqq -K\int_r^s e^{-P(t)} dt \qquad (r, s \in I,\ r \leqq s).$$

The right-hand side being a continuous function of $s \in (r_0, \infty)$, $g(s)$ never goes to $-\infty$ at a finite s. Hence we conclude

$$u(r) \neq 0 \qquad \text{throughout the interval } (r_2, \infty).$$

Moreover, from (2.2) and (2.3) we have

$$\begin{aligned} (e^{P(r)}\dot{g}(r))^{\cdot} &\geqq \frac{2e^{P(r)}}{\varphi(r)}\int_r^{\infty} \psi(s) e^{g(s)-g(r)} ds + \text{summable function} \\ &\geqq \frac{2e^{P(r)}}{\varphi(r)}\int_r^{\infty} \psi(s) \exp\Big(-K\int_r^s e^{-P(t)} dt\Big) ds + \text{summable function} \end{aligned}$$

which yields

$$\begin{aligned} e^{P(R)}\dot{g}(R) &\geqq 2\int_{r_1}^{R} \frac{e^{P(r)}}{\varphi(r)}\int_r^{\infty} \psi(s)\exp\Big(-K\int_r^s e^{-P(t)} dt\Big) ds dr + \text{const.} \\ &= 2\zeta(R) + \text{const.} \end{aligned}$$

Therefore, Condition 3 shows

$$\dot{g}(R) = e^{-P(R)}(2\zeta(R) + \text{const.}) \geqq e^{-P(R)}\zeta(R) \longrightarrow \infty \qquad (R \to \infty),$$

and hence

$$\|u(r)\| = e^{g(r)/2} \longrightarrow \infty \qquad (r \to \infty),$$

which contradicts the fact that

$$\int_{r_0}^{\infty} \psi(s)\|u(s)\|^2 ds < \infty \qquad \text{while} \qquad \int_{r_0}^{\infty} \psi(s) ds = \infty,$$

as were found in (2.2) and Condition 1. Thus, Lemma 4 is established.

LEMMA 5. *Under Conditions* 0, 1, 2 *and* 3, *we can find a positive constant C satisfying*

$$F(r) \geqq C\varphi(r)^{-1} \qquad (r \geqq r_3).$$

PROOF. Lemma 3 shows $(\varphi F)^{\cdot} \geqq 0$ for $r > r_2$. Therefore, $\varphi > 0$ and $F(r_3) > 0$ (by Lemma 4) give

$$\varphi(r)F(r) \geqq \varphi(r_3)F(r_3) = C > 0.$$

LEMMA 6. *Under Conditions* 0, 1, 2, 3 *and* 4, *there exists a positive constant* C *such that*

$$F(r) \geqq C\varphi(r)^{-1} \exp\left\{\int \eta(r)\varphi(r)^{-1}\dot{\varphi}(r)dr\right\} \qquad (r \geqq r_3)$$

holds. In particular, if $\eta(r)$ *is a constant* η, *we have*

$$F(r) \geqq C\varphi(r)^{\eta-1} \qquad (r \geqq r_3).$$

PROOF. Since Condition 4 reads

$$(\varphi F)^{\cdot} \geqq \eta\dot{\varphi}F = \eta\varphi^{-1}\dot{\varphi}(\varphi F),$$

the fact that $\varphi(r)F(r) > 0$ $(r \geqq r_3)$ (by Lemma 4) shows

$$\varphi(r)F(r) \geqq \varphi(r_3)F(r_3)\exp\left\{\int_{r_3}^{r} \eta(s)\varphi(s)^{-1}\dot{\varphi}(s)ds\right\} \qquad (r \geqq r_3).$$

In passing, we refer to the following lemma though it is not used in this paper.

LEMMA 6′. *Under Conditions* 0, 1, 2, 3 *and* 4′, *we can find a positive constant* C *such that*

$$F(r) \geqq C \qquad (r \geqq r_3).$$

PROOF. Condition 4′ means just

$$(\varphi F)^{\cdot} \geqq (\varphi^{-1}\dot{\varphi} - \chi)\varphi F,$$

by Lemma 3. Hence, from Lemma 4 it follows that

$$(\varphi F)^{-1}(\varphi F)^{\cdot} \geqq \varphi^{-1}\dot{\varphi} - \chi$$

for $r \geqq r_3$. Integrating both sides from r_3 to r, we obtain

$$F(r) \geqq F(r_3)\exp\left\{-\int_{r_3}^{r} \chi(s)ds\right\} \geqq C,$$

which proves Lemma 6′.

§ 3. Estimates for multiplication operators.

Let us begin with the proof of Lemmas 1 and 2.

PROOF OF LEMMA 1. Let $v\in C^1([r'-1, r''+1]\times S^{n-1})$. Choose an arbitrary constant a $(0<a<1)$. Let Π be the tangential plane to M at $x\in(r', r'')\times S^{n-1}$ and consider the orthogonal projection of $\mathscr{B}(x, a)=\{y\in M\,|\,\mathrm{dis}(x, y)<a\}$ on Π. We denote by η the projected image of $y\in\mathscr{B}(x, a)$ and set

$$\hat{v}(\eta)=v(y).$$

The distribution of the values of $\hat{v}(\eta)$ is "compressed" compared with that of $v(y)$. But we can easily see from Assumption$_{(\rho,2)}$ and the increasing property of $\rho(r)$ (=the radius of rotation) that the most compressed case occurs when each meridian in $\mathscr{B}(x, a)$ is a part of a circle of radius $1/c$, c being the constant appearing in Assumption$_{(\rho,2)}$. Since the compression of $\hat{v}(\eta)$ means the extension of the gradient $\nabla\hat{v}(\eta)$, we can conclude that there exists a constant C $(0<C<1)$ such that

$$|\nabla\hat{v}(\eta)|\leqq C^{-1}|\nabla v(y)| \qquad \text{for} \quad y\in\mathscr{B}(x, a).$$

(In fact, we can put $C=\cos(ca)$.)

Similarly, we can find a constant C' $(0<C'<1)$ which does not depend on x and satisfies

$$|x-\eta|\geqq C'\,\mathrm{dis}(x, y)$$

as long as $\mathrm{dis}(x, y)<a$. In particular, the image of $\mathscr{B}(x, a)$ contains the ball $|x-\eta|<b$ on Π whose radius b does not depend on x.

Now, in the Euclidean spaces, the following inequality is well-known (cf. [3] Appendix 1):

$$(3.1)\qquad |\hat{v}(x)|^2\leqq\text{const.}\,\delta^{-1}\int_{|x-\eta|<b}|x-\eta|^{2-n-\delta}\{b^\delta|\nabla\hat{v}(\eta)|^2+b^{\delta-2}|\hat{v}(\eta)|^2\}d\eta,$$

where the const. depends only on n. Recalling the previously mentioned estimate of each term and considering the fact that the Jacobian $\leqq 1$ (i.e. $d\eta\leqq dy$), we have

$$|v(x)|^2\leqq\text{const.}\,\delta^{-1}\int_{\mathscr{B}(x,a)}\mathrm{dis}(x, y)^{2-n-\delta}\{C^2b^\delta|\nabla v(y)|^2+b^{\delta-2}|v(y)|^2\}dy,$$

the const. depending only on n again. Thus,

$$
\begin{aligned}
&\int_{r'<r<r''}|q(x)||v(x)|^2dx\\
&\leqq \text{const. } \delta^{-1}\int_{r'<r<r''}|q(x)|dx\\
&\quad\times\int_{\mathscr{B}(x,a)}\operatorname{dis}(x,y)^{2-n-\delta}\{C^2b^\delta|\nabla v(y)|^2+b^{\delta-2}|v(y)|^2\}dy\\
&\leqq \text{const. } \delta^{-1}\int_{r'-a<r(y)<r''+a}\{C^2b^\delta|\nabla v(y)|^2+b^{\delta-2}|v(y)|^2\}dy\\
&\quad\times\int_{\mathscr{B}(y,a)}\frac{|q(x)|}{\operatorname{dis}(x,y)^{n-2+\delta}}dx\\
&=C_1b^\delta\int_{r'-a<r(y)<r''+a}|\nabla v(y)|^2dy+C_2b^{\delta-2}\int_{r'-a<r(y)<r''+a}|v(y)|^2dy,
\end{aligned}
$$

with some $C_1, C_2>0$ ($r(y)$ being r corresponding to y). Putting $C_1b^\delta=\varepsilon$, we obtain (1.2) for C^1-functions. For $v\in H^1_{\text{loc}}$, choose $v_n\in C^1([r'-1, r''+1]\times S^{n-1})$ converging to v in $H^1((r'-1, r''+1)\times S^{n-1})$ and apply the limiting process. This proves the former part. The latter assertion is clear.

PROOF OF LEMMA 2. It is sufficient to consider real-valued q and v. If $\dot q\in S^\delta_{\text{loc}}(M)$ then Property$_{(q,b)}$ follows from Lemma 1. Therefore we suppose (i). Consider $q^{(a)}$ and let $v\in H^2_{\text{loc}}(M)$. Since $q^{(a)}v^2\in L^1_{\text{loc}}(M)$ by Lemma 1, we see that $q^{(a)}v^2|_{\text{fixed } r}\in L^1(S^{n-1})$ for almost every r. Let r' and r'' ($r'<r''$) be such values of r. We can find a set $\Omega\subset S^{n-1}$ such that $S^{n-1}-\Omega$ is a null set and $2v\dot v\in L^1((r', r''))$ for each $\omega\in\Omega$. Therefore, as an indefinite integral, v^2 is absolutely continuous in r for each fixed $\omega\in\Omega$. As was mentioned in the remark before the definition of Property$_{(q,b)}$, the same is said to $q^{(a)}$, where the set Ω can be chosen commonly. These facts yield the integration by parts

$$
\begin{aligned}
(3.2)\qquad \int_{r'}^{r''}\dot q^{(a)}v^2\rho^{n-1}dr&=q^{(a)}v^2\rho^{n-1}|_{r=r''}-q^{(a)}v^2\rho^{n-1}|_{r=r'}\\
&\quad-2\int_{r'}^{r''}q^{(a)}v\dot v\rho^{n-1}dr-(n-1)\int_{r'}^{r''}q^{(a)}v^2\rho^{n-2}\dot\rho dr.
\end{aligned}
$$

The first and the second terms in the right-hand side belong to $L^1(S^{n-1})$. Further, the inequality $2v\dot v\leqq|v|^2+|\dot v|^2$ and $v, \dot v\in H^1_{\text{loc}}(M)$ imply, by virtue of Lemma 1, that the third term also belongs to $L^1(S^{n-1})$ as well as the fourth term. Hence the left-hand side of (3.2) belongs to $L^1(S^{n-1})$. But, because $\dot q^{(a)}$ has definite sign, it follows that $\dot q^{(a)}v^2\in L^1_{\text{loc}}(M)$. The same is said to $q^{(b)}$. Hence, $\dot qv^2\in L^1_{\text{loc}}(M)$, which proves Lemma 2.

The following lemma authorizes the differentiation under integral sign. Namely:

LEMMA 7. *If q has Properties$_{(q,a)(q,b)}$ and $v\in H^2_{\mathrm{loc}}(M)$, then (qu, u) is absolutely continuous in r where $u=\rho(r)^{n-1}v$. Moreover, we have*

$$\frac{d}{dr}(qu, u)=(\dot{q}u, u)+2\operatorname{Re}(qu, \dot{u}) \qquad (a.e.\ r>r_0+1).$$

PROOF. We can easily find by Properties$_{(q,a)(q,b)}$ that

$$\dot{q}u^2,\ q(u^2)^{\cdot}\in L^1_{\mathrm{loc}}((r_0+1, \infty)\times S^{n-1};\ dr d\omega).$$

Hence, their sum $(qu^2)^{\cdot}\in L^1_{\mathrm{loc}}((r_0+1, \infty)\times S^{n-1};\ dr d\omega)$. Accordingly, Fubini's theorem yields

$$\begin{aligned}\int_{S^{n-1}} qu^2|_{r=r''}d\omega-\int_{S^{n-1}} qu^2|_{r=r'}d\omega&=\int_{S^{n-1}}\int_{r'}^{r''}(qu^2)^{\cdot}drd\omega\\&=\int_{r'}^{r''}\int_{S^{n-1}}(qu^2)^{\cdot}d\omega dr.\end{aligned}$$

Fixing r' and replacing r'' by r, we see that

$$(qu, u)=\int_{S^{n-1}}(qu^2)d\omega$$

is the indefinite integral of the L^1_{loc}-function $\int_{S^{n-1}}(qu^2)^{\cdot}d\omega$, and therefore absolutely continuous. Consequently,

$$\begin{aligned}\frac{d}{dr}(qu, u)&=\int_{S^{n-1}}(qu^2)^{\cdot}d\omega\\&=(\dot{q}u, u)+2(qu, \dot{u}) \qquad (\text{a.e. } r>r_0+1).\end{aligned}$$

This completes the proof.

§ 4. Proofs of the theorems.

Let

$$u=u(x)=u(r, \omega)=\rho^{(n-1)/2}f(x)$$

where $f(x)$ is the solution of (1.1). A straightforward calculation shows that u satisfies the equation

$$\ddot{u}+\rho^{-2}\Lambda u+(\lambda-q)u-n_1\rho^{-2}\dot{\rho}^2u-n_2\rho^{-1}\ddot{\rho}u=0 \tag{4.1}$$

on M, where a dot stands for $\partial/\partial r$ and

$$n_1=\frac{(n-1)(n-3)}{4}, \qquad n_2=\frac{n-1}{2}.$$

Now, let

$$\mathfrak{H}=L^2(S^{n-1}) \qquad \text{with} \qquad (v, w)=\int_{S^{n-1}} v(\omega)\overline{w(\omega)}d\omega,$$

and set

$$\mathfrak{D}_0=H^2(S^{n-1}).$$

Our aim is to reduce the problems to the abstract theory described in §2. At first, we note that the fact $f\in H^2_{\mathrm{loc}}(M)$ implies that $u(r)$ satisfies the conditions (i)~(iii) on the solution in §2.

Further, set

$$\begin{aligned} &A_0=\rho^{-2}\Lambda+\lambda-q_1 &&\text{with} \quad \mathfrak{D}(A_0)=\mathfrak{D}_0, \\ &A_1=-q_2-n_1\rho^{-2}\dot{\rho}^2-n_2\rho^{-1}\ddot{\rho} &&\text{with} \quad \mathfrak{D}(A_1)=\mathfrak{H}. \end{aligned} \tag{4.2}$$

We note that Property$_{(q,b)}$ shows the differentiability of (A_0v, v) for $v\in\mathfrak{D}_0$ (Condition 0). Similarly, Lemma 7 realizes condition (iv) on the abstract solution. Hence, rewriting (4.1) as $\ddot{u}+A_0u+A_1u=0$, we find that $u=u(r, \cdot)$ is the solution of (2.1).

We have already obtained several estimates on $F(r)$. But our final need is those for $|f(x)|^2$ or $|u(r, \omega)|^2$. To this end, we present here a lemma which is a modification of [2; Lemma 2] or [12; Lemma 3]. Let $\sigma_R(r)$ be a C^2-function which possesses the following properties where r_4 $(\geqq r_3+1)$ is chosen arbitrarily.

(i) $0\leqq\sigma_R(r)\leqq 1$ $(r_0\leqq r<\infty)$;

(ii) $\sigma_R(r)=0$ $(r_0\leqq r\leqq r_4$ or $R\leqq r)$;

(iii) $\sigma_R(r)=1$ $(r_4+1\leqq r\leqq R-1)$;

(iv) the values of $\sigma_R(r)$ do not depend on R in $r_0\leqq r\leqq r_4+1$;

(v) in $R-1\leqq r\leqq R$, the graph of $\sigma_R(r)$ does not change its shape but for translation.

LEMMA 8. *Let $q=q_1+q_2$ and let $u(r, \cdot)$ be the solution of* (4.1). *Suppose q_1 possesses Properties$_{(q,a)(q,b)}$ and q_2 is bounded. Then there exist positive constants C_1 and C_2 depending only on ρ and n such that*

$$\int_{r_0}^R \sigma_R^2 F dr\leqq C_1\int_{r_0}^R \|u\|^2dr-C_2\int_{r_0}^R \sigma_R^2\rho^{-2}\|\tilde{\nabla}u\|^2dr \qquad (R\geqq r_4+1),$$

where $\tilde{\nabla}$ denotes the gradient on S^{n-1}.

PROOF. Recall that $u=\rho^{(n-1)/2}f$, $|\nabla f|^2=|\partial f/\partial r|^2+\rho^{-2}|\tilde{\nabla} f|^2$, $n_2=(n-1)/2$ and substitute them into (1.2). Then, because $\sigma_R=0$ for $r\geqq R$, we have

$$
\begin{aligned}
(4.3)\qquad \left|\int_{r_0}^R \sigma_R^2(q_1u, u)dr\right| &\leqq \int_{r_4<r<R} |q_1||\sigma_R f|^2dx \\
&\leqq \varepsilon\int_{r_0}^R \{\rho^{n-1}\|(\sigma_R\rho^{-(n-1)/2}u)^{\cdot}\|^2+\sigma_R^2\rho^{-2}\|\tilde{\nabla} u\|^2\}dr \\
&\quad +C_\varepsilon\int_{r_0}^R \sigma_R^2\|u\|^2dr \\
&\leqq 2\varepsilon\int_{r_0}^R \sigma_R^2\|\dot{u}\|^2dr+\varepsilon\int_{r_0}^R \sigma_R^2\rho^{-2}\|\tilde{\nabla} u\|^2dr \\
&\quad +\int_{r_0}^R (2\varepsilon\dot{\sigma}_R^2+2\varepsilon n_2^2\sigma_R^2\rho^{-2}\dot{\rho}^2+C_\varepsilon\sigma_R^2)\|u\|^2dr \\
&\leqq 2\varepsilon\int_{r_0}^R \sigma_R^2\|\dot{u}\|^2dr+\varepsilon\int_{r_0}^R \sigma_R^2\rho^{-2}\|\tilde{\nabla} u\|^2dr+k_1\int_{r_0}^R \|u\|^2dr
\end{aligned}
$$

with

$$k_1=\sup_{r_4<r}|2\varepsilon\dot{\sigma}_R^2+2\varepsilon n_2^2\sigma_R^2\rho^{-2}\dot{\rho}^2+C_\varepsilon\sigma_R^2|.$$

On the other hand, (4.1) and the integration by parts show

$$
\begin{aligned}
(4.4)\qquad \int_{r_0}^R \sigma_R^2\|\dot{u}\|^2dr &= \int_{r_0}^R \sigma_R^2\{\|\dot{u}\|^2+(\ddot{u}, u) \\
&\quad +((\lambda-q_2-n_1\rho^{-2}\dot{\rho}^2-n_2\rho^{-1}\ddot{\rho})u, u)-\rho^{-2}\|\tilde{\nabla} u\|^2-(q_1u, u)\}dr \\
&= \int_{r_0}^R \frac{1}{2}\sigma_R^2\frac{d^2}{dr^2}\|u\|^2dr \\
&\quad +\int_{r_0}^R \sigma_R^2\{((\lambda-q_2-n_1\rho^{-2}\dot{\rho}^2-n_2\rho^{-1}\ddot{\rho})u, u) \\
&\qquad -\rho^{-2}\|\tilde{\nabla} u\|^2-(q_1u, u)\}dr \\
&= \int_{r_0}^R \left(\left\{\frac{1}{2}(\sigma_R^2)^{\cdot\cdot}+\sigma_R^2(\lambda-q_2-n_1\rho^{-2}\dot{\rho}^2-n_2\rho^{-1}\ddot{\rho})\right\}u, u\right)dr \\
&\quad -\int_{r_0}^R \sigma_R^2\rho^{-2}\|\tilde{\nabla} u\|^2dr-\int_{r_0}^R \sigma_R^2(q_1u, u)dr \\
&\leqq k_2\int_{r_0}^R \|u\|^2dr-\int_{r_0}^R \sigma_R^2\rho^{-2}\|\tilde{\nabla} u\|^2dr-\int_{r_0}^R \sigma_R^2(q_1u, u)dr
\end{aligned}
$$

where

$$k_2=\sup_{r_4<r}\left|\frac{1}{2}(\sigma_R^2)^{\cdot\cdot}+\sigma_R^2(\lambda-q_2-n_1\rho^{-2}\dot{\rho}^2-n_2\rho^{-1}\ddot{\rho})\right|.$$

(Note that k_1, k_2 are independent of R.) Thus, from (4.3) we have

$$\int_{r_0}^{R}\sigma_R^2\|\dot{u}\|^2dr \leqq \frac{k_1+k_2}{1-2\varepsilon}\int_{r_0}^{R}\|u\|^2dr - \frac{1-\varepsilon}{1-2\varepsilon}\int_{r_0}^{R}\sigma_R^2\rho^{-2}\|\tilde{\nabla}u\|^2dr. \tag{4.5}$$

Therefore, recalling that $F=\|\dot{u}\|^2-p(\dot{u}, u)+(\lambda+a)\|u\|^2-\rho^{-2}\|\tilde{\nabla}u\|^2-(q_1u, u)$, we see that

$$\int_{r_0}^{R}\sigma_R^2Fdr \leqq \frac{3}{2}\int_{r_0}^{R}\sigma_R^2\|\dot{u}\|^2dr+\int_{r_0}^{R}\sigma_R^2\Big(\frac{1}{2}p^2+\lambda+a\Big)\|u\|^2dr$$
$$-\int_{r_0}^{R}\sigma_R^2\rho^{-2}\|\tilde{\nabla}u\|^2dr-\int_{r_0}^{R}\sigma_R^2(q_1u, u)dr.$$

And hence, by dint of (4.3) and (4.5) again, the assertion of Lemma 8 is now clear.

For the proof of the theorems, we choose as

$$\varphi(r)=\rho(r)^{\alpha}, \qquad p(r)=\beta\rho(r)^{-1}\dot{\rho}(r) \tag{4.6}$$

where α and β are positive constants chosen appropriately in each case. The function $a(r)$ should also be determined later. By substituting these functions into the definition of $B(v, w)$, one immediately verifies the following formula.

PROPOSITION 1.

$$B(v, w)=(\{(\alpha+\beta-2)\rho^{\alpha-3}\dot{\rho}\Lambda+(\alpha+\beta)\rho^{\alpha-1}\dot{\rho}(\lambda-q_1)-\rho^{\alpha}\dot{q}_1$$
$$-\beta\rho^{\alpha-1}\dot{\rho}(q_2+n_1\rho^{-2}\dot{\rho}^2+n_2\rho^{-1}\ddot{\rho})+(\rho^{\alpha}a)^{\cdot}\}v, v)$$
$$+(\alpha-\beta)\rho^{\alpha-1}\dot{\rho}\|w\|^2$$
$$+\rho^{\alpha}(\{2a+2q_2+(2n_1-\beta(\alpha-1))\rho^{-2}\dot{\rho}^2+(2n_2-\beta)\rho^{-1}\ddot{\rho}\}v, w).$$

Although the choices of $\alpha, \beta, p(r)$ and $a(r)$ will be different for each theorem, we shall extensively choose as

$$\psi(r)=\text{const.}\ \rho(r)^{\alpha-1}\dot{\rho}(r)$$

which, as we now show, realizes Condition 3 by itself.

PROPOSITION 2. *If $\alpha>\beta$ and if Assumption$_{(\rho,0)}$ is satisfied, then Condition 3 applies with $\psi=$const. $\rho^{\alpha-1}\dot{\rho}$.*

PROOF. Since $e^{P(r)}=\rho(r)^{\beta}$, we see that

$$\text{const.}\ \zeta(R)=\int_{r_0}^{R}\rho(r)^{\beta-\alpha}\int_{r}^{\infty}\rho(s)^{\alpha-1}\dot{\rho}(s)\exp\Big\{-K\int_{r}^{s}\rho(t)^{-\beta}dt\Big\}dsdr.$$

Putting $Q(r)=\int_{r_0}^{r}\rho(t)^{-\beta}dt$ we have

$$\begin{aligned}
\text{const. }\zeta(R)&=\int_{r_0}^{r}\rho(r)^{\beta-\alpha}\int_{r}^{\infty}\rho(s)^{\alpha-1}\dot\rho(s)\exp\{-K(Q(s)-Q(r))\}dsdr\\
&\geqq\int_{r_0}^{R}\rho(s)^{\alpha-1}\dot\rho(s)e^{-KQ(s)}\int_{r_0}^{s}\rho(r)^{\beta-\alpha}e^{KQ(r)}drds\\
&=\int_{r_0}^{R}\rho(s)^{\alpha-1}\dot\rho(s)e^{-KQ(s)}\int_{r_0}^{s}\rho(r)^{2\beta-\alpha}\rho(r)^{-\beta}e^{KQ(r)}drds\\
&\geqq\begin{cases}\displaystyle\int_{r_0}^{R}\rho(s)^{2\beta-1}\dot\rho(s)e^{-KQ(s)}\int_{r_0}^{s}\rho(r)^{-\beta}e^{KQ(r)}drds & (\text{if } 2\beta\leqq\alpha)\\ \displaystyle\text{const.}\int_{r_0}^{R}\rho(s)^{\alpha-1}\dot\rho(s)e^{-KQ(s)}\int_{r_0}^{s}\rho(r)^{-\beta}e^{KQ(r)}drds & (\text{if } 2\beta\geqq\alpha)\end{cases}\\
&=\begin{cases}\displaystyle\text{const.}\int_{r_0}^{R}\rho(s)^{2\beta-1}\dot\rho(s)(1-e^{-KQ(s)})ds\\ \displaystyle\text{const.}\int_{r_0}^{R}\rho(s)^{\alpha-1}\dot\rho(s)(1-e^{-KQ(s)})ds\end{cases}\\
&\geqq\begin{cases}\text{const. }\rho(R)^{2\beta}-\text{const.}\\ \text{const. }\rho(r)^{\alpha}-\text{const.}\end{cases}
\end{aligned}$$

Hence, if $\beta<\alpha$,

$$\lim_{R\to\infty}\zeta(r)=\infty,\qquad \lim_{R\to\infty}e^{-P(R)}\zeta(R)=\lim_{R\to\infty}\rho(R)^{-\beta}\zeta(R)=\infty.$$

Thus, Proposition 2 is proved.

PROOF OF THEOREM 1. Choose and fix an arbitrarily ε $(0<\varepsilon<\min(\gamma,\lambda-E))$. Set $A_0=\rho^{-2}\Lambda+\lambda-q_1$, $A_1=-q_2-n_1\rho^{-2}\dot\rho^2-n_2\rho^{-1}\ddot\rho$ and put

$$\alpha=\frac{\gamma+\varepsilon}{2},\qquad \beta=\frac{\gamma-\varepsilon}{2},\qquad a(r)=\frac{1}{4}\gamma\varepsilon\rho^{-1}\dot\rho.$$

Substituting them into the formula of Proposition 1, we have

$$\begin{aligned}
B(v,w)=&\Big(\Big\{(\gamma-2)\rho^{\alpha-3}\dot\rho\Lambda+\gamma\rho^{\alpha-1}\dot\rho(\lambda-\gamma^{-1}\rho\dot\rho^{-1}\dot q_1-q_1)\\
&+\rho^{\alpha-1}\dot\rho\Big(\frac{\alpha-1}{4}\gamma\varepsilon\rho^{-1}\dot\rho+\frac{1}{4}\gamma\varepsilon\dot\rho^{-1}\ddot\rho-\beta q_2-\beta n_1\rho^{-2}\dot\rho^2-\beta n_2\rho^{-1}\ddot\rho\Big)\Big\}v,v\Big)\\
&+\varepsilon\rho^{\alpha-1}\dot\rho\|w\|^2\\
&+\rho^{\alpha-1}\dot\rho\Big(\Big\{\frac{1}{2}\gamma\varepsilon+2\rho\dot\rho^{-1}q_2+(2n_1-\beta(\alpha-1))\rho^{-1}\dot\rho+(2n_2-\beta)\dot\rho^{-1}\ddot\rho\Big\}v,w\Big).
\end{aligned}$$

By Assumptions$_{(\rho,1)(q,1)}$, we find that

$$\text{the first inner product} \geqq \gamma\left(\lambda - E - \frac{1}{2}\varepsilon\right)\rho^{\alpha-1}\dot{\rho}\|v\|^2,$$

$$\text{the last inner product} \geqq -\frac{1}{2}\gamma\varepsilon\rho^{\alpha-1}\dot{\rho}(\|w\|^2 + \|v\|^2)$$

for sufficiently large r, say, $r \geqq r_2$. Hence, we can conclude

$$B(v, w) \geqq \gamma(\lambda - E - \varepsilon)\rho^{\alpha-1}\dot{\rho}\|v\|^2 \qquad (r \geqq r_2) \tag{4.7}$$

and thus Condition 1 is fulfiled by $\psi = \text{const.}\, \rho^{\alpha-1}\dot{\rho}$. This ψ also satisfies Condition 3 as was shown in Proposition 2. Next, changing r_2 if necessary, we have

$$a - A_1 = \rho^{-1}\dot{\rho}(\gamma\varepsilon/4 + \rho\dot{\rho}^{-1}q_2 + n_1\rho^{-1}\dot{\rho} + n_2\dot{\rho}^{-1}\ddot{\rho}) \geqq 0 \qquad (r \geqq r_2),$$

which shows Condition 2. Hence, by Lemma 5, $F(r)$ satisfies the inequality

$$F(r) \geqq C\rho(r)^{-\gamma/2-\varepsilon} \qquad (r \geqq r_3), \tag{4.8}$$

and Lemma 8 shows that

$$\begin{aligned}\int_{r_0}^{R} \|u\|^2 dr &\geqq C\int_{r_0}^{R} \sigma_R^2 \rho(r)^{-\gamma/2-\varepsilon} dr \\ &\geqq C\int_{r_4+1}^{R-1} \rho(r)^{-\gamma/2-\varepsilon} dr \\ &= C\int_{r_0}^{R} \rho(r)^{-\gamma/2-\varepsilon} dr + \tilde{C},\end{aligned}$$

with constants $C > 0$ and $\tilde{C}$ where r_4 is the number appearing in the definition of σ_R. On the other hand, from $u = \rho^{(n-1)/2}v$ we have

$$\int_{r_0<r<R} |f(x)|^2 dx = \int_{r_0}^{R}\int_{S^{n-1}} |f|^2 \rho^{n-1} d\omega dr = \int_{r_0}^{R} \|u\|^2 dr.$$

This proves the theorem.

PROOF OF THEOREM 2. Since $\rho = cr$, we see from (4.1) that $u = (cr)^{(n-1)/2}f$ satisfies the equation

$$\ddot{u} + c^{-2}r^{-2}\Lambda u + (\lambda - q)u - n_1 r^{-2}u = 0.$$

Set $\alpha=\gamma/2$ and put[5)]

$$A_0=c^{-2}r^{-2}\Lambda+\lambda-q-n_1r^{-2},$$
$$A_1=0,$$
$$\varphi(r)=r^\alpha, \qquad p(r)=\alpha r^{-1}, \qquad a(r)=\frac{1}{2}\alpha(\alpha-1)r^{-2},$$

and evaluate $B(v, w)$. Then,

$$B(v, w)=\Big(\Big\{2c^{-2}(\alpha-1)r^{\alpha-3}\Lambda+2\alpha r^{\alpha-1}\Big(\lambda-\frac{1}{2\alpha}r\dot{q}-q\Big)$$
$$-2(\alpha-1)n_1r^{\alpha-3}+\frac{1}{2}\alpha(\alpha-1)(\alpha-2)r^{\alpha-3}\Big\}v, v\Big).$$

Hence, it follows from $0<\alpha\leqq 1$, $n_1\geqq 0$ and $\frac{1}{2\alpha}r\dot{q}+q\leqq E+e(r)$ that for any ε $(0<\varepsilon<\lambda-E)$

$$B(v, w)\geqq 2\alpha r^{\alpha-1}(\lambda-E-\varepsilon)\|v\|^2 \qquad (r\geqq {}^{\exists}r_2),$$

which shows that Condition 1 is fulfiled by $\psi=2\alpha(\lambda-E-\varepsilon)r^{\alpha-1}$. Condition 2 is trivial with $b=0$. If $0<\alpha<1$, set $l=K/(1-\alpha)$. Then, noting $e^{P(r)}=r^\alpha$, $\zeta_R(r)$ of Condition 3 is calculated as

$$\zeta(R)=\text{const.}\int_{r_0}^{R}\int_r^\infty s^{\alpha-1}\exp\Big(-K\int_r^s t^{-\alpha}dt\Big)dsdr$$
$$=\text{const.}\int_{r_0}^{R}\exp(lr^{1-\alpha})dr\int_r^\infty s^{\alpha-1}\exp(-ls^{1-\alpha})ds$$
$$=\text{const.}\int_{r_0}^{R}\exp(lr^{1-\alpha})dr\int_{r^{1-\alpha}}^\infty t^{(2\alpha-1)/(1-\alpha)}e^{-lt}dt.$$

Now, let us notice the following inequality. Let a be an arbitrary constant, then

$$\int_x^\infty t^{-a}e^{-lt}dt=\frac{1}{l}x^{-a}e^{-lx}-\frac{a}{l}\int_x^\infty t^{-a-1}e^{-lt}dt$$
$$=\frac{1}{l}x^{-a}e^{-lx}-\frac{a}{l^2}x^{-a-1}e^{-lx}+\frac{a(a+1)}{l^2}\int_x^\infty t^{-a-2}e^{-lt}dt$$
$$\geqq\frac{1}{2l}x^{-a}e^{-lx} \qquad (\text{for large } x),$$

5) The construction of $F(r)$ with these $p(r)$ and $a(r)$ is essentially due to J. Uchiyama [12], though the subsequent treatment is quite different from ours.

irrespective of the sign of a. Accordingly, we have

$$\zeta(R)\geqq \text{const.}\int_{r_0}^{R} r^{2\alpha-1}dr=\text{const. } R^{2\alpha}-\text{const.}$$

If $\alpha=1$, we can take $K>1$, and a direct calculation yields also $\zeta(R)=$ const. R^2-const. Therefore, Condition 3 is satisfied.

The above substitutes the part of the proof of Theorem 1 from the beginning till (4.8), thence finally (4.8) is replaced by

$$F(r)\geqq Cr^{-\gamma/2} \qquad (r\geqq {}^{\exists}r_3),$$

from whence we can derive

$$\int_{r_0}^{R}\|u\|^2 dr\geqq \text{const.}\int_{r_0}^{R} r^{-\gamma/2}dr \qquad (r\geqq {}^{\exists}r_1).$$

Thus, Theorem 2 is established.

PROOF OF THEOREM 2′. Setting

$$\begin{aligned}&A_0=c^{-2}r^{-2}\Lambda+\lambda-q-n_1r^{-2},\\ &A_1=0,\\ &\varphi(r)=r^{\gamma}, \qquad p(r)=0, \qquad a(r)=0,\end{aligned}$$

we have

$$\begin{aligned}B(v, w)&=(\{c^{-2}(\gamma-2)r^{\gamma-3}\Lambda+\gamma r^{\gamma-1}\lambda\\ &\qquad -r^{\gamma}[\dot{q}+\gamma r^{-1}q]-n_1(\gamma-2)r^{\gamma-3}\}v, v)+\gamma r^{\gamma-1}\|w\|^2\\ &\geqq \gamma r^{\gamma-1}(\|w\|^2+\lambda\|v\|^2)\\ &\geqq \gamma r^{\gamma-1}(\|w\|^2+(A_0v, v)-p(v, w)+a\|v\|^2),\end{aligned}$$

where we have used $q\geqq 0$ in the last inequality.

This inequality shows that Condition 4 is fulfiled with $\eta=1$. And hence it follows from Lemma 6 that

$$F(r)\geqq C \qquad (r\geqq {}^{\exists}r_3).$$

Replacing (4.8) by this inequality, we have only to follow the rest of the proof of Theorem 1.

PROOF OF THEOREM 3. If we put $\gamma=0$ in the proof of Theorem 2, it follows from $\dot{q}\leqq 0$ that

$$B(v, w)=((-2c^{-2}r^{-3}\Lambda-\dot{q}+2n_1r^{-3})v, v)\geqq 0.$$

But this means, as is seen by Lemma 3, that

$$\dot{F}(r)=B(u,\dot{u})=2c^{-2}r^{-3}\|\tilde{\nabla}u\|^2-(\dot{q}u,u)+2n_1r^{-3}\|u\|^2 \\ \geqq 0.$$

Now, since $u=(cr)^{(n-1)/2}f$ and $|\nabla f|^2=|\partial f/\partial r|^2+r^{-2}|\tilde{\nabla}f|^2$, we see that

$$\begin{aligned} F&=\|\dot{u}\|^2+c^{-2}r^{-2}(\Lambda u,u)+((\lambda-q)u,u) \\ &=(cr)^{n-1}\{\|\dot{f}\|^2-c^{-2}(\|\nabla f\|^2-\|\dot{f}\|^2)+\frac{(n-1)^2}{4}r^{-2}\|f\|^2 \\ &\qquad +(n-1)r^{-1}(\dot{f},f)+\lambda\|f\|^2-(qf,f)\}. \end{aligned}$$

But since f and ∇f remain bounded as $r\to 0$, we find

$$\begin{aligned} \lim_{r\to 0}F(r)&=\limsup_{r\to 0}F(r) \\ &\geqq -\text{const.}\liminf_{r\to 0} r^{n-1}\int_{S^{n-1}}q_+d\omega, \end{aligned}$$

where $q_+=\max(q,0)$ is the positive part of q which also belongs to $S^{\delta}_{\mathrm{loc}}(M)$. Therefore, if we assume

$$\liminf_{r\to 0} r^{n-1}\int_{S^{n-1}}q_+d\omega=2a>0,$$

then

$$\begin{aligned} \int_{|y|<\varepsilon}\frac{q_+(y)}{|y|^{n-2+\delta}}dy&=\int_0^{\varepsilon}r^{-n+2-\delta}r^{n-1}\int_{S^{n-1}}q_+d\omega dr \\ &\geqq a\int_0^{\varepsilon}r^{-n+2-\delta}dr=\infty \end{aligned}$$

which contradicts that $q_+\in S^{\delta}_{\mathrm{loc}}(M)$. Thus,

$$F(0)\geqq 0$$

and hence $F(r)$ is nonnegative for $r\geqq 0$.

If $n\geqq 4$, we have $n_1>0$ so that $\dot{F}(r)>0$ at the point where $u\neq 0$. That is, there exists an r_3 such that $F(r_3)>0$. Consequently,

$$F(r)\geqq C>0 \qquad (r\geqq r_3).$$

Therefore, Lemma 8 together with the above inequality yields

$$\int_{r_0}^{R}\|u\|^2dr\geqq CR \qquad (r\geqq {}^{\exists}r_1).$$

If $n=3$, then $n_1=0$. The case $F(r)\not\equiv 0$ does not bring about new questions. On the other hand, $F(r)\equiv 0$ implies $\dot{F}(r)\equiv 0$. But since $\dot{q}\leqq 0$, it follows that $\tilde{\nabla}u\equiv 0$ and $(\dot{q}u, u)\equiv 0$. Hence u must depend solely on r while q must be constant. In this case, however, we obtain the above estimate which is clear by seeing the solution u with $u(0)=0$. The proof is completed.

(Indeed if $n>3$, f and ∇f are not necessarily bounded at $r=0$.)

PROOF OF THEOREM 4. We introduce a variable τ by

$$\tau=\int_{r_0}^{r}\rho(r)^{-1}dr,$$

and set $u=f$. (As has been noted, $\tau\to\infty$ as $r\to\infty$.) Then we have[6]

$$\ddot{u}+\Lambda u+\rho^2(\lambda-q)u=0 \qquad (0<\tau<\infty) \tag{4.9}$$

where the dot represents $d/d\tau$.

Let us now set

$$A_0=\Lambda+\rho^2(\lambda-q), \qquad A_1=0,$$
$$\varphi(\tau)=1, \qquad p(\tau)=0, \qquad a(\tau)=0,$$

and substitute them into Definition of $B(v, w)$ (with r replaced by τ). Then we obtain from Assumption$_{(q,4)}$ that for an arbitrary $\varepsilon>0$,

$$\begin{aligned} B(v, w) &= (\dot{A}_0 v, v) \\ &= 2\rho\dot{\rho}\left(\lambda-q-\frac{1}{2}\rho\left(\frac{d\rho}{dr}\right)^{-1}\frac{\partial q}{\partial r}\right)\|v\|^2 \\ &\geqq 2\rho\dot{\rho}(\lambda-E-e(r))\|v\|^2 \\ &\geqq \varepsilon\rho\dot{\rho}\|v\|^2 \end{aligned}$$

for large τ, say, $\tau\geqq\tau_2$. Hence, Condition 1 applies with

$$\phi(\tau)=\varepsilon\rho(\tau)\dot{\rho}(\tau).$$

Condition 2 is trivial with $b=0$. Condition 3 is a conclusion of

$$\zeta(T)=\int_0^T\int_\tau^\infty \text{const. } \rho(s)\dot{\rho}(s)e^{-K(s-\tau)}ds\,d\tau$$

6) If $n\neq 2$, it seems difficult to find τ and u which satisfy an equation free of $\dot{\rho}$ and $\ddot{\rho}$, the coefficient of Λ being kept decreasing.

$$\geqq \int_0^T \text{const.}\ \rho(s)\dot{\rho}(s)e^{-Ks}\int_0^s e^{K\tau}d\tau$$
$$\longrightarrow \infty \qquad (T\to\infty).$$

Hence, it follows from Lemma 5 that

$$F(\tau)\geqq \exists C \qquad (\tau\geqq \exists\tau_3) \tag{4.10}$$

for $F(\tau)=\|\dot{u}\|^2+(A_0u, u)$.

We claim now

$$\int_0^T \sigma_T^2(\tau)F(\tau)d\tau\leqq \text{const.}\int_0^T \rho^2\|u\|^2 d\tau.$$

This inequality can be shown by modifying the proof of Lemma 8 as follows. Recall that $u=f$, $d\tau=\rho^{-1}dr$, $dx=\rho dr d\omega=\rho^2 d\tau d\omega$ and $|\nabla f|^2=\rho^{-2}\left(\left|\frac{\partial f}{\partial\tau}\right|^2+|\tilde{\nabla}f|^2\right)$. Put $T=\int_{r_0}^R \rho(r)^{-1}dr$ and let $\sigma_T(\tau)$ stand for $\sigma_R(r)$. Then we obtain

$$\left|\int_0^T \sigma_T^2\rho^2(qu, u)d\tau\right|\leqq 2\varepsilon\int_0^T \sigma_T^2\|\dot{u}\|^2 d\tau+2\varepsilon\int_0^T \dot{\sigma}_T^2\|u\|^2 d\tau$$
$$+\varepsilon\int_0^T \sigma_T^2\|\tilde{\nabla}u\|^2 d\tau+C_\varepsilon\int_0^T \sigma_T^2\rho^2\|u\|^2 d\tau$$

instead of (4.3). On the other hand, (4.9) shows

$$\int_0^T \sigma_T^2\|\dot{u}\|^2 d\tau=\int_0^T\left(\left\{\frac{1}{2}(\sigma_T^2)^{\cdot\cdot}+\sigma_T^2\rho^2\lambda\right\}u, u\right)d\tau$$
$$-\int_0^T \sigma_T^2\|\tilde{\nabla}u\|^2 d\tau-\int_0^T \sigma_T^2\rho^2 q\|u\|^2 d\tau$$

instead of (4.4). Thus similarly to Lemma 8, we are led to

$$\int_0^T \sigma_T^2 F d\tau\leqq C_1\int_0^T \rho^2\|u\|^2 d\tau-C_2\int_0^T \sigma_T^2\|\tilde{\nabla}u\|^2 d\tau \qquad (T\geqq\exists\tau_0).$$

This inequality together with (4.10) establishes Theorem 4, since

$$\int_{r_0<r<R}|f(x)|^2 dx=\int_{r_0}^R \|u\|^2\rho dr$$
$$=\int_0^T \rho^2\|u\|^2 d\tau$$
$$\geqq \text{const.}\ T$$

$$=\text{const.}\int_{r_0}^{R}\rho(r)^{-1}dr \qquad (R\geqq{}^{\exists}r_1).$$

PROOF OF COROLLARY TO THEOREM 4. Putting

$$r=\int_{\rho_0}^{\rho}\sqrt{1+t'(\rho)^2}d\rho, \qquad R=\int_{\rho_0}^{P}\sqrt{1+t'(\rho)^2}d\rho$$

and writing $\rho=\rho(r)$, we have $\frac{1}{2}\rho\left(\frac{d\rho}{dr}\right)^{-1}\frac{\partial q}{\partial r}+q=\frac{1}{2}\rho\frac{\partial q}{\partial\rho}+q\leqq E+e(\rho)$ besides

$$\int_{r_0}^{R}\rho(r)^{-1}dr=\int_{\rho_0}^{P}\frac{\sqrt{1+t'(\rho)^2}}{\rho}d\rho\geqq C\log P\longrightarrow\infty.$$

Thus, writing $e(\rho)$ as $e(r)$, we verify Assumption$_{(q,4)}$. Accordingly, Theorem 4 shows

$$\begin{aligned}\int_{\rho_0<\rho<P}|f(\rho,\omega)|^2d\sigma&=\int_{r_0<r<R}|f(x)|^2dx\\&\geqq C\int_{r_0}^{R}\rho(r)^{-1}dr\\&=C\int_{\rho_0}^{P}\rho^{-1}\sqrt{1+t'(\rho)^2}d\rho,\end{aligned}$$

for $P\geqq\rho_1$, which proves the corollary.

References

[1] Agmon, S., Uniqueness results for solutions of differential equations in Hilbert space with applications to problems in partial differential equations, Lectures in Differential Equations Vol. II (Edited by A. K. Aziz), Van Nostrand Mathematical Studies No. 19, 1969.

[2] Agmon, S., Lower bounds for solutions of Schrödinger equations, J. Analyse Math. **23** (1970), 1-25.

[3] Eastham, M. S. P. and H. Kalf, Schrödinger-type operators with continuous spectra, Res. Notes in Math. No. 65, Pitman, Boston-London-Melbourn, 1982.

[4] Konno, R., Estimate for eigenfunctions of the Laplace-Beltrami operator on surfaces of revolution, Memoirs of the Institute of Sciences and Technology Meiji Univ. **11** (1972), 65-71 (in Japanese).

[5] Konno, R., Asymptotic behavior of solutions of the equation $-\Delta u+qu=\lambda u$ on spherically symmetric Riemannian manifolds, Memoirs of the Institute of Sciences and Technology Meiji Univ. **13** (1974), 1-8 (in Japanese).

[6] McKean, H. P., An upper bound to the spectrum of Δ on a manifold of negative curvature, J. Differential Geom. **4** (1970), 359-366.

[7] Masuda, K., On the positive eigen-values of the Schrödinger operators and the unique

continuation theorems, Sûrikaisekikenkyûsho Kôkyûroku **160** (1972), 61-67 (in Japanese).

[8] Masuda, K., Lower bounds for solutions of differential inequalities I, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **22** (1975), 221-227.

[9] Mochizuki, K. and J. Uchiyama, On eigenvalues in the continuum of 2-body or many-body Schrödinger operators, Nagoya Math. J. **70** (1978), 125-141.

[10] Simon, B., On positive eigenvalues of one-body Schrödinger operators, Comm. Pure Appl. Math. **22** (1967), 531-538.

[11] Tayoshi, T., The continuous spectra of elliptic differential operators in unbounded domains, to appear in J. Math. Kyoto Univ.

[12] Uchiyama, J., Lower bounds of growth order of solutions of Schrödinger equations with homogeneous potentials, Publ. Res. Inst. Math. Sci. **10** (1975), 425-444.

[13] Weidmann, J., On the continuous spectrum of Schrödinger operators, Comm. Pure Appl. Math. **19** (1966), 107-110.

[14] Weidmann, J., The virial theorem and its application to the spectral theory of Schrödinger operators, Bull. Amer. Math. Soc. **73** (1967), 452-456.

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