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On convergence and rates of the convergence of motions of incompressible fluids in $\mathbf{R}^2$ *as viscosity goes to* 0

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§ 1. Introduction

Let $T>0$ be arbitrarily fixed. In this paper we study certain properties of solutions $\boldsymbol{u}^{(\nu)}(x,t):=(u_1^{(\nu)}(x,t), u_2^{(\nu)}(x,t))$ of the following equations with a parameter $\nu\geq 0$:

$$(1.1)_\nu \qquad \begin{cases} \dfrac{\partial}{\partial t}\boldsymbol{u}^{(\nu)}-\nu\Delta\boldsymbol{u}^{(\nu)}+(\boldsymbol{u}^{(\nu)}\cdot\nabla)\boldsymbol{u}^{(\nu)}+\nabla p^{(\nu)}=0, & \boldsymbol{R}^2\times(0,T), \\ \operatorname{div}\boldsymbol{u}^{(\nu)}=0, & \boldsymbol{R}^2\times(0,T), \\ \boldsymbol{u}^{(\nu)}|_{t=0}=\boldsymbol{u}_0, & \boldsymbol{R}^2. \end{cases}$$

On $\boldsymbol{u}^{(\nu)}$, we impose an asymptotic condition

$$(1.2) \qquad \boldsymbol{u}^{(\nu)}(x,t)\to 0 \quad \text{as} \quad |x|\to\infty,\ 0\leq t<T.$$

We assume a compatibility condition: $\boldsymbol{u}_0(x)\to 0$ as $|x|\to\infty$. By a suitable reduction, equations $(1.1)_\nu$ can be solved solely for $\boldsymbol{u}^{(\nu)}$. Then $p^{(\nu)}$ can be found using this $\boldsymbol{u}^{(\nu)}$ so that $\{\boldsymbol{u}^{(\nu)}, p^{(\nu)}\}$ satisfies the first equality of $(1.1)_\nu$. Hence we call $\boldsymbol{u}^{(\nu)}$ "a solution of $(1.1)_\nu$". The parameter ν is called viscosity. If $\nu>0$ then $(1.1)_\nu$ is the Navier-Stokes equations, describing motions of incompressible viscous fluids. If $\nu=0$ then $(1.1)_\nu$ is the Euler equations, describing motions of ideal fluids. In both of these cases, the existence and uniqueness of solutions $\boldsymbol{u}^{(\nu)}$ of $(1.1)_\nu$, (1.2) are proved under various conditions on the initial value $\boldsymbol{u}_0$. See [5], [6], [10], [12], and also see [1], [3], [4], [8], which deal with equations $(1.1)_\nu$ in bounded domains of $\boldsymbol{R}^2$ or on 2-dim. manifolds under suitable boundary conditions. The convergence of the solutions $\boldsymbol{u}^{(\nu)}$ as $\nu\to 0$ is also studied in many papers under various assumptions on $\boldsymbol{u}_0$: [1], [2], [5], [9] and [10]. One of the aims of this paper is to prove the convergence under weaker assumptions on $\boldsymbol{u}_0$: $\boldsymbol{u}_0\in C^{1+\alpha}(\boldsymbol{R}^2)$ and $\operatorname{rot}\boldsymbol{u}_0\in L^1(\boldsymbol{R}^2)$. Another aim is to give as good rates of the convergence as possible. In [11], the author reported some results on these subjects without proof. This paper improves them and gives the proofs.

We state our results after introducing some notations. We use the following norms and semi-norms of functions on $\boldsymbol{R}^2$ or $\boldsymbol{R}^2\times[0, T)$. Let $f:\boldsymbol{R}^2\to\boldsymbol{R}^j$, $j=1$ or 2. Then,

$$\|f\|_0=\sup_{x\in\boldsymbol{R}^2}|f(x)|, \qquad \|f\|_{L^1}=\iint_{\boldsymbol{R}^2}|f(x)|d^2x,$$

$$\|f\|_\beta=\sup_{\substack{x,y\in\boldsymbol{R}^2\\ x\neq y}}\frac{|f(x)-f(y)|}{|x-y|^\beta},\ 0<\beta<1, \quad \|f\|_{1+\beta}=\|\nabla f\|_\beta,\ 0\leq\beta<1,$$

$$\|f\|_{\beta,\mathrm{loc}}=\sup_{\substack{x,y\in\boldsymbol{R}^2\\ 0<|x-y|<1}}\frac{|f(x)-f(y)|}{|x-y|^\beta},\ 0<\beta<1,$$

$$\|f\|_{B^0}=\|f\|_0+\|f\|_{L^1}, \qquad \|f\|_{B^\beta}=\|f\|_\beta+\|f\|_0+\|f\|_{L^1},\ 0<\beta<1.$$

Let $g:\boldsymbol{R}^2\times[0, T)\to\boldsymbol{R}^j$, $j=1, 2$. Then,

$$\|g\|_{\beta,T}=\sup_{0\leq t<T}\|g(\cdot, t)\|_\beta, \quad 0\leq\beta<2,$$

$$\|g\|_{B^\beta,T}=\sup_{0\leq t<T}\|g(\cdot, t)\|_{B^\beta},\ 0\leq\beta<1.$$

We use the following function spaces. For $0\leq\beta<1$, B^β and B_T^β are Banach spaces of scalar valued continuous functions with norms $\|\cdot\|_{B^\beta}$ and $\|\cdot\|_{B^\beta,T}$, respectively. We also use the following spaces.

$$V=\{\boldsymbol{v}\in(C^1(\boldsymbol{R}^2))^2:\operatorname{div}\boldsymbol{v}=0,\ \|\boldsymbol{v}\|_0+\|\boldsymbol{v}\|_1<\infty\}$$

$$V_T=\{\boldsymbol{u}\in(C(\boldsymbol{R}^2\times[0, T)))^2:\boldsymbol{u}(\cdot, t)\in V,\ 0\leq t<T,\ \|\boldsymbol{u}\|_{0,T}+\|\boldsymbol{u}\|_{1,T}<\infty\}$$

$$V^\beta=V\cap(C^{1+\beta}(\boldsymbol{R}^2))^2, \qquad 0\leq\beta<1,$$

$$V_T^\beta=\{\boldsymbol{u}\in V_T:\boldsymbol{u}(\cdot, t)\in V^\beta,\ 0\leq t<T\}, \qquad 0\leq\beta<1.$$

THEOREM 1. *Let* $0<\alpha<1$. *Suppose* $\boldsymbol{u}_0\in V$, $\boldsymbol{u}_0(x)\to 0$ *as* $|x|\to\infty$, *and* $\operatorname{rot}\boldsymbol{u}_0\in B^\alpha$. *Then there are constants* $\nu_1>0$ *and* M *depending only on* α, T, *and* $\|\operatorname{rot}\boldsymbol{u}_0\|_{B^\alpha}$ *such that there exists a solution* $\boldsymbol{u}^{(\nu)}$ *of* $(1.1)_\nu$, (1.2) *in* V_T^α *with* $\|\operatorname{rot}\boldsymbol{u}^{(\nu)}\|_{B^\alpha,T}\leq M$ *for each* $0\leq\nu\leq\nu_1$.

THEOREM 2. *Under the same assumptions of Theorem* 1, *a solution* $\boldsymbol{u}^{(\nu)}$ *of* $(1.1)_\nu$, (1.2) *is unique in the space* $\bigcup_{0<\beta<1}V_T^\beta$ *for each* $\nu\geq 0$.

THEOREM 3. *Under the same assumptions of Theorem* 1, *the family of solutions* $\boldsymbol{u}^{(\nu)}$ *converges to* $\boldsymbol{u}^{(0)}$ *in the space* $(C([0, T):C^{1+\gamma}(\boldsymbol{R}^2)))^2$ *for any* $0\leq\gamma<\alpha$. *Moreover, there is a constant* C *depending only on* α, T, *and* $\|\operatorname{rot}\boldsymbol{u}_0\|_{B^\alpha}$ *such that the following estimates hold for* $\nu\in[0, \nu_1]$:

$$\|\boldsymbol{u}^{(\nu)}(\cdot, t)-\boldsymbol{u}^{(0)}(\cdot, t)\|_0 \leq C(\nu t)^{(1+\alpha)/2}, \qquad 0\leq t<T, \tag{1.3}$$

$$\|\operatorname{rot} \boldsymbol{u}^{(\nu)}(\cdot, t)-\operatorname{rot} \boldsymbol{u}^{(0)}(\cdot, t)\|_\beta \leq C(\nu t)^{(\alpha-\beta)/2}, \qquad 0\leq t<T, \tag{1.4}$$

where $0\leq\beta\leq\alpha$ *is arbitrary and* ν_1 *is a constant given in Theorem* 1.

REMARK 1. The existence and the uniform boundedness for $\nu\in[\varepsilon, \infty)$, where $\varepsilon>0$ is arbitrary, follow with standard arguments. We can, therefore, conclude the existence of solutions $\boldsymbol{u}^{(\nu)}$ of $(1.1)_\nu$, (1.2) with $\|\operatorname{rot} \boldsymbol{u}^{(\nu)}\|_{B^\alpha, T}\leq M'$, $\nu\in[0, \infty)$, choosing M' suitably. Furthermore, we can show (1.3) and (1.4) for $\nu\in[0, \infty)$, taking another value of C. We omit the details.

REMARK 2. The asymptotic condition (1.2) can be generalized. We can consider the following asymptotic condition and give Theorem 4.

$$\lim_{|x|\to\infty} \boldsymbol{u}^{(\nu)}(x, t)=\boldsymbol{u}_\infty(t), \qquad 0\leq t<T. \tag{1.5}$$

THEOREM 4. *Let* $\alpha>0$. *Suppose* $\boldsymbol{u}_0\in V$ *and* $\operatorname{rot} \boldsymbol{u}_0\in B^\alpha$. *Let* $\boldsymbol{u}_\infty\in (C^1[0, T))^2$ *be given and* $\boldsymbol{u}_\infty(0)=\lim_{|x|\to\infty} \boldsymbol{u}_0(x)$. *Then for equations* $(1.1)_\nu$, (1.5), *the same conclusions as those of Theorem* 1–3 *hold.*

Theorem 4 can be shown by using Proposition 2.1 given in § 2. But we omit the detailed proof.

REMARK 3. Let us compare our results with those of preceding works. Golovkin [5] first proved the convergence and Kato [9] refined it: he showed, if $\boldsymbol{u}_0\in H^s(\boldsymbol{R}^2)$ with $s>2$, then $\boldsymbol{u}^{(\nu)}$ converges to $\boldsymbol{u}^{(0)}$ in $L^\infty(0, T; H^s(\boldsymbol{R}^2))$. McGrath [10] assumed $\operatorname{rot} \boldsymbol{u}_0\in L^1(\boldsymbol{R}^2)\cap C^{2+\alpha}(\boldsymbol{R}^2)$ with some $\alpha>0$ and showed $\operatorname{rot} \boldsymbol{u}^{(\nu)}$ converges to $\operatorname{rot} \boldsymbol{u}^{(0)}$ in $L^\infty(0, T; C^1(\boldsymbol{R}^2))$. Not only showing the convergence, Beale and Majda [2] also gave a rate of the convergence assuming high regularity on $\boldsymbol{u}_0$. Our theorems obviously improve the results of [10]. Our class of initial value is essentially wider than those of [2], [5] and [9]. Indeed, they all assume $\boldsymbol{u}_0\in H^s(\boldsymbol{R}^2)$ with $s>2$. This implies $\boldsymbol{u}_0\in B^\alpha$, $0<\alpha<s-2$, owing to the Sobolev imbedding theorem. Besides, the condition $\boldsymbol{u}_0\in L^2(\boldsymbol{R}^2)$ induces a restriction on $\boldsymbol{u}_0$: $\iint \operatorname{rot} \boldsymbol{u}_0(x) d^2x$ must vanish if $\operatorname{rot} \boldsymbol{u}_0$ has a compact support. The rates of convergence given by (1.3) and (1.4) seem to be optimal.

REMARK 4. There are some errors and ambiguities in the preceding paper [11]. In the statement of Theorem 2 of [11], the conclusion "$u^{(\nu)}\rightarrow u^{(0)}$" is an error and the correct one is "$\omega^{(\nu)}\rightarrow\omega^{(0)}$". In Theorem 3 of [11], h is arbitrary, but h must satisfy $0\leq h<\beta$. In addition, the asymptotic condition (1.2) of this paper must be assumed in both of these theorems. On the other hand, assumptions (3) and (4) in [11] are eliminated in this paper. We also note that the notations used here differ slightly from those in [11].

REMARK 5. It is interesting to know the existence and uniqueness of weak solutions of the Euler equations under assumptions of less regularity on initial values. The problem of the convergence under such assumptions is also interesting. On these subjects, [1] and [6] give partial answers and important suggestions.

The plan of this paper is as follows. We prove Theorem 1 in § 2, Theorem 2 in § 3, and Theorem 3 in § 4. In Appendix we prove a lemma used in § 4.

Before proceeding to § 2, we prepare some notational conventions. We use the same c to denote fixed constants of various values. On the other hand we use the same C to denote various constants depending on certain quantities like T, $\|\mathrm{rot}\, u_0\|_{B^\alpha}$, etc.. By signs $\iint\cdot\cdot d^2x$, we denote an integral over $\boldsymbol{R}^2$.

§ 2. Proof of Theorem 1

In this section, we first reduce equations $(1.1)_\nu$, (1.2) to equations for vorticity in Proposition 2.1. To solve the vorticity equations, we prepare Propositions 2.2–2.4 and use the Schauder fixed point theorem. Then Theorem 1 will follow immediately.

We introduce some operators to give the reduction stated above. We define $F_1: \bigcup_{0<\beta<1} B^\beta\rightarrow V$, by

$$F_1(f)(x)=\iint k(x-y)f(y)d^2y,\qquad x\in\boldsymbol{R}^2,\tag{2.1}$$

for any $f\in\bigcup_{0<\beta<1} B^\beta$ where $k(x)=(2\pi)^{-1}x^\perp/|x|^2$, $x^\perp:=(-x_2, x_1)$. We define $F_{1,T}: \bigcup_{0<\beta<1} B_T^\beta\rightarrow V_T$, by $F_{1,T}[g](x, t)=F_1[g(\cdot, t)](x)$, $0\leq t<T$, for any $g\in\bigcup_{0<\beta<1} B_T^\beta$. For each $\nu\geq 0$, we define $F_2^{(\nu)}: V_T\rightarrow B^0$, by $F_2^{(\nu)}[u]=\omega$ for $u\in V_T$, where ω is the solution of the equation

$$(2.2)_\nu \qquad \begin{cases} \dfrac{\partial}{\partial t}\omega - \nu\Delta\omega + \boldsymbol{u}\cdot\nabla\omega = 0 & \text{in } \boldsymbol{R}^2\times(0, T), \\ \omega|_{t=0} = \omega_0 & \text{in } \boldsymbol{R}^2. \end{cases}$$

Here $\omega_0 = \operatorname{rot} \boldsymbol{u}_0$. Finally we put $F^{(\nu)} := F_2^{(\nu)} \circ F_{1,T}$, $\nu \geq 0$. We note that the operators defined here are introduced in [10]. The following lemma is due to [10] and [8].

LEMMA 2.1. *Let $0 \leq \beta' \leq \beta < 1$, $0 < \beta$. Then F_1 is continuous as an operator $B^\beta \to V^\beta$. Furthermore, the following estimate holds*:

$$(2.3) \qquad \|F_1[f]\|_{1+\beta'} \leq \frac{c}{\beta\cdot(1-\beta)^{\beta'/\beta}} \|f\|_{B^\beta}, \qquad f \in B^\beta.$$

PROPOSITION 2.1. *Let $\nu \geq 0$ be arbitrarily fixed. Let $\boldsymbol{u}^{(\nu)}$ be a classical solution of* $(1.1)_\nu$ *with $\boldsymbol{u}^{(\nu)} \in V_T$ and $\operatorname{rot} \boldsymbol{u}^{(\nu)} \in B_T^\beta$ for some $0 < \beta < 1$. Then $\boldsymbol{u}_\infty(t) := \lim_{|x|\to\infty} \boldsymbol{u}^{(\nu)}(x, t)$, $0 \leq t < T$, exists and moreover, $\boldsymbol{u}_\infty \in (C[0, T))^2 \cap (C^1(0, T))^2$. Furthermore, $\omega := \operatorname{rot} \boldsymbol{u}^{(\nu)}$ satisfies the following equation*:

$$(2.4) \qquad \omega = F_3^{(\nu)}[F_{1,T}[\omega] + \boldsymbol{u}_\infty].$$

Conversely, let $\boldsymbol{u}_\infty \in (C[0, T))^2 \cap (C^1(0, T))^2$ be given and let $\omega \in B_T^\beta$ satisfy (2.4). *Then $\boldsymbol{u}^{(\nu)} := F_{1,T}[\omega] + \boldsymbol{u}_\infty$ is a solution of the equation* $(1.1)_\nu$, *satisfying the asymptotic condition* (1.5).

PROOF. Put $w := \boldsymbol{u}^{(\nu)} - F_{1,T}[\operatorname{rot} \boldsymbol{u}^{(\nu)}]$. Then we easily get $\operatorname{div} \boldsymbol{w} = \operatorname{rot} \boldsymbol{w} = 0$. Hence $\boldsymbol{w}(\cdot, t)$ is harmonic in $\boldsymbol{R}^2$. Since $\boldsymbol{w}$ is bounded, $\boldsymbol{w}(\cdot, t)$ is a constant; we put $\boldsymbol{u}_\infty(t) := \boldsymbol{w}(x, t)$. On the other hand, we get $\lim_{|x|\to\infty} F_{1,T}[\operatorname{rot} \boldsymbol{u}^{(\nu)}](x, t) = 0$, since $\operatorname{rot} \boldsymbol{u}^{(\nu)} \in B_T^\beta$. Hence we have $\lim_{|x|\to\infty} \boldsymbol{u}^{(\nu)}(x, t) = \boldsymbol{u}_\infty(t)$ and $\boldsymbol{u}^{(\nu)} = F_{1,T}[\operatorname{rot} \boldsymbol{u}^{(\nu)}] + \boldsymbol{u}_\infty$. Taking the rotation of $(1.1)_\nu$ we conclude $\operatorname{rot} \boldsymbol{u}^{(\nu)} = F_2^{(\nu)}[\boldsymbol{u}^{(\nu)}]$. This implies (2.4). The regularity of $\boldsymbol{u}_\infty$ follows from the identity $\boldsymbol{u}_\infty = \boldsymbol{u}^{(\nu)} - F_{1,T}[\operatorname{rot} \boldsymbol{u}^{(\nu)}]$. Thus we complete the proof of the first half. The latter half of Proposition 2.1 can be shown with a similar argument to that of the proof of Lemma 3.3 in [8]. Q.E.D.

Henceforth, we assume $\boldsymbol{u}_\infty = 0$, corresponding to the asymptotic condition (1.2), although the general case can be treated in a similar manner. Now, we define $\mathcal{S}_{\beta,M} := \{\zeta \in B_T^\beta : \|\zeta\|_{B^0,T} \leq \|\omega_0\|_{B^0}, \|\zeta\|_{\beta,T} \leq M\}$ for given $0 < \beta < 1$ and $M > 0$. In the following propositions we assume that the same assumptions of Theorem 1 hold.

PROPOSITION 2.2. *There are constants* $\nu_1>0$, $0<\beta<\alpha$, and $M>0$ *depending on* α, T *and* $\|\omega_0\|_{B^\alpha}$ *such that* $F^{(\nu)}[\mathcal{S}_{\beta,M}]\subset\mathcal{S}_{\beta,M}$ *for* $\nu\in[0,\nu_1]$.

PROPOSITION 2.3. *For any constants* $\nu_1>0$, $\beta>0$ *and* $M>0$, $\bigcup_{0\leq\nu\leq\nu_1} F^{(\nu)}[\mathcal{S}_{\beta,M}]$ *is bounded in* B_T^α.

PROPOSITION 2.4. *Let* $\beta>0$ *and* $M>0$ *be arbitrary. Then, for each* $\nu\geq 0$, $F^{(\nu)}$ *is continuous as an operator* $B_T^\beta\to C([0,T)\times\boldsymbol{R}^2)$ *and moreover,* $F^{(\nu)}[\mathcal{S}_{\beta,M}]$ *is relatively compact in* $C([0,T)\times\boldsymbol{R}^2)$.

Before proving these propositions, we prove Theorem 1 using Propositions 2.1–2.4.

PROOF OF THEOREM 1. Let β, M and ν_1 be as in Proposition 2.2. We can apply the Schauder fixed point theorem to the operators $F^{(\nu)}$, $\nu\in[0,\nu_1]$, in the space $C([0,T)\times\boldsymbol{R}^2)$, by virtue of Propositions 2.2 and 2.4. Hence there is a fixed point $\omega^{(\nu)}$ of $F^{(\nu)}$ in $\mathcal{S}_{\beta,M}$ for each $\nu\in[0,\nu_1]$. In addition, it follows from Proposition 2.3 that $\bigcup_{0\leq\nu\leq\nu_1}\{\omega^{(\nu)}\}$ is bounded in B_T^α. This, together with the latter half of Proposition 2.1, implies Theorem 1. Q.E.D.

Here we use the Lagrangian coordinates $U_{t,s}[u]:\boldsymbol{R}^2\to\boldsymbol{R}^2$, $0\leq t,s<T$, $\boldsymbol{u}\in V_T$, given by the following: $U_{t,s}[u](x)$, $x\in\boldsymbol{R}^2$, are the solutions of the ordinary differential equations

$$\begin{cases}\dfrac{d}{dt}U_{t,s}[\boldsymbol{u}](x)=\boldsymbol{u}(U_{t,s}[\boldsymbol{u}](x),t)\\[2ex] U_{s,s}[\boldsymbol{u}](x)=x.\end{cases}\tag{2.5}$$

We write simply $U_{t,s}$ for $U_{t,s}[\boldsymbol{u}]$, if no confusion occurs. Then, as is well known, it follows from $\boldsymbol{u}\in V_T$ that the maps $U_{t,s}$ are volume preserving diffeomorphisms on $\boldsymbol{R}^2$ and $U_{t,s}^{-1}=U_{s,t}$. Using Gronwall's inequality, we get the following lemma.

LEMMA 2.2. *Let* $0\leq\beta<1$ *and let* $\boldsymbol{u}\in V_T^\beta$. *Then,*

$$\|U_{t,s}\|_{1+\beta}\leq c(1+T)(1+\|\boldsymbol{u}\|_{1+\beta,T})\exp(cT\|\boldsymbol{u}\|_{1,T}),\qquad 0\leq t,s<T.\tag{2.6}$$

The following lemma is proved in [10].

LEMMA 2.3. *Let* $R>0$. *Then there exist constants* $\delta>0$ *and* $M_1>0$

depending only on R and T such that for any $\zeta \in B_T^0$ with $\|\zeta\|_{B^0,T} \leq R$, the following estimate holds:

$$\|U_{t,s}[F_{1,T}[\zeta]]\|_{\delta,\mathrm{loc}} \leq M_1, \qquad 0 \leq s, t < T. \tag{2.7}$$

Let us introduce some operators. Let λ be a parameter taking every value in $(0, \infty)$. Let h_λ be $h_\lambda(\rho) = (\pi\lambda)^{-1}\exp(-\rho^2/\lambda)$, $\rho \in \boldsymbol{R}$. Given $\boldsymbol{v} \in V$, let $\phi_{\lambda,v}$ be functions on $\boldsymbol{R}^2 \times \boldsymbol{R}^2$ given by $\phi_{\lambda,v}(x, y) = 2\lambda^{-1}(x-y)\cdot(\boldsymbol{v}(y) - \boldsymbol{v}(x))h_\lambda(|x-y|)$, $x, y \in \boldsymbol{R}^2$. We define operators I_λ and $J_{\lambda,v} : B^0 \to B^0$ by

$$I_\lambda[f](x) = \iint h_\lambda(|x-y|)f(y)d^2y, \tag{2.8}$$

$$J_{\lambda,v}[f](x) = \iint \phi_{\lambda,v}(x, y)f(y)d^2y, \tag{2.9}$$

for any $f \in B^0$. Given $\boldsymbol{u} \in V_T$, we define operators $\Phi_{\nu,u}$ and $\Psi_{\nu,u} : B_T^0 \to B_T^0$ for $\nu > 0$ by

$$\Phi_{\nu,u}[g](x, t) = \int_0^t I_{4\nu(t-s)}[g(U_{t,s}^{-1}(\cdot), s)](x)ds, \tag{2.10}$$

$$\Psi_{\nu,u}[g](x, t) = \int_0^t J_{4\nu(t-s),u(\cdot,t)}[g(U_{t,s}^{-1}(\cdot), s)](x)ds, \tag{2.11}$$

for any $g \in B_T^0$. From a classical theorem given by Il'in-Kalashnikov-Oleinik [7 : § 4], we get the following identities:

$$(\partial/\partial t - \nu\Delta + \boldsymbol{u}\cdot\nabla)I_{4\nu t}[f \circ U_{t,s}^{-1}] = J_{4\nu t,u(\cdot,t)}[f \circ U_{t,s}^{-1}], \tag{2.12}$$

$$(\partial/\partial t - \nu\Delta + \boldsymbol{u}\cdot\nabla)\Phi_{\nu,u}[g] = g + \Psi_{\nu,u}[g]. \tag{2.13}$$

PROPOSITION 2.5. *Let $\boldsymbol{u} \in V_T$. If $\nu > 0$, then $F_2^{(\nu)}[\boldsymbol{u}]$ is given by*

$$F_2^{(\nu)}[\boldsymbol{u}] = I_{4\nu t}[\omega_0 \circ U_{t,0}^{-1}] + \sum_{j=0}^{\infty} (-1)^{j+1}\Phi_{\nu,u}(\Psi_{\nu,u})^j W_{\nu,u}, \tag{2.14}$$

where $W_{\nu,u} : \boldsymbol{R}^2 \times [0, T) \to \boldsymbol{R}$, given by $W_{\nu,u}(x, t) = J_{4\nu t,u(\cdot,t)}[\omega_0 \circ U_{t,0}^{-1}](x)$. If $\nu = 0$, then $F_2^{(\nu)}[\boldsymbol{u}]$ is given by

$$F_2^{(0)}[\boldsymbol{u}](x, t) = \omega_0(U_{t,0}^{-1}(x)). \tag{2.15}$$

For the proof, see [7 : section 4] ($\nu > 0$) and see [10] ($\nu = 0$). Using Proposition 2.5 we give some estimates for $F_2^{(\nu)}$.

LEMMA 2.4. *Let* $0\leq\beta<1$ *and* $v\in V^{\beta}$. *Then* $\phi_{\lambda,v}$ *satisfies the followings*:

$$|\phi_{\lambda,v}(x, y)|\leq c\|v\|_{1}h_{2\lambda}(|x-y|), \tag{2.16}$$

$$\oint_{|x-y|=\rho}\phi_{\lambda,v}(x, y)dx=0, \qquad \rho>0, \tag{2.17}$$

$$\oint_{|x-y|=\rho}\phi_{\lambda,v}(x, y)dy=0, \qquad \rho>0, \tag{2.18}$$

$$\left|\oint_{|x-y|=\rho}(x-y)\phi_{\lambda,v}(x, y)dx\right|\leq c\|v\|_{1+\beta}\rho^{4+\beta}\lambda^{-1}h_{\lambda}(|x-y|), \quad \rho>0, \tag{2.19}$$

$$\left|\oint_{|x-y|=\rho}(x-y)\phi_{\lambda,v}(x, y)dy\right|\leq c\|v\|_{1+\beta}\rho^{4+\beta}\lambda^{-1}h_{\lambda}(|x-y|), \quad \rho>0. \tag{2.20}$$

Here $y\in \boldsymbol{R}^{2}$ $(x\in \boldsymbol{R}^{2})$ *is arbitrarily fixed in* (2.17) *and* (2.19) (*in* (2.18) *and* (2.20), *respectively*).

PROOF. The inequality (2.16) clearly holds. To prove (2.17), change the variables $x\rightarrow z$, $z=x-y$. Then, the Gauss theorem yields

$$\begin{aligned}\oint_{|x-y|=\rho}\phi_{\lambda,v}(x, y)dx&=2\lambda^{-1}h_{\lambda}(\rho)\iint_{|z|<\rho}\rho \operatorname{div} v(z+y)d^{2}z\\&=0,\end{aligned}$$

since $v\in V$. The equality (2.18) can be shown similarly. By the mean value theorem, we not $v(x)-v(y)=(x-y)\cdot\nabla v(y)+r(x, y)$, where r satisfies $|r(x, y)|\leq\|v\|_{1+\beta}|x-y|^{\beta}$. Hence, changing the variables $z:=x-y$, we get

$$\begin{aligned}\oint_{|x-y|=\rho}(x-y)\phi_{\lambda,v}(x, y)dx&=2\lambda^{-1}h_{\lambda}(\rho)\oint_{|z|=\rho}z\times z\cdot(z\cdot\nabla)v(y)dz\\&\quad+2\lambda^{-1}h_{\lambda}(\rho)\oint_{|z|=\rho}z\times z\cdot r(z+y, y)dz.\end{aligned} \tag{2.21}$$

The first term of the right-hand side of (2.21) vanishes since the integrand is a homogeneous polynomial of degree three. It is easy to show

$$\left|2\lambda^{-1}h_{\lambda}(\rho)\oint_{|z|=\rho}z\times z\cdot r(x, y)d\rho\right|\leq c\|v\|_{1+\beta}\rho^{4+\beta}\lambda^{-1}h_{\lambda}(\rho).$$

Hence we have (2.19). The inequality (2.20) can be shown similarly.

Q.E.D.

LEMMA 2.5. *Let* $0\leq\beta'\leq\beta<1$, $\beta\neq 0$. *Let* $v\in V$. *Then, for any* $f\in B^{\beta}$,

$I_\lambda[f]$ and $J_{\lambda,v}[f]$ satisfy the following estimates:

$$\|I_\lambda[f]\|_\beta \leq \|f\|_\beta, \tag{2.22}$$

$$\|I_\lambda[f]\|_{\beta,\mathrm{loc}} \leq \|f\|_{\beta,\mathrm{loc}}, \tag{2.23}$$

$$\|J_{\lambda,v}[f]\|_{\beta'} \leq c\lambda^{(\beta-\beta')/2}\|v\|_1\|f\|_\beta. \tag{2.24}$$

PROOF. The inequalities (2.22) and (2.23) clearly hold. We prove (2.24) for $\beta'=0$ and $\beta'=\beta$. Then the case $0<\beta'<\beta$ follows from interpolation. First we show the case $\beta'=0$. Using (2.16), (2.18) and the Hölder continuity of f, we get

$$\begin{aligned} |J_{\lambda,v}[f](x)| &= \left|\iint \phi_{\lambda,v}(x, y)(f(y)-f(x))d^2y\right| \\ &\leq c\|f\|_\beta\|v\|_1\int_0^\infty h_{2\lambda}(\rho)\rho^{\beta+1}d\rho \\ &\leq c\|f\|_\beta\|v\|_1\lambda^{\beta/2}, \end{aligned} \tag{2.25}$$

which gives (2.24) for $\beta'=0$. To prove (2.24) for $\beta'=\beta$, fix x and x' arbitrarily. Then

$$J_{\lambda,v}[f](x')-J_{\lambda,v}[f](x)=\iint \phi_{\lambda,v}(x', y)f(y)d^2y-\iint \phi_{\lambda,v}(x, y)f(y)d^2y. \tag{2.26}$$

Put $z:=x'-y$ in the first term of the right-hand side and $z:=x-y$ in the second, and change the variables $y\rightarrow z$ in the both. We get:

$$\begin{aligned} &J_{\lambda,v}[f](x')-J_{\lambda,v}[f](x) \\ &=\iint \phi_{\lambda,v}(x', z+x')(f(z+x')-f(z+x))d^2z \\ &\quad+\iint \{\phi_{\lambda,v}(x', z+x')-\phi_{\lambda,v}(x, z+x)\}f(z+x)d^2z \\ &(=J_1+J_2). \end{aligned} \tag{2.27}$$

Using (2.16) and the assumption $f\in B^\beta$, we have

$$|J_1| \leq c\|v\|_1\|f\|_\beta|x'-x|^\beta\iint h_{2\lambda}(|z|)d^2z \leq c\|v\|_1\|f\|_\beta|x'-x|^\beta. \tag{2.28}$$

From (2.18) we obtain the following identity:

$$J_2=\iint\{\phi_{\lambda,v}(x', z+x')-\phi_{\lambda,v}(x, z+x)\}\{f(z+x)-f(x)\}d^2z. \tag{2.29}$$

We shall give estimates for J_2 the cases $|x'-x|>\lambda^{1/2}$ and $|x'-x|\leq\lambda^{1/2}$

separately. When $|x'-x|>\lambda^{1/2}$, using (2.16) we get

$$|J_2|\leq c\|v\|_1\|f\|_\beta\iint|z|^\beta h_{2\lambda}(|z|)d^2z \leq c\|v\|_1\|f\|_\beta\lambda^{\beta/2}\leq c\|v\|_1\|f\|_\beta|x'-x|^\beta. \tag{2.30}$$

The remained case: By the mean value theorem, we note

$$|\phi_{\lambda,v}(x',z+x')-\phi_{\lambda,v}(x,z+x)|=|2\lambda^{-1}z\cdot\{v(x')-v(z+x')-v(x)+v(z+x)\}|h_\lambda(|z|) \leq c\lambda^{-1}\|v\|_1|x'-x||z|h_\lambda(|z|).$$

Thus we get

$$\|J_2\|\leq c\lambda^{-1}\|v\|_1|x'-x|\|f\|_\beta\iint|z|^{1+\beta}h_\lambda(|z|)d^2z \leq c\|v\|_1\|f\|_\beta\lambda^{(\beta-1)/2}|x'-x| \leq c\|v\|_1\|f\|_\beta|x'-x|^\beta, \tag{2.31}$$

since $|x'-x|<\lambda^{1/2}$. Thus we have $|J_2|\leq c\|v\|_1\|f\|_\beta|x'-x|^\beta$ in both cases. This, together with (2.27) and (2.28), yields

$$|J_{\lambda,v}[f](x')-J_{\lambda,v}[f](x)|\leq c\|v\|_1\|f\|_\beta|x'-x|^\beta, \tag{2.32}$$

which shows (2.24) for $\beta'=\beta$. This completes the proof. Q.E.D.

From this lemma and Lemma 2.2, we obtain the following corollary.

COROLLARY 1. *Let $u\in V_T$ and let $0\leq\beta'\leq\beta<1$. Then, for any $g\in B_T^\beta$ and for $\nu>0$, the following estimates hold for $0\leq s<t<T$:*

$$\|I_{4\nu(t-s)}[g(U_{t,s}^{-1}(\cdot),s)]\|_\beta\leq\|U_{t,s}^{-1}\|_1^\beta\|g(\cdot,s)\|_\beta, \tag{2.33}$$

$$\|\Phi_{\nu,u}[g](\cdot,t)\|_\beta\leq\exp(c\beta T\|u\|_{1,T})\int_0^t\|g(\cdot,s)\|_\beta ds, \tag{2.34}$$

$$\|J_{4\nu(t-s),u(\cdot,t)}[g(U_{t,s}^{-1}(\cdot),s)]\|_{\beta'}\leq c(\nu(t-s))^{(\beta-\beta')/2} \times\|u\|_{1,T}\exp(c\beta T\|u\|_{1,T})\|g(\cdot,s)\|_\beta, \tag{2.35}$$

$$\|\Psi_{\nu,u}[g](\cdot,t)\|_\beta\leq c\|u\|_{1,T}\exp(c\beta T\|u\|_{1,T})\int_0^t\|g(\cdot,s)\|_\beta ds. \tag{2.36}$$

The following corollary also holds.

COROLLARY 2. *Let $u\in V_T$. Let $0\leq\beta\leq\alpha$. Then, for $\nu>0$, the following estimates hold for $t\in[0,T)$.*

$$\|\Phi_{\nu,u}(\Psi_{\nu,u})^j W_{\nu,u}(\cdot,t)\|_\beta\leq c(\nu t)^{(\alpha-\beta)/2}\|\omega_0\|_\alpha\frac{t^{j+1}}{(j+1)!} \times\exp(c\alpha T\|u\|_{1,T})\{c\|u\|_{1,T}\exp(c\beta T\|u\|_{1,T})\}^{j+1},\quad j=0,1,2,\cdots, \tag{2.37}$$

$$\|(\Psi_{\nu,u})^j W_{\nu,u}(\cdot,t)\|_\beta \le c(\nu t)^{(\alpha-\beta)/2}\|\omega_0\|_\alpha \frac{t^j}{j!} \times\{c\|u\|_{1,T}\exp(c\beta T\|u\|_{1,T})\}^{j+1}, \quad j=0,1,2,\cdots. \tag{2.38}$$

PROOF. Using (2.36) j times repeatedly and (2.34), we get

$$\begin{aligned}\|\Phi_{\nu,u}(\Psi_{\nu,u})^j W_{\nu,u}\|_\beta \le& \exp(c\beta T\|u\|_{1,T})\{c\|u\|_{1,T}\exp(c\beta T\|u\|_{1,T})\}^j \\ &\int_0^t ds\int_0^s ds_1\cdots\int_0^{s_{j-1}}\|W_{\nu,u}(\cdot,s_j)\|_\beta ds_j \\ \le& \exp(c\beta T\|u\|_{1,T})\{c\|u\|_{1,T}\exp(c\beta T\|u\|_{1,T})\}^j \\ &\times\frac{t^{j+1}}{(j+1)!}\sup_{0\le s<t}\|J_{4\nu s,u(\cdot,s)}[\omega_0\circ U_{s,0}^{-1}]\|_\beta. \end{aligned}\tag{2.39}$$

Using (2.35) and (2.6), we get

$$\sup_{0\le s<t}\|J_{4\nu s,u(\cdot,s)}[\omega_0\circ U_{s,0}^{-1}]\|_\beta \le c(\nu t)^{(\alpha-\beta)/2}\|\omega_0\|_\alpha\|u\|_{1,T}\exp(c\alpha T\|u\|_{1,T}). \tag{2.40}$$

From (2.39) and (2.40) we get (2.37). The inequality (2.38) can be shown similarly. Q.E.D.

Using these lemmas and corollaries, we obtain a certain estimate for the operator $F_2^{(\nu)}$.

PROPOSITION 2.6. *Let* $0<\gamma\le 1$. *Let* $u\in V_T$. *Then, for* $\nu\ge 0$, $F_2^{(\nu)}[u]$ *satisfies the following estimate*:

$$\|F_2^{(\nu)}[u]\|_{\alpha\gamma,T} \le \sup_{0\le t<T}\|\omega_0\circ U_{t,0}^{-1}\|_{\alpha\gamma} + c(\nu T)^{(\alpha-\alpha\gamma)/2}\|\omega_0\|_\alpha \times\exp(c\alpha T\|u\|_{1,T})\{\exp(cT\|u\|_{1,T}\exp(cT\|u\|_{1,T}))-1\}. \tag{2.41}$$

PROOF. From (2.14), (2.15), (2.22) and (2.37), we get

$$\|F_2^{(\nu)}[u]\|_{\alpha\gamma,T} \le \sup_{0\le t<T}\|\omega_0\circ U_{t,0}^{-1}\|_{\alpha\gamma} + c(\nu T)^{(\alpha-\alpha\gamma)/2}\|\omega_0\|_\alpha \times\sum_{j=0}^\infty\frac{t^{j+1}}{(j+1)!}\exp(c\alpha T\|u\|_{1,T})\{c\|u\|_{1,T}\exp(c\alpha\gamma T\|u\|_{1,T})\}^{j+1} \tag{2.42}$$

for $\nu\ge 0$. Computing the series, we have (2.41). Q.E.D.

PROOF OF PROPOSITION 2.2. First we use Lemma 2.3 setting $R=\|\omega_0\|_{B^0}$. Let δ and M_1 be the constants given in this lemma. Then we have

$$\|U_{t,s}[F_{1,T}[\zeta]]\|_{\delta,\mathrm{loc}} \le M_1 \qquad 0\le t,s<T, \tag{2.43}$$

for any $\zeta\in B^0_T$ with $\|\zeta\|_{B^0,T}\le\|\omega_0\|_{B^0}$. We take $\beta=\alpha\delta$ and $M=2(M_1^\alpha+1)\times\|\omega_0\|_{B^0}$. Let $\zeta\in\mathcal{S}_{\beta,M}$ be fixed arbitrarily and put $u:=F_{1,T}[\zeta]$. The B^0 norm of solutions of the equations $(2.2)_\nu$, $\nu\ge 0$, does not increase in time:

$$\|F^{(\nu)}[\zeta]\|_{B^0,T}\le\|\omega_0\|_{B^0}. \tag{2.44}$$

We show $\|F^{(\nu)}[\zeta]\|_{\alpha\delta}\le M$ for small ν. Using Proposition 2.6 and (2.43) and noting $\|\omega_0\circ U_{t,0}^{-1}\|_{\alpha\delta}\le\|U_{t,0}^{-1}\|^\alpha_{\delta,\mathrm{loc}}\|\omega_0\|_\alpha+2\|\omega_0\|_0$, we have

$$\|F_2^{(\nu)}[u]\|_{\alpha\delta,T}\le M_1^\alpha\|\omega_0\|_\alpha+2\|\omega_0\|_0+c_1(\nu T)^{(\alpha-\alpha\delta)/2}\|\omega_0\|_\alpha q(c_2T\|u\|_{1,T}), \tag{2.45}$$

where $q(\rho)=e^\rho\{e^{\rho e^\rho}-1\}$, $\rho\in R$, and c_j, $j=1,2$, are fixed constants. From Lemma 2.1, we get $\|u\|_{1,T}\le c_3(M+\|\omega_0\|_{B^0})/\alpha\delta$, where c_3 is a fixed constant. Thus we have

$$\begin{aligned}\|F_2^{(\nu)}[u]\|_{\alpha\delta,T}\le(M_1^\alpha+2)\|\omega_0\|_{B^\alpha}&+c_1(\nu T)^{(\alpha-\alpha\delta)/2}\|\omega_0\|_\alpha\\ &\times q(c_2c_3T(M+\|\omega_0\|_{B^0})/\alpha\delta).\end{aligned} \tag{2.46}$$

Let $\nu_1=T^{-1}\{(c_1q(c_2c_3T(M+\|\omega_0\|_{B^0})/\alpha\delta)^{-1}M_1^\alpha\}^{2/(\alpha-\alpha\delta)}$. Then we get, for $0\le\nu\le\nu_1$,

$$\|F_2^{(\nu)}[u]\|_{\alpha\delta,T}\le(M_1^\alpha+2)\|\omega_0\|_{B^\alpha}+M_1^\alpha\|\omega_0\|_\alpha\le M. \tag{2.47}$$

From (2.44) and (2.47) we conclude $F_2^{(\nu)}[u]\in\mathcal{S}_{\beta,M}$ for $\nu\in[0,\nu_1]$. Since $u=F_{1,T}[\zeta]$ and $\zeta\in\mathcal{S}_{\beta,M}$ is arbitrary, we get $F^{(\nu)}[\mathcal{S}_{\beta,M}]\subset\mathcal{S}_{\beta,M}$, $\nu\in[0,\nu_1]$.

Q.E.D.

PROOF OF PROPOSITION 2.3. Proposition 2.3 follows from Proposition 2.6, (2.41) with $\gamma=1$, and Lemmas 2.1-2.2. Q.E.D.

LEMMA 2.6. *Let $\nu>0$ be arbitrarily fixed. Let $0<\beta<1$ and $M>0$. Then $\sup_{|x|\ge L}|F^{(\nu)}[\zeta](x,t)|\to 0$ as $L\to\infty$, uniformly in $t\in[0,T)$ and in $\zeta\in\mathcal{S}_{\beta,M}$.*

PROOF. Let $\zeta\in\mathcal{S}_{\beta,M}$ and $u=F_{1,T}[\zeta]$. Then, by (2.34), (2.35), (2.36) and Lemma 2.1, there is a constant C depending only on β, T, M and $\|\omega_0\|_{B^0}$ such that

$$\Big\|\sum_{j=n}^{\infty}(-1)^{j+1}\Phi_{\nu,u}(\Psi_{\nu,u})^jW_{\nu,u}\Big\|_{0,T}\le C\sum_{j=n}^{\infty}\frac{C^{j+1}}{(j+1)!},\quad n=0,1,2,\cdots. \tag{2.48}$$

Hence we have

$$\Big\|\sum_{j=n}^{\infty}(-1)^{j+1}\Phi_{\nu,u}(\Psi_{\nu,u})^jW_{\nu,u}\Big\|_{0,T}\to 0 \text{ as } n\to\infty, \text{ uniformly in } \zeta\in\mathcal{S}_{\beta,M}. \tag{2.49}$$

We note $\lim_{|x|\to\infty} \omega_0(x)=0$ since $\omega_0 \in B^\beta$. We also note $\|u\|_0 \le c\|\zeta\|_{B^0}$ and $\sup_{x\in R^2} |U_{t,0}^{-1}(x)-x| \le cT\|\zeta\|_{B^0}$. Thus we deduce

$$|I_{4\nu t}[\omega_0 \circ U_{t,0}^{-1}](x,t)| \to 0 \text{ as } |x|\to\infty, \text{ uniformly in } t\in[0,T) \text{ and in } \zeta \in \mathcal{S}_{\beta,M}, \tag{2.50}$$

and

$$|\Phi_{\nu,u}(\Psi_{\nu,u})^j W_{\nu,u}(x,t)| \to 0 \text{ as } |x|\to\infty, \text{ uniformly in } t\in[0,T) \text{ and in } \zeta\in\mathcal{S}_{\beta,M}, \quad j=0,1,2,\cdots. \tag{2.51}$$

Combining (2.49), (2.50) and (2.51) with Proposition 2.5, we get the conclusion. Q.E.D.

PROOF OF PROPOSITION 2.4. The continuity of $F^{(\nu)}$ for fixed $\nu \ge 0$ follows from a classical theory of continuous dependence of solutions of $(2.2)_\nu$ on its coefficients. The relative compactness of $F^{(\nu)}[\mathcal{S}_{\beta,M}]$ for fixed $\nu\ge 0$ follows from Lemma 2.6 and (2.44), (2.45) (see [10], Lemma 2.2). Q.E.D.

§ 3. Proof of Theorem 2

We show the uniqueness of fixed point of $F^{(\nu)}$ for each $\nu\ge 0$. Under the assumptions of Theorem 1, the following propositions hold.

PROPOSITION 3.1. *The fixed point of* $F^{(\nu)}$ *in* $\bigcup_{0<\beta<1} B_T^\beta$ *is unique for each* $\nu>0$.

PROPOSITION 3.2. *The fixed point of* $F^{(0)}$ *in* $\bigcup_{0<\beta<1} B_T^\beta$ *is unique.*

PROOF OF THEOREM 2. Let $0<\beta<1$. Given $\nu\ge 0$, let $u^{(\nu)}, v^{(\nu)} \in V_T^\beta$ be solutions of $(1.1)_\nu$, (1.2) with $\operatorname{rot} u^{(\nu)}, \operatorname{rot} v^{(\nu)} \in B_T^\beta$. Then Proposition 2.1 implies that $\operatorname{rot} u^{(\nu)}$ and $\operatorname{rot} v^{(\nu)}$ are fixed points of $F^{(\nu)}$. Hence $\operatorname{rot} u^{(\nu)} = \operatorname{rot} v^{(\nu)}$ follows from Propositions 3.1 and 3.2. Then, by Proposition 2.1, we get $u^{(\nu)} = F_{1,T}[\operatorname{rot} u^{(\nu)}] = F_{1,T}[\operatorname{rot} v^{(\nu)}] = v^{(\nu)}$. Q.E.D.

To prove Proposition 3.1, we use L^1 estimate for solutions of inhomogeneous parabolic equation:

$$\begin{cases} \dfrac{\partial}{\partial t}\omega - \nu\Delta\omega + u\cdot\nabla\omega = g, \\ \omega|_{t=0} = f. \end{cases} \tag{3.1}$$

We know by [7 : section 4],

LEMMA 3.1. *Let* $\nu>0$, $\boldsymbol{u}\in V_T$ *be fixed arbitrarily. If* $g=0$ *and* $f\in B^0$, *then the solution* ω *of* (3.1) *satisfies*

$$\|\nabla\omega(\cdot,t)\|_{B^0}\leq Ct^{-1/2}\|f\|_{B^0}, \tag{3.2}$$

where C *depends only on* ν, $\|\boldsymbol{u}\|_{0,T}$ *and* $\|\boldsymbol{u}\|_{1,T}$. *While, if* $f=0$ *and* $g\in L^1(0,T;B^0)$ *then*

$$\|\omega(\cdot,t)\|_{B^0}\leq C\int_0^t\|g(\cdot,s)\|_{B^0}ds, \tag{3.3}$$

where C *depends only on* ν, $\|\boldsymbol{u}\|_{0,T}$ *and* $\|\boldsymbol{u}\|_{1,T}$.

PROOF OF PROPOSITION 3.1. Let $0<\beta<1$. Let ζ_1 and ζ_2 be fixed points of $F^{(\nu)}$ in B_T^β. Put $\boldsymbol{v}:=F_{1,T}[\zeta_2]$. Then ζ_2 satisfies $(\partial/\partial t-\nu\Delta+\boldsymbol{v}\cdot\nabla)\zeta_2=0$. Hence by Lemma 3.1, we get $\|\nabla\zeta_2(\cdot,t)\|_{B^0}\leq C\|\omega_0\|B_{B^0}t^{-1/2}$ where C depends only on ν, β and $\|\zeta_2\|_{B^\beta,T}$. Put $\boldsymbol{u}:=F_{1,T}[\zeta_1]$ and $\omega:=\zeta_1-\zeta_2$. Then ω satisfies $(\partial/\partial t-\nu\Delta+\boldsymbol{u}\cdot\nabla)\omega=(\boldsymbol{v}-\boldsymbol{u})\cdot\nabla\zeta_2$, $\omega|_{t=0}=0$. Hence from Lemma 3.1 and (2.44) we get

$$\begin{aligned}\|\omega(\cdot,t)\|_{B^0}&\leq C\int_0^t\|(\boldsymbol{u}-\boldsymbol{v})\cdot\nabla\zeta_2(\cdot,s)\|_{B^0}ds\\&\leq C\int_0^t\|\nabla\zeta_2(\cdot,s)\|_{B^0}\|(\boldsymbol{u}-\boldsymbol{v})(\cdot,s)\|_0ds\\&\leq C\|\omega_0\|_{B^0}\int_0^t s^{-1/2}\|\omega(\cdot,s)\|_{B^0}ds,\end{aligned} \tag{3.4}$$

where C depends only on ν, β and $\|\zeta_i\|_{B^\beta,T}$, $i=1,2$. With standard arguments, we conclude $\omega=0$ from (3.4). Thus we have $\zeta_1=\zeta_2$. Q.E.D.

Let $\mathcal{D}$ be the set of all volume preserving diffeomorphisms $U:\boldsymbol{R}^2\to\boldsymbol{R}^2$ with $\|U\|_1+\|U^{-1}\|_1<\infty$. Given $f\in B^0$, we define an operator $A_f:\mathcal{D}\times\mathcal{D}\to(C(\boldsymbol{R}^2))^2$ by

$$A_f(U,V)(x)=\iint k(Ux-Vy)f(y)d^2y,\qquad x\in\boldsymbol{R}^2, \tag{3.5}$$

for any U, $V\in\mathcal{D}$. Here and in what follows, we write Ux for $U(x)$.

LEMMA 3.2. *Let* $\beta>0$ *and* $f\in B^\beta$. *Then* A_f *is Lipschitz continuous in the sense*

$$\|A_f(U,V)-A_f(W,V)\|_0\leq C\|f\|_{B^\beta}(1+\|V^{-1}\|_1^\beta)\|U-W\|_0, \tag{3.6}$$

(3.7) $$\|A_f(U, V)-A_f(U, W)\|_0 \leq C\|f\|_{B^\beta}(1+\|V^{-1}\|_1)\|V-W\|_0,$$

hold for any $U, V, W \in \mathcal{D}$, where C depends only on β.

PROOF. First we note $A_f[U, V](x)=F_1[f\circ V^{-1}](Ux)$. Hence from Lemma 2.1 we get $|A_f[U, V](x)-A_f[W, V](x)| \leq \frac{c}{\beta}\|f\circ V^{-1}\|_{B^\beta}|Ux-Wx|$. Since $\|f\circ V^{-1}\|_{B^\beta} \leq \|f\|_{B^0}+\|f\|_\beta\|V^{-1}\|_1^\beta \leq \|f\|_{B^\beta}(1+\|V^{-1}\|_1^\beta)$, we have (3.6). Put $d:=\|V-W\|_0$. Since $\|A_f(U, V)\|_0 \leq c\|f\|_{B^0}$ holds, we may assume $d<1/4$ without loss of generality. Using volume preserving property of V and W, and using $\iint_{|y|<1} k(y)d^2y=0$, we get

(3.8)
$$\begin{aligned} A_f(U, V)(x)-A_f(U, W)(x) &= \iint_{|Ux-Vy|<1} k(Ux-Vy)\{f(y)-f(V^{-1}Ux)\}d^2y \\ &\quad +\iint_{|Vx-Vy|\geq 1} k(Ux-Vy)f(y)d^2y \\ &\quad -\iint_{|Ux-Wy|<1} k(Ux-Wy)\{f(y)-f(V^{-1}Ux)\}d^2y \\ &\quad -\iint_{|Ux-Wy|\geq 1} k(Ux-Wy)f(y)d^2y. \end{aligned}$$

Using this, we give a decomposition of $A_f[U, V](x)-A_f[U, W](x)$, $x\in R^2$ arbitrarily fixed. Let D_j, $j=1, 2, \cdots, 5$, be the following subsets of R^2: $D_1=\{y : \min(|Ux-Vy|, |Ux-Wy|)\leq 2d\}$, $D_2=\{y : 2d<|Ux-Vy|<1,\ 2d<|Ux-Wy|<1\}$, $D_3=\{y : |Ux-Vy|<1,\ |Ux-Wy|\geq 1\}$, $D_4=\{|Ux-Vy|\geq 1,\ |Ux-Wy|<1\}$ and $D_5=\{y : |Ux-Vy|\geq 1, |Ux-Wy|\geq 1\}$. Then the decomposition is

(3.9) $$A_f[U, V](x)-A_f[U, W](x)=\sum_{j=1}^{5}\mathcal{K}_j,$$

where

(3.10) $$\mathcal{K}_1=\iint_{D_1}\{k(Ux-Vy)-k(Ux-Wy)\}\{f(y)-f(V^{-1}Ux)\}d^2y,$$

(3.11) $$\mathcal{K}_2=\iint_{D_2}\{k(Ux-Vy)-k(Ux-Wy)\}\{f(y)-f(V^{-1}Ux)\}d^2y,$$

(3.12) $$\mathcal{K}_3=\iint_{D_3}\{k(Ux-Vy)(f(y)-f(V^{-1}Ux))-k(Ux-Wy)f(y)\}d^2y,$$

(3.13) $$\mathcal{K}_4=\iint_{D_4}\{k(Ux-Wy)(f(y)-f(V^{-1}Ux))-k(Ux-Vy)f(y)\}d^2y,$$

$$(3.14)\qquad \mathcal{K}_5=\iint_{D_5}\{k(Ux-Vy)-k(Ux-Wy)\}f(y)d^2y.$$

Let us give estimates for $\mathcal{K}_j, j=1, 2, \cdots, 5$. Since $|k(x)|\leq c|x|^{-1}$, we have

$$(3.15)\qquad |\mathcal{K}_1|\leq c\|f\|_0 d.$$

Since

$$(3.16)\qquad |k(x)-k(y)|=\frac{1}{2\pi}\frac{|x-y|}{|x||y|},\qquad x, y\in \boldsymbol{R}^2,$$

and since $|f(y)-f(V^{-1}Ux)|\leq\|f\|_\beta|V^{-1}Vy-V^{-1}Ux|^\beta\leq\|f\|_\beta\|V^{-1}\|_1^\beta|Vy-Ux|^\beta$, we get

$$(3.17)\qquad |\mathcal{K}_2|\leq c\|f\|_\beta\|V^{-1}\|_1^\beta d/\beta.$$

Since $|Ux-Vy|>|Ux-Wy|-|Vy-Wy|\geq 1-d\geq 3/4$ in D_3, the integrand of $\mathcal{K}_3$ is estimated by $c\|f\|_0$. Furthermore, since $1+d\geq|Ux-Wy|>1$ in D_3, Lebesgue measure of D_3 is less than cd. Hence we deduce

$$(3.18)\qquad |\mathcal{K}_3|\leq c\|f\|_0 d.$$

Similarly we have

$$(3.19)\qquad |\mathcal{K}_4|\leq c\|f\|_0 d.$$

Using (3.16) we easily see

$$(3.20)\qquad |\mathcal{K}_5|\leq c\|f\|_{L^1} d.$$

Combining (3.15), (3.17)-(3.20) with (3.9), we have

$$(3.21)\qquad |A_f[U, V](x)-A_f[U, W](x)|\leq\frac{c}{\beta}\|f\|_{B^\beta}(1+\|V^{-1}\|_1^\beta)d.$$

Since $d=\|V-W\|_0$ and x is arbitrary, we get (3.7) from (3.21). Q.E.D.

PROOF OF PROPOSITION 3.2. Let $\zeta_i\in B_T^0$, $i=1, 2$, be fixed points of $F^{(0)}$. Put $\boldsymbol{u}:=F_{1,T}[\zeta_1]$ and $\boldsymbol{v}:=F_{1,T}[\zeta_2]$. We write $U_t:=U_{t,0}[\boldsymbol{u}]$ and $V_t:=U_{t,0}[\boldsymbol{v}]$, $0\leq t<T$. Note that $\zeta_1=\omega_0\circ U_t^{-1}$ and $\zeta_2=\omega_0\circ V_t^{-1}$, by Proposition 2.5. Hence it suffices to show $U_t=V_t$, $0\leq t<T$. Let d^+/dt be the right upper derivative. We get

$$\begin{aligned}\frac{d^+}{dt}\|U_t-V_t\|_0 &\le \left\|\frac{d}{dt}(U_t-V_t)\right\|_0\\ &=\|A_{\omega_0}[U_t,U_t]-A_{\omega_0}[V_t,V_t]\|_0\\ &\le C\|\omega_0\|_{B^\alpha}\|U_t-V_t\|_0,\end{aligned}$$

by Lemma 3.2, where C depends only on α,β and $\|\zeta_i\|_{B^\beta_T}$, $i=1,2$. Hence, from Gronwall's inequality, we obtain $\|U_t-V_t\|=0$, $0\le t<T$. Q.E.D.

§ 4. Proof of Theorem 3

Let $u^{(\nu)}$, $\nu\in[0,\nu_1]$, be the unique solution of $(1.1)_\nu$, (1.2). Put $\omega^{(\nu)}:=\operatorname{rot}u^{(\nu)}$, and $U^{(\nu)}_{t,s}:=U_{t,s}[u^{(\nu)}]$, $0\le s,t<T$. The following proposition is the key to the proof of Theorem 3.

PROPOSITION 4.1. *There is a constant C depending only on α, T, $\|\omega_0\|_{B^\alpha}$ and ν_1 such that for $0\le\nu\le\nu_1$,*

$$\|U^{(\nu)}_{t,0}-U^{(0)}_{t,0}\|_0\le Ct(\nu t)^{(1+\alpha)/2},\qquad 0\le t<T. \tag{4.1}$$

We shall give the proof of Proposition 4.1 after showing some lemmas and a proposition.

LEMMA 4.1. *There is a constant C depending on α, $\|\omega_0\|_{B^\alpha}$, T and ν_1 such that for $0\le\nu\le\nu_1$,*

$$\|u^{(\nu)}\|_{1+\alpha,T}\le C, \tag{4.2}$$

$$\|U^{(\nu)}_{t,s}\|_{1+\alpha}\le C,\qquad 0\le s,t<T. \tag{4.3}$$

PROOF. The estimate (4.2) follows from Theorems 1, 2, and Lemma 2.1. Then (4.3) follows from (4.2) and Lemma 2.2. Q.E.D.

LEMMA 4.2. *Let $U\in\mathcal{D}$. Then,*

$$\iint|x-Uy|^{-1}h_\lambda(|y-z|)d^2y\le c(\|U\|_1+\|U^{-1}\|_1)|z-U^{-1}x|^{-1},\quad x,z\in\boldsymbol{R}^2. \tag{4.4}$$

PROOF. First we note

$$\|U^{-1}\|_1^{-1}|z-U^{-1}x|\le|Uz-x|\le\|U\|_1|z-U^{-1}x|,\quad x,z\in\boldsymbol{R}^2. \tag{4.5}$$

Fix x and z arbitrarily and put $D:=\{y\in\boldsymbol{R}^2:|x-Uy|>(2\|U^{-1}\|_1)^{-1}\times|z-U^{-1}x|\}$. Then by simple calculations we get

$$\iint_D |x-Uy|^{-1}h_\lambda(|y-z|)d^2y \leq \sup_{y\in D} |x-Uy|^{-1}\iint h_\lambda(|y-z|)d^2y \leq c\|U^{-1}\|_1|z-U^{-1}x|^{-1}. \tag{4.6}$$

For $y\in \boldsymbol{R}^2\backslash D$, we have $|y-z|\geq|z-U^{-1}x|-|y-U^{-1}x|\geq|z-U^{-1}x|/2$ using (4.5). This implies

$$\begin{aligned}\iint_{R^2\backslash D}|x-Uy|^{-1}h_\lambda(|y-z|)d^2y &\leq \sup_{y\notin D} h_\lambda(|y-z|)\iint_{R^2\backslash D}|x-Uy|^{-1}d^2y \\ &\leq ch_{4\lambda}(|z-U^{-1}x|)\|U^{-1}\|_1^{-1}|z-U^{-1}x| \\ &\leq c\|U\|_1|z-U^{-1}x|.\end{aligned} \tag{4.7}$$

Then combining (4.6) with (4.7), we get (4.4). Q.E.D.

LEMMA 4.3. *Let $0<\beta<1$. Suppose that $U\in \mathscr{D}$, $\|U\|_{1+\beta}<\infty$ and $\boldsymbol{v}\in V^{1+\beta}$. Then, for any $f\in B^\beta$,*

$$\left|\iiiint k(x-Uy)\phi_{\lambda,v}(y,z)f(z)d^2yd^2z\right|\leq C\|f\|_{B^\beta}(\lambda+\lambda^{(1+\beta)/2}), \tag{4.8}$$

where C depends only on β, $\|\boldsymbol{v}\|_1$, $\|\boldsymbol{v}\|_{1+\beta}$, $\|U\|_1$, $\|U\|_{1+\beta}$, and $\|U^{-1}\|_1$.

Proof of Lemma 4.3 is given in Appendix. The following corollary is an immediate consequence of Lemma 4.3.

COROLLARY. *Under the same assumptions of Lemma 4.3, there exists a constant C depending only on β, $\|\boldsymbol{v}\|_1$, $\|\boldsymbol{v}\|_{1+\beta}$, $\|U\|_1$, $\|U\|_{1+\beta}$, and $\|U^{-1}\|_1$ such that for any $f\in B^\beta$,*

$$\|F_1[J_{\lambda,v}[f]\circ U^{-1}]\|_0\leq C\|f\|_{B^\beta}(\lambda+\lambda^{(1+\beta)/2}). \tag{4.9}$$

LEMMA 4.4. *There exists a constant C depending only on α such that for any $f\in B^\alpha$,*

$$\|F_1[I_\lambda[f]]-F_1[f]\|_0\leq C\|f\|_{B^\alpha}\lambda^{(1+\alpha)/2}. \tag{4.10}$$

PROOF. Since $\iint h_\lambda(|z|)d^2z=1$, we get

$$F_1[I_\lambda[f]](x)-F_1[f](x)=\iint k(x-y)\left\{\iint h_\lambda(|y-z|)(f(z)-f(y))d^2z\right\}d^2y. \tag{4.11}$$

We put $\eta:=y-z$, and change the variables $(y,z)\to(\eta,z)$. Then from Fubini's theorem we get

$$
\begin{aligned}
(4.12)\qquad & F_1[I_\lambda[f]](x)-F_1[f](x)\\
&=\iint h_\lambda(|\eta|)\left\{\iint k(x-z-\eta)(f(z)-f(z+\eta))d^2z\right\}d^2\eta\\
&=\iint h_\lambda(|\eta|)\{F_1[f](x-\eta)-F_1[f](x)\}d^2\eta.
\end{aligned}
$$

Using the mean value theorem and Lemma 2.1, we have

$$
(4.13)\qquad |F_1[f](x-\eta)-F_1[f](x)+\eta\cdot\nabla F_1[f](x)|\le C\|f\|_{B^\alpha}|\eta|^{1+\alpha},
$$

where C depends only on α. Noting that $\iint h_\lambda(|\eta|)\eta\cdot\boldsymbol{a}\,d^2\eta=0$, for any $\boldsymbol{a}\in\boldsymbol{R}^2$, we get

$$
\begin{aligned}
(4.14)\qquad & \left|\iint h_\lambda(|\eta|)\{F_1[f](x-\eta)-F_1[f](x)\}d^2\eta\right|\\
&=\left|\iint h_\lambda(|\eta|)\{F_1[f](x-\eta)-F_1[f](x)+\eta\cdot\nabla F_1[f](x)\}d^2\eta\right|\\
&\le C\|f\|_{B^\alpha}\iint|\eta|^{1+\alpha}h_\lambda(|\eta|)d^2\eta\\
&\le C\|f\|_{B^\alpha}\lambda^{(1+\alpha)/2},
\end{aligned}
$$

where C depends only on α. Combining (4.14) and (4.12), we establish (4.10). Q.E.D.

Using these lemmas, we get following Proposition 4.2.

PROPOSITION 4.2. *Let $\nu\in[0,\nu_1]$. Then there is a constant C depending only on α, T, ν_1 and $\|\omega_0\|_{B^\alpha}$ such that*

$$
(4.15)\qquad \|\boldsymbol{u}^{(\nu)}(\cdot,t)-\boldsymbol{u}^{(0)}(\cdot,t)\|_0\le C(\nu t)^{(1+\alpha)/2}+C\|U_t^{(\nu)}-U_t^{(0)}\|_0.
$$

PROOF. By Proposition 2.1, $\boldsymbol{u}^{(\nu)}=F_{1,T}[\mathrm{rot}\,\boldsymbol{u}^{(\nu)}]$ and $\mathrm{rot}\,\boldsymbol{u}^{(\nu)}=F^{(\nu)}[\mathrm{rot}\,\boldsymbol{u}^{(\nu)}]$ for each $\nu\in[0,\nu_1]$. Hence Proposition 2.5 yields

$$
(4.16)\qquad \boldsymbol{u}^{(\nu)}(\cdot,t)=F_1[I_{4\nu t}[\omega_0\circ U_t^{(\nu)^{-1}}]]-F_1\left[\sum_{j=0}^{\infty}(-1)^{j+1}\Phi_\nu(\Psi_\nu)^jW_\nu(\cdot,t)\right]
$$

for $\nu>0$, where $\Phi_\nu=\Phi_{\nu,\boldsymbol{u}^{(\nu)}}$, $\Psi_\nu=\Psi_{\nu,\boldsymbol{u}^{(\nu)}}$ and $W_\nu=W_{\nu,\boldsymbol{u}^{(\nu)}}$. For $\nu=0$,

$$
(4.17)\qquad \boldsymbol{u}^{(0)}(\cdot,t)=F_1[\omega_0\circ U_t^{(0)^{-1}}].
$$

Hence using Lebesgue's theorem, we have

$$(4.18)\qquad u^{(\nu)}(\cdot,t)-u^{(0)}(\cdot,t)=F_1[I_{4\nu t}[\omega_0\circ U_t^{(\nu)^{-1}}]]-F_1[\omega_0\circ U_t^{(0)^{-}}]$$
$$-F_1[\Phi_\nu W_\nu(\cdot,t)]+F_1\Big[\Phi_\nu\Psi_\nu\Big[\sum_{j=0}^{\infty}(-\Psi_\nu)^j W_\nu\Big](\cdot,t)\Big]$$

for $\nu\in(0,\nu_1]$. We evaluate the right-hand side of (4.18). Using the triangle inequality, and using Lemma 4.4 (4.10) and Lemma 3.2, (3.7), we get

$$(4.19)\qquad \begin{aligned}&\|F_1[I_{4\nu t}[\omega_0\circ U_t^{(\nu)^{-1}}]]-F_1[\omega_0\circ U_t^{(0)^{-1}}]\|_0\\ &\le\|F_1[I_{4\nu t}[\omega_0\circ U_t^{(\nu)^{-1}}]]-F_1[\omega_0\circ U_t^{(\nu)^{-1}}]\|_0+\|F_1[\omega_0\circ U_t^{(\nu)^{-}}]-F_1[\omega_0\circ U_t^{(0)^{-1}}]\|_0\\ &\le C\|\omega_0\|_{B^\alpha}(\nu t)^{(1+\alpha)/2}+C\|\omega_0\|_{B^\alpha}\|U_t^{(\nu)}-U_t^{(0)}\|_0,\end{aligned}$$

where C depends only on α, T, $\|\omega_0\|_{B^\alpha}$ and ν_1. Repeating the arguments used to deduce (4.12) from (4.11), we have

$$(4.20)\qquad \begin{aligned}F_1[\Phi_\nu W_\nu(\cdot,t)]&=F_1\Big[\int_0^t I_{4\nu(t-s)}[W_\nu(U_{t,s}^{(\nu)^{-1}}(\cdot),s)]ds\Big]\\ &=\int_0^t I_{4\nu(t-s)}[F_1[W_\nu(U_{t,s}^{(\nu)^{-1}}(\cdot),s)]]ds.\end{aligned}$$

Using the volume preserving property of $U_{t,s}^{(\nu)^{-1}}$, we get

$$(4.21)\qquad F_1[W_\nu(U_{t,s}^{(\nu)^{-1}},s)](x)=\iiiint k(x-U_{t,s}^{(\nu)}y)\phi_{4\nu s,u}(\nu)_{(\cdot,s)}(y,z)\omega_0(U_s^{(\nu)^{-}}z)d^2zd^2y.$$

Hence by Lemma 4.3 we have

$$(4.22)\qquad \begin{aligned}\|F_1[W_\nu(U_{t,s}^{(\nu)^{-1}},s)]\|_0&\le C\|\omega_0\circ U_s^{(\nu)^{-1}}\|_\alpha(4\nu s)^{(1+\alpha)/2}\\ &\le C(\nu s)^{(1+\alpha)/2},\end{aligned}$$

where C depends only on α, T, ν_1 and $\|\omega_0\|_{B^\alpha}$. Integrating the both sides with respect to s, we get

$$(4.23)\qquad \|F_1[\Phi_\nu W_\nu(\cdot,t)]\|\le\int_0^t C(\nu s)^{(1+\alpha)/2}ds\le Ct(\nu t)^{(1+\alpha)/2},$$

where C depends only on α, T, $\|\omega_0\|_{B^\alpha}$ and ν_1. Similarly, from (2.38) we deduce

$$(4.24)\qquad \|F_1[\Phi_\nu\Psi_\nu\sum_{j=0}^{\infty}(-\Psi_\nu)^jW_\nu(\ ,t)]\|_0\le Ct^2(\nu t)^{(1+\alpha)/2},$$

where C depends only on α, T, $\|\omega_0\|_{B^\alpha}$ and ν_1. Combining (4.19), (4.23) and (4.24) with (4.18), we conclude that (4.15) holds. Q.E.D.

PROOF OF PROPOSITON 4.1. Using the triangle inequality, we get

$$d/dt^{+}\|U_{t,0}^{(\nu)}-U_{t,0}^{(0)}\|_{0}\leq\|u^{(\nu)}(\cdot,t)-u^{(0)}(\cdot,t)\|_{0}+\|u^{(0)}(\cdot,t)\|_{1}\|U_{t,0}^{(\nu)}-U_{t,0}^{(0)}\|_{0}. \tag{4.25}$$

By Proposition 4.2 and Lemma 4.1 (4.2), we get

$$d/dt^{+}\|U_{t,0}^{(\nu)}-U_{t,0}^{(0)}\|_{0}\leq C(\nu t)^{(1+\alpha)/2}+C\|U_{t,0}^{(\nu)}-U_{t,0}^{(0)}\|_{0},\quad 0<t<T, \tag{4.26}$$

where C depends only on α, T, $\|\omega_0\|_{B^\alpha}$ and ν_1. Using Gronwall's inequality, we conclude

$$\|U_t^{(\nu)}-U_t^{(0)}\|_0\leq Ct(\nu t)^{(1+\alpha)/2} \tag{4.27}$$

for $0\leq\nu\leq\nu_1$, where C depends only on α, T, $\|\omega_0\|_\alpha$ and ν_1. Q.E.D.

PROOF OF THEOREM 3. The inequality (1.3) is an immediate consequence of Propositions 4.1 and 4.2. Let us prove the inequality (1.4) for $\beta=0$. Since $\omega^{(\nu)}=F_2^{(\nu)}[u^{(\nu)}]$, Proposition 2.5 yields

$$\begin{aligned}\omega^{(\nu)}(\cdot,t)-\omega^{(0)}(\cdot,t)=&\{I_{4\nu t}[\omega_0\circ U_t^{(\nu)^{-1}}]-\omega_0\circ U_t^{(\nu)^{-1}}\}\\ &+\{\omega_0\circ U_t^{(\nu)^{-1}}-\omega_0\circ U_t^{(0)^{-1}}\}+\sum_{j=0}^{\infty}(-1)^{j+1}\Phi_\nu(\Psi_\nu)^jW_\nu\end{aligned} \tag{4.28}$$

for $\nu>0$. Using (2.3) and Lemma 4.1 (4.3), we get

$$\|I_{4\nu t}[\omega_0\circ U_t^{(\nu)^{-1}}]-\omega_0\circ U_t^{(\nu)^{-1}}\|_0\leq c\|\omega_0\circ U_t^{(\nu)^{-1}}\|_\alpha(\nu t)^{\alpha/2}\leq C(\nu t)^{\alpha/2} \tag{4.29}$$

for $0\leq\nu\leq\nu_1$, where C depends only on α, T, $\|\omega_0\|_\alpha$ and ν_1. Using Proposition 4.1 and Lemma 4.1 (4.3), we have

$$\begin{aligned}\|\omega_0\circ U_t^{(\nu)^{-1}}-\omega_0\circ U_t^{(0)^{-1}}\|_0&\leq\|\omega_0\|_\alpha\|U_t^{(\nu)^{-1}}-U_t^{(0)^{-1}}\|_0^\alpha\\ &=\|\omega_0\|_\alpha\|U_t^{(0)^{-1}}(U_t^{(0)}-U_t^{(\nu)})U_t^{(\nu)^{-1}}\|_0^\alpha\\ &\leq\|\omega_0\|_\alpha\|U_t^{(0)^{-1}}\|_1^\alpha\|U_t^{(0)}-U_t^{(\nu)}\|_0^\alpha\\ &\leq C(\nu t)^{\alpha(1+\alpha)/2}\end{aligned} \tag{4.30}$$

for $0\leq\nu\leq\nu_1$, where C depends only on α, T, $\|\omega_0\|_\alpha$ and ν_1. By (2.37) with $\beta=0$ and Lemma 4.1, we get

$$\|\sum_{j=0}^{\infty}(-1)^{j+1}\Phi_\nu(\Psi_\nu)^jW_\nu(\cdot,t)\|_0\leq C(\nu t)^{\alpha/2} \tag{4.31}$$

for $0\leq\nu\leq\nu_1$, where C depends only on α, T, $\|\omega_0\|_{B^\alpha}$ and ν_1. Combining (4.29), (4.30) and (4.31) with (4.28) we get

$$\|\omega^{(\nu)}(\cdot,t)-\omega^{(0)}(\cdot,t)\|_0\leq C(\nu t)^{\alpha/2} \tag{4.32}$$

for $0\leq\nu\leq\nu_1$, where C depends only on α, T, $\|\omega_0\|_\alpha$ and ν_1. Thus we have proved (1.4) for $\beta=0$. For $\beta=\alpha$, (1.4) is a consequence of Theorems 1 and 2. For the case $0<\beta<\alpha$, (1.4) follows from interpolation. The convergence of $\boldsymbol{u}^{(\nu)}$ in $(C([0, T); C^{1+\gamma}(\boldsymbol{R}^2)))^2$ follows from (1.4) and Lemma 2.1.

Q.E.D.

Appendix. Proof of Lemma 4.3

Let $\mathcal{J}$ be the integral in the right-hand side of (4.8):

$$\mathcal{J}=\iiiint k(x-Uy)\phi_{\lambda,v}(y, z)f(z)d^2yd^2z. \tag{4.33}$$

Put $K=(\|U\|_1\|U^{-1}\|_1)^{-1}$. Let D_j, $j=0,1,\cdots,5$ be subsets of $\boldsymbol{R}^2\times\boldsymbol{R}^2$ given by

$$\begin{aligned}
&D_0=\bigcup_{j=0}^{3}D_j=\{(y, z): |z-U^{-1}x|<1,\ |y-z|<K/2\},\\
&D_1=\{(y, z): |z-U^{-1}x|<\sqrt{\lambda},\ |y-z|<K/2\},\\
&D_2=\left\{(y, z): \sqrt{\lambda}\leq|z-U^{-1}x|<1,\ |y-z|<\frac{K}{2}|z-U^{-1}x|\right\},\\
&D_3=\left\{(y, z): \sqrt{\lambda}\leq|z-U^{-1}x|<1,\ \frac{K}{2}|z-U^{-1}x|\leq|y-z|<K/2\right\},\\
&D_4=\{(y, z): |z-U^{-1}x|\geq 1,\ |y-z|<K/2\},\\
&D_5=\{(y, z): |y-z|\geq K/2\}.
\end{aligned}$$

Let $\mathcal{J}_j$, $j=0,1,\cdots,5$, be given by

$$\mathcal{J}_0=f(U^{-1}x)\iiiint_{D_0}k(x-Uy)\phi_{\lambda,v}(y, z)d^2yd^2z \tag{4.34}$$

$$\mathcal{J}_j=\iiiint_{D_j}k(x-Uy)\phi_{\lambda,v}(y, z)(f(z)-f(U^{-1}x))d^2yd^2z,\quad j=1,2,3 \tag{4.35}$$

$$\mathcal{J}_j=\iiiint_{D_j}k(x-Uy)\phi_{\lambda,v}(y, z)f(z)d^2yd^2z,\qquad j=4,5. \tag{4.36}$$

Then we have

$$\mathcal{J}=\sum_{j=0}^{5}\mathcal{J}_j. \tag{4.37}$$

Let us give estimates for $\mathcal{J}_j$, $j=0,1,2,\cdots,5$.

Estimate for $\mathcal{J}_0$. Put $D_0^*:=\{(y, z): (z, y)\in D_0\}$. Since $\phi_{\lambda,v}(y, z)=$

$\phi_{\lambda,v}(z, y)$, we get

$$\mathcal{J}_0=f(U^{-1}x)\iiiint_{D_0} k(x-Uy)\phi_{\lambda,v}(y, z)d^2yd^2z \\ +f(U^{-1}x)\iiiint_{D_0\setminus D_0^*} (k(x-Uy)-k(x-Uz))\phi_{\lambda,v}(y, z)d^2yd^2z. \tag{4.38}$$

By virtue of (2.18), the first term of the right-hand side vanishes. To evaluate the second term, we first note $|(k(x-Uy)-k(x-Uz))|\leq c\|U^{-1}\|_1^2\|U\|_1|y-z|$ in $D_0\setminus D_0^*$. Put $\xi:=y+z-2U^{-1}x$ and $\eta:=y-z$. We change the variables $(y, z)\rightarrow(\xi, \eta)$. The image of $D_0\setminus D_0^*$ by this transformation is included in $D^\sharp:=\{(\xi, \eta):2-|\eta|\leq|\xi|\leq 2+|\eta|, |\eta|<1/2\}$. Hence using (2.16) we get

$$\left|\iiiint_{D_0\setminus D_0^*} (k(x-Uy)-k(x-Uz))\phi_{\lambda,v}(y, z)d^2yd^2z\right| \\ \leq\iiiint_{D^\sharp} c\|U^{-1}\|_1^2\|U\|_1|\eta|\,\|v\|_1h_{2\lambda}(\eta)d^2\xi d^2\eta \\ \leq c\|U^{-1}\|_1^2\|U\|_1\|v\|_1\lambda. \tag{4.39}$$

Thus we have

$$|\mathcal{J}_0|\leq c\|U^{-1}\|_1^2\|U\|_1\|v\|_1\|f\|_0\lambda. \tag{4.40}$$

Estimate for $\mathcal{J}_1$. Using the estimate (2.16) and Lemma 4.2, we get

$$|\mathcal{J}_1|\leq\|f\|_\beta\iint_{|z-U^{-1}x|\leq\sqrt{\lambda}}|z-U^{-1}x|^\beta d^2z\left\{\iint|x-Uy|^{-1}|\phi_{\lambda,v}(y, z)|d^2y\right\} \\ \leq c\|f\|_\beta\|v\|_1(\|U\|_1+\|U^{-1}\|_1)\lambda^{(1+\beta)/2}. \tag{4.41}$$

Estimates for $\mathcal{J}_2$ and $\mathcal{J}_4$. We expand v at $y=z$:

$$v(y)-v(z)=((y-z)\cdot\nabla)v(z)+r(y, z), \tag{4.42}$$

where r satisfies

$$|r(y, z)|\leq\|v\|_{1+\beta}\cdot|y-z|^{1+\beta}. \tag{4.43}$$

We also expand $k(x-Uy)$ at $y=z$:

$$k(x-Uy) \\ =k(x-Uz)-\{((y-z)\cdot\nabla)Uz+r_1(y, z)\}\cdot\{\nabla k(x-Uz)+r_2(x; y, z)\}, \tag{4.44}$$

where r_j, $j=1, 2$, satisfy

$$|r_1(y, z)|\leq\|U\|_{1+\beta}|y-z|^{1+\beta}, \qquad y, z\in \boldsymbol{R}^2, \tag{4.45}$$

$$|r_2(x;y,z)|\le c\|U\|_1\|U^{-1}\|_1^3|y-z||z-U^{-1}x|^{-3},\qquad x\in\boldsymbol{R}^2,\ (y,z)\in D_2, \tag{4.46}$$

$$|r_2(x;y,z)|\le c\|U\|_1\|U^{-1}\|_1^3|y-z|,\qquad x\in\boldsymbol{R}^2,\ (y,z)\in D_4. \tag{4.47}$$

Using the expansions (4.42) and (4.44), we have

$$\begin{aligned}\mathcal{J}_2=&\iiiint_{D_2}k(x-Uz)\phi_{\lambda,v}(y,z)(f(z)-f(U^{-1}x))d^2yd^2z\\ &-\iiiint_{D_2}\{((y-z)\cdot\nabla)Uz\}\nabla k(x-Uz)(z-y)\cdot\{((y-z))\cdot\nabla v(z)\}\\ &\qquad\times 2\lambda^{-1}h_\lambda(|y-z|)(f(z)-f(U^{-1}x))d^2yd^2z\\ &-\iiiint_{D_2}\{((y-z)\cdot\nabla)Uz+r_1(y,z)\}\{\nabla k(x-Uz)+r_2(x;y,z)\}\\ &\qquad\times(z-y)\cdot r(y,z)2\lambda^{-1}h_\lambda(|y-z|)(f(z)-f(U^{-1}x))d^2yd^2z\\ &+\iiiint_{D_2}\{\{((y-z)\cdot\nabla)Uz+r_1(y,z)\}r_2(x;y,z)+r_1(y,z)\nabla k(x-Uz)\}\\ &\qquad\times(z-y)\cdot\{((y-z)\cdot\nabla)v(z)\}2\lambda^{-1}h_\lambda(|y-z|)(f(z)-f(U^{-1}x))d^2yd^2z.\end{aligned}\tag{4.48}$$

The first term vanishes owing to (2.18). The second term also vanishes since the integrand is a homogeneous polynomial of $y-z$ of order three for fixed z. We give estimates for the last two terms using (4.43), (4.5) and (4.46), concluding

$$|\mathcal{J}_2|\le C\|f\|_\beta\lambda^{(1+\beta)/2}, \tag{4.49}$$

where C depends only on β, $\|v\|_1$, $\|v\|_{1+\beta}$, $\|U\|_1$, $\|U\|_{1+\beta}$, and $\|U^{-1}\|_1$. In a similar manner, we can evaluate $\mathcal{J}_4$ using (4.47), obtaining

$$|\mathcal{J}_4|\le C\|f\|_{L^1}(\lambda+\lambda^{(1+\beta)/2}), \tag{4.50}$$

where C depends only on β, $\|v\|_1$, $\|v\|_{1+\beta}$, $\|U\|_1$, $\|U\|_{1+\beta}$, and $\|U^{-1}\|_1$.

Estimate for $\mathcal{J}_3$. In D_3, $\phi_{\lambda,v}$ satisfies

$$|\phi_{\lambda,v}(y,z)|\le c\|v\|_1h_{4\lambda}(|y-z|)\exp\Big(-\frac{K^2|z-U^{-1}x|^2}{16\lambda}\Big). \tag{4.51}$$

Thus, using Lemma 4.2, we get

$$\begin{aligned}|\mathcal{J}_3|\le&\, c\|v\|_1\|f\|_\beta\iiiint_{D_3}|x-Uy|^{-1}h_{4\lambda}(|y-z|)|z-U^{-1}x|^\beta\\ &\qquad\times\exp\Big(-\frac{K^2|z-U^{-1}x|^2}{16\lambda}\Big)d^2yd^2z\\ \le&\, c\|v\|_1\|f\|_\beta(\|U\|_1+\|U^{-1}\|_1)K^{-(1+\beta)}\lambda^{(1+\beta)/2}.\end{aligned}\tag{4.52}$$

Estimate for $\mathcal{J}_5$. In D_5, $\phi_{\lambda,v}$ satisfies

$$|\phi_{\lambda,v}(y,z)|\leq c\|v\|_1 h_{4\lambda}(|y-z|)e^{-1/4\lambda}. \tag{4.53}$$

Thus, using Lemma 4.2, we get

$$|\mathcal{J}_5|\leq c\|v\|_1\iiiint_{D_5}|x-Uy|^{-1}h_{4\lambda}(|y-z|)|f(z)|d^2yd^2z\leq C\|v\|_1\|f\|_{L^1}\lambda. \tag{4.54}$$

Combining (4.40), (4.41), (4.49), (4.50), (4.52) and (4.54) with (4.37), we get

$$|\mathcal{J}|\leq C\|f\|_{B^\beta}(\lambda+\lambda^{(1+\beta)/2}), \tag{4.55}$$

where C depends only on β, $\|v\|_1$, $\|v\|_{1+\beta}$, $\|U\|_1$, $\|U\|_{1+\beta}$, and $\|U^{-1}\|_1$.

Q.E.D.

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