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Positive scalar curvature and higher $\hat{A}$-genus

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0. Introduction and the statement of main results.

The scalar curvature, which is obtained from the Riemannian curvature by taking two-times contractions, is certainly one of the simplest invariants of a Riemannian manifold. Thus it will be interesting to know how the behavior of scalar curvature is related to the global topology of manifolds. However, scalar curvature is a much weaker invariant than Riemannian curvature, it appears to be not so much related to the global topology of a manifold as Riemannian curvature is. Nevertheless, it is known that there are still certain interesting results in this direction. For instance, the result due to Lichnerowicz states that a closed spin Riemannian manifold with positive scalar curvature has the vanishing $\hat{A}$-genus. This result shows that there is a topological obstruction for a closed spin manifold to admit a Riemannian metric with positive scalar curvature.

In recent years further results have been obtained in this direction, see [1]. In particular Gromov and Lawson found other topological obstructions for a closed spin manifold to admit a Riemannian metric with positive scalar curvature and proved that torus never admits such a metric. The obstruction, so called higher $\hat{A}$-genus, is defined in the following way.

Let X be a closed spin manifold. Let Γ be an arbitrary discrete group and u a rational cohomology class of $K(\Gamma, 1)$. Given a homomorphism from $\pi_1(X)$ to Γ, we obtain the corresponding map

$$f:\ X \longrightarrow K(\Gamma, 1).$$

Then we consider the following number

$$\hat{A}(u)(X) = \langle \hat{\mathfrak{A}}(X) f^*(u), [X] \rangle$$

where $\hat{\mathfrak{A}}(X)$ denotes the total $\hat{A}$-class of X and $[X]$ is the fundamental homology class of X. We call this number $\hat{A}(u)(X)$ a higher $\hat{A}$-genus of X associated with u.

Then, Gromov and Lawson proved the following:

THEOREM [2]. *Let X be a closed spin manifold of even-dimension. We set $\Gamma = \boldsymbol{Z}^k$ and fix a homomorphism from $\pi_1(X)$ to $\boldsymbol{Z}^k$. Then, if X admits a Riemannian metric with positive scalar curvature, the higher $\hat{A}$-genus $\hat{A}(u)(X)$ vanishes for each $u \in H^*(T^k; \boldsymbol{Q})$.*

We note that Theorem also holds for an odd-dimensional closed spin manifold X considering $X \times S^1$ and $f \times 1 : X \times S^1 \to T^k \times S^1$.

Putting $X = T^k$ and $f = \mathrm{id}$ in Theorem, Gromov and Lawson proved that torus does not admit any metric with positive scalar curvature. Then these results suggests the following conjectures due to Gromov, Lawson and Rosenberg [3] [10]:

CONJECTURE A. Suppose that M is a closed aspherical manifold. Then M does not admit any Riemannian metric with positive scalar curvature.

CONJECTURE B. Let X be a closed spin manifold and let us fix a discrete group Γ and a homomorphism from $\pi_1(X)$ to Γ. Then, if X admits a Riemannian metric with positive scalar curvature, the higher $\hat{A}$-genus $\hat{A}(u)(X)$ of X vanishes for each $u \in H^*(K(\Gamma, 1); \boldsymbol{Q})$.

In this paper we shall prove Theorem A and B described below which support Conjecture A and B respectively under some geometric conditions. We first describe them below.

Let M be a closed aspherical manifold with $\pi_1(M) = \Gamma$ and let us fix a Riemannian metric on M. Then Γ acts on the universal covering $\tilde{M}$ by isometries with respect to the induced metric from M. Furthermore in Condition B below we assume that there is a vector bundle V which is a Γ-vector bundle over $\tilde{M}$ and is equipped with Γ-equivariant spin^c structure and Γ-invariant euclidian metric.

CONDITION A. There exists a smooth map $\varphi : \tilde{M} \to \boldsymbol{R}^n$ satisfying the following conditions:

1) φ is a proper map with non-zero mapping degree;

2) We denote by r the Euclidian distance from the origin of $\boldsymbol{R}^n$ to $\varphi(x)$. Then, there are constants $C > 0$, $N > 0$ and $\varepsilon < 1$ such that

$$\|\varphi_*(X)\| \leq C \cdot r^{\varepsilon} \|X\|$$

for $r>N$ and for any tangent vector $X\in T_x\tilde{M}$.

CONDITION B. There exists a smooth map $\varphi: \tilde{M}\times\tilde{M}\to V$ satisfying the following conditions:

1) When we restrict φ to $\{x\}\times\tilde{M}$, it induces a proper map from $\{x\}\times\tilde{M}$ into some fibre of V with non-zero mapping degree;

2) It is equivariant with the diagonal action on $\tilde{M}\times\tilde{M}$, namely,

$$\varphi(\gamma x_1, \gamma x_2)=\gamma\cdot\varphi(x_1, x_2) \qquad \gamma\in\Gamma;$$

3) We denote by r the distance from the origin of the fibre of V to $\varphi(x_1, x_2)$. Then, there are constants $C>0$, $N>0$ and $\varepsilon<1$ such that

$$\|\varphi_*(X)\|\leq C\cdot r^{\varepsilon}\|X\|$$

for $r>N$ and for any tangent vector X along the slice $\{x_1\}\times\tilde{M}$ at (x_1, x_2).

We note that the properties with M in Condition A and B are independent of the Riemannian metric chosen. In fact, let g and h be two metrics on M and suppose that the property 2) of Condition A holds for g. Then there are constants $K>0$ and $L>0$ such that $g(X, X)\leq K^2\cdot h(X, X)$ and $h(X, X)\leq L^2\cdot g(X, X)$ for any $X\in TM$ since M is compact. Thus the property 2) of Condition A holds for h if we replace the constant C by $C\cdot K\cdot L$. The same argument holds for the property 3) of Condition B.

THEOREM A. *Suppose that M is a closed spin manifold satisfying Condition* A. *Then M does not admit any Riemannian metric with positive scalar curvature.*

THEOREM B. *Suppose that M is a closed manifold satisfying Condition* B *with $\pi_1(M)=\Gamma$. Let X be a closed spin manifold. If X admits a Riemannian metric with positive scalar curvature, then the higher $\hat{A}$-genus $\hat{A}(u)(X)$ of X vanishes for each $u\in H^*(K(\Gamma, 1); \mathbf{Q})$ and for each homomorphism from $\pi_1(X)$ to Γ.*

As a corollary of Theorem B we obtain the following results which can be obtained combining the results of [7] [9] [10].

COROLLARY. *Conjecture* B *holds for the groups which are realized as the fundamental groups of the following spaces*:

1) *A closed* spin^c *Riemannian manifold with non-positive sectional curvature;*

2) *The Seifert fibre space* $M(a)=(T^k\times W)/Q$ *which corresponds to a special Bieberbach extension* $a: 1\rightarrow Z^k\rightarrow \Gamma\rightarrow Q\rightarrow 1$ *where* W *is a simply connected* spin^c *Riemannian manifold with non-positive sectional curvature,* Q *is a discrete subgroup of* Isom(W) *and acts on* W *properly discontinuously with compact quotient* W/Q *preserving the* spin^c *structure of* W. *(For the terminologies, we refer the reader to* [9]. *They will be also recalled in* 6.3.*)*

Gromov and Lawson have already proved Conjecture A when M has a contracting proper map with non-zero mapping degree from its universal covering to the Euclidian space. (They obtain more general results, see [3].) This corresponds to the case when $\varepsilon=0$ in Theorem A if M is a spin manifold. However, the way to prove Theorem A is different from theirs. It is rather similar to that of Miščenko [7], in which the Novikov conjecture is proved for the fundamental groups of closed manifolds with non-positive sectional curvature with the aid of Fredholm complex theory. (See also [4].) We use this in the proof of Theorem A and B. The geometric conditions described above play crucial roles there.

This paper is organized as follows. In Section 1 we shall review the definition of Fredholm complexes and the relation between Fredholm complexes and K-theory. Section 2 is devoted to the proof of a vanishing theorem which is similar to that of Lichnerowicz. In Section 3 we state the generalized Atiyah-Singer index theorem which is our principal tool in this paper. In Section 4 we construct Fredholm complexes starting from Condition A or B above. Section 5 and 6 are devoted to the proof of Theorem A, B and Corollary.

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1. Fredholm complexes and K-theory.

This section is devoted to recalling the definition of Fredholm complexes and the relation to K-theory. The main references of this section will be [4], [12]. We shall later need this relation when we identify the analytical index of a Fredholm complex with its topological index. This identification is nothing but a generalized Atiyah-Singer index theorem stated in Section 3.

1.1. Let H^0 and H^1 be Banach spaces and we shall consider bounded operators from H^0 to H^1. An operator F is called a Fredholm operator if it has a closed range and finite-dimensional kernel and cokernel. An operator K is called a compact operator if the image of the unitball in H^0 is relatively compact, namely, its closure is compact. If K is a compact operator on a Banach space H, the operator $1_H - K$ becomes a Fredholm operator on it. Furthermore if the image of an operator is finite-dimensional, it is a compact operator. More generally the following lemma holds; see [8].

1.2. LEMMA. *Let H be a Hilbert space and K be a bounded operator on H. We denote by $\{e_i\}_{i=1}^{\infty}$ a complete orthonormal basis of H and by $H(j)$ the subspace generated by $\{e_i\}_{i\geq j}$. If the norm of the restricted operator $K|_{H(j)}$ tends to zero as j goes to infinity*:

$$\lim_{j\to\infty} \|K|_{H(j)}\| = 0,$$

then K is a compact operator.

1.3. DEFINITION. Let X be a compact topological space and let E^i denote vector bundles over X whose fibres are Banach spaces. Suppose that we have a finite complex (a sequence of bundle homomorphisms)

$$A^* : \ 0 \longrightarrow E^0 \xrightarrow{A^0} E^1 \xrightarrow{A^1} \cdots \longrightarrow E^n \longrightarrow 0$$

over X. Then A^* is called a Fredholm complex if the following two conditions are satisfied.

1) Compactness condition: The operators $A^{i+1}\cdot A^i$ are compact operators when they are restricted on each fibre.

2) Existence of parametrices: There exist bundle homomorphisms $B^i : E^i \to E^{i-1}$ for which $1-(B^{i+1}\cdot A^i + A^{i-1}\cdot B^i)$ are compact operators on each fibre.

We call such operators B^i parametrices of A^*.

Note that if E^i are finite-dimensional vector bundles, the conditions 1) and 2) are automatically satisfied and hence it defines a Fredholm complex. In this case the alternating sum $\sum(-1)^i E^i$ determines an element of $K(X)$. Furthermore a generalized Fredholm complex similarly determines an element of $K(X)$. We shall describe it in the following.

Let $P^* : 0 \longrightarrow E^0 \xrightarrow{P^0} E^1 \longrightarrow \cdots \longrightarrow E^n \longrightarrow 0$ denote a Fredholm complex over X. Then P^* is called acyclic if its restriction on each fibre is exact. The following proposition is due to Segal [12].

1.4. PROPOSITION. *Suppose that X is a compact topological space. Then if A^* is a Fredholm complex over X, one can find a complex B^* consisting of finite-dimensional vector bundles F^i over X and acyclic Fredholm complexes P^* and Q^* such that $A^*\oplus P^*$ is homotopically equivalent to $B^*\oplus Q^*$, where homotopy equivalence means that two Fredholm complexes can be deformed each other preserving the conditions* 1) *and* 2) *of* 1.3.

Furthermore the alternating sum $\sum(-1)^i F^i$ in $K(X)$ depends only on A^*.

We refer to the element $\sum(-1)^i F^i$ as the analytical index of A^* and denote it by index A^*.

As to properties of Fredholm complexes we only need the following one in this paper.

1.5. LEMMA. *Let X be a compact topological space and A^* be a Fredholm complex over X. If the operators $B^{i+1}\cdot A^i+A^{i-1}\cdot B^i$, which appear in* 2) *of* 1.3, *have trivial kernels, the analytical index of A^* vanishes.*

The proof is similar to that of [12].

In the sequel we shall not distinguish a Fredholm complex from the element of $K(X)$ it defines and denote them by the same letter.

2. Dirac operator and vanishing theorem.

In this section we consider a flat Hilbert bundle and define the corresponding Dirac operator. Furthermore we prove a vanishing theorem similar to that of Lichnerowicz. In the sequel M will be a closed Riemannian manifold with a spin structure and S will denote the spinor bundle over M.

2.1. Let E be a vector bundle over M whose typical fibre is a Hilbert space. Suppose that the transition functions of E take values in a discrete subgroup of all invertible unitary operators on Hilbert space. We call such a vector bundle E a flat Hilbert bundle over M. Then we denote by $\Gamma(S\otimes E)$ the smooth sections of $S\otimes E$, which are infinitely differentiable in the sense of Fréchet. Using the local trivializations this can be described as follows. Let U be a coordinate neighborhood of E and f_U be a trivialization over U:

$$f_U : E|_U \cong U \times H$$

where H denotes the Hilbert space with a complete orthonormal basis $\{e_i\}_{i=1}^{\infty}$. Under the identification through f_U, a continuous section s of $S \otimes E$ locally has the form

$$s = \sum_{i=1}^{\infty} s_i \otimes e_i$$

over U where s_i are continuous sections of S over U. Now we denote by ∇ the covariant differentiation of S. Then s is a smooth section of $S \otimes E$ over U if and only if each s_i is smooth and

$$\sum_{i=1}^{\infty} \|\nabla^k s_i\|^2 < \infty \qquad (k = 0, 1, 2, \cdots).$$

2.2. We shall define the Dirac operator D^E on $\Gamma(S \otimes E)$. Let D: $\Gamma(S) \to \Gamma(S)$ denote the Dirac operator over M. For a smooth section $s \in \Gamma(S \otimes E)$ we define $D^E(s)$ to be

$$D^E\left(\sum_{i=1}^{\infty} s_i \otimes e_i\right) = \sum D(s_i) \otimes e_i$$

using the local representation of s in 2.1. It is well-defined since E is a flat Hilbert bundle and D is a differential operator. Then D^E preserves the condition for smooth sections in 2.1 and therefore it defines an operator on $\Gamma(S \otimes E)$. We call D^E a Dirac operator with coefficient bundle E.

2.3. Let h denote the hermitian metric of $S \otimes E$ which is induced from those of S and E. It defines an inner product $(\ ,\ )$ on $\Gamma(S \otimes E)$ by

$$(s, t) = \int_M h(s, t) dvol$$

for $s, t \in \Gamma(S \otimes E)$. Then D^E is formally self-adjoint with respect to $(\ ,\ )$:

$$(D^E s, t) = (s, D^E t) \qquad s, t \in \Gamma(S \otimes E).$$

Furthermore the following Weitzenböck formula holds:

$$((D^E)^2 s, t) = (\nabla s, \nabla t) + \left(\frac{\kappa}{4} s, t\right) \qquad s, t \in \Gamma(S \otimes E)$$

where ∇s means $\sum \nabla s_i \otimes e_i$ with the local representation of s in 2.1 and

κ denotes the scalar curvature over M. If we replace D^E by D, it is the usual Weitzenböck formula in [3], and the above formula can be proved by using it without difficulty.

If M is an even-dimensional manifold, the complexified spinor bundle $S\otimes\boldsymbol{C}$ splits into the half-spinor bundles:

$$S\otimes\boldsymbol{C}=S_+\oplus S_-$$

and the Dirac operator D also splits correspondingly:

$$D=\begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}: \begin{matrix}\Gamma(S_+)\\ \oplus \\ \Gamma(S_-)\end{matrix} \longrightarrow \begin{matrix}\Gamma(S_+)\\ \oplus \\ \Gamma(S_-)\end{matrix}.$$

The operator D_+ is similarly extended to the operator from $\Gamma(S_+\otimes E)$ to $\Gamma(S_-\otimes E)$ and it is denoted by D_+^E.

In the left of this section we fix a flat vector bundle E and denote D^E and D_+^E simply by D and D_+, respectively.

2.4. We define Sobolev norms $\|\ \|_n$ $(n=0, 1, 2, \cdots)$ on $\Gamma(S\otimes E)$ by

$$\|s\|_n^2=((1+D^2)^n s, s) \qquad s\in\Gamma(S\otimes E)$$

and call the completion of $\Gamma(S\otimes E)$ with respect to $\|\ \|_n$ a Sobolev space of order n. We denote it by $H^n(S\otimes E)$. Note that $H^0(S\otimes E)$ is the Hilbert space of L^2-sections of $S\otimes E$.

Then we extend D on $\Gamma(S\otimes E)$ to a bounded operator between Banach spaces:

$$D: H^{n+1}(S\otimes E) \longrightarrow H^n(S\otimes E) \qquad n=0, 1, 2, \cdots.$$

Actually, it suffices to show that it is bounded on $\Gamma(S\otimes E)$. To do this we note that

$$\|Ds\|_n^2+\|s\|_n^2=\|s\|_{n+1}^2 \qquad s\in\Gamma(S\otimes E).$$

Therefore we obtain

$$\|Ds\|_n^2\leq\|s\|_{n+1}^2,$$

which proves the boundedness of D.

2.5. VANISHING THEOREM. *Let E be a flat Hilbert bundle over M. Suppose that the scalar curvature of M is bounded from below by a positive constant c. Then the Dirac operator $D: H^{n+1}(S\otimes E)\rightarrow H^n(S\otimes E)$*

with coefficient bundle E has trivial kernel and cokernel.

PROOF. Let s be a section of $\ker D$ in $H^{n+1}(S\otimes E)$. Then there exists a sequence $\{s_k\}$ of smooth sections such that s_k tends to s in $H^{n+1}(S\otimes E)$. Applying the Weitzenböck formula in 2.3, we obtain

$$\begin{aligned}\|Ds_k\|_n^2 &= ((1+D^2)^n \cdot Ds_k, Ds_k) \\ &= \left(\sum_{j=0}^{n}\binom{n}{j}D^{2j}\cdot Ds_k,\ Ds_k\right) \\ &= \sum\binom{n}{j}(D^2\cdot D^j s_k, D^j s_k) \\ &= \sum\binom{n}{j}\left\{(\nabla D^j s_k, \nabla D^j s_k) + \left(\frac{\kappa}{4}D^j s_k, D^j s_k\right)\right\} \\ &\geq \sum\binom{n}{j}\frac{c}{4}(D^j s_k, D^j s_k) \\ &= \frac{c}{4}((1+D^2)^n s_k, s_k) \\ &= \frac{c}{4}\|s_k\|_n^2.\end{aligned}$$

Since $Ds=0$, $\|Ds_k\|_n$ tends to zero and hence $\|s_k\|_n$ tends to zero. On the other hand we have

$$\|Ds_k\|_n^2 + \|s_k\|_n^2 = \|s_k\|_{n+1}^2$$

by the equality in 2.4. Therefore $\|s_k\|_{n+1}^2$ tends to zero and s should be zero in $H^{n+1}(S\otimes E)$.

To show that cokernel of D is trivial, we note that there is a formally self-adjoint operator T such that $T^2=1+D^2$; see [11, p. 171], [8]. Then we can verify that T induces isomorphisms from $H^{k+1}(S\otimes E)$ to $H^k(S\otimes E)$ $(k=0, 1, 2, \cdots)$ and preserves the smooth sections $\Gamma(S\otimes E)$.

Now consider the bounded self-adjoint operator $T^n\cdot D\cdot T^{-n-1}: H^0(S\otimes E) \to H^0(S\otimes E)$. The self-adjointness is well-defined since $H^0(S\otimes E)$ is a Hilbert space. Then we obtain

$$\begin{aligned}\operatorname{coker} D &\cong \operatorname{coker}(T^n\cdot D\cdot T^{-n-1}) \\ &= \ker(T^n\cdot D\cdot T^{-n-1})^* \\ &= \ker(T^n\cdot D\cdot T^{-n-1}) \\ &\cong \ker D\end{aligned}$$

which has already been proved to be zero above. Hence D has trivial cokernel and the proof is completed.

2.6. We conclude this section by recalling the work of Kazdan and Warner [5], see also [1]. Let M be a compact manifold. Suppose that there is a Riemannian metric of M with positive scalar curvature, namely, a metric whose scalar curvature is non-negative everywhere and strictly positive at some point. Then they proved that M carries a conformally equivalent metric with strictly positive scalar curvature. Therefore, by virtue of this result we can replace the condition with M in the vanishing theorem by the following: M admits a Riemannian metric with positive scalar curvature.

3. Index theorem.

In this section we shall state a generalized Atiyah-Singer index theorem and show that the analytical index of a certain Fredholm complex vanishes over a closed spin Riemannian manifold with positive scalar curvature.

3.1. Let X be an even-dimensional closed spin manifold and M be a compact topological space. Consider a Fredholm complex

$$A^*\colon\ 0 \longrightarrow E^0 \xrightarrow{\ A\ } E^1 \longrightarrow 0$$

over $M\times X$. We assume that E^0 and E^1 are flat Hilbert bundles along $\{y\}\times X$ for each $y\in M$, namely, the transition functions are locally constant along $\{y\}\times M$. Then we can construct two families of Dirac operators with coefficient bundles $\{E_y^0\}_{y\in M}$ and $\{E_y^1\}_{y\in M}$. We denote these families of operators by $\{D_+^0(y)\}_{y\in M}$ and $\{D_+^1(y)\}_{y\in M}$, respectively. Then we shall define a complex $D\otimes A^*$ over M in the following way:

$$0 \longrightarrow \{H^{n+1}(S_+\otimes E_y^0)\}_{y\in M} \xrightarrow{\ A^0\ } \begin{matrix}\{H^n(S_-\otimes E_y^0)\}_{y\in M}\\ \oplus \\ \{H^{n+1}(S_+\otimes E_y^1)\}_{y\in M}\end{matrix} \xrightarrow{\ A^1\ } \{H^n(S_-\otimes E_y^1)\}_{y\in M} \longrightarrow 0$$

where

$$A^0=\left\{\begin{pmatrix} D_+^0(y) \\ 1\otimes A_y \end{pmatrix}\right\}_{y\in M} \qquad A^1=\{(-1\otimes A_y \quad D_+^1(y))\}_{y\in M}.$$

Here we denote $E^i|_{\{y\}\times M}$, $A|_{\{y\}\times M}$ by E_y^i, A_y respectively.

Note that the above complex is not necessarily a Fredholm complex.

However, if it is the case, following holds.

3.2. A generalized Atiyah-Singer index theorem. *If $D\otimes A^*$ is a Fredholm complex over M, its analytical index is cohomologically given by*

$$\mathrm{ch}(\mathrm{index}(D\otimes A^*))=\hat{\mathfrak{A}}(X)\mathrm{ch}(A^*)/[X]$$

in $H^*(M; Q)$, where ch *is the Chern character homomorphism from $K^*(M)$ to $H^*(M; Q)$.*

The proof is similar to that of [7, p. 102].

Now we shall investigate the above index when X admits a Riemannian metric with positive scalar curvature.

3.3. PROPOSITION. *Let X be an even-dimensional closed spin manifold which admits a Riemannian metric with positive scalar curvature. If Compactness condition in Definition* 1.3 *is satisfied for $D\otimes A^*$, then it is a Fredholm complex over M and furthermore the analytical index vanishes.*

PROOF. The vanishing theorem 2.5 shows that $D^i_+(y)$ are invertible. Thus the family $\{D^i_+(y)\}_{y\in M}$ has the inverse bundle homomorphism, which is denoted by L^i. We set

$$B^1=(L^0, 0)$$
$$B^2=\begin{pmatrix} 0 \\ L^1 \end{pmatrix}.$$

Then,

$$B^1\cdot A^0=(L^0, 0)\begin{pmatrix} \{D^0_+(y)\} \\ 1\otimes A \end{pmatrix}=1$$

$$A^0\cdot B^1+B^2\cdot A^1=\begin{pmatrix} \{D^0_+(y)\} \\ 1\otimes A \end{pmatrix}(L^0,\ 0)+\begin{pmatrix} 0 \\ L^1 \end{pmatrix}(-1\otimes A, \{D^1_+(y)\})$$
$$=\begin{pmatrix} 1 & 0 \\ (1\otimes A)L^0-L^1(1\otimes A) & 1 \end{pmatrix}$$

$$A^1\cdot B^2=(-1\otimes A, \{D^1_+(y)\})\begin{pmatrix} 0 \\ L^1 \end{pmatrix}=1.$$

Since $A^1_y\cdot A^0_y=D^1_+(y)\cdot(1\otimes A_y)-(1\otimes A_y)\cdot D^0_+(y)$ is a compact operator by Compactness condition, so is $L^1_y\cdot A^1_y\cdot A^0_y\cdot L^0_y=(1\otimes A_y)\cdot L^0_y-L^1_y\cdot(1\otimes A_y)$ is also a compact operator. Thus the above calculation shows that $1-B_y^1\cdot A_y^0$, $1-(A_y^0\cdot B_y^1+B_y^2\cdot A_y^1)$ and $1-A_y^1\cdot B_y^2$ are compact operators and therefore we can take B^i as parametrices of $D\otimes A^*$. It also shows that $B^1\cdot A^0$, $A^0\cdot B^1+B^2\cdot A^1$ and $A^1\cdot B^2$ have trivial kernels. Hence $D\otimes A^*$ determines a trivial element in $K(M)$ by Lemma 1.5, which completes the proof.

4. Construction of Fredholm complexes.

In this section we shall construct Fredholm complexes A^* and B^* under Condition A and B, respectively. It will then turn out that Compactness condition holds for the complex $D\otimes A^*$, which makes Proposition 3.3 useful. The Fredholm complex B^* plays a crucial role in proving the vanishing of the higher $\hat{A}$-genus.

4.1. Construction of a Fredholm complex A^*. We assume that M is a manifold of even-dimension $m=2k$ with Condition A. Let $f_0: S^{m-1}\to U(k)$ be a map representing the generator of $\pi_{m-1}(U(k))$. Here we regard S^{m-1} as the unit sphere in $\boldsymbol{R}^m$ and $U(k)$ as a subset of the square matrices $M(k,\boldsymbol{C})$ of order k. Then we shall extend f_0 to $f: \boldsymbol{R}^m\to M(k,\boldsymbol{C})$ by the following formula:

$$f(r,\theta^1,\cdots,\theta^{m-1})=\tau(r)f_0(\theta^1,\cdots,\theta^{m-1})$$

where $(r,\theta^1,\cdots,\theta^{m-1})$ denotes the polar coordinate of $\boldsymbol{R}^m$ and $\tau:[0,\infty)\to\boldsymbol{R}$ is a smooth function such that $\tau(r)=0$ if and only if $r=0$, and $\tau(r)=1$ if $r\geq 1$. Then the Bott generator of $K(\boldsymbol{R}^m)$ is represented by $\{V\xrightarrow{f}V\}$ where V denotes a k-dimensional trivial complex vector bundle over $\boldsymbol{R}^m$ and f is regarded as an endmorphism of V.

Now we shall construct the complex A^* over M. Let W denote the pull-back φ^*V and h denote the pull-back φ^*f. Equipping V with the canonical flat connection and hermitian metric, we obtain the induced connection and metric on W. Denoting by π the covering projection $\tilde{M}\to M$, we set

$$E_y=\sum\oplus_{\pi(x)=y}W_x$$
$$A_y=\sum\oplus_{\pi(x)=y}h_x$$

where $\sum\oplus$ means a Hilbert completion of a direct sum. Now we define a complex A^* to be

$$0\longrightarrow E\xrightarrow{A}E\longrightarrow 0$$

where $E=\{E_y\}_{y\in M}$ and $A=\{A_y\}_{y\in M}$. Then, the vector bundle E is a flat bundle over M and the bundle homomorphism A consists of Fredholm operators. In fact, flatness of E is obvious from the above construction and Fredholmness of A_y follows from the fact that h_x is invertible except over $x\in\varphi^{-1}(0)$. Therefore, A^* determines a Fredholm complex over M.

Next we shall prove the following lemma which will ensure Compactness condition for $D\otimes A^*$.

4.2. LEMMA. *Consider the situation in* 4.1. *Then there is a constant* $c>0$ *such that*

$$\|\nabla h\|_x \leq c\cdot r^{\varepsilon}\cdot \|\nabla f\|_{\varphi(x)}$$

for $r\geq N$. *Furthermore,* $\|\nabla h\|_x$ *tends to zero as* x *goes to infinity in* $\tilde{M}$, *or equivalently,* r *goes to infinity.*

PROOF. Let $\{X_j\}$ be an orthonormal basis of $T_x\tilde{M}$. Then the norm $\|\nabla h\|_x$ is given by

$$\|\nabla h\|_x^2=\sum_j\|\nabla_{X_j}h\|_x^2.$$

Since W is equipped with the induced metric from V, we obtain

$$\begin{aligned}\sum\|\nabla_{X_j}h\|_x^2 &= \sum\|\nabla_{\varphi_*(X_j)}f\|_{\varphi(x)}^2\\ &= \sum\|\varphi_*(X_j)\|^2\|\nabla_{\varphi_*(X_j)/\|\varphi_*(X_j)\|}f\|_{\varphi(x)}^2\\ &\leq \sum\|\varphi_*(X_j)\|^2\|\nabla f\|_{\varphi(x)}^2\\ &= \sum(C\cdot r^{\varepsilon}\cdot\|X_j\|)^2\|\nabla f\|_{\varphi(x)}^2\\ &= m\cdot C^2\cdot r^{2\varepsilon}\cdot\|\nabla f\|_{\varphi(x)}^2\end{aligned}$$

which yields the inequality if we set $c=\sqrt{m}\cdot C$.

To prove the latter statement, it suffices to show that $r^{\varepsilon}\cdot\|\nabla f\|$ tends to zero as r goes to infinity. Now let $f^{\alpha\beta}(x)$ denote the (α,β)-component of $f(x)\in M(k,\boldsymbol{C})$. By the definition, f is equal to f_0 for $r\geq 1$ and hence f is independent of r there.

Recall that the polar coordinate are given by

$$\begin{aligned}x^1 &= r\cdot\cos\theta^1\\ x^2 &= r\cdot\sin\theta^1\cdot\cos\theta^2\\ &\vdots\\ x^{m-1} &= r\cdot\sin\theta^1\cdot\sin\theta^2\cdots\sin\theta^{m-2}\cdot\cos\theta^{m-1}\\ x^m &= r\cdot\sin\theta^1\cdot\sin\theta^2\cdots\sin\theta^{m-2}\cdot\sin\theta^{m-1}\end{aligned}$$

and their Jacobian matrix is given as follows:

$$\begin{pmatrix} \cos\theta^1 & \sin\theta^1\cdot\cos\theta^2 & \cdots \\ -r\cdot\sin\theta^1 & r\cdot\cos\theta^1\cdot\cos\theta^2 & \\ 0 & -r\cdot\sin\theta^1\cdot\sin\theta^2 & \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \end{pmatrix}.$$

Thus the inverse of the Jacobian matrix has the form

$$A\cdot\begin{pmatrix} r^{m-1}\cdot a_{11} & r^{m-2}\cdot a_{12} & \cdots & r^{m-2}\cdot a_{1m} \\ r^{m-1}\cdot a_{21} & r^{m-2}\cdot a_{22} & \cdots & r^{m-2}\cdot a_{2m} \\ \vdots & \vdots & & \vdots \\ r^{m-1}\cdot a_{m1} & & \cdots & r^{m-2}\cdot a_{mm} \end{pmatrix}$$

where $A=(r^{m-1}\cdot\sin^{m-2}\theta^1\cdots\sin\theta^{m-2})^{-1}$ and a_{ij} denotes a smooth function independent of r.

Since the differentials $\partial f^{\alpha\beta}/\partial r$ vanish for $r\geq 1$, we obtain

$$\begin{pmatrix} \dfrac{\partial f^{\alpha\beta}}{\partial x^1} \\ \\ \dfrac{\partial f^{\alpha\beta}}{\partial x^m} \end{pmatrix} = r^{m-1}\cdot A\cdot\begin{pmatrix} a_{11} & r^{-1}\cdot a_{12} & \cdots & r^{-1}\cdot a_{1m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & r^{-1}\cdot a_{m2} & \cdots & r^{-1}\cdot a_{mm} \end{pmatrix}\begin{pmatrix} 0 \\ \dfrac{\partial f^{\alpha\beta}}{\partial\theta^1} \\ \dfrac{\partial f^{\alpha\beta}}{\partial\theta^{m-1}} \end{pmatrix}$$
$$= r^{-1}\cdot V^{\alpha\beta}$$

for $r\geq 1$ where $V^{\alpha\beta}$ denote smooth mapping into $\boldsymbol{R}^m$ independent of r. Therefore it follows that

$$\begin{aligned} \|\nabla f\|^2 &= \sum_j \left\|\frac{\partial f}{\partial x^j}\right\|^2 \\ &= \sum_{j,\alpha,\beta}\left|\frac{\partial f^{\alpha\beta}}{\partial x^j}\right|^2 \\ &= \sum_{\alpha,\beta} r^{-2}\cdot\|V^{\alpha\beta}\|^2 \end{aligned}$$

and hence

$$r^{2\varepsilon}\cdot\|\nabla f\|^2 = r^{2\varepsilon-2}\cdot\sum_{\alpha,\beta}\|V^{\alpha\beta}\|^2 \longrightarrow 0$$

as $r\to\infty$. This completes the proof of Lemma.

We can now state the key proposition:

4.3. PROPOSITION. *If an even-dimensional closed spin manifold M satisfies Condition* A, *then Compactness condition holds for the complex $D\otimes A^*$ constructed as in* 3.1.

PROOF. Consider the following diagram:

$$\begin{array}{ccc} H^{n+1}(S_+\otimes E) & \xrightarrow{D_+} & H^n(S_-\otimes E) \\ {\scriptstyle 1\otimes A}\downarrow & & \downarrow{\scriptstyle 1\otimes A} \\ H^{n+1}(S_+\otimes E) & \xrightarrow[D_+]{} & H^n(S_-\otimes E). \end{array}$$

Then Compactness condition for $D\otimes A^*$ means that the above diagram commutes modulo compact operators. We shall prove this fact in the following.

Let U be a contractible open set in M and y_0 a point of U. Then $E|_U$ is isomorphic to the product bundle $U\times \sum\bigoplus_{\pi(x)=y_0} W_x$ over U through the trivializations of W over $\pi^{-1}(U)$. Then we can represent a local smooth section $s\in\Gamma(U, S_+\otimes E)$ as

$$s=\sum_{i=1}^{\infty} s_i\otimes e_i$$

where $s_i\in\Gamma(U, S_+)$, and $\{e_i\}_{i=1}^{\infty}$ denotes a complete orthonormal basis of $\sum\bigoplus_{\pi(x)=y_0} W_x$ consisting of the orthonormal basis of each direct summand W_x. Then the difference $D_+\cdot(1\otimes A)-(1\otimes A)\cdot D_+$ is locally calculated as follows:

$$\begin{aligned} &\{D_+\cdot(1\otimes A)-(1\otimes A)\cdot D_+\}\Big(\sum_i s_i\otimes e_i\Big) \\ &=\Big\{D_+\cdot\Big(1\otimes \sum_x\bigoplus h_x\Big)-\Big(1\otimes \sum_x\bigoplus h_x\Big)\cdot D_+\Big\}\Big(\sum_i s_i\otimes e_i\Big) \\ &=D_+\cdot\Big(\sum_{i,x} s_i\otimes h_x e_i\Big)-\Big(1\otimes \sum_x\bigoplus h_x\Big)\Big(\sum_i D_+ s_i\otimes e_i\Big) \\ &=\sum_{i,x} D_+\cdot(s_i\otimes h_x e_i)-\sum_{i,x} D_+ s_i\otimes h_x e_i \\ &=\sum_{i,x}\sum_{j=1}^{m} c(X_j)\nabla_{X_j}(s_i\otimes h_x e_i)-\sum_{i,x}\Big(\sum_{j=1}^{m} c(X_j)\nabla_{X_j} s_i\Big)\otimes h_x e_i \end{aligned}$$

$$= \sum_{i,x} \sum_{j=1}^{m} c(X_j)\{\nabla_{X_j} s_i \otimes h_x e_i + s_i \otimes (\nabla_{X_j} h)_x e_i\} - \sum_{i,x} \sum_{j} c(X_j) \nabla_{X_j} s_i \otimes h_x e_i$$
$$= \sum_{j} c(X_j)\Big(\sum_{i,x} s_i \otimes (\nabla_{X_j} h)_x e_i\Big)$$
$$= \sum_{j} c(X_j)\Big(1 \otimes \sum_x \oplus (\nabla_{X_j} h)_x\Big)\Big(\sum_i s_i \otimes e_i\Big)$$

where $\{X_j\}_{j=1}^{m}$ is an orthonormal basis of TM at $y=\pi(x) \in U$ and $c(X_j)$ denotes the Clifford multiplication by X_j.

Suppose that we are given a partition of unity subordinate to a finite covering consisting of coordinate neighborhoods of M. Using this, we can split $D_+ \cdot (1 \otimes A) - (1 \otimes A) \cdot D_+$ into a finite sum of local operators as described above. Thus it is sufficient to prove that the operator

$$1 \otimes \sum \oplus (\nabla_{X_j} h)_x : H^{n+1}(U; S_+ \otimes E) \longrightarrow H^n(U; S_+ \otimes E)$$

is a compact operator. Here we note that $H^k(U; S_+ \otimes E)$ is isomorphic to $H^k(U; S_+) \otimes \sum \oplus W_x$ and that $1 \otimes \sum \oplus (\nabla_{X_j} h)_x$ can be considered as a tensor product of two operators,

$$1_U : H^{n+1}(U; S_+) \longrightarrow H^n(U; S_+)$$

and

$$\sum \oplus (\nabla_{X_j} h)_x : \sum \oplus W_x \longrightarrow \sum \oplus W_x.$$

Now we choose $\{e_i\}_{i=1}^{\infty}$ such that $x \to \infty$ as $i \to \infty$ when e_i is contained in a direct summand W_x, and denote by $H(k)$ the subspace generated by $\{e_i\}_{i \geq k}$. Then the operator norm $\| \sum \oplus (\nabla_{X_j} h)_x |_{H(k)} \|$ tends to zero as k goes to infinity by Lemma 4.2. Hence, it follows that $\sum \oplus (\nabla_{X_j} h)_x$ is a compact operator by Lemma 1.2. On the other hand, $1_U : H^{n+1}(U; S_+) \to H^n(U; S_+)$ is also a compact operator, which is nothing but the Rellich lemma. Therefore, $1 \otimes \sum \oplus (\nabla_{X_j} h)_x$ is a compact operator since a tensor product of compact operators is again a compact operator. This completes the proof of Proposition.

4.4. Construction of a Fredholm complex B^*. Suppose that M is an even-dimensional closed aspherical manifold satisfying Condition B. Then we shall construct a Fredholm complex B^* over $M \times M$.

First we note that there is a map

$$\bar{\varphi} : \tilde{M} \underset{\Gamma}{\times} \tilde{M} \longrightarrow V/\Gamma$$

by the property 2) of Condition B, where $\tilde{M} \underset{\Gamma}{\times} \tilde{M}$ denotes the Γ-orbit space with the diagonal action and V/Γ denotes the quotient vector bundle over M. We also note that $\tilde{M} \underset{\Gamma}{\times} \tilde{M}$ can be considered as a fibre bundle over M with the fibre $\tilde{M}$. Then the property 1) in Condition B implies that $\bar{\varphi}$ maps each fibre of $\tilde{M} \underset{\Gamma}{\times} \tilde{M}$ into some fibre of V/Γ.

Now we shall discuss the Thom class of V as we discussed the Bott generator in 4.1. The Thom class is represented by

$$\begin{aligned} &p_0^* S_+ \xrightarrow{\ f\ } p_0^* S_-, \\ &f(\xi) = \tau(\|\xi\|) \cdot c(\xi/\|\xi\|) \qquad \text{on} \quad \xi \in V \end{aligned}$$

where S_+ and S_- denote the half-spinor bundles of V, p_0 denotes the bundle projection of V, τ is the smooth function defined in 4.1 and $c(\eta)$ denotes the Clifford multiplication by η. Since V has a equivariant spin^c structure, f induces a homomorphism

$$f/\Gamma : (p_0^* S_+)/\Gamma \longrightarrow (p_0^* S_+)/\Gamma$$

over V/Γ. Then we set

$$\begin{aligned} W_\pm &= \bar{\varphi}^*((p_0^* S_\pm)/\Gamma) \\ h &= \bar{\varphi}^*(f/\Gamma) \end{aligned}$$

and we define a Fredholm complex B^* over $M \times M$ to be

$$B^* : 0 \longrightarrow E_+ \xrightarrow{\ B\ } E_- \longrightarrow 0$$

where

$$\begin{aligned} E_{\pm(y_1, y_2)} &= \sum \bigoplus_{p(x) = (y_1, y_2)} (W_\pm)_x \\ B_{(y_1, y_2)} &= \sum \bigoplus_{p(x) = (y_1, y_2)} h_x . \end{aligned}$$

Here we denote by p the covering projection $\tilde{M} \underset{\Gamma}{\times} \tilde{M} \to M \times M$. Then we obtain the following commutative diagram.

$$\begin{array}{ccccc} (p_0^* S_\pm)/\Gamma & \longleftarrow & W_\pm & \longrightarrow & E_\pm \\ \downarrow & & \downarrow & & \Big| \\ V/\Gamma & \xleftarrow{\ \bar{\varphi}\ } & \tilde{M} \underset{\Gamma}{\times} \tilde{M} & & \Big| \\ \downarrow & & \downarrow{\scriptstyle p} & & \downarrow \\ M & \xleftarrow{\ p_1\ } & M \times M & = & M \times M \end{array}$$

where $p_1(y_1, y_2)=y_1$.

Recall that $(p_0^* S_\pm)/\Gamma$ admit connections which are flat along the fibres of V/Γ, namely, the transition functions are locally constant along the fibres. Hence we can equip $W_\pm$ with the induced connections by $\bar{\varphi}$ which are flat along the fibres of the bundle $p_1 \cdot p : \tilde{M} \underset{\Gamma}{\times} \tilde{M} \rightarrow M$ mentioned previously. On the other hand, $p : \tilde{M} \underset{\Gamma}{\times} \tilde{M} \rightarrow M \times M$ yields the covering projection if we restrict p on the fibres of the bundle $\tilde{M} \underset{\Gamma}{\times} \tilde{M} \rightarrow M$. Hence, from the construction, it follows that $E_\pm$ are flat Hilbert bundles along the slices $\{y\} \times M$. Furthermore B is a homomorphism consisting of Fredholm operators, which is proved by the same argument as in 4.1. Thus B^* determines a Fredholm complex over $M \times M$ with Hilbert bundles $E_\pm$ which are flat along the slices $\{y\} \times M$.

5. Proof of Theorem A and B.

To prove Theorem A and B we need the following lemma, see [4].

5.1. Lemma. *We assume that the aspherical manifold M in Condition* A *or* B *is even-dimensional, say $2k$. Now we consider the Fredholm complexes A^* and B^* constructed in the previous section. Then, the following holds*:

1) $\quad \mathrm{ch}(A^*) = d \cdot u_M$

$\quad \mathrm{ch}(B^*|_{\{y\} \times M}) = d \cdot u_M \qquad (y \in M)$

where u_M denotes the generator of $H^{2k}(M; \boldsymbol{Q})$ such that $\langle u_M, [M] \rangle = 1$, and d denotes the mapping degree of φ or $\varphi|_{\{x\} \times \tilde{M}}$;

2) *Let p_1 and p_2 be projections from $M \times M$ to M such that $p_1(y_1, y_2) = y_1$ and $p_2(y_1, y_2) = y_2$. Then for any finite-dimensional vector bundle F over M, it follows that*

$$p_1^*(F) \otimes B^* = p_2^*(F) \otimes B^*.$$

5.2. Proof of Theorem A. We first consider an even-dimensional manifold M. Suppose that M admits a metric with positive scalar curvature. Then Compactness condition holds for $D \otimes A^*$ by Proposition 4.3 and hence the analytical index of $D \otimes A^*$ vanishes by Proposition 3.3. On the other hand, we obtain

$$\mathrm{ch}(\mathrm{index}(D \otimes A^*)) = \langle \hat{\mathfrak{A}}(M) \mathrm{ch}(A^*), [M] \rangle$$

$$
\begin{aligned}
&=\langle \hat{\mathfrak{A}}(M)du_M, [M]\rangle \\
&=\langle du_M, [M]\rangle \\
&\neq 0
\end{aligned}
$$

by Theorem 3.2 and Lemma 5.1, 1). However this formula contradicts the previous statement. Therefore M does not admit any metric with positive scalar curvature.

For an odd-dimensional manifold M, consider $M\times S^1$ and apply the same argument to this. Then, it follows that $M\times S^1$ does not admit any metric with positive scalar curvature. However, a metric on M with positive scalar curvature induces such a metric on $M\times S^1$. Hence M does not admit any metric with positive scalar curvature. This completes the proof of Theorem A.

Next we shall show Theorem B. For the proof we need the following lemma to calculate $\mathrm{ch}(B^*)$.

Let M be a $2k$-dimensional closed manifold and u be an element of $H^{\mathrm{even}}(M\times M; Q)$. We denote by $\{e_i\}_{i=0}^N$ a basis of $H^{\mathrm{even}}(M; Q)$ such that $e_0=1\in H^0(M; Q)$, $e_N\in H^{2k}(M; Q)$ and $\langle e_N, [M]\rangle=1$. Furthermore we denote by $\{e_i^*\}$ the dual basis of $\{e_i\}$, i.e.

$$\langle e_i\cup e_j^*, [M]\rangle=\delta_{ij}.$$

5.3. LEMMA. *Suppose that*

$$
\begin{aligned}
&u|_{\{pt\}\times M}=e_N \\
&(a\times 1)\cup u=(1\times a)\cup u
\end{aligned}
$$

for any $a\in H^{\mathrm{even}}(M; Q)$. *Then,* u *has the form*

$$u=\sum_j (e_j^*\cup a_N)\times e_j$$

modulo $H^{\mathrm{odd}}(M; Q)\otimes H^{\mathrm{odd}}(M; Q)$ *where* a_N *is an invertible element of* $H^{\mathrm{even}}(M; Q)$.

PROOF. We represent u in the following way:

$$u=\sum_i a_i\times e_i+d$$

where $a_i\in H^{\mathrm{even}}(M; Q)$ and $d\in H^{\mathrm{odd}}(M; Q)\otimes H^{\mathrm{odd}}(M; Q)$. Then it follows that

$$\begin{aligned}((e_j^*\times 1)\cup u)/[M] &= \Big(\sum_i (e_j^*\cup a_i)\times e_i\Big)\Big/[M]+((e_j^*\times 1)\cup d)/[M]\\ &= \sum_i e_j^*\cup a_i\langle e_i, [M]\rangle\\ &= e_j^*\cup a_N\end{aligned}$$

and

$$\begin{aligned}((1\times e_j^*)\cup u)/[M] &= \sum_i (a_i\times(e_j^*\cup e_i))/[M]+((1\times e_j^*)\cup d)/[M]\\ &= \sum_i a_i\langle e_j^*\cup e_i, [M]\rangle\\ &= \sum_i a_i\delta_{ij}\\ &= a_j.\end{aligned}$$

Considering the assumption, the above calculations yield

$$e_j^*\cup a_N=a_j$$

and hence

$$u=\sum_j (e_j^*\cup a_N)\times e_j$$

modulo $H^{\mathrm{odd}}(M;\ \boldsymbol{Q})\otimes H^{\mathrm{odd}}(M;\ \boldsymbol{Q})$.

Furthermore it follows that

$$\begin{aligned}e_N &= u|_{\{pt\}\times M}\\ &= \sum_i a_i\times e_i|_{\{pt\}\times M}+d|_{\{pt\}\times M}\\ &= \sum_i \langle a_i, [pt]\rangle e_i.\end{aligned}$$

Since $\{e_i\}$ forms a basis of $H^{\mathrm{even}}(M;\ \boldsymbol{Q})$, we then obtain

$$\langle a_N, [pt]\rangle=1.$$

Now, a cohomology class $a\in H^*(M;\ \boldsymbol{Q})$ is invertible if and only if $\langle a, [pt]\rangle\neq 0$. Hence this fact completes the proof of Lemma.

5.4. PROOF OF THEOREM B. First we assume that both M and X are even-dimensional. Then we consider the Fredholm complex B^* over $M\times M$ constructed in 4.5 and pull it back over $M\times X$ by the map $1\times f$. Since the Hilbert bundles $E_\pm$ are flat along the slices $\{y\}\times M$, $(1\times f)^*E_\pm$ are also flat along $\{y\}\times X$. Then we can verify that the complex $D\otimes(1\times f)^*B^*$ satisfies Compactness condition by the same argument as in Proposition 4.3. Hence it follows that the analytical index of $D\otimes(1\times f)^*B^*$ vanishes from Proposition 3.3.

On the other hand, the Chern character of B^* has the following form by Lemma 5.3 together with Lemma 5.1.

$$\mathrm{ch}(B^*)=\sum_j e_j^* a_N \times e_j.$$

Hence a generalized Atiyah-Singer index theorem yields

$$\begin{aligned}\mathrm{ch}(\mathrm{index}(D\otimes(1\times f)^*B^*)) &= \hat{\mathfrak{A}}(X)\mathrm{ch}(1\times f)^*B^*)/[X]\\ &= \hat{\mathfrak{A}}(X)\sum_j e_j^* a_N \times f^* e_j/[X]\\ &= \sum_j e_j^* a_N \langle \hat{\mathfrak{A}}(X) f^* e_j, [X]\rangle.\end{aligned}$$

Here we note that $\{e_j^* a_N\}$ also forms a basis since a_N is invertible. Therefore we obtain

$$\langle \hat{\mathfrak{A}}(X) f^* e_j, [X]\rangle = 0$$

for each e_j. Hence the higher $\hat{A}$-genus $\hat{A}(u)(X)$ vanishes for each $u\in H^*(M;\boldsymbol{Q})$ because $\{e_j\}$ forms a basis of $H^{\mathrm{even}}(M;\boldsymbol{Q})$ and $\hat{\mathfrak{A}}(X)$ is a polynomial of $4i$-dimensional cohomology classes.

When M and X are of general dimensions n and k, respectively, we consider the manifolds $M\times T^{2k+n}$ and $X\times T^k$. Then, applying the above argument to these manifolds and a map $f\times\iota: X\times T^k \to M\times T^{2k+n}$ where ι denotes an embedding of T^k into the first k-components of T^{2k+n}, we obtain

$$\langle \hat{\mathfrak{A}}(X\times T^k)(f\times\iota)^* u, [X\times T^k]\rangle = 0$$

for each $u\in H^*(M\times T^{2k+n};\boldsymbol{Q})$. Now we choose $u=u_0\times u_k$ such that $u_0\in H^*(M;\boldsymbol{Q})$, $u_k\in H^k(T^{2k+n};\boldsymbol{Q})$ and $\langle u_k, \iota_*[T^k]\rangle=1$. Then, noticing $\hat{\mathfrak{A}}(T^k)=1$, it follows that

$$\begin{aligned}0 &= \langle \hat{\mathfrak{A}}(X\times T^k)(f\times\iota)^*(u_0\times u_k), [X\times T^k]\rangle\\ &= \langle (\hat{\mathfrak{A}}(X)\times\hat{\mathfrak{A}}(T^k))(f^*u_0\times\iota^*u_k), [X]\times[T^k]\rangle\\ &= \langle \hat{\mathfrak{A}}(X) f^* u_0, [X]\rangle\langle \iota^* u_k, [T^k]\rangle\\ &= \langle \hat{\mathfrak{A}}(X) f^* u_0, [X]\rangle\end{aligned}$$

for each $u_0\in H^*(X;\boldsymbol{Q})$. Thus the proof of Theorem B is completed.

6. Proof of Corollary.

In this section we shall show that Condition B holds for closed spin^c manifolds with non-positive sectional curvature and some Seifert fibre spaces described in [9]. Then, Corollary follows from this applying Theorem B to these manifolds.

6.1. Let M be a closed spin^c Riemannian manifold with non-positive sectional curvature. We set

$$V = T\tilde{M}$$
$$\varphi : \tilde{M}\times\tilde{M} \longrightarrow T\tilde{M}$$
$$\varphi(x_1, x_2) = \exp_{x_1}^{-1}(x_2).$$

Then, the properties 1) and 2) in Condition B are easily verified with M and furthermore it is well known that the property 3) holds for M with $\varepsilon=0$; see e.g. [6]. Thus we obtain the following.

6.2. PROPOSITION. *A closed* spin^c *Riemannian manifold with non-positive sectional curvature satisfies Condition* B.

6.3. Next we show that Condition B holds for some Seifert fibre spaces. However, before doing so, we shall recall the construction of such Seifert fibre spaces according to [9].

Let W be a simply connected Riemannian manifold with non-positive sectional curvature and let Q be a discrete subgroup of $\mathrm{Isom}(W)$. We assume that the action of Q on W is properly discontinuous and the quotient space W/Q is compact. We further assume that Q acts on $T^k\times W$ by covering transformations such that the following diagram commutes:

$$\begin{array}{ccc} T^k\times W & \xrightarrow{T^k\backslash} & W \\ {\scriptstyle /Q}\downarrow & & \downarrow{\scriptstyle /Q} \\ (T^k\times W)/Q & \longrightarrow & W/Q. \end{array}$$

Then, the action of Q on $T^k\times W$ is given explicitly by

$$q : T^k\times W \longrightarrow T^k\times W$$
$$q(t, w) = (t\cdot m(q^{-1}, w), q(w))$$

for $q\in Q$, where m denotes a map

$$m : Q\times W \longrightarrow T^k.$$

Considering this covering $T^k\times W\to(T^k\times W)/Q$, we obtain the following exact sequence:

$$\underline{a}: \ 1 \longrightarrow Z^k \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1$$

where Γ denotes the fundamental group of $(T^k \times W)/Q$.

Conversely, suppose that we are given a exact sequence $\underline{a}: 1 \to Z^k \to \Gamma \to Q \to 1$. It is known that such an extension determines a map

$$m: \ Q \times W \longrightarrow T^k$$

and m induces a Q-action on $T^k \times W$ by the previous formula. Then, Conner and Raymond showed that $(T^k \times W)/Q$ is a closed manifold if and only if Γ is torsion-free. They called such a extension a Bieberbach extension. In the sequel we shall denote $(T^k \times W)/Q$ by $M(\underline{a})$.

Then, Rees [9] singled out particular types of the Bieberbach extensions and called them special extensions. Here we do not discuss them in detail. However, we only need the following property with them in the sequel.

P) Let $\underline{a}$ be a special extension and let m denote the resulting map from $\underline{a}$:

$$m: \ Q \times W \longrightarrow T^k.$$

Then, there exists a constant $c > 0$ independent of $q \in Q$ and $w \in W$ such that

$$\| m(q, \cdot)_*(Y) \| \leq c \| Y \|$$

for any $Y \in T_w W$.

6.4. While we have discussed only Seifert fibre spaces in the above, we here consider a general aspherical manifold M as in Introduction. We use the same notation as there.

Let ϕ be a map from $\tilde{M}$ to V such that

i) it is a proper map into some fibre of V with non-zero mapping degree, and

ii) there exist $C > 0$, $N > 0$ and $\varepsilon < 1$ such that

$$\| \phi_*(X) \| \leq C \cdot r^{\varepsilon} \| X \|$$

for $r \geq N$ and for any tangent vector $X \in T_x \tilde{M}$. Here we denote by r the distance from the origin to $\phi(x)$ as usual.

Now we shall denote by $\mathcal{S}$ the set of all above maps and define a Γ-action on $\mathcal{S}$ by the formula $\gamma(\phi) = \gamma \cdot \phi \cdot \gamma^{-1}$ for $\gamma \in \Gamma$. Then the following diagram commutes:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\ \phi\ } & V \\ \gamma\downarrow & & \downarrow\gamma \\ \tilde{M} & \xrightarrow{\ \gamma(\phi)\ } & V . \end{array}$$

6.5. LEMMA. *Suppose that there is a Γ-invariant contractible subset $\mathcal{S}_0 \subset \mathcal{S}$. Then, we can find a smooth map $\varphi : \tilde{M} \times \tilde{M} \to V$ satisfying all properties in Condition* B.

PROOF. We shall construct φ over skelton by skelton. Suppose that M is equipped with Γ-equivariant simplicial decomposition. We then choose representative simplexes $\{\sigma_i^{(n)}\}_i$ of the Γ-orbits for each n-skelton $\tilde{M}^{(n)}$.

First we construct φ over $\tilde{M}^{(0)} \times \tilde{M}$. We define φ to be an arbitrary $\phi \in \mathcal{S}_0$ on each $\sigma_i^{(0)} \times \tilde{M}$ and extend them over $\tilde{M}^{(0)} \times \tilde{M}$ in such a way that φ is compatible with the diagonal action, i.e. the following diagram is commutative:

$$\begin{array}{ccc} \sigma_i^{(0)} \times \tilde{M} & \xrightarrow{\ \varphi=\phi\ } & V \\ \gamma\times 1\downarrow & & \downarrow\gamma \\ \gamma(\sigma_i^{(0)}) \times \tilde{M} & \xrightarrow{\ \varphi=\gamma(\phi)\ } & V . \end{array}$$

This map satisfies Condition B over $\tilde{M}^{(0)} \times \tilde{M}$. Note that $\varphi|_{\{x\}\times \tilde{M}}$ $(x \in \tilde{M}^{(0)})$ is also an element of $\mathcal{S}_0$ since $\mathcal{S}_0$ is Γ-invariant. Then we can extend φ over $\sigma_i^{(1)} \times \tilde{M}$ satisfying Condition B because $\mathcal{S}_0$ is connected. Then we can also define φ over $\tilde{M}^{(1)} \times \tilde{M}$ in a similar way as above. Continuing the above step we can construct φ over $\tilde{M} \times \tilde{M}$ because $\mathcal{S}_0$ is a Γ-invariant contractible set.

Now we return to the situation in 6.3.

6.6. We consider a manifold $M(\underline{a}) = (T^k \times W)/Q$ in 6.3. Let σ be a map from W to $\boldsymbol{R}^k$ such that

$$\|\sigma_*(Y)\| \leq c \cdot \|Y\| \qquad (Y \in TW)$$

for some constant $c \geq 0$. We denote by $\mathcal{D}$ the set of all above maps. Furthermore we set

$$\phi_{\sigma,v} : \ \boldsymbol{R}^k \times W \longrightarrow T\boldsymbol{R}^k \times TW$$
$$\phi_{\sigma,v}(t, w) = (t + \sigma(w), \exp_v^{-1}(w)) \in T_t\boldsymbol{R}^k \times T_v W$$

where $\sigma \in \mathcal{D}, v \in W$. Here we identify TR^k with $R^k \times R^k$ in canonical manner. Now we denote by $\mathcal{S}_0$ the set of all $\phi_{\sigma,v}$ parametrized by $\mathcal{D} \times W$. Note that $\mathcal{S}_0$ is contractible since $\mathcal{D} \times W$ is contractible.

Here we define a Γ-action on $TR^k \times TW$ by

$$\begin{aligned} \gamma : \ & TR^k \times TW \longrightarrow TR^k \times TW \\ & \gamma(Y, Z) = (Y, \gamma(Z)). \end{aligned}$$

Then we obtain the following:

6.7. Lemma. *$\mathcal{S}_0$ is a Γ-invariant contractible subset of $\mathcal{S}$ in* 6.4.

Proof. First we verify that $\mathcal{S}_0$ is a Γ-invariant set. Actually, it follows that

$$\begin{aligned} \gamma \cdot \phi_{\sigma,v} \cdot \gamma^{-1}(t, w) &= \gamma \cdot \phi_{\sigma,v}(t + \tilde{m}(\gamma, w), \gamma^{-1}(w)) \\ &= \gamma(t + \tilde{m}(\gamma, w) + \sigma(\gamma^{-1}(w)), \exp_v^{-1}(\gamma^{-1}(w))) \\ &= \gamma(t + \tilde{m}(\gamma, w) + \sigma \cdot \gamma^{-1}(w), \gamma^{-1} \cdot \exp_{\gamma(v)}^{-1}(w)) \\ &= (t + \tilde{m}(\gamma, w) + \sigma \cdot \gamma^{-1}(w), \exp_{\gamma(v)}^{-1}(w)), \end{aligned}$$

namely,

$$\gamma \cdot \phi_{\sigma,v} \cdot \gamma^{-1} = \phi_{\tilde{m}(\gamma,\cdot) + \sigma \cdot \gamma^{-1}, \gamma(v)}$$

where $\tilde{m}(\gamma, \ ) : W \to R^k$ denotes a lifting of $m(\gamma, \ ) : W \to T^k$. Since $\mathcal{D}$ is invariant under the addition by $\tilde{m}(\gamma, \ )$ and since γ acts on W by an isometry, the above formula has shown that $\mathcal{S}_0$ is a Γ-invariant set.

Next we verify that $\mathcal{S}_0$ is a subset of $\mathcal{S}$. In fact the property P) in 6.3 yields this. Hence it follows that $\mathcal{S}_0$ is a Γ-invariant contractible subset of $\mathcal{S}$.

Therefore, if $TR^k \times TW$ admits a equivariant spin^c structure, the manifold $M(\underline{a})$ corresponding to a special extension $\underline{a}$ satisfies Condition B by Lemma 6.5 and Lemma 6.7. Thus the following holds.

6.8. Proposition. *Let $M(\underline{a})$ be a manifold corresponding to a special extension $\underline{a}$ and we further assume that W is a* spin^c *manifold and that Q acts on W preserving the* spin^c *structure. Then $M(\underline{a})$ satisfies Condition* B.

Finally we note that there is a manifold which satisfies the condition in Proposition 6.8 but not admit any metric with non-positive sectional curvature. Actually the same argument as in [9] holds in our case although we put more assumptions on $M(\underline{a})$ than in [9].

References

[1] Bérard-Bergery, L., La courbure scalaire des variétés riemanniennes, Sém. Bourbaki, exposé n° 556, juin 1980, Lecture Notes in Math. vol. 842, Springer-Verlag, Berlin-Heidelberg-New York, 1981, 225-245.

[2] Gromov, M. and H. B. Lawson, Jr., Spin and scalar curvature in the presence of a fundamental group, I, Ann. of Math. **111** (1980), 423-434.

[3] Gromov, M. and H. B. Lawson, Jr., Positive scalar curvature and the Dirac operator on complete Riemannian manifolds, Publ. Math. I.H.E.S. **58** (1983), 83-196.

[4] Hsiang, W. C. and H. D. Rees, Miscenko's work on Novikov's conjecture, Contemp. Math. **10** (1982), 77-98.

[5] Kazdan, J. L. and F. W. Warner, A direct approach to the determination of Gaussian and scalar curvature function, Invent. Math. **28** (1975), 227-230.

[6] Kobayashi, S. and K. Nomizu, Foundations of Differential Geometry, Volume II, Interscience, New York, 1969.

[7] Miščenko, A. S., Infinite-dimensional representations of discrete groups, and higher signatures, Math. USSR-Izv. **8** (1974), 85-111.

[8] Miščenko, A. S. and A. T. Fomenko, The index of elliptic operators over C^*-algebras, Math. USSR-Izv. **15** (1980), 87-112.

[9] Rees, H. D., Special manifolds and Novikov's conjecture, Topology **22** (1983), 365-378.

[10] Rosenberg, J., C^*-algebras, positive scalar curvature and the Novikov conjecture, Publ. Math. I.H.E.S. **58** (1983), 197-212.

[11] Seely, R. T., Integro-differential operators on vector bundles, Trans. Amer. Math. Soc. **117** (1965), 167-204.

[12] Segal, G., Fredholm complexes, Quart. J. Math. Oxford Ser. (2) **21** (1970), 385-402.

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