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On the Riemann-Hilbert-Birkhoff problem for ordinary differential equations containing a parameter

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Since Birkhoff [9, 10], several authors have studied the asymptotic analysis of linear meromorphic differential equations with an irregular singular point (see, for example, Wasow [35], Majima [22] and the references therein for a lot of papers concerning this topic). Stokes phenomena around an irregular singular point are of our central concern in this theory. The treatment of this theory is, roughly speaking, devided into the following three stages.

(i) *the direct problem*: to establish the formal and analytic reduction of a differential equation, to compute Stokes coefficients arising from a given differential equation and to investigate the properties of Stokes phenomena arising from differential equations.

(ii) *the inverse problem*: to characterize Stokes phenomena arising from differential equations and to construct and classify differential equations giving rize to a prescribed Stokes phenomenon.

(iii) *the moduli problem*: to study geometrical structures of the space of an isoformal family (see Babbitt-Varadarajan [2-6]) of meromorphic differential equations up to an action of the analytic gauge transformation group. Remark that a Stokes phenomenon is a complete invariant under this action.

There are so many papers concerning the direct problem (i) that we cannot quote all of them here. In addition to Birkhoff [9, 10], we only refer to the classical works of Hukuhara [16, 17] and Turrittin [33, 34] (the Hukuhara-Turrittin formal reduction theorem). As for the inverse problem (ii), the works of Sibuya [30, 31] and Malgrange [24] are important. They introduced a new concept of the sheaf of germs of asymptotically developable functions, established a kind of cohomology vanishing theorem (the Sibuya-Malgrange theorem) and applied it to a study of construction and classification problem of systems of meromorphic linear differential equations with an irregular singular point. Birkhoff's original work is also important (the theorem of Birkhoff's canonical form). The recent work of Babbitt-Varadarajan [2-6] deals with the moduli problem (iii) deeply on the basis of the Sibuya-Malgrange theory.

See Jurkat-Lutz *et al.* [7, 8, 18], Sibuya [29], Trjitzinsky [32] for other aspects of the theory of Stokes phenomena and Majima [22] for an extension to several variables.

We call the inverse problem (ii) and the moduli problem (iii) in the theory of Stokes phenomena *the Riemann-Hilbert-Birkhoff problem* (*the R-H-B problem* for short). This is an analogue of the Riemann-Hilbert problem for differential equations with regular singular points (see Röhrl [28]).

It is well-known (see *e.g.* Wasow [35]) that the direct problem in the asymptotic analysis of differential equations with a singular parameter around its singularity is basically analogous to that of meromorphic differential equations with an irregular singular point. So there are a lot of papers concerning the direct problem on Stokes phenomena with respect to a parameter. As far as we are aware, however, no one has considered the R-H-B problem for Stokes phenomena with respect to a parameter. So we shall study this problem in the present paper. Our theory requires the notion of the sheaf of germs of the asymptotically developable functions containing a parameter, Sibuya-Malgrange type vanishing theorems for this sheaf, the study of a certain group action on this sheaf and the deformation theory of holomorphic vector bundles over the Riemann sphere.

On the other hand, our original interest lies in the spectral theory for ordinary differential equations (see *e.g.* Naimark [27] and Dunford-Schwartz [13]) as well as the R-H-B problem for Stokes phenomena with respect to a parameter itself. On one side, there is the celebrated Weyl-Stone-Titchmarsh-Kodaira theory for self-adjoint ordinary differential equations as a general theory. On the other side, there are many particular (and deeper) analysis of spectra for particular classes of differential equations, such as the Schrödinger equations with a decaying potential. We know that asymptotic analysis with respect to a spectral parameter was very often used in the latter type of researches. Combining these observations, we are naturally led to an idea that we should develop an asymptotic analysis of differential equations containing a spectral parameter in a sufficiently general setting which will be adequate to develop a general spectral theory as a supplement of the Weyl-Stone-Tichmarsh-Kodaira theory. We hope that our result in this paper will help further development of such a spectral theory.

In this paper, we shall first consider the following germs of differential equations containing a parameter k which is singular at $k=\infty$,

$$\frac{dY}{dx}=\{kP+q(x,k)\}Y \quad (\text{around } k=\infty), \tag{0.1}$$

where $P\in\mathfrak{gl}(n,\boldsymbol{C})$ is a fixed constant semi-simple matrix and

$$q(x,k)\in\mathfrak{gl}(n,\mathfrak{a}\{1/k\}). \tag{0.2}$$

Here $\mathfrak{a}\{1/k\}$ is, roughly speaking, the totality of convergent power series of $1/k$ with coefficients depending real-analytically on the variable x (see Section 1 for the rigorous definition). Next, we shall consider the equation (0.1) in the case where an action of a cyclic group, called A-cyclic action, is present on the equation (0.1). Namely we consider (0.1) under the condition that the matrix P satisfies a certain condition (Hypothesis 5.1) and that

$$q(x,k)\in\mathfrak{gl}(n,\mathfrak{a}\{1/k\})_{A\text{-cyclic}} \qquad (\text{see Section 5}). \tag{0.3}$$

Finally we shall consider the germs of single n-th order differential equations containing a spectral parameter,

$$\{\partial^n+a_1(x)\partial^{n-1}+\cdots+a_n(x)\}f=k^nf, \tag{0.4}$$

where $\partial=d/dx$ and $a_j(x)$ are germs of real-analytic functions. This class of equations is closely related to a particular case of (0.3) (cf. Theorem III).

As is shown later, any equation of the form (0.1) with (0.2) (resp. (0.3)) can be reduced to the following simple equation

$$\frac{dY}{dx}=kPY \tag{0.5}$$

by an appropriate *formal* gauge transformation (see Sections 5–6)

$$\begin{gathered} Y\longmapsto g(x,k)Y, \\ g(x,k)\in GL_0(n,\mathfrak{a}[[1/k]]) \quad (\text{resp. } \in GL_0(n,\mathfrak{a}[[1/k]])_{A\text{-cyclic}}). \end{gathered} \tag{0.6}$$

Thus, in this paper, we shall only deal with the simplest isoformal families represented by (0.5). This is because we want to concentrate our attension on making clear the mechanism of solving the R-H-B problem (the inverse and moduli problem) for the simplest isoformal families, for which the direct problem becomes easy to handle, and because consideration of such isoformal families are sufficient for application to the asymptotic analysis of differential equations containing a spectral parameter.

Now we shall state our main results. Just after that, we shall roughly explain the notation and terminology apearing in the statement of the main results.

THEOREM I (see Theorem 14.2). *Let $P\in\mathfrak{gl}(n, \boldsymbol{C})$ be a semi-simple matrix. Then we have the following commutative diagram of sheaves over $\boldsymbol{R}^1$:*

$$\begin{tikzcd} Diff_{\infty,P} \arrow[r, "s"] & Stokes_{\infty,P} \\ Diff_P^{(0)} \arrow[u, "i"] \arrow[ur, "t"'] & \end{tikzcd} \tag{0.7}$$

where i is a natural inclusion map and s is a map defined by assigning an element $D-q\in Diff_{\infty,P}$ to a Stokes phenomenon with respect to a parameter k which arises from the asymptotic behaviours of solutions of the differential equation $DW=qW$. Moreover, what is most important, the map t is surjective.

THEOREM II (see Theorem 16.2). *Suppose that $P\in\mathfrak{gl}(n, \boldsymbol{C})$ is a semi-simple matrix and $A\in GL(n, \boldsymbol{C})$ is a matrix such that $A^N=$*id. *and $\omega P=APA^{-1}$, where* id. *is the identity matrix in $GL(n, \boldsymbol{C})$ and ω is a primitive N-th root of unity. Then we have the following commutative diagram of sheaves over $\boldsymbol{R}^1$:*

$$\begin{tikzcd} Diff_{\infty,P,A\text{-}cyclic} \arrow[r, "s"] & Stokes_{\infty,P,A\text{-}cyclic} \\ Diff_{P,A\text{-}cyclic}^{(0)} \arrow[u, "i"] \arrow[ur, "t"'] & \end{tikzcd} \tag{0.8}$$

where i is a natural inclusion map and s is a map defined by assigning an element $D-q\in Diff_{\infty,P,A\text{-}cyclic}$ to an A-cyclic Stokes phenomenon with respect to a parameter k which arises from the asymptotic behaviours of cyclic solutions of the differential equation $DW=qW$. Moreover, what is most important, the map t is surjective.

THEOREM III (see Theorem 17.4). *We have the following commutative diagram of sheaves over $\boldsymbol{R}^1$.*

$$\begin{tikzcd} Diff_{\infty,P,A\text{-}cyclic} \arrow[r, "s"] & Stokes_{\infty,P,A\text{-}cyclic} \\ Diff^{(\mathfrak{m})}_{P,A\text{-}cyclic} \arrow[u, "i"] & \\ diff \arrow[u, "i"] \arrow[uur, "t"'] & \end{tikzcd} \tag{0.9}$$

where the maps s and i are the same as in Theorem II *and the map t is defined by assigning an element $L \in diff$ to an A-cyclic Stokes phenomenon with respect to a spectral parameter k which arises from the asymptotic behaviours of solutions of the differential equation*

$$Lf = k^n f \qquad (\text{see } (0.11)). \tag{0.10}$$

Here, $\mathfrak{m}=(0, 1, \cdots, n-1)$ *and, moreover, P and A are n-by-n matrices given by*

$$P=\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}, \qquad A=\begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{n-1} \end{pmatrix},$$

ω: *a primitive n-th root of unity.*

Furthermore, what is most important, the map t is surjective.

We shall roughly explain the notation and terminology appearing in the statement of the above theorems, whose precise definitions will be given in the subsequent sections. We denote by $\mathcal{F} \longrightarrow X$ a sheaf $\mathcal{F}$ over a topological space X. In this paper X is one of $\boldsymbol{R}^1, \boldsymbol{P}^1, \boldsymbol{P}^1_\infty, \boldsymbol{R}^1\times S^1$, $\boldsymbol{R}^1\times\boldsymbol{P}^1$ or $\boldsymbol{R}^1\times\boldsymbol{P}^1_\infty$, where $\boldsymbol{P}^1_\infty$ is the real blow-up of the projective line at the infinity.

$\mathfrak{a} \longrightarrow \boldsymbol{R}^1$: the sheaf of germs of real analytic functions over $\boldsymbol{R}^1$.
$\mathfrak{a}\{1/k\} \longrightarrow \boldsymbol{R}^1$: the sheaf of germs of convergent power series in $1/k$ containing a parameter x (see §1 for definition)
$Diff_{\infty,P} \longrightarrow \boldsymbol{R}^1$: the sheaf of germs of differential operators defined by

$$Diff_{\infty,P}=\left\{D-q(x, k); q(x, k) \in \mathfrak{gl}(n, \mathfrak{a}\{1/k\})\right\},$$

where $D=\partial-k\,\mathrm{ad}(P)$ is a derivation (see Definition 4.2).

$\mathit{Diff}_{\infty,P,A\text{-}cyclic}\longrightarrow\boldsymbol{R}^1$: the subsheaf of $\mathit{Diff}_{\infty,P}$ whose elements admit a certain action of a cyclic group $\boldsymbol{Z}_N\simeq\langle A\rangle$, called A-cyclic action (see Definition 5.2).

$\mathit{Diff}_P^{(\mathfrak{m})}\longrightarrow\boldsymbol{R}^1$: the sheaf of germs of certain type of partial connections on the vector bundle $\boldsymbol{E}_{\mathfrak{m}}$ over $\boldsymbol{R}^1\times\boldsymbol{P}^1$, where, for $\mathfrak{m}=(m_1,\cdots,m_n)$, $m_1\leqq\cdots\leqq m_n$, $m_j\in\boldsymbol{Z}$,

$$\boldsymbol{E}_{\mathfrak{m}}=pr^*\mathcal{O}(m_1)\oplus\cdots\oplus\mathcal{O}(m_n).$$

Here $\mathcal{O}(m)$ denotes the line bundle over $\boldsymbol{P}^1$ with degree m and $pr:\boldsymbol{R}^1\times\boldsymbol{P}^1\longrightarrow\boldsymbol{R}^1$ is the projection (see Definition 11.4).

$\mathit{Diff}_{P,A\text{-}cyclic}^{(\mathfrak{m})}\longrightarrow\boldsymbol{R}^1$: the sheaf of germs of a certain type of partial connections on $\boldsymbol{E}_{\mathfrak{m}}$ admitting A-cyclic action (see Definition 15.4).

$\mathit{diff}\longrightarrow\boldsymbol{R}^1$: the sheaf of germs of single n-th order linear ordinary differential operators, i.e., the sheaf associated with the presheaf

$$\boldsymbol{R}^1\supset J\longmapsto\mathit{diff}(J)=\{L=\partial^n+a_1(x)\partial^{n-1}+\cdots+a_n(x);\ a_j(x)\in\mathfrak{a}(J),\ j=1,\cdots,n\}, \tag{0.11}$$

where $\partial=d/dx$ (see Definition 17.2).

$\mathcal{A}\longrightarrow\boldsymbol{R}^1\times S^1$: the sheaf of germs of asymptotically developable functions containing a parameter, where $\boldsymbol{R}^1$ denotes the parameter space (see Definition 1.2).

$GL(n,\mathcal{A})_{\mathrm{id}}\longrightarrow\boldsymbol{R}^1\times S^1$: the sheaf of n-by-n matrices with entries in $\mathcal{A}$ which are asymptotic to the identity matrix.

$\mathit{Stokes}_{\infty,P}\longrightarrow\boldsymbol{R}^1$: this sheaf is defined by (see Definition 8.1)

$$\mathit{Stokes}_{\infty,P}=\mathrm{Image}\,[\mathcal{R}^1\pi(GL(n,\mathcal{A})_{\mathrm{id}}\cap\mathcal{K}erD)\xrightarrow{\text{incl.}}\mathcal{R}^1\pi(GL(n,\mathcal{A})\cap\mathcal{K}erD)],$$

where $\pi:\boldsymbol{R}^1\times S^1\longrightarrow\boldsymbol{R}^1$ is the projection and $D=\partial-k\,\mathrm{ad}(P)$ is a derivation (see Notation 4.1).

$\mathit{Stokes}_{\infty,P,A\text{-}cyclic}\longrightarrow\boldsymbol{R}^1$: a subsheaf of $\mathit{Stokes}_{\infty,P}$ which admits an action of the cyclic group $\boldsymbol{Z}_n\simeq\langle A\rangle\simeq\langle\omega\rangle$ of order n (see Definition 9.1).

The point in the above theorems is as follows. The sheaves of differential operators $\mathit{Diff}_{\infty,P}$, $\mathit{Diff}_{\infty,P,A\text{-}cyclic}$ and the sheaves of Stokes phenomena $\mathit{Stokes}_{\infty,P}$, $\mathit{Stokes}_{\infty,P,A\text{-}cyclic}$ are *local object* with respect to a parameter k which make sense only around $k=\infty$. On the other hand, the sheaves of partial connections $\mathit{Diff}_P^{(\mathfrak{m})}$, $\mathit{Diff}_{P,A\text{-}cyclic}^{(\mathfrak{m})}$ and the sheaves of differential operators *diff* are *global objects* with respect to a parameter k which are defined for all $k \in \boldsymbol{P}^1$. We have natural correspondences between these two types of local objects which assign an operator in $\mathit{Diff}_{\infty,P}$ (resp. $\mathit{Diff}_{\infty,P,A\text{-}cyclic}$) to its Stokes phenomenon in $\mathit{Stokes}_{\infty,P}$ (resp. $\mathit{Stokes}_{\infty,P,A\text{-}cyclic}$). Our theorems assert that these correspondences are surjective and, moreover, in certain cases, all Stokes phenomena are realized even by elements of the global objects mentioned above. We note that $\mathit{Diff}_{\infty,P}$, $\mathit{Diff}_{\infty,P,A\text{-}cyclic}$ and *diff* are much more restricted than $\mathit{Diff}_P^{(m)}$ and $\mathit{Diff}^{(m)}_{P,A\text{-}cyclic}$ because the formers are global objects and the latters are local objects. We can regard the formers are the "Birkhoff's canonical forms" of the latters.

In the present paper the word "parameter" is used in two different senses. If we consider a differential equation (with space variable x) containing a parameter k, then we regard k as a "parameter". On the contrary, if we consider a Stokes phenomenon with respect to a parameter k, then we regard the space variable x as a "parameter" (especially as a *deformation parameter* of certain holomorphic vector bundles over $\boldsymbol{P}^1$ associated with a Stokes phenomenon).

As mentioned earlier, the author writes this paper expecting to apply its results (especially Theorem III) to the spectral theory of linear ordinary differential equations. For this purpose, it would be preferable to work with the sheaf $\mathcal{E} \longrightarrow \boldsymbol{R}^1$ of germs of smooth functions rather than the real analytic one $\mathfrak{a} \longrightarrow \boldsymbol{R}^1$, namely, to deal with a differential equation

$$\{\partial^n + a_1(x)\partial^{n-1} + \cdots + a_n(x)\} f = k^n f$$

with coefficients $a_1(x), \cdots, a_n(x)$ in the sheaf $\mathcal{E}$ rather than $\mathfrak{a}$. In fact, adoption of the sheaf $\mathcal{E}$ instead of $\mathfrak{a}$ would still turn out to be all right, but demand some technical modifications which would make an exposition of our paper much more illegible. Thus we shall work with the sheaf $\mathfrak{a}$.

The outline of this paper is as follows. In §1 we introduce the sheaf $\mathcal{A} \longrightarrow \boldsymbol{R}^1 \times S^1$ of germs of asymptotically developable functions containing a parameter, define other related sheaves and state some preliminary properties of them. In §2 we state a version of the Sibuya-Malgrange theorem (Theorem 2.5, cf. Sibuya [30], Malgrange [24], Majima [21]) for our sheaf $\mathcal{A}$ following Malgrange's method. In §3 we establish another version of the Sibuya-Malgrange theorem (Theorem 3.4) when a cyclic group Z_N acts on the sheaves with which we are concerned. A key is Lemma 3.6 which is an analog of the Hilbert Satz 90 (see also Messing and Sibuya [26]). In § 4 we define a sheaf $Diff_\infty \longrightarrow \boldsymbol{R}^1$, a collection of certain differential operators containing a parameter k which make sense only around $k=\infty$, i.e. a *local object* with respect to k, and show the existence of a formal solution of a differential equation in $Diff_\infty$, (Theorem 4.5). In § 5 we define a sheaf $Diff_{\infty,A\text{-}cyclic} \longrightarrow \boldsymbol{R}^1$, a subsheaf of $Diff_\infty$ which admits an action of the group $\langle A\rangle \simeq Z_N$, and show the existence of A-cyclic formal solution for its elements (Theorem 5.3). In § 6 (resp. § 7) we prove the existence of a fundamental system of (resp. A-cyclic) analytic solutions asymptotic to a given (resp. A-cyclic) formal solution (Theorem 6.1 and 7.1). In the non-cyclic case (§ 6) we follow the standard method (see e.g. Wasow [35]) to prove such an existence theorem. In § 8 (resp. § 9) we define a sheaf $Stokes_\infty \longrightarrow \boldsymbol{R}^1$ (resp. $Stokes_{\infty,A\text{-}cyclic} \longrightarrow \boldsymbol{R}^1$) of germs of (resp. A-cyclic) Stokes phenomena and a sheaf $Gauge_\infty \longrightarrow \boldsymbol{R}^1$ (resp. $Gauge_{\infty,A\text{-}cyclic} \longrightarrow \boldsymbol{R}^1$) of germs of (resp. A-cyclic) gauge transformation group acting on $Diff_\infty$ (resp. $Diff_{\infty,A\text{-}cyclic}$). Then we establish a bijection

$$
(0.3) \qquad \begin{gathered} Gauge_\infty \backslash Diff_\infty \xrightarrow{\sim} Stokes_\infty \\ (\text{resp. } Gauge_{\infty,A\text{-}cyclic} \backslash Diff_{\infty,A\text{-}cyclic} \xrightarrow{\sim} Stokes_{\infty,A\text{-}cyclic}) \end{gathered}
$$

by utilizing the Sibuya-Malgrange type theorem (of cyclic version), (Theorem 8.4 and 9.3). Note that sheaves so far defined are local objects making sense only around $k=\infty$.

The sheaf $Diff_\infty$, which is identified, as a set, with $\mathfrak{gl}(n, \mathfrak{a}\{1/k\})$ (see Definition 4.2), is so "large" that the following problem naturally arises: *Find a subset* $\mathcal{X} \subset Diff_\infty$ *as "small" as possible so that*

$$
(0.4) \qquad Gauge_\infty \cdot \mathcal{X} = Diff_\infty
$$

holds, (Problem 11.1). If this is solved, then, in view of the bijection (0.3), we can conclude that an arbitrary Stokes phenomenon arises from a differential equation of the class $\mathcal{X}$, much smaller than $Diff_\infty$. The latter half of this paper is devoted to this problem and its A-cyclic version (§ 11~§ 17). In § 11 we formulate this problem precisely. Our choice of the subset $\mathcal{X}$ is as follows: Let $\boldsymbol{E}_{\mathfrak{m}} \longrightarrow \boldsymbol{R}^1 \times \boldsymbol{P}^1$ be a certain vector bundle such that $\boldsymbol{E}_{\mathfrak{m}}|_{\{x\}\times \boldsymbol{P}^1}$ $(x \in \boldsymbol{R}^1)$ is a holomorphic vector bundle over $\boldsymbol{P}^1$, where $\mathfrak{m}$ is a parameter specifying the type of vector bundle (an n-tuple of Chern classes). We define $\mathcal{X} = Diff^{(\mathfrak{m})}$ to be a family of partial connections acting on sections of $E_{\mathfrak{m}}$ whose restriction to $E_{\mathfrak{m}}|_{R^1\times(\text{a neighbourhood of } k=\infty \text{ in } P^1)}$ are identified with elements of $\mathit{Diff}^{(\mathfrak{m})}$. Then $\mathit{Diff}^{(\mathfrak{m})}$ can be regarded as a subsheaf of Diff_∞ in a natural way. Contrary to Diff_∞, $\mathit{Diff}^{(\mathfrak{m})}$ is a *global object*, so the latter is much smaller than the former. In fact, their "sizes" are

$$\mathit{Diff}_\infty \simeq \mathfrak{a}^\infty,\ \mathit{Diff}^{(\mathfrak{m})} \simeq \mathfrak{a}^l \text{ for some } l < +\infty,$$

(see Remark 11.9). In § 12 we define the sheaf $\mathit{Stokes}^{(\mathfrak{m})} \longrightarrow R^1$ of global Stokes phenomena and the sheaf $\mathit{Gauge}^{(\mathfrak{m})} \longrightarrow R^1$ of global gauge transformation group, where an element of $\mathit{Stokes}^{(\mathfrak{m})}$ is a cohomology class defined over $J\times P^1_\infty$ with J an open set in R^1. We establish the injection $\mathit{Gauge}^{(\mathfrak{m})}\backslash\mathit{Diff}^{(\mathfrak{m})} \longrightarrow \mathit{Stokes}^{(\mathfrak{m})}$ (not surjection). After a preparation of some results from deformation theory for holomorphic vector bundles over P^1 (§ 13), we shall establish the identity:

$$\mathit{Gauge}_\infty \cdot \mathit{Diff}^{(0)} = \mathit{Diff}_\infty \qquad \text{(Theorem 14.2)}$$

in § 14. § 15–17 is devoted to the A-cyclic version of the above problem. We formulate the problem in § 15 and establish in § 16 the following equality:

$$\mathit{Gauge}_{\infty,A\text{-}cyclic} \cdot \mathit{Diff}^{(0)}_{A\text{-}cyclic} \simeq \mathit{Stokes}_{\infty,A\text{-}cyclic}.$$

Finally, in § 17, under the condition (0.1) we establish

$$\mathit{Gauge}_{\infty,A\text{-}cyclic} \cdot \mathit{diff} \simeq \mathit{Stokes}_{\infty,A\text{-}cyclic},$$

which contains the following equality as a simple corollary,

$$\mathit{Gauge}_{\infty,A\text{-}cyclic} \cdot \mathit{Diff}^{(\mathfrak{m})} \simeq \mathit{Stokes}_{\infty,A\text{-}cyclic},$$

where $\mathfrak{m} = (0, 1, \cdots, n-1)$.

§ 1. The sheaf of germs of asymptotically developable functions containing a parameter

The sheaf $\mathcal{A} \longrightarrow S^1$ of germs of asymptotically developable functions of one variable was defined essentially by Sibuya [30] and definitively by Malgrange [24]. In a similar manner, we shall define the sheaf $\mathcal{A} \longrightarrow R^1\times S^1$ of germs of asymptotically developable functions containing a parameter, where $R^1 = \{x\}$ denotes a parameter space. In this paper we always assume that an asymptotic expansion is uniform with respect to a parameter on any compact subset, so we do not indicate it explicitly in what follows. Although the concept of uniform asymptotic expansion of a function containing a parameter has been developed and used by several authors (see e.g. Wasow [35], Majima [21]), it is still worthwhile to define it in a manner suitable for our aim.

As usual, let $\mathcal{O}$ be the sheaf of germs of holomorphic functions over any complex spaces, and $\mathfrak{a}\longrightarrow\boldsymbol{R}^1$ be the sheaf of germs of real analytic functions over $\boldsymbol{R}^1$, i.e. $\mathfrak{a}$ is the sheaf associated with the presheaf $\boldsymbol{R}^1\supset J\longmapsto\mathfrak{a}(J)$ with

$$\mathfrak{a}(J)=\varinjlim_{V\supset J}\mathcal{O}(V),$$

where V is a complex neighbourhood of J, and the inductive limit is taken in terms of the natural restriction mappings. Let $\mathfrak{a}[[t]]\longrightarrow\boldsymbol{R}^1$ be defined to be the sheaf associated with the presheaf

$$\boldsymbol{R}^1\supset J\longmapsto\mathfrak{a}[[t]](J)=\varinjlim_{V\supset J}\mathcal{O}(V)[[t]],$$

where $\mathcal{O}(V)[[t]]$ denotes the ring of formal power series with coefficients in the ring $\mathcal{O}(V)$ and the inductive limit is as above. The sheaf $\mathfrak{a}[[t]]$ will be called the *sheaf of formal power series containing a parameter*. Moreover let $\mathfrak{a}\{t\}\longrightarrow\boldsymbol{R}^1$ be the sheaf associated with the presheaf

$$\boldsymbol{R}^1\supset J\longmapsto\mathfrak{a}\{t\}(J)=\varinjlim_{U\supset J,V\ni 0}\mathcal{O}(U\times V),$$

where the variable t is regarded as a (complex) coordinate at the origin in V. The sheaf $\mathfrak{a}\{t\}$ will be called the *sheaf of convergent power series containing a parameter*. Note that the sheaves $\mathfrak{a}[[t]]$ and $\mathfrak{a}\{t\}$ are defined so that their elements depend "uniformly" on the parameter $x\in\boldsymbol{R}^1$. Indeed, for an element $f=\sum c_m(x)t^m\in\mathfrak{a}[[t]]$, all the (real analytic) coefficients $c_m(x)$ are analytically continuable to a *common* complex neighbourhood and, if $f\in\mathfrak{a}\{t\}$, then it converges uniformly in x in this *common* neighbourhood.

Remark 1.1. The sheaves $\mathfrak{a}[[t]]$, $\mathfrak{a}\{t\}$ and the projection $\pi:\boldsymbol{R}^1\times S^1\longrightarrow\boldsymbol{R}^1$ induce the sheaves $\pi^*\mathfrak{a}[[t]]\longrightarrow\boldsymbol{R}^1\times S^1$ and $\pi^*\mathfrak{a}\{t\}\longrightarrow\boldsymbol{R}^1\times S^1$. In spite of an abuse of notations, but for simplicity, we also denote the latters by $\mathfrak{a}[[t]]$ and $\mathfrak{a}\{t\}$ respectively.

Now we define the sheaf of germs of asymptotically developable functions containing a parameter. Let us set

$$S^1=\{t\in \boldsymbol{C};\ |t|=1\},$$
$$S(r,U)=\{t\in \boldsymbol{C}\setminus\{0\};\ |t|<r,\ t/|t|\in U\},$$

where $r>0$ and U is an open set in S^1.

DEFINITION 1.2 ($\mathcal{A}\longrightarrow R^1\times S^1$). Let J be an open set in $\boldsymbol{R}^1$, V a complex neighbourhood of J, r a positive number and U an open set in S^1. We say $f(x,t)\in\mathcal{A}(V,r,U)$ iff the following conditions are fulfilled.
(i) $f(x,t)\in\mathcal{O}(V\times S(r,U))$,
(ii) There exists a formal power series $\hat{f}=\sum_{m=0}^{\infty}f_m(x)t^m\in\mathcal{O}(V)[[t]]$ such that, for arbitrary $M\in N$, an uniform estimate

$$|f(x,t)-\sum_{m=0}^{M}f_m(x)t^m|\leqq C_M|t|^{M+1},$$

holds for $(x,t)\in V\times S(r,V)$, where C_M is a constant depending only on M.

Then we say that $f(x,t)$ is asymptotically developable to $\hat{f}$ in $V\times S(r,V)$ as t tends to zero, and denote this situation by

$$f(x,t)\sim\hat{f}=\sum_{m=0}^{\infty}f_m(x)t^m\quad (S(r,V)\ni t\longrightarrow 0,\ x\in V).$$

The formal power series $\hat{f}$ is called the asymptotic expansion of $f(x,t)$. Next, let us set

$$\mathcal{A}(J\times U)=\varinjlim_{r>0,\,V\supset J}\mathcal{A}(V,r,U),$$

where the inductive limit is taken in terms of the natural restriction map $\mathcal{A}(V',r',U)\longrightarrow\mathcal{A}(V,r,U)$ with $r'>r>0$ and $V'\supset V\supset J$, complex neighbourhoods of J. Since the set of all subsets in $\boldsymbol{R}^1\times S^1$ of the form $J\times U$ with J open in $\boldsymbol{R}^1$ and U open in S^1 forms a basis of open sets in $\boldsymbol{R}^1\times S^1$, the correspondence $J\times U\longmapsto\mathcal{A}(J\times U)$ defines a presheaf over $\boldsymbol{R}^1\times S^1$ which satisfies the sheaf condition. The associated sheaf will be denoted by

$$\mathcal{A}\longrightarrow\boldsymbol{R}^1\times S^1,$$

and called the *sheaf of germs of asymptotically developable functions containing a parameter*. For an element $f\in\mathcal{A}(J\times U)$, its asymptotic expansion $\hat{f}\in\mathfrak{a}[[t]](J)$ is uniquely determined. This situation is denoted by

$$f\sim\hat{f}\quad(U\ni t\longrightarrow 0,\, x\in J).$$

REMARK 1.3. $\qquad \mathfrak{a}\{t\}(J)=\mathcal{A}(J\times S^1).$

PROPOSITION 1.4 (Wasow [Theorem 9.4, 35]). *If V and V' are complex neighbourhoods of an open set $J\subset \boldsymbol{R}^1$ and $V'\Subset V$ (relatively compact), then the following diagram is well-defined and commutative*:

$$\begin{array}{ccc}
\mathcal{A}(V,r,U) & \xrightarrow{\text{asympt.}} & \mathcal{O}(V)[[t]] \\
\partial\big\downarrow & & \big\downarrow\partial \\
\mathcal{A}(V',r,U) & \xrightarrow{\text{asympt.}} & \mathcal{O}(V')[[t]],
\end{array}$$

where $\partial=d/dx$ is a differentiation with respect to the parameter.

COROLLARY 1.5. *By taking an inductive limit, we obtain from Proposition* 1.4 *the following commutative diagram of sheaves*:

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\text{asympt.}} & \mathfrak{a}[[t]] \\
\partial\big\downarrow & & \big\downarrow\partial \\
\mathcal{A} & \xrightarrow{\text{asympt.}} & \mathfrak{a}[[t]]
\end{array}$$

PROPOSITION 1.6 (Borel-Ritt type theorem, Wasow [Theorem 9.6, 35]). *Let V and V' be complex neighbourhoods of an open set $J\subset\boldsymbol{R}^1$ with $V'\Subset V$. Let $r>0$ and U be open in S^1. Then, for an arbitrary formal power series $\hat{f}(x,t)=\sum_{m=0}^{\infty}f_m(x)t^m\in\mathcal{O}(V)[[t]]$, there exists an element $f(x,t)\in\mathcal{A}(V',r,U)$ such that*

$$f(x,t)\sim\hat{f}(x,t)\quad(S(r,U)\ni t\longrightarrow 0,\, x\in V').$$

COROLLARY 1.7 (Borel-Ritt short exact sequence). *From Proposition* 1.6, *there is induced a short exact sequence of sheaves*

$$1\longrightarrow GL(n,\mathcal{A})_{\mathrm{id}}\xrightarrow{\text{incl.}}GL(n,\mathcal{A})\xrightarrow{\text{asympt.}}GL(n,\mathfrak{a}[[t]])\longrightarrow 1,$$

where

$$GL(n, \mathcal{A})_{\mathrm{id}}=\{F \in GL(n, \mathcal{A});\, F \sim \textit{the identity matrix in } GL(n, \boldsymbol{C})\}.$$

REMARK 1.8. Corollary 1.5 and 1.7 are fundamental for asymptotic analysis for functions containing a parameter. Recall that a key to Proposition 1.4 is the Cauchy's inequality. Thus, if we defined a sheaf $\mathcal{A} \longrightarrow \boldsymbol{R}^1 \times S^1$ (smoothly dependent on a parameter) naively by using the sheaf $\mathcal{E} \longrightarrow \boldsymbol{R}^1$ instead of $\mathfrak{a} \longrightarrow \boldsymbol{R}^1$, then Corollary 1.5 would break down. Hence, when we discuss it in a *smooth* category, we must modify the definition of $\mathcal{A}$ so that Corollary 1.5 and Corollary 1.7 are simultaneously valid. It is one of the difficulties in working with $\mathcal{E}$, which, in fact, can be overcome. However, we shall not go further to this matter (see § 0).

§ 2. Sibuya-Malgrange theorem for the sheaf $\mathcal{A} \longrightarrow \boldsymbol{R}^1 \times S^1$

Sibuya [30] and Malgrange [24] proved a kind of vanishing theorem which plays a central role in asymptotic analysis, (see also Majima [21]). They used it, among other things, for a study of a classification problem of systems of linear ordinary differential equations with an irregular singular point, or the Riemann-Hilbert-Birkhoff problem of local version concerning Stokes phenomena. In this section, following the idea of Malgrange [24], we shall state a similar theorem for our sheaf $\mathcal{A} \longrightarrow \boldsymbol{R}^1 \times S^1$, which we shall utilize for a study of a classification problem of systems (or single higher-order) linear ordinary differential equations containing a singular parameter, or the Riemann-Hilbert-Birkhoff problem of local version concerning Stokes phenomena with respect to a parameter.

To state the theorem we make some preparations. We start with defining the sheaf $\mathcal{D} \longrightarrow \boldsymbol{R}^1 \times S^1$ in a similar manner as in the sheaf $\mathcal{A} \longrightarrow \boldsymbol{R}^1 \times S^1$.

DEFINITION 2.1 ($\mathcal{D} \longrightarrow \boldsymbol{R}^1 \times S^1$). Let J be an open set in $\boldsymbol{R}^1$, V a complex neighbourhood of J, r a positive number and U an open set in S^1. We say $f(x, t) \in \mathcal{D}(V, r, U)$ iff the following conditions are fulfilled.

(i) $f \in C^{\infty}(V \times S(r, U))$,
(ii) $f(\cdot, t) \in \mathcal{O}(V)$ for arbitrary fixed $t \in S(r, U)$,
(iii) There exists a formal power series

$$\hat{f}(x, t)=\sum_{m, m'=0}^{\infty} f_{m, m'}(x) t^m \bar{t}^{m'} \in \mathcal{O}(V)[[t, \bar{t}]]$$

such that, for arbitrary $M\in N$, a uniform estimate

$$|f(x, t)-\sum_{m+m'\leqq M} f_{m,m'}(x)t^{m}\bar{t}^{m'}|\leqq C_{M}|t|^{M+1}$$

holds for every $(x, t)\in V\times S(r, U)$, where C_{M} is a constant depending only on M.

We call $\hat{f}(x, t)\in\mathcal{O}(V)[[t,\bar{t}]]$ the *asymptotic expansion* of the function $f(x, t)$ and denote this situation by

$$f(x, t)\sim\hat{f}(x, t)\quad(S(r, U)\ni t\longrightarrow 0, x\in V).$$

Now let us set

$$\mathcal{D}(J\times U)=\varinjlim_{r>0, V\supset J}\mathcal{D}(V, r, U),$$

where we take the inductive limit in the same way as above. Hence we obtain a presheaf $R^{1}\times S^{1}\supset J\times U\longmapsto\mathcal{D}(J\times V)$. The associated sheaf will be denoted by

$$\mathcal{D}\longrightarrow R^{1}\times S^{1}.$$

To an element $f\in\mathcal{D}(J\times U)$ there corresponds an element $\hat{f}(x, t)\in\mathfrak{a}[[t,\bar{t}]](J)$ as its asymptotic expansion. We denote this situation by

$$f\sim\hat{f}\quad(U\ni t\longrightarrow 0, x\in J).$$

Notation 2.2.

$$\mathfrak{gl}(n, \mathcal{D})_{0}=\{f\in\mathfrak{gl}(n, \mathcal{D}); f\sim\text{the zero matrix in } \mathfrak{gl}(n, \mathcal{D})\},$$
$$GL(n, \mathcal{D})_{\mathrm{id}}=\{F\in GL(n, \mathcal{D}); F\sim\text{the unit matrix in } GL(n, \mathcal{D})\}.$$

In what follows, "π" always denotes the projection

$$\pi: R^{1}\times S^{1}\longrightarrow R^{1}.$$

Recall that, from a (non-commutative) sheaf $\mathcal{F}\longrightarrow R^{1}\times S^{1}$, the sheaves $\mathcal{R}^{i}\pi(\mathcal{F})\longrightarrow R^{1}$ $(i=0,1)$ are derived.

DEFINITION 2.3. To state the Sibuya-Malgrange theorem, we define an equivalence relation between two elements in $\Gamma(J, \mathcal{R}^{0}\pi(\mathfrak{gl}(n, \mathcal{D})_{0}))$ as follows, where J is an interval in R^{1}: For A and $B\in\Gamma(J, \mathcal{R}^{0}\pi(\mathfrak{gl}(n, \mathcal{D})_{0}))$, they are said to be equivalent iff there exist Φ and $\Psi\in\Gamma(J, \mathcal{R}^{0}\pi(GL(n, \mathcal{D})))$ such that

(i) $(\partial/\partial\bar{t})\Phi = A\Phi$, $(\partial/\partial\bar{t})\Psi = B\Psi$,
(ii) $\Phi^{-1}\Psi \in \Gamma(J, \mathcal{R}^0\pi(GL(n, \mathcal{D})_{\mathrm{id}}))$.

REMARK 2.4. In what follows, we shall mainly use the variable k instead of t, where we always assume the relation $t = k^{-1}$. Thus $t = 0$ corresponds to $k = \infty$ and $f \sim \hat{f}$ ($U \ni k \longrightarrow \infty, x \in J$) stands for $f \sim \hat{f}$ ($U^{-1} \ni t \longrightarrow 0, x \in J$), where U is open in S^1 and $U^{-1} = \{\xi; \xi^{-1} \in U\}$.

THEOREM 2.5 (Sibuya-Malgrange type theorem). *Let $I \subset \boldsymbol{R}^1$ be a compact interval, then the following commutative diagram of bijections is valid:*

$$\begin{tikzcd}
GI(n, \mathfrak{a}\{1/k\})(I) \backslash GL(n, \mathfrak{a}[[1/k]])(I) \arrow[d, "\beta"'] \arrow[dr, "\alpha"] & \\
\Gamma(I, \mathcal{R}^0\pi(\mathfrak{gl}(n, \mathcal{D})_0))/\sim \arrow[r, "\gamma"'] & \Gamma(I, \mathcal{R}^1\pi(GL(n, \mathcal{A})_{\mathrm{id}}))
\end{tikzcd}$$

Here α, β and γ are maps defined as follows: Let $\mathcal{U} = \{U_i\}_{i \in \mathcal{J}}$ be a various covering of S^1 which may be chosen according to our necessity.
α) For an arbitrary coset $GL(n, \mathfrak{a}\{1/k\})(I) \cdot F(x, k)$, it is shown by Proposition 1.6 *that there exists $\Phi_i(x, k) \in \Gamma(I \times U_i, GL(n, \mathcal{A}))$ $(i \in \mathcal{J})$ such that*

$$\Phi_i(x, k) \sim F(x, k) \quad (U_i \ni k \longrightarrow \infty, x \in I).$$

Then we assign this coset to a cohomology class to which the cocycle

$$\{\Phi_i^{-1}\Phi_j \text{ on } I \times U_i \cap I \times U_j\} \in Z^1(I \times \mathcal{U}, GL(n, \mathcal{A})_{\mathrm{id}} | I \times S^1)$$

belongs. This correspondence does not depend on a choice of $\{\Phi_i\}$ and a representative F of the coset, and defines a map α.
β) For an arbitrary coset $GL(n, \mathfrak{a}\{1/k\})(I) \cdot F(x, k)$, it is shown that there exists a $\Psi(x, k) \in \Gamma(I \times S^1, GL(n, \mathcal{D}))$ such that

$$\Psi(x, k) \sim F(x, k) \quad (S^1 \ni k \longrightarrow \infty, x \in I).$$

Then we assign this coset to an equivalence class defined in Definition 2.2 *to which*

$$A = -\Psi^{-1}(\partial/\partial\bar{t})\Psi \in \Gamma(I, \mathcal{R}^0\pi(\mathfrak{gl}(n, \mathcal{D})_0))$$

belongs. This correspondence does not depend on a choice of Ψ and a representative F of the coset, and defines a map β.

γ) *Let A be a representative of an arbitrary equivalence class in $\Gamma(I, \mathcal{R}^0\pi(\mathfrak{gl}(n, \mathcal{D})_0))/\sim$. Then it is shown that there exist $\Psi_i \in \Gamma(I\times U_i, GL(n, \mathcal{D})_{\mathrm{id}})$ $(i \in \mathcal{I})$ such that*

$$(\partial/\partial\bar{t})\Psi_i = A(x, k)\Psi_i.$$

Note that $\Psi_i^{-1}\Psi_j$ is holomorphic in $k \in U_i \cap U_j$. Then we assign the equivalence class containing A to a cohomology class to which the cocycle

$$\{\Psi_i^{-1}\Psi_j \text{ on } I\times U_i \cap I\times U_j\} \in Z^1(I\times\mathcal{U},\ GL(n, \mathcal{A})_{\mathrm{id}}|I\times S^1)$$

belongs. This correspondence does not depend on a choice of $\{\Psi_i\}$ and a representative A, and defines a map γ.

COROLLARY 2.6. *The statement of Theorem 2.5 is equivalent to the assertion that the inclusion map*

$$\Gamma(I, \mathcal{R}^1\pi(GL(n, \mathcal{A})_{\mathrm{id}})) \xrightarrow{\text{incl.}} \Gamma(I, \mathcal{R}^1\pi(GL(n, \mathcal{A})))$$

is trivial, and the both are valid.

It is valuable for later argument to recall how the maps α and γ are derived.

α) By the Borel-Ritt short exact sequence (Corollary 1.7), we deduce a cohomology long exact sequence (see Giraud [14])

$$\begin{aligned} &1 \longrightarrow 1 \longrightarrow GL(n, \mathfrak{a}\{1/k\})(I) \longrightarrow GL(n, \mathfrak{a}[[1/k]]) \\ &\xrightarrow{\delta} H^1(I\times S^1, GL(n, \mathcal{A})_{\mathrm{id}}|I\times S^1) \longrightarrow H^1(I\times S^1, GL(n, \mathcal{A})|I\times S^1) \longrightarrow \cdots \end{aligned}$$

From the coboundary homomorphism δ, the map

$$\alpha : GL(n, \mathfrak{a}\{1/k\})(I)\backslash GL(n, \mathfrak{a}[[1/k]])(I) \longrightarrow \Gamma(I, \mathcal{R}^1\pi(GL(n, \mathcal{A})_{\mathrm{id}}))$$

is induced. In particular, α is injective. The above long exact sequence also proves the first half of Corollary 2.6.

γ) We have a short exact sequence of sheaves:

$$1 \longrightarrow GL(n, \mathcal{A})_{\mathrm{id}} \xrightarrow{\text{incl.}} GL(n, \mathcal{D})_{\mathrm{id}} \xrightarrow{j} \mathfrak{gl}(n, \mathcal{D})_0 \longrightarrow 0,$$

where the map j is defined by

$$j(\Psi) = \{(\partial/\partial\bar{t})\Psi\}\Psi^{-1} \in \mathfrak{gl}(n, \mathcal{D})_0 \quad \text{for } \Psi \in GL(n, \mathcal{D})_{\mathrm{id}}.$$

This is a fine resolution of $GL(n, \mathcal{A})_{\mathrm{id}}$ and in particular

$$H^1(I\times S^1, GL(n, \mathcal{D})_{\mathrm{id}}|I\times S^1)=\{\text{trivial element}\}.$$

Thus, from the above short exact sequence, we deduce a cohomology long exact sequence

$$1\longrightarrow\Gamma(I\times S^1, GL(n, \mathcal{D})_{\mathrm{id}}|I\times S^1)\xrightarrow{j}\Gamma(I\times S^1, \mathfrak{gl}(n, \mathcal{D})_0|I\times S^1)$$
$$\xrightarrow{\delta}H^1(I\times S^1, GL(n, \mathcal{A})_{\mathrm{id}}|I\times S^1)\longrightarrow 1.$$

The coboundary homomorphism δ induces a map

$$\gamma:\Gamma(I, \mathcal{R}^0\pi(\mathfrak{gl}(n, \mathcal{D})_0))/\sim\longrightarrow\Gamma(I, \mathcal{R}^1\pi(GL(n, \mathcal{A})_{\mathrm{id}})).$$

The long exact sequence shows that γ is bijective.

§ 3. Sibuya-Malgrange theorem of cyclic version

In this section we shall establish a version of the Sibuya-Malgrange Theorem (Theorem 3.4) when a cyclic group Z_N acts on sheaves with which we are concerned. The precise definition of this action will be given in Definition 3.1 and 3.3. This theorem will play an important role, among other things, in a classification of *higher-order single* ordinary differential operators of the form: $L=\partial^n+a_1(x)\partial^{n-1}+\cdots+a_n(x)$ (Definition 17.2). The argument in this section remind us of Hilbert Satz 90 (cf. Artin [1]). See also Sibuya and Camaron [12] and Massing and Sibuya [26].

Let $N\geqq 1$ be a fixed integer and A be a fixed matrix such that

$$A\in GL(n, C), \qquad A^N=\mathrm{id}.$$

DEFINITION 3.1. $GL(n, \mathfrak{a}[[1/k]])_{A\text{-cyclic}}$
$$=\{f(x, k)\in GL(n, \mathfrak{a}[[1/k]]); f(x, \omega k)=Af(x, k)A^{-1}\},$$
$$GL(n, \mathfrak{a}\{1/k\})_{A\text{-cyclic}}=GL(n, \mathfrak{a}[[1/k]])_{A\text{-cyclic}}\cap GL(n, \mathfrak{a}\{1/k\}).$$

Here ω is a primitive N-th root of unity.

DEFINITION 3.2. We say that $\mathcal{U}$, a covering of S^1, is *N-cyclic* iff

$$\omega U\in\mathcal{U} \qquad \text{for any } U\in\mathcal{U}.$$

DEFINITION 3.3. Let $\sigma\in\mathcal{R}^1\pi(GL(n, \mathcal{A}))$ and suppose that σ is a germ at $x_0\in R^1$. Then σ is said *A-cyclic* iff there exist an N-cyclic covering $\mathcal{U}$ of S^1 and a cocycle $\{S_{U,V}(x, k)$ on $I\times U\cap I\times V; U, V\in\mathcal{U}\}\in Z^1(I\times\mathcal{U},$

$GL(n, \mathcal{A}))$ which represents the cohomology class σ, where I is an interval containing x_0, such that the following condition holds:

$$S_{\omega U, \omega V}(x, \omega k) = A S_{U,V}(x, k) A^{-1} \quad \text{in } I \times U \cap I \times V.$$

We employ the following notations

$$\mathcal{R}^1\pi(GL(n, \mathcal{A}))_{A\text{-cyclic}} = \{\sigma \in \mathcal{R}^1\pi(GL(n, \mathcal{A})); \sigma \text{ is } A\text{-cyclic}\},$$
$$\mathcal{R}^1\pi(GL(n, \mathcal{A})_{\mathrm{id}})_{A\text{-cyclic}} = \mathcal{R}^1\pi(GL(n, \mathcal{A})_{\mathrm{id}}) \cap \mathcal{R}^1\pi(GL(n, \mathcal{A}))_{A\text{-cyclic}}.$$

THEOREM 3.4 (Sibuya-Malgrange theorem of cyclic version). *Let $I \subset \boldsymbol{R}^1$ be a compact interval, then we have a canonical bijection:*

$$GL(n, \mathfrak{a}\{1/k\})_{A\text{-cyclic}} \backslash GL(n, \mathfrak{a}[[1/k]])_{A\text{-cyclic}} \xrightarrow{\alpha} \Gamma(I, \mathcal{R}^1\pi(GL(n, \mathcal{A})_{\mathrm{id}})_{A\text{-cyclic}}).$$

Here the map α is defined as follows: Let $\mathcal{U}$ be an appropriate N-cyclic covering. For an arbitrary coset $GL(n, \mathfrak{a}\{1/k\})_{A\text{-cyclic}}(I)\cdot F(x, k)$, it is shown that there exist $\Phi_U(x, k) \in \Gamma(I \times U, GL(n, \mathcal{A}))$ $(U \in \mathcal{U})$ such that

$$\Phi_U(x, k) \sim F(x, k) \qquad (U \ni k \longrightarrow \infty, x \in I)$$
$$\Phi_{\omega U}(x, \omega k) = A \Phi_U(x, k) A^{-1} \qquad \text{in } I \times U.$$

We assign this coset to a cohomology class in $\mathcal{R}^1\pi(GL(n, \mathcal{A})_{\mathrm{id}})_{A\text{-cyclic}}$ to which the cocycle

$$\{\Phi_U^{-1}\Phi_V \text{ on } I \times U \cap I \times V\} \in Z^1(I \times \mathcal{U}, GL(n, \mathcal{A})_{\mathrm{id}})_{A\text{-cyclic}}$$

belongs. This correspondence does not depend on a choice of $\{\Phi_U\}$ and a representative F of the coset, and defines a map α.

COROLLARY 3.5. *The statement of Theorem 3.4 is equivalent to the assertion that the inclusion map*

$$\Gamma(I, \mathcal{R}^1\pi(GL(n, \mathcal{A})_{\mathrm{id}})_{A\text{-cyclic}}) \xrightarrow{\text{incl.}} \Gamma(I, \mathcal{R}^1\pi(GL(n, \mathcal{A}))_{A\text{-cyclic}})$$

is trivial, and both are valid.

To prove the theorem, we use the following lemma.

LEMMA 3.6. *Let $I \subset \boldsymbol{R}^1$. If $G(x, k) \in GL(n, \mathfrak{a}\{t\})(I)$ satisfies*

$$G(x, \omega^{N-1}t) G(x, \omega^{n-2}t) \cdots G(x, t) = \mathrm{id}, \tag{3.1}$$

then there exists an $F(x, t) \in GL(n, \mathfrak{a}\{t\})(I)$ such that

(i) $F(x, \omega t)G(x, t)=G(x, 0)F(x, t)$,
(ii) $F(x, 0)=\mathrm{id}$.

PROOF. $$F(x, t)=\frac{1}{N}\sum_{j=1}^{N} G(x, 0)^{N-j}G(x, \omega^{j-1}t)G(x, \omega^{j-2}t)\cdots G(x, t)$$

is an answer. Indeed, the condition (i) is directly checked by using (3.1). Letting $t=0$ in (3.1), we obtain $G(x, 0)^N=\mathrm{id}.$ and hence $F(x, 0)=\mathrm{id}.$, the condition (ii).

PROOF OF THEOREM 3.4 AND COROLLARY 3.5. By using Borel-Ritt theorem and doing some extra work, we can easily show that the map α "defined" in the statement of the theorem is actually well-defined. A composition of α with the inclusion $\Gamma(I, \mathcal{R}^1\pi(GL(n, \mathcal{A})_{\mathrm{id}})_{A\text{-cyclic}} \longrightarrow \Gamma(I, \mathcal{R}^1\pi(GL(n, \mathcal{A})_{\mathrm{id}}))$ coincides with a map which was denoted by α in Theorem 2.6. Hence α is injective. Now we shall show that α is surjective. Given an arbitrary element σ of $\Gamma(I, \mathcal{R}^1\pi(GL(n, \mathcal{A})_{\mathrm{id}})_{A\text{-cyclic}})$, there exists a representative cocycle

$$\{S_{U,V}(x, k) \text{ on } I\times U\cap I\times V\}\in Z^1(I\times\mathcal{U}, GL(n, \mathcal{A})_{\mathrm{id}})_{A\text{-cyclic}},$$

where $\mathcal{U}$ is an N-cyclic covering of S^1. Then, by Theorem 2.5, there exist $\Psi_U(x, k)\in\Gamma(I\times U, GL(n, \mathcal{A})_{\mathrm{id}})$ $(U\in\mathcal{U})$ such that

$$\Psi_U(x, k)S_{U,V}(x, k)=\Psi_V(x, k) \qquad \text{in } I\times U\cap I\times V. \tag{3.2}$$

Since $S_{U,V}\sim\mathrm{id}.$, we have $\lim_{U\ni k\to\infty}\Psi_U(x, k)=\lim_{V\ni k\to\infty}\Psi_V(x, k)$ $(=: \Psi(x)\in GL(n, \mathfrak{a})(I))$. Replacing $\Psi_U(x, k)$ with $\Psi^{-1}(x)\Psi_U(x, k)$, if necessary, we may assume

$$\Psi_U(x, \infty)=\lim_{U\ni k\to\infty}\Psi_U(x, k)=\mathrm{id}.$$

On the other hand, we have

$$\Psi_{\omega U}(x, \omega k)AS_{U,V}(x, k)=\Psi_{\omega U}(x, \omega k)S_{\omega U,\omega V}(x, \omega k)A=\Psi_{\omega V}(x, \omega k)A \quad \text{in } I\times U\cap I\times V. \tag{3.3}$$

Combining (3.2) and (3.3) we find that

$$\Psi_{\omega U}(x, \omega k)A\Psi_U(x, k)^{-1}=\Psi_{\omega V}(x, \omega k)A\Psi_V(x, k)^{-1} \quad \text{in } I\times U\cap I\times V.$$

Hence, an element $G(x, k)\in GL(n, \mathfrak{a}\{1/k\})(I)$ is defined by setting $G(x, k)=\Psi_{\omega U}(x, \omega k)A\Psi_U(x, k)^{-1}$ if $(x, k)\in I\times U$, whence

$$\Psi_{\omega U}(x, \omega k)A=G(x, k)\Psi_U(x, k) \quad (k\in U\in\mathcal{U}, x\in I). \tag{3.4}$$

From this we have $G(x, 0)=A$ since $\Psi_U(x, \infty)=\mathrm{id}$. It follows from the assumption $A^N=\mathrm{id}$ that $\Psi_U(x, k)=\Psi_{\omega^N U}(x, \omega^N k)A^N=G(x, \omega^{N-1}k)$. $\Psi_{\omega^{N-1}U}(x, \omega^{N-1}k)A^{N-1}=\cdots=G(x, \omega^{N-1}k)\cdots G(x, k)\cdot\Psi_U(x, k)$, i.e.,

$$G(x, \omega^{N-1}k)G(x, \omega^{N-2}k)\cdots G(x, k)=\mathrm{id}.$$

Therefore, applying Lemma 3.6 to (3.4), we find that there exists an $F(x, k)\in GL(n, \mathfrak{a}\{1/k\})(I)$ such that

$$F(x, \omega k)G(x, k)=F(x, k).$$

Now let us define $\Phi_U(x, k)\in\Gamma(I\times U, GL(n, \mathcal{A}))$ $(U\in\mathcal{U})$ by

$$\Phi_U(x, k)=F(x, k)\Psi_U(x, k) \qquad \text{in } I\times U.$$

Then we find that

$$\begin{aligned}
\Phi_U(x, k)S_{U,V}(x, k)&=\Phi_V(x, k) \qquad \text{in } I\times U\cap I\times V,\\
\Phi_{\omega U}(x, \omega k)&=F(x, \omega k)\Psi_{\omega U}(x, \omega k)\\
&=AF(x, k)G(x, k)^{-1}\cdot G(x, k)\Psi_U(x, k)A^{-1}\\
&=A\Phi_U(x, k)A^{-1} \qquad \text{in } I\times U.
\end{aligned}$$

Hence the asymptotic expansion $\Phi(x, k)\in GL(n, \mathfrak{a}[[1/k]])(I)$ of $\Phi_U(x, k)$ is independent of $U\in\mathcal{U}$ and A-cyclic. In view of the method of construction, it is evident that α maps the coset $GL(n, \mathfrak{a}\{1/k\})(I)\cdot\Phi(x, k)$ to the cohomology class σ, which shows that α is surjective. Hence the theorem is proved.

§ 4. The sheaf $Diff_\infty \longrightarrow R^1$ and formal solutions

In this section we shall define the sheaf $Diff_\infty \longrightarrow R^1$ of germs of matricial differential operators containing a parameter k which have a simple pole at $k=\infty$ with a given residue P. We also discuss formal solutions of differential equations defined by an elements of $Diff_\infty$. In the following argument, we always keep a matrix P fixed, so that we does not indicate explicitly the dependence on P. We assume that

(4.1) $$P\in\mathfrak{gl}(n, \boldsymbol{C}): \ \text{semi-simple}.$$

Note that, then, for any $\boldsymbol{C}$-algebra R,

$$\mathrm{ad}(P)\in End(\mathfrak{gl}(n, R)): \ \text{semi-simple}.$$

Let us set

$$\mathfrak{gl}(n, R)' = \text{Kernel}[\text{ad}(P) : \mathfrak{gl}(n, R) \longrightarrow \mathfrak{gl}(n, R)],$$
$$\mathfrak{gl}(n, R)'' = \text{Image}[\text{ad}(P) : \mathfrak{gl}(n, R) \longrightarrow \mathfrak{gl}(n, R)].$$

Since $\text{ad}(P)$ is semi-simple, we have

$$\mathfrak{gl}(n, R) = \mathfrak{gl}(n, R)' \oplus \mathfrak{gl}(n, R)'',$$
$$\text{ad}(P)'' = \text{ad}(P)|_{\mathfrak{gl}(n,R)''} \in GL(\mathfrak{gl}(n, R)'').$$

Let

$$\Pi' : \mathfrak{gl}(n, R) \longrightarrow \mathfrak{gl}(n, R)', \quad \Pi'' : \mathfrak{gl}(n, R) \longrightarrow \mathfrak{gl}(n, R)''$$

be projections with respect to the above direct sum decomposition and denote

$$F' = \Pi' F, \quad F'' = \Pi'' F \qquad \text{for } F \in \mathfrak{gl}(n, R).$$

Moreover we note that

$$\mathfrak{gl}(n, R)' \mathfrak{gl}(n, R)'' = \mathfrak{gl}(n, R)'' \mathfrak{gl}(n, R)' = \mathfrak{gl}(n, R)'',$$
$$\mathfrak{gl}(n, R)' : \text{ a sub-algebra of } \mathfrak{gl}(n, R).$$

Notation 4.1 (Derivation D). We define a derivation D acting on a space of n-by-n matrices with entries depending on x and k:

$$D = \partial - k\, \text{ad}(P), \quad \partial = d/dx.$$

DEFINITION 4.2 ($\mathit{Diff}_\infty \longrightarrow \boldsymbol{R}^1$). For $J \subset \boldsymbol{R}^1$, let $f\text{-}\mathit{Diff}_\infty(J)$ and $\mathit{Diff}_\infty(J)$ be the following sets of differential operators:

$$f\text{-}\mathit{Diff}_\infty(J) = \{D - q(x, k);\, q(x, k) \in \mathfrak{gl}(n, \mathfrak{a}[[1/k]])(J)\}$$
$$\mathit{Diff}_\infty(J) = \{D - q(x, k);\, q(x, k) \in \mathfrak{gl}(n, \mathfrak{a}\{1/k\})(J)\}.$$

Since $D - q(x, k)$ is determined uniquely by $q(x, k)$, the both will be identified. Sheaves associated with the presheaves $J \longmapsto f\text{-}\mathit{Diff}_\infty(J)$ and $J \longmapsto \mathit{Diff}_\infty(J)$ are denoted by $f\text{-}\mathit{Diff}_\infty \longrightarrow \boldsymbol{R}^1$ and $\mathit{Diff}_\infty \longrightarrow \boldsymbol{R}^1$ respectively. Note that Diff_∞ depends on the matrix P. Thus we denote it by $\mathit{Diff}_{\infty,P}$ if we have to clarify the dependence on P.

Next we shall define a sheaf to which formal solutions to our differential equations (see (4.2)) will belong.

DEFINITION 4.3.

$$GL_0(n, \mathfrak{a}[[1/k]])=\left\{W=\sum_{m=0}^{\infty} W_m(x)k^{-m} \in GL(n, \mathfrak{a}[[1/k]]);\ W_0(x) \in GL(n, \mathfrak{a})'\right\}.$$

Here, for a C-algebra R, we define

$$GL(n, R)'=GL(n, R) \cap \mathfrak{gl}(n, R)'.$$

LEMMA 4.4. *Let $D-q(x, k) \in f\text{-}Diff_\infty(J)$ and $W(x, k) \in GL(x, \mathfrak{a}[[1/k]])(J)$. If $W(x,k)$ is a formal solution of the differential equation*

$$DW=q(x, k)W, \tag{4.2}$$

then $W(x, k) \in GL_0(n, \mathfrak{a}[[1/k]])(J)$. Moreover, if we develop q and W into formal power series of $1/k$:

$$q(x, k)=\sum_{m=0}^{\infty} q_m(x)k^{-m}, \quad W(x, k)=\sum_{m=0}^{\infty} W_m(x)k^{-m},$$

then we have the following reccurence formulas.

$$W_0 \in \mathfrak{gl}(n, \mathfrak{a})', \quad \partial W_0=q_0' W_0,$$
$$\begin{cases} (\partial-q_0')W_m'=(q_0 W_m'')'+\sum_{\nu=0}^{m-1}(q_{m-\nu}W_\nu)' & (m \geqq 1), \\ \mathrm{ad}(P)'' W_{m+1}''=\partial W_m''-\sum_{\nu=0}^{m}(q_{m-\nu}W_\nu)'' & (m \geqq 0), \end{cases}$$

where the order of successive determination is

$$W_0,\ W_1'',\ W_1',\ W_2'',\ W_2',\ \cdots,\ W_m'',\ W_m',\ \cdots.$$

Transition from W_m' to W_{m+1}'' is unique since $\mathrm{ad}(P)''$ is injective, whereas that from W_m'' to W_M' depends on a value $W_m'(x_0) \in C$ at $x_0 \in J$.

THEOREM 4.5. *For an arbitrary $q(x, k) \in f\text{-}Diff_\infty(J)$, there exists a formal solution of the equation $DW=q(x, k)W$, which belongs to $GL_0(n, \mathfrak{a}[[1/k]])(J)$. Let $W(x, k)$ be an arbitrary formal solution, then the set of all formal solutions coincides with the coset $W(x, k) \cdot GL(n, C[[1/k]])'$. Conversely, for an arbitrary coset of $GL_0(n, \mathfrak{a}[[1/k]])(J)/GL(n, C[[1/k]])'(J)$, let W be a representative of this coset and set $q(x, k)=(DW)W^{-1}$. Then $q(x, k)$ is determined only by the coset, independent of a choice representative W and belongs to $f\text{-}Diff_\infty(J)$. Namely, we have the following bijection:*

$$\begin{array}{ccc} f\text{-}Diff_\infty(J) & \rightarrowtail\!\!\!\twoheadrightarrow & GL_0(n, \mathfrak{a}[[1/k]])(J)/GL(n, \mathfrak{a}[[1/k]])'(J) \\ \Psi & & \Psi \\ q(x,k) & \longmapsto & \textit{set of all formal solutions of } DW=qW. \end{array}$$

PROOF. We shall check the fact that any formal solution is just obtained by multiplying an element of $GL(n, \boldsymbol{C}[[1/k]])'$ on the right of a given formal solution W. Indeed it is clear that $\widetilde{W}=WC$ with $C\in GL(n, \boldsymbol{C}[[1/k]])'$ is again a formal solution. Conversely, for any other formal solution $\widetilde{W}$, there exists a matrix $C=C(x,k)$ such that

$$\widetilde{W}=WC,\ C(x,k)=\sum_{m=0}^{\infty} C_m(x)k^{-m}\in GL(n, \mathfrak{a}[[1/k]])(J)\cap \mathcal{K}erD.$$

Thus, if x_0 is an interior point of J, then we have

$$C(x,k)=C(x_0,k)'+\exp\{k(x-x_0)P\}C(x_0,k)''\exp\{-k(x-x_0)P\}. \tag{4.3}$$

Let $\boldsymbol{C}\langle t_1,\cdots,t_l\rangle$ be the set of formal Laurent series in $t_1,\cdots,t_l$. The above equality make no sense in $\mathfrak{gl}(n, \boldsymbol{C}\langle k\rangle)$, where $x\in J$ being fixed, but make sense as an equality in $\mathfrak{gl}(n, \boldsymbol{C}\langle k, x-x_0\rangle)$. Under this interpretation, it is rewritten as

$$\sum_{n\geqq 0}\sum_{m\leqq 0}\frac{C_{-m}^{(n)}(x_0)}{n!}(x-x_0)^n k^m=\sum_{m\leqq 0}C_{-m}(x_0)'k^m + \sum_{n\geqq 0}\sum_{n-m\geqq 0}D_{n,n-m}(x-x_0)^n k^m, \tag{4.4}$$

where

$$D_{n,m}=\sum_{\nu=0}^{n}\frac{(-1)^\nu}{\nu!(n-\nu)!}P^{n-\nu}C_m(x_0)''P^\nu \qquad (n\geqq m\geqq 0).$$

By comparing the coefficients of $(x-x_0)^n k^m$ $(n\geqq m\geqq 0)$ in (4.4), we find that

$$D_{n,m}=0 \qquad (n\geqq m\geqq 0).$$

On the other hand, we can easily see that $\exp\{t\,\mathrm{ad}(P)\}C_m(x_0)''=\sum_{n=0}^{\infty}D_{n,m}t^n$, whence $\exp\{t\,\mathrm{ad}(P)\}C_m(x_0)''$ is a polynomial in t of degree at most m. Therefore,

$$\mathrm{ad}(P)^{m+1}C_m(x_0)''=(d/dt)^{m+1}\exp\{t\,\mathrm{ad}(P)\}C_m(x_0)''|_{t=0}=0,$$

Since $\mathrm{ad}(P)$ is injective on $\mathfrak{gl}(n, \boldsymbol{C})''$, this shows that $C_m(x_0)''=0$ for $m\geqq 0$

and then $C(x_0, k)''=0$. Hence, by (4.3), $C(x, k)=C(x_0, k)'\in GL(n, \boldsymbol{C}[[1/k]])'$. This proves the theorem.

§ 5. The sheaf $Diff_{\infty,A\text{-}cyclic}\longrightarrow R^1$ and A-cyclic formal solution

In this section, we shall make a similar argument as in § 4, when a Z_N-action is present. We have introduced matrices $A\in GL(n, \boldsymbol{C})$ with $A^N=\mathrm{id}$, (§ 3) and $P\in \mathfrak{gl}(n, \boldsymbol{C})$, semi-simple, (§ 4).

HYPOTHESIS 5.1. We assume that the matrices A and P satisfy the following relation:

$$\omega P=APA^{-1},$$

where ω is a primitive N-th root of unity.

For the derivation D introduced in Notation 4.1, we set

$$D^{\omega}=\partial-\omega k\,\mathrm{ad}(P).$$

Moreover, let i_A be defined by

$$i_A\in GL(\mathfrak{gl}(n, R)),\quad i_AF=AFA^{-1}\ \ (F\in \mathfrak{gl}(n, R)).$$

Then, under the Hypothesis 5.1, we have

$$D^{\omega}=i_ADi_A^{-1}.$$

DEFINITION 5.2.

$f\text{-}Diff_{\infty,A\text{-}cyclic}=\{q(x, k)\in Diff_{\infty};\ q(x, \omega k)=Aq(x, k)A^{-1}\}$,

$Diff_{\infty,A\text{-}cyclic}=Diff_{\infty}\cap f\text{-}Diff_{\infty,A\text{-}cyclic}$,

$GL_0(n, \mathfrak{a}[[1/k]])_{A\text{-}cyclic}=\{W(x, k)\in GL_0(n, \mathfrak{a}[[1/k]]);\ W(x, \omega k)=AW(x, k)A^{-1}\}$,

$GL(n, \boldsymbol{C}[[1/k]])'_{A\text{-}cyclic}=GL(n, \boldsymbol{C}[[1/k]])'\cap GL_0(n, \mathfrak{a}[[1/k]])_{A\text{-}cyclic}$.

For $q\in f\text{-}Diff_{\infty,A\text{-}cyclic}$, if a formal solution W of the equation $DW=qW$ belongs to $GL_0(n, \mathfrak{a}[[1/k]])_{A\text{-}cyclic}$, then W will be called an *A-cyclic* formal solution.

THEOREM 5.3. *Let J be an open interval. For an arbitrary $q(x, k)\in f\text{-}Diff_{\infty,A\text{-}cyclic}(J)$, there exists an A-cyclic formal solution $W(x, k)\in GL_0(n, \mathfrak{a}[[1/k]])_{A\text{-}cyclic}(J)$ of the equation $DW=qW$. As that in Theorem* 4.5, *we have the following* (*well-defined*) *bijection*

$$\begin{array}{ccc} f\text{-}Diff_{\infty,A\text{-}cyclic}(J) & \rightarrowtail\!\!\!\twoheadrightarrow & GL_0(n, \mathfrak{a}[[1/k]])_{A\text{-}cyclic}(J)/GL(n, \boldsymbol{C}[[1/k]])_{A\text{-}cyclic}(J) \\ \Psi & & \Psi \\ q(x, k) & \longmapsto & \text{The set of all } A\text{-cyclic formal solutions of} \\ & & \text{the equation } DW = qW. \end{array}$$

PROOF. It suffices to show the first half of the assertion of the theorem. For an arbitrary element $q(x, k) \in f\text{-}Diff_{\infty,A\text{-}cyclic}(J)$, at any rate, Theorem 4.5 assures the existence of a formal solution $\tilde{W}(x, k) \in GL_0(n, \mathfrak{a}[[1/k]])(J)$ of the equation $D\tilde{W} = q\tilde{W}$. From the recurrence formula in Lemma 4.4, we may assume that $\tilde{W}(x_0, \infty) = \text{id.}$ at an $x_0 \in J$. Replacing k with ωk and taking $D^{\omega} = i_A D i_A^{-1}$ into account, we obtain from the above equation

$$i_A D i_A^{-1} \tilde{W}(x, \omega k) = Aq(x, k)A^{-1}\tilde{W}(x, \omega k) = i_A q(x, k) i_A^{-1}\tilde{W}(x, \omega k).$$

Hence $i_A^{-1}\tilde{W}(x, \omega k)$ is also a formal solution of $DW = q(x, k)W$. It follows from Theorem 4.4 that there exists a matrix $\tilde{C}(k) \in GL(n, \boldsymbol{C}[[1/k]])'$ such that

$$\tilde{W}(x, \omega k) = A\tilde{W}(x, k)\tilde{C}(k)A^{-1}.$$

In particular, by $\tilde{W}(x_0, \infty) = \text{id.}$, we have $\tilde{C}(\infty) = \text{id.}$ Thus, if we set $\hat{C}(k) = \tilde{C}(k)A^{-1}$, then we can rewrite these equalities as follows

$$\tilde{W}(x, \omega k) = A\tilde{W}(x, k)\hat{C}(k), \quad \hat{C}(\infty) = A^{-1}. \tag{5.1}$$

Since $A^N = \text{id.}$, this shows that $\tilde{W}(x, k)A^{-N}\tilde{W}(x, \omega^N k) = A^{1-N}$. $\tilde{W}(x, \omega^{N-1}k)$ $\hat{C}(\omega^{N-1}k) = \cdots = \tilde{W}(x, k)\hat{C}(x)\cdots\hat{C}(\omega^{N-2}k)\hat{C}(\omega^{N-1}k)$, whence

$$\hat{C}(k)\hat{C}(\omega k)\cdots\hat{C}(\omega^{N-2}k)\hat{C}(\omega^{N-1}k) = \text{id.} \tag{5.2}$$

Thus, if we set

$$C(k) = \frac{1}{N}\sum_{j=1}^{N}\hat{C}(k)\cdots\hat{C}(\omega^{j-2}k)\hat{C}(\omega^{j-1}k)A^j \in \mathfrak{gl}(n, \boldsymbol{C}[[1/k]]),$$

then we find that (cf. Lemma 3.6)

$$\begin{aligned} &\hat{C}(k)C(\omega k) = C(k)A^{-1}, \\ &C(\infty) = \text{id.}, \quad \text{i.e. } C(k) \in GL(n, \boldsymbol{C}[[1/k]]). \end{aligned} \tag{5.3}$$

On the other hand, by Hypothesis 5.1, $\hat{C}(k) = \tilde{C}(k)A^{-1}$ and $\tilde{C}(k) \in \mathcal{K}er\,\mathrm{ad}(P)$, we obtain the commutation relation $P\hat{C}(k) = \omega\hat{C}(k)P$. Together with

$PA=\omega^{-1}AP$ (Hypothesis 5.1), this shows that

$$\begin{aligned}P\cdot C(k)&=\frac{1}{N}\sum_{j=1}^{N}P\hat{C}(k)\cdots\hat{C}(\omega^{j-1}k)A^j\\&=\frac{1}{N}\sum_{j=1}^{N}\omega\hat{C}(k)P\hat{C}(\omega k)\cdots\hat{C}(\omega^{j-1})A^j\\&=\cdots=\frac{1}{N}\sum_{j=1}^{N}\omega^j\hat{C}(k)\cdots\hat{C}(\omega^{j-1}k)PA^j\\&=\frac{1}{N}\sum_{j=1}^{N}\omega^j\cdot\omega^{-j}\hat{C}(k)\cdots\hat{C}(\omega^{j-1}k)A^jP\\&=C(k)\cdot P.\end{aligned}$$

This shows that $C(k)\in GL(n, C[[1/k]])'$. Hence Theorem 4.5 implies that $W(x, k)=\tilde{W}(x, k)C(k)$ is also a formal solution of $DW=qW$. Moreover (5.1) and (5.3) implies that $W(x, \omega k)=AW(x, k)A^{-1}$ holds. Therefore $W(x, k)$ is an A-cyclic formal solution to be desired. This completes the proof of the theorem.

§6. Existence of an analytic solution asymptotic to a formal solution

We show in this section that, for any formal solution $W(x, k)\in GL_0(n, \mathfrak{a}[[1/k]])$ of an arbitrary $q(x, k)\in \mathit{Diff}_\infty$, there exists an analytic solution which is asymptotically developable to $W(x, k)$ as $k\longrightarrow\infty$ in an appropriate sector containing an arbitrary direction.

THEOREM 6.1. *Let $J\subset \boldsymbol{R}^1$ be an open interval and $x_0\in J$ be a point. Let $W(x, k)\in GL_0(n, \mathfrak{a}[[1/k]])(J)$ be any formal solution for an arbitrary $q(x, k)\in \mathit{Diff}_\infty(J)$. Then there exist an open interval $I\subset J$ containing x_0, an open covering $\mathcal{U}$ of S^1 and*

$$W_U(x, k)\in\Gamma(I\times U, GL(n, \mathcal{A}))\qquad \text{for } U\in\mathcal{U}$$

such that the following conditions hold:

$$\begin{aligned}&DW_U=q(x, k)W_U &&\text{in } (x, k)\in I\times U,\\&W_U(x, k)\sim W(x, k) &&(U\ni k\longrightarrow\infty,\ x\in I).\end{aligned}$$

We call such a system $\{W_U(x, k); U\in\mathcal{U}\}$ a fundamental system of solutions for $q(x, k)\in \mathit{Diff}_\infty$.

DEFINITION 6.2 (Characteristic sector). Consider the set

$\{k \in \boldsymbol{C} \setminus \{0\}$; real parts of all eigenvalues of $k\,\mathrm{ad}(P)'' \in GL(\mathfrak{gl}(n, \boldsymbol{C})'') \neq 0\}$.

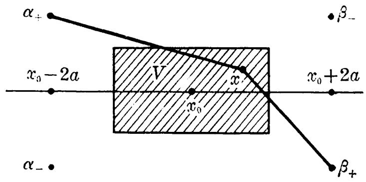
This is clearly a disjoint union of a finite number of open sectors S with vertex at the origin. Each member of these sectors will be called a *characteristic sector*. The set of all characteristic sectors is denoted by $\mathcal{S}$.

PROOF OF THEOREM. Since $q(x, k) \in \mathit{Diff}_\infty(J)$, we may assume that, for sufficiently small $a_0, b_0 > 0$ and a sufficiently large $r > 0$, there exists an open set $\tilde{V}$ in $\boldsymbol{C}$ such that

$$[x_0 - 2a_0, x_0 + 2a_0] \times \sqrt{-1}[-2a_0 b_0, 2a_0 b_0] \subset \tilde{V},$$
$$q(x, k) \in \mathfrak{gl}(n, \mathcal{O}(\tilde{V} \times \{|k| > r\})).$$

Let a and b are numbers such that $0 < a < a_0$ and $0 < b < b_0$, which will be definitively chosen later. Moreover let us set

$$\alpha_\pm = x_0 - 2a(1 \mp \sqrt{-1}b),$$
$$\beta_\pm = x_0 + 2a(1 \mp \sqrt{-1}b),$$
$$V = (x_0 - a, x_0 + a) + \sqrt{-1}(-b, b).$$



Let $\varphi_1 > \varphi_0 > 0$ be defined by $\tan \varphi_1 = 3b$ and $\tan \varphi_0 = 3/b$ respectively, then the following is valid: (See Figure)

$$\pm\varphi_0 \lesseqgtr \arg(x - \alpha_\mp) \lesseqgtr \pm\varphi_1 \qquad (x \in V),$$
$$\pm\varphi_0 \lesseqgtr \arg(\beta_\mp - x) \lesseqgtr \pm\varphi_1 \qquad (x \in V).$$

For $\rho > 0$ and $\xi < \eta$ we define

$$S(\rho; \xi, \eta) = \{k \in \boldsymbol{C}; \xi < \arg k < \eta,\ |k| > \rho\},$$
$$S(\xi, \eta) = S(0; \xi, \eta).$$

In the following argument, we keep an arbitrary characteristic sector $S = S(\theta_0, \theta_1)$ fixed. Let us set

$$S_+ = S(\theta_0 + \varphi_1, \theta_1 + \varphi_0), \quad S_- = S(\theta_0 - \varphi_0, \theta_1 - \varphi_1),$$

$[\alpha_\pm, x]$: a line segment joining $\alpha_\pm$ with x,
$[x, \beta_\pm]$: a line segment joining x with $\beta_\pm$,

for $x\in V$. Then we observe that

$$k(x-t)\in S \text{ if } k\in S_{\pm},\ x\in V,\ t\in[\alpha_{\pm}, x]\cup[x, \beta_{\pm}].$$

Since φ_1 tends to zero as b does, we can take $b=b(S)$ so small that

$$2\varphi_1<\theta_1-\theta_0.$$

In what follows, we keep such a $b=b(S)$ fixed. Then we see that

$$S_+\cup S_-=S(\theta_0-\varphi_0, \theta_1+\varphi_0)\supset S,$$

namely, S is a proper subsector of $S_+\cup S_-$.

We proceed to the second step. Let $\mathfrak{gl}(n, R)''_+$ (resp. $\mathfrak{gl}(n, R)''_-$) be the direct sum of eigenspaces corresponding to eigenvalues of $k\,\mathrm{ad}(P)''\in GL(\mathfrak{gl}(n, R)'')$ $(k\in S)$ whose real parts are positive (resp. negative), where R is a $\boldsymbol{C}$-algebra. Note that $\mathfrak{gl}(n, R)''_{\pm}$ are independent of a choice of $k\in S$ and depend only on S. Moreover,

$$\mathfrak{gl}(n, R)''=\mathfrak{gl}(n, R)''_+\oplus\mathfrak{gl}(n, R)''_-.$$

Now we define projections $\Pi_{\pm}$ as follows:

$$\begin{aligned}\Pi_-:\mathfrak{gl}(n, R)&\longrightarrow\mathfrak{gl}(n, R)'\oplus\mathfrak{gl}(n, R)''_-,\\ \Pi_+:\mathfrak{gl}(n, R)&\longrightarrow\mathfrak{gl}(n, R)''_+,\end{aligned}$$

and we set $S_{\pm}(\rho)=S_{\pm}\cap\{|k|>\rho\}$.

Let $W(x, k)\in\mathfrak{gl}(n, \mathcal{O}(\tilde{V})[[1/k]])(J)$ be an arbitrary formal solution for $q(x, k)\in \mathit{Diff}_{\infty}(J)$. If we take an open set V_1 such that $V\Subset V_1\subset\tilde{V}$, then by Borel-Ritt theorem (Proposition 1.6) we can find $\Psi(x, k)\in GL(V_1\times S_{\pm}(R_0))$ such that

$$\Psi(x, k)\sim W(x, k) \qquad (S_{\pm}(R_0)\ni k\longrightarrow\infty, x\in V_1),$$

where R_0 is a sufficiently large number. Therefore, if we put

$$f(x, k)=D\Psi-q(x, k)\Psi\in\mathfrak{gl}(n, \mathcal{O}(V_1\times S_{\pm}(R_0))),$$

then we obtain by Proposition 1.4,

$$f(x, k)\sim 0 \qquad (S_{\pm}(R_0)\ni k\longrightarrow\infty, x\in V). \tag{6.1}$$

Next we shall show that there exists $\Phi(x, k)\in\mathfrak{gl}(n, \mathcal{O}(V\times S_{\pm}(R_0)))$ such that

$$
\begin{aligned}
&D\Phi = q(x,k)\Phi + f(x,k), \\
&\Phi(x,k) \sim 0 \qquad (S_\pm(R_0) \ni k \longrightarrow \infty,\ x \in V).
\end{aligned} \tag{6.2}
$$

To do this we define an operator L by

$$
\begin{aligned}
(L\Phi)(x,k) = &\int_{\alpha_\pm}^{x} \exp\{k(x-t)\mathrm{ad}(P)\}\Pi_-\Phi(t,k)dt \\
&+ \int_{\beta_\pm}^{x} \exp\{k(x-t)\mathrm{ad}(P)\}\Pi_+\Phi(t,k)dt, \\
&\qquad (x \in V,\ k \in S_\pm(R_0)),
\end{aligned} \tag{6.3}
$$

where the paths of integration are line segments joining $\alpha_\pm$ with x and x with $\beta_\pm$ respectively. Moreover, we define an operator M by

$$M\Phi = L(q(x,k)\Phi).$$

If we set $F = Lf$, then, by (6.1) and (6.3), we see that

$$
\begin{aligned}
&F(x,k) \in \mathfrak{gl}(n, \mathcal{O}(V \times S_\pm(R_0))), \\
&F(x,k) \sim 0 \quad (S_\pm(R_0) \ni k \longrightarrow \infty,\ x \in V).
\end{aligned} \tag{6.4}
$$

We provide $\mathfrak{gl}(n, \mathcal{O}(V \times S_\pm(R)))$ with the supremum norm $\|\cdot\|_R$ $(R > R_0)$ for it to be a Banach space. The lengths of the paths of integration involved in the operator M are proportional to a. So the operator norm of M as a linear operator acting on $\mathfrak{gl}(n, \mathcal{O}(V \times S_\pm(R)))$ admits an estimate:

$$\|M\|_R \leqq C\|q\|_R\, a, \quad (R \geqq R_0),$$

where C is a constant. Let $a = a(S)$ be a fixed number such that $0 < a < 1/(C\|q\|_{R_0})$. Then we have

$$\|M\|_R \leqq \|M\|_{R_0} < 1, \quad (R \geqq R_0).$$

Therefore the integral equation

$$\Phi = M\Phi + F, \quad \Phi \in \mathfrak{gl}(n, \mathcal{O}(V \times S_\pm(R_0)))$$

has a unique solution, for which an estimate

$$\|\Phi\|_R \leqq (1 - \|M\|_{R_0})^{-1}\|F\|_R \quad (R \geqq R_0)$$

holds. Hence (6.4) shows that $\Phi(x,k) \sim 0$ $(S_\pm(R_0) \ni k \longrightarrow \infty,\ x \in V)$, which proves (6.2). Now (6.2) shows that $W_{S_\pm} = \Psi - \Phi$ satisfies

$$W_{S_\pm}(x,k) \in GL(n, \mathcal{O}(V \times S_\pm(R_1))),$$

$$\begin{aligned}&DW_{S_\pm}=q(x,k)W_{S_\pm},\\&W_{S_\pm}(x,k)\sim W(x,k)\quad (S_\pm(R_1)\ni k\longrightarrow\infty,\ x\in V),\end{aligned}$$

for sufficiently large $R_1=R(S)>R_0$.

So far, we have kept an $S\in\mathcal{S}$ fixed. But, from now on, let S run over the set $\mathcal{S}$ of all characteristic sectors. Set

$$I=(x_0-a,\ x_0+a),\quad a=\min\{a(S);\ S\in\mathcal{S}\}.$$

For $S\in\mathcal{S}$, let $U_\pm(S)$ be open sets of S^1 such that $S_\pm(R(S))=S(R(S),U_\pm(S))$, and let $\mathcal{U}$ be the set $\{U_\pm(S);\ S\in\mathcal{S}\}$. Then $\mathcal{U}$ is an open covering of S^1. Finally we define W_U $(U\in\mathcal{U})$ as follows:

$$W_U=W_{S_+}\ (\text{resp. } W_{S_-})\ \text{ if } U=U_+(S)\ (\text{resp. } U_-(S)).$$

Then it is clear that the interval I, the covering $\mathcal{U}$ of S^1 and the system $\{W_U(x,k);\ U\in\mathcal{U}\}$ are what are to be sought. This proves the theorem.

§ 7. A-cyclic analytic solution asymptotic to an A-cyclic formal solution

In this section we assume Hypothesis 5.1 and discuss a similar problem as in § 6 when a cyclic action is present.

THEOREM 7.1. *Let $J\subset \boldsymbol{R}^1$ be an open interval and $x_0\in J$ be a point. Suppose that $W(x,k)\in GL_0(n,\mathfrak{a}[[1/k]])_{A\text{-cyclic}}(J)$ be an arbitrary A-cyclic formal solution for $q(x,k)\in Diff_{\infty,A\text{-cyclic}}(J)$. Then there exist an interval $I\subset J$ containing x_0, an N-cyclic covering $\mathcal{U}$ of S^1 and $W_U(x,k)\in\Gamma(I\times U, GL(n,\mathcal{A}))$ $(U\in\mathcal{U})$ such that*

$$\begin{aligned}&DW_U=q(x,k)W_U,\quad (x,k)\in I\times U,\\&W_U(x,k)\sim W(x,k)\qquad (U\ni k\longrightarrow\infty,\ x\in I),\\&W_{\omega U}(x,\omega k)=AW_U(x,k)A^{-1},\quad (x,k)\in I\times U.\end{aligned}$$

PROOF. By Theorem 6.1, there exist an interval $I\subset J$, a covering $\tilde{\mathcal{U}}$ of S^1 and $\tilde{W}_{\tilde{U}}(x,k)\in\Gamma(I\times\tilde{U}, GL(n,\mathcal{A}))$ $(\tilde{U}\in\tilde{\mathcal{U}})$ for which the first and the second properties of the theorem are satisfied. Let $\mathcal{U}$ be the minimal N-cyclic covering of S^1 which contains $\tilde{\mathcal{U}}$. Then the cyclic group $\boldsymbol{Z}_N\simeq\langle\omega\rangle$ acts on $\mathcal{U}$ in an obvious way. So we devide $\mathcal{U}$ into orbits with respect to this action. By the minimality of $\mathcal{U}$ over $\tilde{\mathcal{U}}$, each orbit contains at least one element of $\tilde{\mathcal{U}}$. Thus we can specify an element of $\tilde{\mathcal{U}}$ as a representative of an arbitrary orbit. If $U\in\mathcal{U}$ and

$\tilde{U}\in\tilde{\mathcal{U}}$ is the representative of the orbit through U, then U is expressed as $U=\omega^j\tilde{U}$ for some $j\in \boldsymbol{Z}$. Then we define $W(x, k)\in\Gamma(I\times U, GL(n, \mathcal{A}))$ as follows

$$W_U(x, k)=A^j\tilde{W}_{\tilde{U}}(x, \omega^{-j}k)A^{-j}.$$

Since the formal solution W satisfies $W(x, \omega k)=AW(x, k)A^{-1}$, we get

$$\begin{aligned}W_U(x, k)\sim A^jW(x, \omega^{-j}k)A^{-j}&=A^j\cdot A^{-j}W(x, k)A^j\cdot A^{-j}\\&=W(x, k)\quad (U\ni k\longrightarrow\infty, x\in I).\end{aligned}$$

Moreover, from the fact that $D^{\omega}=i_ADi_A^{-1}$ and $q(x, k)$ is A-cyclic, it follows that

$$\begin{aligned}DW_U&=i_{A^j}D^{\omega^{-j}}(i_{A^j})^{-1}(A^j\tilde{W}_{\tilde{U}}(x, \omega^{-j}k)A^{-j})\\&=A^j(D^{\omega^{-j}}\tilde{W}_{\tilde{U}}(x, \omega^{-j}k))A^{-j}\\&=A^jq(x, \omega^{-j}k)\tilde{W}_{\tilde{U}}(x, \omega^{-j})A^{-j}\\&=A^j\cdot A^{-j}q(x, k)A^j\cdot\tilde{W}_{\tilde{U}}(x, \omega^{-j}k)A^{-j}\\&=q(x, k)W_U(x, k).\end{aligned}$$

Therefore, the interval I, the N-cyclic covering $\mathcal{U}$ of S^1 and the system $\{W_U(x, k); U\in\mathcal{U}\}$ satisfy the first and the second assertions of the theorem. Moreover, it is clear from the definition of $W_U(x, k)$ that they also satisfy the third assertion. Thus the theorem is proved.

§ 8. Stokes phenomena and gauge transformation group

In this section, we define two sheaves: $Stokes_{\infty}\longrightarrow\boldsymbol{R}^1$, the sheaf of germs of Stokes phenomena, and $Gauge_{\infty}\longrightarrow\boldsymbol{R}^1$, the sheaf of germs of gauge transformation groups acting on $Diff_{\infty}$. We shall obtain a natural map $Diff_{\infty}\longrightarrow Stokes_{\infty}$ by assigning $q(x, k)\in Diff_{\infty}$ to a Stokes phenomenon arising from a fundamental system of solutions for $q(x, k)$. This map induces an injection $Gauge_{\infty}\diagdown Diff_{\infty}\longrightarrow Stokes_{\infty}$. Our goal in this section is to show that it is a bijection.

DEFINITION 8.1 ($Stokes_{\infty}\longrightarrow\boldsymbol{R}^1$). We define the *sheaf of germs of Stokes phenomena* by

$$Stokes_{\infty}=\mathrm{Image}\,[\mathcal{R}^1\pi(GL(n, \mathcal{A})_{\mathrm{id}}\cap\mathcal{K}erD)\xrightarrow{i}\mathcal{R}^1\pi(GL(n, \mathcal{A})\cap\mathcal{K}erD)]$$

where i is the natural inclusion map and $\pi:\boldsymbol{R}^1\times S^1\longrightarrow\boldsymbol{R}^1$ is the projection.

DEFINITION 8.2 ($Gauge_\infty \longrightarrow \boldsymbol{R}^1$). Consider the sheaf of multiplicative groups over $\boldsymbol{R}^1$:

$$GL_0(n, \mathfrak{a}\{1/k\}) = \left\{ g(x, k) = \sum_{m=0}^{\infty} g_m(x) k^{-m} \in GL(n, \mathfrak{a}\{1/k\});\ g_0(x) \in GL(n, \mathfrak{a})' \right\}.$$

This group acts on $Diff_\infty \simeq \mathfrak{gl}(n, \mathfrak{a}\{1/k\})$ in the following manner: For an element $g \in GL_0(n, \mathfrak{a}\{1/k\})$, let $\rho(g)$ be defined by

$$\begin{array}{ccc} \rho(g) : Diff_\infty & \longrightarrow & Diff_\infty \\ \cup\!\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\!\shortmid \\ q & \longmapsto & gqg^{-1} + (Dg)g^{-1} \end{array}$$

This action is called a gauge transformation. The group $GL_0(n, \mathfrak{a}\{1/k\})$ regarded as a transformation group in the above sense acting on $Diff_\infty$ is called the *gauge* (*transformation*) *group* and denoted by

$$Gauge_\infty = GL_0(n, \mathfrak{a}\{1/k\}) \longrightarrow \boldsymbol{R}^1.$$

REMARK 8.3. For a differential equation $DW = q(x, k)W$, if a new dependent variable $\tilde{W}$ is introduced by $W = g^{-1}\tilde{W}$, $g \in Gauge_\infty$, then the above equation is transformed into

$$D\tilde{W} = \{\rho(g)q\}\tilde{W}.$$

THEOREM 8.4. *We have the following bijection of sheaves over* $\boldsymbol{R}^1$:

$$Gauge_\infty \backslash Diff_\infty \rightarrowtail\!\!\!\!\rightarrow Stokes_\infty,$$

where the bijection is defined as follows: Let $q(x, k)$ *be an arbitrary element of* $Diff_\infty$ *and be a germ at* $x_0 \in \boldsymbol{R}^1$. *We may assume that* $q(x, k) \in Diff_\infty(J)$ *with* $x_0 \in J \subset \boldsymbol{R}^1$. *Choose a formal solution* $W(x, k) \in GL_0(n, \mathfrak{a}[[1/k]])(J)$ *of the equation* $DW = q(x, k)W$. *Then, by Theorem* 6.1, *there exist an interval* I ($x_0 \in I \subset J$), *a covering* $\mathcal{U}$ *of* S^1 *and sections* $W_U(x, k) \in \Gamma(I \times U, GL(n, \mathcal{A}))$ ($U \in \mathcal{U}$) *such that*

$$DW_U = q(x, k)W_U, \quad W_U \sim W \qquad (U \ni k \longrightarrow \infty, x \in I).$$

From $\{W_U\}$ *we have a cocycle*

$$\{S_{U,V} = W_U^{-1} W_V \text{ on } I \times U \cap I \times V\} \in Z^1(I \times \mathcal{U}, GL(n, \mathcal{A})_{\mathrm{id}} \cap \mathcal{K}erD),$$

which represents a cohomology class in $\Gamma(I, \mathcal{R}^1\pi(GL(n, \mathcal{A})_{\mathrm{id}} \cap \mathcal{K}erD))$. *Let* σ *be the image of this cohomology class by the inclusion map*

$$\mathcal{R}^1\pi(GL(n, \mathcal{A})_{\mathrm{id}} \cap \mathcal{K}erD) \longrightarrow \mathcal{R}^1\pi(GL(n, \mathcal{A}) \cap \mathcal{K}erD) = Stokes_\infty.$$

Then a map $Diff_\infty \longrightarrow Stokes_\infty$ is defined by assigning $q(x, k)$ to σ. We note that this map is well-defined, i.e. independent of possible choices of a formal solution W and a fundamental system of analytic solutions $\{W_U\}$ asymptotic to W. The meaning of the above bijection is that the map $Diff_\infty \longrightarrow Stokes_\infty$ is surjective and that two elements of $Diff_\infty$ give rise to a same Stokes phenomenon iff they are transformed into one another by a gauge transformation.

PROOF. We devide the proof into three steps: (i) well-definedness of the map $Diff_\infty \longrightarrow Stokes_\infty$, (ii) well-definedness and injectivity of the map $Gauge_\infty \backslash Diff_\infty \longrightarrow Stokes_\infty$, (iii) surjectivity.

(i): Let W and $\tilde{W}$ be two arbitrary formal solutions of $DW = q(x, k)W$. Let $\{W_U\}_{U\in\mathcal{U}}$ and $\{\tilde{W}_U\}_{U\in\mathcal{U}}$ be fundamental systems of solutions asymptotic to the formal solutions W and $\tilde{W}$ respectively, where $\mathcal{U}$ is a sufficiently refined covering of S^1. Then, if we set $\tilde{W}_U = W_U C_U$, we have $C_U \in \Gamma(I\times U, GL(n, \mathcal{A}) \cap \mathcal{K}erD)$ and

$$\tilde{S}_{U,V} := \tilde{W}_U^{-1}\tilde{W}_V = C_U^{-1}W_U^{-1}W_V C_V = C_U^{-1}S_{U,V}C_V \text{ on } I\times U \cap I\times V.$$

Therefore the two cocycles $\{S_{U,V}\}$ and $\{\tilde{S}_{U,V}\}$ give rise to the same cohomology class in $\mathcal{R}^1\pi(GL(n, \mathcal{A}) \cap \mathcal{K}erD)$. This shows that the map $Diff_\infty \longrightarrow Stokes_\infty$ is well-defined.

(ii): We first show that, for an arbitrary $q \in Diff_\infty$ and $g \in Gauge_\infty$, q and $\tilde{q} = \rho(g)q$ give rise to the same Stokes phenomenon. Indeed, if we choose $\{\tilde{W}_U\}_{U\in\mathcal{U}}$ as a fundamental system of solutions for q, then we can take $\{\tilde{W}_U = gW_U\}_{U\in\mathcal{U}}$ as that for $\tilde{q}$. Hence we have $\tilde{S}_{U,V} = \tilde{W}_U^{-1}\tilde{W}_V = W_U^{-1}W_V = S_{U,V}$, which implies that q and $\tilde{q}$ give rise to the same Stokes phenomenon. Conversely, suppose that q and $\tilde{q} \in Diff_\infty$ give rise to the same Stokes phenomenon. Namely, we assume that, in the following two process

$$q \longmapsto W \longmapsto \{W_U\} \longmapsto \{S_{U,V} = W_U^{-1}W_V\},$$
$$\tilde{q} \longmapsto \tilde{W} \longmapsto \{\tilde{W}_U\} \longmapsto \{\tilde{S}_{U,V} = \tilde{W}_U^{-1}\tilde{W}_V\},$$

the two terminal cocycles in $Z^1(I\times\mathcal{U}, GL(n, \mathcal{A})_{\mathrm{id}} \cap \mathcal{K}erD)$ represent the same element of $\mathcal{R}^1\pi(GL(n, \mathcal{A}) \cap \mathcal{K}erD)$. Then there exist $C_U \in \Gamma(I\times U, GL_0(n, \mathcal{A}) \cap \mathcal{K}erD)$ $(U\in\mathcal{U})$ such that

$$\tilde{S}_{U,V} = C_U^{-1}S_{U,V}C_V \qquad \text{in } I\times U \cap I\times V.$$

This relation is rewritten as follows

$$\widetilde{W}_U(W_U C_U)^{-1} = \widetilde{W}_V(W_V C_V)^{-1} \qquad \text{in } I\times U\cap I\times V.$$

Hence we can define an element $g\in\Gamma(I\times S^1, GL_0(n, \mathcal{A}))=GL_0(n, \mathfrak{a}\{1/k\})(I)=Gauge_\infty(I)$ by putting $g=\widetilde{W}_U(W_U C_U)^{-1}$ in $I\times U$, and we have

$$\widetilde{W}_U = g\cdot(W_U C_U).$$

Taking the fact $C_U\in KerD$ into account, we obtain

$$\begin{aligned}\tilde{q} &= (D\widetilde{W}_U)\widetilde{W}_U^{-1} = g(DW_U)\cdot C_U(W_U C_U)^{-1}g^{-1} + (Dg)(W_U C_U)(W_U C_U)^{-1}g^{-1}\\ &= g(DW_U\cdot W_U^{-1})g^{-1} + (Dg)g^{-1} = gqg^{-1} + (Dg)g^{-1} = \rho(g)q.\end{aligned}$$

This shows that the map $Diff_\infty \longrightarrow Stokes_\infty$ induces an injection $Gauge_\infty \backslash Diff_\infty \longrightarrow Stokes_\infty$.

(iii): Let $\{S_{U,V} \text{ on } I\times U\cap I\times V\}\in Z^1(I\times\mathcal{U}, GL(n, \mathcal{A})_{\mathrm{id}}\cap\mathcal{K}erD)$ be a representative cocycle of an arbitrary Stokes phenomenon σ. Then, by Theorem 2.5 and Corollary 2.6, after passing to a refinement of the covering if necessary, we can find $W_U\in\Gamma(I\times U, GL(n, \mathcal{A}))$ for $U\in\mathcal{U}$ such that

$$W_U S_{U,V} = W_V \qquad \text{in } I\times U\cap I\times V.$$

Since $S_{U,V}(x, k)\longrightarrow \mathrm{id}$ $(U\cap V\ni k\longrightarrow\infty)$, it follows that $W_U(x, \infty)=W_V(x, \infty)$. Hence, by replacing $W_U(x, k)$ by $W_U(x, \infty)^{-1}W_U(x, k)$ if necessary, we may assume that $W_U(x, \infty)=\mathrm{id}$. i.e. $W_U(x, k)\in\Gamma(I\times U, GL_0(n, \mathcal{A}))$. Since $S_{U,V}\in KerD$, we find that

$$(DW_V)W_V^{-1} = (DW_U)S_{U,V}(W_U S_{U,V})^{-1} = (DW_U)W_U^{-1} \qquad \text{in } I\times U\cap I\times V.$$

Therefore an element $q(x, k)\in\Gamma(I\times S^1, \mathfrak{gl}(n, \mathcal{A}))=\mathfrak{gl}(n, \mathfrak{a}\{1/k\})(I)=Diff_\infty(I)$ can be defined by putting $q(x, k)=(DW_U)W_U^{-1}$ for $(x, k)\in I\times U$. By the construction, it is evident that $q(x, k)$ gives rise to the Stokes phenomenon σ. This implies that the map $Diff_\infty\longrightarrow Stokes_\infty$ is surjective and the proof is completed.

§ 9. *A*-cyclic Stokes phenomena and *A*-cyclic gauge transformation group

In this section we assume Hypothesis 5.1.

DEFINITION 9.1 ($Stokes_{\infty, A\text{-}cyclic}\longrightarrow \boldsymbol{R}^1$). We define the *sheaf of germs of A-cyclic Stokes phenomena* by

$$Stokes_{\infty, A\text{-cyclic}} = \text{Image}\left(\mathcal{R}^1\pi(GL(n, \mathcal{A})_{\text{id}} \cap \mathcal{K}erD)_{A\text{-cyclic}} \xrightarrow{\text{incl.}} \mathcal{R}^1\pi(GL(n, \mathcal{A}) \cap \mathcal{K}erD)_{A\text{-cyclic}} \right)$$

where $\pi: \boldsymbol{R}^1 \times S^1 \longrightarrow \boldsymbol{R}^1$ is the projection.

DEFINITION 9.2 ($Gauge_{\infty, A\text{-cyclic}} \longrightarrow \boldsymbol{R}^1$). We define the *sheaf of A-cyclic gauge transformation groups* by

$$Gauge_{\infty, A\text{-cyclic}} = GL_0(n, \mathfrak{a}\{1/k\})_{A\text{-cyclic}}.$$

We regard $Gauge_{\infty, A\text{-cyclic}}$ as a transformation group on $Diff_{\infty, A\text{-cyclic}}$, where the action of $Gauge_{\infty, A\text{-cyclic}}$ on $Diff_{\infty, A\text{-cyclic}}$ is defined to be the restriction of that of $Gauge_{\infty}$ on $Diff_{\infty}$ (see Definition 8.1).

THEOREM 9.3. *We have the following bijection of sheaves over* $\boldsymbol{R}^1$:

$$Gauge_{\infty, A\text{-cyclic}} \backslash Diff_{\infty, A\text{-cyclic}} \longrightarrow Stokes_{\infty, A\text{-cyclic}},$$

where the bijection as defined as follows: Let $q(x, k)$ be an arbitrary element of $Diff_{\infty, A\text{-cyclic}}$ and be a germ at $x_0 \in \boldsymbol{R}^1$. We may assume that $q(x, k) \in Diff_{\infty, A\text{-cyclic}}(J)$ with $x_0 \in J \subset \boldsymbol{R}^1$. By Theorem 5.3, *there exists an A-cyclic formal solution $W(x, k) \in GL_0(n, \mathfrak{a}[[1/k]])_{A\text{-cyclic}}(J)$ of the equation $DW = qW$. Then Theorem* 7.1 *implies that, for some interval I ($x_0 \in I \subset J$) and some N-cyclic covering $\mathcal{U}$, there exist $W_U(x, k) \in \Gamma(I \times U, GL(n, \mathcal{A}))$ for $U \in \mathcal{U}$ such that*

$$\begin{aligned} &DW_U = q(x, k) W_U \quad \text{in } I \times U, \\ &W_U(x, k) \sim W(x, k) \quad (U \ni k \longrightarrow \infty, x \in I), \\ &W_{\omega U}(x, \omega k) = A W_U(x, k) A^{-1} \quad \text{in } I \times U. \end{aligned}$$

Then an associated cocycle $\{S_{U,V}(x, k) = W_U^{-1} W_V$ on $I \times U \cap I \times V\} \in Z^1(I \times \mathcal{U}, GL(n, \mathcal{A})_{\text{id}} \cap \mathcal{K}erD)$ is A-cyclic, i.e.

$$S_{\omega U, \omega V}(x, \omega k) = A S_{U,V}(x, k) A^{-1} \quad \text{in } I \times U \cap I \times V.$$

Hence this cocycle represents a cohomology class in $\mathcal{R}^1\pi(GL(n, \mathcal{A})_{\text{id}} \cap \mathcal{K}erD)_{A\text{-cyclic}}$. Let σ be the image of this cohomology class by the inclusion map $\mathcal{R}^1\pi(GL(n, \mathcal{A}) \cap \mathcal{K}erD)_{A\text{-cyclic}} \longrightarrow \mathcal{R}^1\pi(GL(n, \mathcal{A}) \cap \mathcal{K}erD)_{A\text{-cyclic}}$. Then a map $Diff_{\infty, A\text{-cyclic}} \longrightarrow Stokes_{\infty, A\text{-cyclic}}$ is defined by associating $q(x, k)$ to σ. Indeed, this map is well-defined, i.e. independent of possible choices of an A-cyclic formal solution W and a fundamental system of A-cyclic analytic solutions $\{W_U\}$ asymptotic to W. The above map induces a bijection

$$Gauge_{\infty,A\text{-}cyclic}\backslash Diff_{\infty,A\text{-}cyclic}\longrightarrow Stokes_{\infty,A\text{-}cyclic}.$$

PROOF. The proof is almost the same as that of Theorem 8.1, except for an utilization of the cyclic version of Sibuya-Malgrange theorem (Theorem 3.4) instead of Theorem 2.5 in proving the surjectivity of the map $Diff_{\infty,A\text{-}cyclic}\longrightarrow Stokes_{\infty,A\text{-}cyclic}$. Hence we omit the proof.

§ 10. A Reduction

We introduce the following notations:

$$Diff''_{\infty}=\mathrm{Image}\,[\mathrm{ad}(P):Diff_{\infty}\longrightarrow Diff_{\infty}]=\{q'';\,q\in Diff_{\infty}\},$$
$$Diff''_{\infty,A\text{-}cyclic}=\{q'';\,q\in Diff_{\infty,A\text{-}cyclic}\},$$
$$Gauge'_{\infty}=Gauge_{\infty}\cap \mathcal{K}er\,\mathrm{ad}(P),$$
$$Gauge'_{\infty,A\text{-}cyclic}=Gauge_{\infty,A\text{-}cyclic}\cap \mathcal{K}er\,\mathrm{ad}(P).$$

Then we have the following:

THEOREM 10.1. $$Gauge'_{\infty}\cdot Diff''_{\infty}=Diff_{\infty},$$
$$Gauge'_{\infty,A\text{-}cyclic}\cdot Diff''_{\infty,A\text{-}cyclic}=Diff_{\infty,A\text{-}cyclic}.$$

In particular, any Stokes phenomenon arises from an element of $Diff''_{\infty}$. *Similarly, any A-cyclic Stokes phenomenon arises from an element of* $Diff''_{\infty,A\text{-}cyclic}$.

REMARK 10.2. $Diff''_{\infty,A\text{-}cyclic}=Diff''_{\infty}\cap Diff_{\infty,A\text{-}cyclic}$.

PROOF OF REMARK. Indeed it is clear that the first set contains the second. Thus we shall show $Diff''_{\infty,A\text{-}cyclic}\supset Diff_{\infty,A\text{-}cyclic}\cap Diff''_{\infty}$. Let $q(x,k)''\in Diff''_{\infty,A\text{-}cyclic}$ be an arbitrary element, where $q(x,k)\in Diff_{\infty,A\text{-}cyclic}$. Then there exists an $r(x,k)\in Diff_{\infty}$ such that $q(x,k)''=\mathrm{ad}(P)r(x,k)$. In the cyclic case, we assume the relation $\omega P=APA^{-1}$ (Hypothesis 5.1). So we have

$$Aq(x,k)''A^{-1}=\mathrm{ad}(APA^{-1})(Ar(x,k)A^{-1})=\mathrm{ad}(P)\{\omega Ar(x,k)A^{-1}\}.$$

Hence $Aq(x,k)''A^{-1}\in Diff''_{\infty}$. On the other hand, since $\mathrm{ad}(P)q(x,k)''=\mathrm{ad}(P)q(x,k)$, we observe that

$$\begin{aligned}&\mathrm{ad}(P)q(x,\omega k)''=\mathrm{ad}(P)q(x,\omega k)=\mathrm{ad}(P)Aq(x,k)A^{-1}\\&=A\{\mathrm{ad}(A^{-1}PA)q(x,k)\}A^{-1}=\omega^{-1}A\{\mathrm{ad}(P)q(x,k)\}A^{-1}\\&=\omega^{-1}A\{\mathrm{ad}(P)q(x,k)''\}A^{-1}=\omega^{-1}\,\mathrm{ad}(APA^{-1})Aq(x,k)''A^{-1}\end{aligned}$$

$$=\omega^{-1}\operatorname{ad}(\omega P)Aq(x,k)''A^{-1}=\operatorname{ad}(P)\{Aq(x,k)''A^{-1}\}.$$

Since $\operatorname{ad}(P)$ is injective on $Diff''_\infty$, we obtain

$$q(x,\omega k)''=Aq(x,k)''A^{-1}.$$

This shows that $q(x,k)''\in Diff_{\infty,A\text{-cyclic}}$, whence $Diff''_{\infty,A\text{-cyclic}}\subset Diff''_\infty\cap Diff_{\infty,A\text{-cyclic}}$ as desired.

Proof of Theorem. Clearly it suffices to show that $Diff_\infty\subset Gauge'_\infty\cdot Diff''_\infty$ and $Gauge'_{\infty,A\text{-cyclic}}\cdot Diff''_{\infty,A\text{-cyclic}}\supset Diff_{\infty,A\text{-cyclic}}$. To do this, for any $q\in Diff_\infty$, let us find $\tilde q\in Diff''_\infty$ and $g\in Gauge'_\infty$ such that $q=\rho(g)\tilde q$ holds. This condition is rewritten as

$$Dg=qg-g\tilde q,\quad g\in Gauge'_\infty,\quad \tilde q\in Diff''_\infty.$$

Further, it is equivalent to the condition

$$\partial g=q'g,\quad \tilde q=g^{-1}q''g.$$

Since $Gauge'_\infty$ forms a (topological) group, the equation $\partial g=q'g$ has a solution g in $Gauge'_\infty$. If we set $\tilde q=g^{-1}q''g$, then we have $q=\rho(g)\tilde q$. This proves the first assertion.

Secondly suppose $q\in Diff_{\infty,A\text{-cyclic}}$, then Remark 10.2 implies that q' and q'' are A-cyclic. Thus we solve an initial value problem $\partial g=q'g$, $g|_{x=x_0}=\mathrm{id}.$ to obtain a solution g in $Gauge'_{\infty,A\text{-cyclic}}$. Then $\tilde q=g^{-1}q''g$ is also A-cyclic. Hence $g\in Gauge'_{\infty,A\text{-cyclic}}$ and $\tilde q\in Diff''_{\infty,A\text{-cyclic}}$ are what are to be sought. This proves the second assertion.

§ 11. Vector bundles over $R^1\times S^1$ and partial connections on them

We established in § 8 the bijection $Gauge_\infty\backslash Diff_\infty\overset{\sim}{\to}Stokes_\infty$ (Theorem 8.4). However, the sheaf $Diff_\infty$ is "too large", so the following problem is the next to be considered:

Problem 11.1. *Find a representative, which is as "simple" as possible, of each orbit of the action of $Gauge_\infty$ on $Diff_\infty$. In other words, find a subset $\mathcal{X}\subset Diff_\infty$ as "small" as possible so that*

$$Gauge_\infty\cdot\mathcal{X}=Diff_\infty$$

holds. If this is solved, then it turns out from the above bijection that an arbitrary Stokes phenomenon does arise from much smaller class

$\mathcal{X}$ *of differential operators in* $Diff_\infty$.

Hereafter, we devote ourselves to consider this problem. We start with considering the meaning that $\mathcal{X}$ is as "small" as possible. Recall that $Diff_\infty \simeq \mathfrak{gl}(n, \mathfrak{a}\{1/k\})$ is an object whose elements make sense only around $k=\infty$ with respect to a parameter k. Thus, if we let $\mathcal{X}$ to be a set of "global" operators defined on $\boldsymbol{R}^1 \times \boldsymbol{P}^1 = \{x\} \times \{k\}$ whose restriction into $\boldsymbol{R}^1 \times \{$a neighbourhood of $k=\infty\}$ belong to $Diff_\infty$, then we can think of $\mathcal{X}$ as a "small" subset of $Diff_\infty$. More precisely, we let $\mathcal{X}$ to be a set of partial connections satisfying certain conditions which act on sections of some vector bundle $\boldsymbol{E}_{\mathfrak{m}}$ over $\boldsymbol{R}^1 \times S^1$ introduced below:

Let

$$\mathfrak{m} = (m_1, \cdots, m_n), \quad m_j \in \boldsymbol{Z}, \quad m_1 \leqq m_2 \leqq \cdots \leqq m_n$$

be an n-tuple of integers and be

$$pr : \boldsymbol{R}^1 \times \boldsymbol{P}^1 \longrightarrow \boldsymbol{P}^1 : \text{projection}.$$

A vector bundle over $\boldsymbol{R}^1 \times \boldsymbol{P}^1$ which is the pull-back of a holomorphic vector bundle $\mathcal{O}(m_1) \oplus \cdots \oplus \mathcal{O}(m_n)$ over $\boldsymbol{P}^1$ by the above projection will be denoted by

$$\boldsymbol{E}_{\mathfrak{m}} = pr^* \mathcal{O}(m_1) \oplus \cdots \oplus \mathcal{O}(m_n) \longrightarrow \boldsymbol{R}^1 \times \boldsymbol{P}^1,$$

where $\mathcal{O}(m)$ is the line bundle with Chern class $m \in \boldsymbol{Z} \simeq H^2(\boldsymbol{P}^1, \boldsymbol{Z})$.

DEFINITION 11.2 ($\tilde{\mathcal{O}} \longrightarrow \boldsymbol{R}^1 \times \boldsymbol{P}^1$). Let $J \subset \boldsymbol{R}^1$ and $D \subset \boldsymbol{P}^1$ be open sets. Put

$$\tilde{\mathcal{O}}(J \times D) = \varinjlim_{V \supset J} \mathcal{O}(V \times D),$$

where V runs over all complex neighbourhoods of J. The sheaf associated with the presheaf $J \times D \longmapsto \tilde{\mathcal{O}}(J \times D)$ will be denoted by $\tilde{\mathcal{O}}$.

Write $\boldsymbol{P}^1$ in the form:

$$\boldsymbol{P}^1 = D_0 \cup D_\infty, \quad D_\nu : \text{a neighbourhood of } k=\nu \text{ in } \boldsymbol{P}^1 \ (\nu = 0, \infty).$$

Then $\boldsymbol{E}_{\mathfrak{m}}$ is defined to be a vector bundle whose transition relation in terms of an appropriate fiber coordinates, $\xi = {}^t(\xi_1, \cdots, \xi_n)$ over $\boldsymbol{R}^1 \times D_0$ and $\eta = {}^t(\eta_1, \cdots, \eta_n)$ over $\boldsymbol{R}^1 \times D_\infty$, is given by

$$\xi=\Omega(k)\eta, \quad \Omega(k)=\begin{pmatrix} k^{m_1} & & & & \\ & k^{m_2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & k^{m_n} \end{pmatrix}.$$

DEFINITION 11.3. Let $\mathfrak{m}=(m_1, \cdots, m_n)$, $m_j \in \boldsymbol{Z}$, $m_1 \leqq \cdots \leqq m_n$ be an n-tuple of integers and $P=(P_{ij}) \in \mathfrak{gl}(n, \boldsymbol{C})$ be a matrix. Then the pair $(\mathfrak{m}, P)$ is said to be *admissible* iff

$$m_i - m_j < -1 \text{ implies } P_{ij}=0.$$

The meaning of the admissibility will be clear in Definition 11.4 below (see also Remark 11.5).

DEFINITION 11.4 ($\mathit{Diff}^{(\mathfrak{m})} \longrightarrow \boldsymbol{R}^1$). Suppose that $(\mathfrak{m}, P)$ is an admissible pair. For an open $J \subset \boldsymbol{R}^1$, set

$$\mathit{Diff}^{(\mathfrak{m})}(J)=\left\{ q=\begin{pmatrix} q_0(x, k) \\ q_\infty(x, k) \end{pmatrix};\ \begin{array}{l} q_0(x, k) \in \mathfrak{gl}(n, \widetilde{\mathcal{O}})(J \times D_0) \\ q_\infty(x, k) \in \mathfrak{gl}(n, \widetilde{\mathcal{O}})(J \times D_\infty) \\ q_0(x, k)\Omega(k)=\Omega(k)(kP+q_\infty(x, k)) \end{array} \right\}.$$

We denote by $\mathit{Diff}^{(\mathfrak{m})}$ the sheaf associated with the presheaf $J \longmapsto \mathit{Diff}^{(\mathfrak{m})}(J)$.

REMARK 11.5. In order that $\mathit{Diff}^{(\mathfrak{m})}$ is not empty, it is necessary and sufficient that the pair $(\mathfrak{m}, P)$ is admissible.

REMARK 11.6. For $q \in \mathit{Diff}^{(\mathfrak{m})}(J)$, we can define an operator ∇_q acting on sections of $E_{\mathfrak{m}}|_{J \times P^1}$ by

$$\begin{array}{ll} \xi \longmapsto (\partial - q_0(x, k))\xi & \text{in } J \times D_0, \\ \eta \longmapsto (\partial - kP - q_\infty(x, k))\eta & \text{in } J \times D_\infty, \end{array}$$

where ξ and η are fiber coordinates of $E_{\mathfrak{m}}$ over $J \times D_0$ and $J \times D_\infty$, respectively. The operator ∇_q is a partial connection whose exterior differential part acts only along the parameter space $\boldsymbol{R}^1$. Note that ∇_q has a single pole at $k=\infty$ with a prescribed residue P.

REMARK 11.7. An element $q={}^t(q_0, q_\infty) \in \mathit{Diff}^{(\mathfrak{m})}$ is uniquely determined by a specification of the first (or second) entry q_0 (or q_∞). Note that the second entry q_∞ belongs to Diff_∞ in the sense that q_∞ determines an element $D-q_\infty(x, k) \in \mathit{Diff}_\infty$ and they are identified. Therefore we can regard that

$$Diff^{(\mathfrak{m})}\subset Diff_{\infty}.$$

Thus we arrive at the following problem:

Problem 11.8. *Can we conclude that*

$$Gauge_{\infty,P}\cdot Diff_{P}^{(\mathfrak{m})}=Diff_{\infty,P}\ ?$$

Or, for which pair $(\mathfrak{m}, P)$ *does this equality hold?*

REMARK 11.9. In view of the definitions of $Diff_{\infty,P}$ and $Diff_{P}^{(\mathfrak{m})}$ (Definition 4.2 and 11.4), we can see that their "sizes" are measured as follows.

$$Diff_{\infty}\simeq(\mathfrak{a}\{1/k\})^{n^{2}}\simeq\mathfrak{a}^{\infty},\quad Diff^{(\mathfrak{m})}\simeq\mathfrak{a}^{l},$$

for some $l\in \boldsymbol{Z}$ such that $0<l\leqq n^{2}(m_{n}-m_{1}+1)$, where $\mathfrak{m}=(m_{1},\cdots,m_{n})$, $m_{1}\leqq\cdots\leqq m_{n}$. This shows that $Diff^{(\mathfrak{m})}$ is much smaller than $Diff_{\infty}$ and that Problem 11.8 is an exceedingly meaningful problem.

§ 12. Fundamental system of solutions for $Diff^{(\mathfrak{m})}$, the sheaf $Stokes^{(\mathfrak{m})}\longrightarrow \boldsymbol{R}^{1}$ and a map $Gauge^{(\mathfrak{m})}\backslash Diff^{(\mathfrak{m})}\longrightarrow Stokes^{(\mathfrak{m})}$

In this section, we shall define the sheaf $Stokes^{(\mathfrak{m})}\longrightarrow\boldsymbol{R}^{1}$ of "global" Stokes phenomena and consider a map $Diff^{(\mathfrak{m})}\longrightarrow Stokes^{(\mathfrak{m})}$. This map is defined by assigning a partial connection in $Diff^{(\mathfrak{m})}$ to a "global" Stokes phenomenon which is associated with a fundamental system of solutions of this connection.

Let $p:\boldsymbol{P}_{\infty}^{1}\longrightarrow\boldsymbol{P}^{1}$ be a real blow-up of $\boldsymbol{P}^{1}$ at the infinity. For simplicity, we also denote by $p:\boldsymbol{R}^{1}\times\boldsymbol{P}_{\infty}^{1}\longrightarrow\boldsymbol{R}^{1}\times\boldsymbol{P}^{1}$ the projection. Then we can consider that

$$\boldsymbol{P}_{\infty}^{1}=(\boldsymbol{P}^{1}\backslash\{\infty\})\cup S^{1},\quad p|_{\boldsymbol{P}^{1}\backslash\{\infty\}}=\mathrm{id}_{\boldsymbol{P}^{1}\backslash\{\infty\}},\quad p(S^{1})=\{\infty\}.$$

DEFINITION 12.1 ($\widetilde{\mathcal{A}}\longrightarrow\boldsymbol{R}^{1}\times\boldsymbol{P}_{\infty}^{1}$). We define a sheaf $\widetilde{\mathcal{A}}$ over $\boldsymbol{R}^{1}\times\boldsymbol{P}_{\infty}^{1}$ as follows.

$$\widetilde{\mathcal{A}}|_{\boldsymbol{R}^{1}\times(\boldsymbol{P}^{1}\backslash\{\infty\})}=\widetilde{\mathcal{O}}|_{\boldsymbol{R}^{1}\times(\boldsymbol{P}^{1}\backslash\{\infty\})},\quad \widetilde{\mathcal{A}}|_{\boldsymbol{R}^{1}\times S^{1}}=\mathcal{A}.$$

The sheaf $GL(n,\widetilde{\mathcal{A}})$ is defined in an obvious manner. Moreover the sheaf $GL(n,\widetilde{\mathcal{A}})_{\mathrm{id}}$ is defined by

$$GL(n,\widetilde{\mathcal{A}})_{\mathrm{id}|\boldsymbol{R}^{1}\times(\boldsymbol{P}^{1}\backslash\{\infty\})}=GL(n,\widetilde{\mathcal{O}})|_{\boldsymbol{R}^{1}\times(\boldsymbol{P}^{1}\backslash\{\infty\})},$$
$$GL(n,\widetilde{\mathcal{A}})_{\mathrm{id}|\boldsymbol{R}^{1}\times S^{1}}=GL(n,\mathcal{A})_{\mathrm{id}}.$$

Let $q={}^t(q_0, q_\infty)$ be an element of $Diff^{(m)}$ and be a germ at $x_0\in \boldsymbol{R}^1$. We consider an equation $\nabla_q s=0$ for local sections s of the vector bundle $\boldsymbol{E}_m$. This equation is rewritten as

$$(12.1)\qquad \begin{aligned} &\{\partial - q_0(x, k)\}\xi=0 && \text{in } J\times D_0, \\ &\{\partial - kP - q_\infty(x, k)\}\eta=0 && \text{in } J\times D_\infty, \\ &\qquad \xi=\Omega(k)\eta, \end{aligned} \qquad (x_0\in J\subset \boldsymbol{R}^1)$$

where $\xi={}^t(\xi_1, \cdots, \xi_n)$ and $\eta={}^t(\eta_1, \cdots, \eta_n)$ are fiber coordinates of $\boldsymbol{E}_m$ over $\boldsymbol{R}^1\times D_0$ and $\boldsymbol{R}^1\times D_\infty$ respectively, moreover, $D_0=\boldsymbol{C}$ and D_∞ is a neighbourhood of $k=\infty$. Consider a pair (Ξ, H) of fundamental solutions of (12.1) and transform it into (Z, W) by

$$\Xi=Z\exp(kxP), \quad H=W\exp(kxP).$$

Then we obtain the following system of equations,

$$(12.2)\qquad \begin{aligned} &DZ=\{q_0(x, k)-kP\}Z && \text{in } J\times D_0, \\ &DW=q_\infty(x, k)W && \text{in } J\times D_\infty, \\ &\qquad Z=\Omega(k)W. \end{aligned}$$

Since the first equation of (12.2) has no singularity with respect to the parameter k in D_0, it has a solution

$$Z(x, k)\in \Gamma(J\times D_0, GL(n, \tilde{\mathcal{O}})).$$

It follows from § 5 and § 6 that the second equation (12.2) has a formal solution $W(x, k)\in GL_0(n, \mathfrak{a}[[1/k]])(J)$ and then has a fundamental system of analytic solutions $\{W_U\}$ asymptotic to $W(x, k)$, i.e.

$$\begin{aligned} &W_U(x, k)\in \Gamma(I\times U, GL(n, \mathcal{A})) && (U\in \mathcal{U}), \\ &W_U(x, k)\sim W(x, k) && (U\ni k\longrightarrow \infty, x\in I), \end{aligned}$$

where I is some open interval such that $x_0\in I\subset J$ and $\mathcal{U}$ is a covering of S^1.

Notation 12.2. For a covering $\mathcal{U}$ of S^1, we set $\mathcal{U}_0=\{D_0\}\cup\mathcal{U}$. Then $\mathcal{U}_0$ is regarded as a covering of $\boldsymbol{P}^1_\infty$.

DEFINITION 12.3. Such a system $\{Z, W_U\ (U\in\mathcal{U})\}$ mentioned above is called a *fundamental system* of solutions for $q\in Diff^{(m)}$.

The system $\{Z, W_U\ (U\in\mathcal{U})\}$ satisfies a transition relations

$$(12.3)\qquad \begin{array}{ll} Z(x,k)S_U(x,k)=\Omega(k)\,W_U(x,k) & \text{in } I\times D_0\cap I\times U, \\ W_U(x,k)S_{U,V}(x,k)=W_V(x,k) & \text{in } I\times U\cap I\times V, \end{array}$$

for $U, V\in \mathcal{U}$, where

$$S_U(x,k)\in\Gamma(I\times D_0\cap I\times U, GL(n,\tilde{\mathcal{O}})\cap \mathcal{K}erD),$$
$$S_{U,V}(x,k)\in\Gamma(I\times U\cap I\times V, GL(n,\tilde{\mathcal{O}})\cap \mathcal{K}erD).$$

These satisfy a cocycle condition:

$$\begin{array}{ll} S_U S_{U,V}=S_V & \text{in } I\times D_0\cap I\times U\cap I\times V, \\ S_{U,U'}S_{U',U''}=S_{U,U''} & \text{in } I\times U\cap I\times U'\cap I\times U''. \end{array}$$

In other words, we have

$$\{S_U, S_{U,V};\, U, V\in\mathcal{U}\}\in Z^1(I\times U_0, GL(n,\tilde{\mathcal{A}})_{\mathrm{id}}\cap\mathcal{K}erD).$$

This cocycle determines an element of $\Gamma(I, \mathcal{R}^1\tilde{\pi}(GL(n,\tilde{\mathcal{A}})_{\mathrm{id}}\cap\mathcal{K}erD))$. Let $s(q)$ be the image of this element by the inclusion map $\mathcal{R}^1\tilde{\pi}(GL(n,\tilde{\mathcal{A}})_{\mathrm{id}}\cap \mathcal{K}erD)\longrightarrow\mathcal{R}^1\tilde{\pi}(GL(n,\tilde{\mathcal{A}})\cap\mathcal{K}erD)$, where $\tilde{\pi}: \boldsymbol{R}^1\times\boldsymbol{P}^1_\infty\longrightarrow\boldsymbol{R}^1$ is the projection.

DEFINITION 12.4 ($Stokes^{(m)}\longrightarrow\boldsymbol{R}^1$). We define the *sheaf* $Stokes^{(m)}$ *of germs of global Stokes phenomena* by

$$Stokes^{(m)}=\mathrm{Image}\,[\mathcal{R}^1\tilde{\pi}(GL(n,\tilde{\mathcal{A}})_{\mathrm{id}}\cap\mathcal{K}erD)\xrightarrow{\text{incl.}}\mathcal{R}^1\tilde{\pi}(GL(n,\tilde{\mathcal{A}})\cap\mathcal{K}erD)].$$

An element $s(q)\in Stokes^{(m)}$ is called the *global Stokes phenomenon* of $q\in Diff^{(m)}$.

REMARK 12.5. The map $Diff^{(m)}\ni q\longmapsto s(q)\in Stokes^{(m)}$ is well-defined. Namely, the Stokes phenomenon $s(q)$ does not depend on a choice of a formal solution W and a fundamental system of solutions $\{Z, W_U\}$. A proof of this assertion is similar to that of Theorem 8.4 and is omitted.

DEFINITION 12.6 ($Gauge^{(m)}\longrightarrow\boldsymbol{R}^1$). We define the *sheaf* $Gauge^{(m)}$ *of gauge transformation groups* to be a sheaf associated with the presheaf $J\longmapsto Gauge^{(m)}(J)$, where

$$Gauge^{(m)}(J)=\{g={}^t(g_0, g_\infty)\in GL(n,\tilde{\mathcal{O}})(J\times D_0)\times GL_0(n,\mathfrak{a}\{1/k\})(J);\ g_0\Omega(k)=\Omega(k)g_\infty\},$$

where J is an open set in $\boldsymbol{R}^1$. This is a group acting on $Diff^{(m)}$ by

$$\begin{array}{ccccc} \rho(g): Diff^{(m)} & \longrightarrow & Diff^{(m)} & & (g \in Gauge^{(m)}) \\ \cup & & \cup & & \\ q=\begin{pmatrix} q_0 \\ q_\infty \end{pmatrix} & \longmapsto & \begin{pmatrix} g_0 q_0 g_0^{-1} + (\partial g_0) g_0^{-1} \\ g_\infty q_\infty g_\infty^{-1} + (\partial g_\infty) g_\infty^{-1} \end{pmatrix}. & & \end{array}$$

PROPOSITION 12.7. *The map* $Diff^{(m)} \longrightarrow Stokes^{(m)}$, $q \longmapsto s(q)$ *induces an injection*

$$Gauge^{(m)} \backslash Diff^{(m)} \rightarrowtail Stokes^{(m)}.$$

PROOF. Given q and $\tilde{q}$, suppose that $s(q)=s(\tilde{q})$. Let $\{Z, W_U\ (U \in \mathcal{U})\}$ be a fundamental system of solutions for q and $\{S_U, S_{U,V}\ (U, V \in \mathcal{U})\}$ be a cocycle representing the Stokes phenomenon $s(q)$. The counterparts for $\tilde{q}$ are denoted by $\{Z, W_U\ (U \in \mathcal{U})\}$ and $\{\tilde{S}_U, \tilde{S}_{U,V}\ (U, V \in \mathcal{U})\}$ respectively. Note that, after passing to a refinement of a covering if necessary, we can take the covering $\mathcal{U}_0 = \{D_0\} \cup \mathcal{U}$ of $\boldsymbol{P}^1_\infty$ in common for q and $\tilde{q}$. Then we have the transition relations,

$$\text{(12.4)} \qquad \begin{array}{ll} ZS_U = \Omega W_U, \quad \tilde{Z}\tilde{S}_U = \Omega \tilde{W}_U & \text{in } I \times D_0 \cap I \times U, \\ W_U S_{U,V} = W_U, \quad \tilde{W}_U \tilde{S}_{U,V} = \tilde{W}_V & \text{in } I \times U \cap I \times V. \end{array}$$

It follows from the condition $s(q)=s(\tilde{q})$ that there exist B and C_U $(U \in \mathcal{U})$ such that

$$\tilde{S}_U = B S_U C_U^{-1}, \quad \tilde{S}_{U,V} = C_U S_{U,V} C_V^{-1},$$

$$B \in \Gamma(I \times D_0, GL(n, \tilde{\mathcal{O}}) \cap \mathcal{K}er D), \quad C_U \in \Gamma(I \times U, GL_0(n, \tilde{\mathcal{A}}) \cap \mathcal{K}er D).$$

By these relations, we find that $\tilde{W}_U C_U W_U^{-1} = \tilde{W}_V C_V W_V^{-1}$ in $I \times U \cap I \times V$. Hence an element $g_\infty \in GL_0(n, \mathfrak{a}\{1/k\})(I)$ can be defined by putting $g_\infty = \tilde{W}_U C_U W_U^{-1}$ in $I \times U$. Then we have $\tilde{W}_U C_U = g_\infty W_U$. Since $\tilde{W}_U C_U$ is also a solution of $D\tilde{W}_U = \tilde{q}_\infty \tilde{W}_U$ and has an asymptotic expansion

$$\tilde{W}_U C_U \sim g_\infty W \qquad (U \ni k \longrightarrow \infty, U \in \mathcal{U}),$$

we can replace $\tilde{W}_U C_U$ with $\tilde{W}_U$. Similarly we can replace $\tilde{Z}B$ with $\tilde{Z}$. After this replacement, we have $\tilde{S}_{U,V} = S_{U,V}$ and $\tilde{S}_U = S_U$. Therefore it follows from (12.4) that

$$\tilde{Z} S_U = \Omega \tilde{W}_U \text{ in } I \times D_0 \cap I \times U, \quad \tilde{W}_U = g_\infty W_U \text{ in } I \times U.$$

Combined with the relation $ZS_U = \Omega W_U$ (see (12.4)), we observe that $g_0 = \tilde{Z} Z^{-1}$ satisfies

$$g_0 = \tilde{Z} Z^{-1} = \Omega \tilde{W}_U W_U^{-1} \Omega^{-1} = \Omega g_\infty \Omega^{-1} \in GL(n, \tilde{\mathcal{O}})(I \times D_0).$$

Hence, $g={}^t(g_0, g_\infty)\in Gauge^{(\mathfrak{m})}$ and $\tilde{Z}=g_0 Z$, $\widetilde{W}_U=g_\infty W_U$. Taking into account the relations $q_0=\partial(Ze^{kxP})\cdot(Ze^{kxP})^{-1}$, $q_\infty=(DW_U)W^{-1}$ in $I\times U$ and their counterparts for $\tilde{q}$, we obtain $\rho(g)(q_0, q_\infty)=(\tilde{q}_0, q_\infty)$. Conversely, if this relation holds, then it is easily seen that $s(q)=s(\tilde{q})$ holds. Hence the map $Diff^{(\mathfrak{m})}\longrightarrow Stokes^{(\mathfrak{m})}$ induces an injection $Gauge^{(\mathfrak{m})}\diagdown Diff^{(\mathfrak{m})}\longrightarrow Stokes^{(\mathfrak{m})}$.

§ 13. Deformation of holomorphic vector bundles over $\boldsymbol{P}^1$

In order to solve Problem 11.8, we shall require some results concerning deformation of holomorphic vector bundles over the projective line $\boldsymbol{P}^1$. We start with a well-known theorem on the structures of holomorphic vector bundles over $\boldsymbol{P}^1$.

THEOREM 13.1 (Birkhoff [9, 11], Grothendieck [15]). *Any holomorphic vector bundle of rank n over $\boldsymbol{P}^1$ is equivalent to a direct sum of n holomorphic line bundles. Any holomorphic line bundle over $\boldsymbol{P}^1$ is determined by its Chern class $\in H^2(\boldsymbol{P}^1, \boldsymbol{Z}^2)\simeq\boldsymbol{Z}$. Hence, for any holomorphic vector bundle V of rank n, there exists an n-tuple of integers $\mathfrak{m}=(m_1, \cdots, m_n)$ such that*

$$V\simeq\mathcal{O}(m_1)\oplus\cdots\oplus\mathcal{O}(m_n),\quad m_j\in\boldsymbol{Z},\quad m_1\leqq\cdots\leqq m_n.$$

For a family of holomorphic vector bundles, we have

THEOREM 13.2 (see Kodaira [20]). *Let X_x $(x\in\boldsymbol{R})$ be a smooth family of compact complex manifolds and let $V_x\longrightarrow X_x$ $(x\in\boldsymbol{R})$ be a smooth family of holomorphic vector bundles. Then the dimension of $H^p(X_x, \mathcal{O}(V_x))$ is upper semi-continuous in x for any p.*

THEOREM 13.3. *Let $V_x\longrightarrow\boldsymbol{P}^1$ $(|x|<x_0)$ be a smooth family of holomorphic vector bundle over $\boldsymbol{P}^1$. Suppose that*

$$V_0\simeq\mathcal{O}(m_1)\oplus\cdots\oplus\mathcal{O}(m_n),\quad m_1\leqq\cdots\leqq m_n,\quad m_n-m_1\leqq 1.$$

Then there exists a constant x_1 $(0<x_1\leqq x_0)$ such that

$$V_x\simeq V_0\quad(|x|<x_1).$$

PROOF. By considering the deformation $V_x\otimes\mathcal{O}(-m_n-1)$ instead of V_x if necessary, we may assume that $m_n=-1$. Then V_0 has no holomorphic section, i.e. $\dim H^0(\boldsymbol{P}^1, \mathcal{O}(V_0))=0$. Hence it follows from Theorem 13.2 that there exists a constant x_2 $(0<x_2\leqq x_0)$ such that $\dim H^0(\boldsymbol{P}^1, \mathcal{O}(V_x))=0$ for $|x|<x_2$. Therefore, if we write V_x for a fixed x $(|x|<x_2)$ in the

form $V_x \simeq \mathcal{O}(l_1) \oplus \cdots \oplus \mathcal{O}(l_n)$, $l_1 \leqq \cdots \leqq l_n$, then we have $l_n \leqq -1$. On the other hand, since the Chern class of a line bundle does not change under a smooth deformation, the Chern class of the determinant line bundle $\det V_x$ does not change as x varies: $c(\det V_x) = c(\det V_0)$. Namely, $m_1 + \cdots + m_n = l_1 + \cdots + l_n$. Let $n_1 = \sharp\{j; m_j = -1\}$ and $n_1' = \sharp\{j; l_j = -1\}$. Then $n_1' - 2n = -2(n - n_1') - n_1' \geqq l_1 + \cdots + l_n = -2(n - n_1) - n_1 = n_1 - 2n$, since $m_j = -2$ $(j < n - n_1 + 1)$. Hence we have $n_1' \geqq n_1$. Again, by Theorem 13.2, there exists a constant x_1 $(0 < x_1 \leqq x_2)$ such that $\dim H^0(\boldsymbol{P}^1, \mathcal{O}(V_x) \otimes \mathcal{O}(1)) \leqq \dim H^0(\boldsymbol{P}^1, \mathcal{O}(V_0) \otimes \mathcal{O}(1))$ for $|x| < x_1$. This means that $n_1' = \sharp\{j; l_j = -1\} \leqq \sharp\{j; m_j = -1\} = n_1$. Combined with $n_1' \geqq n_1$, it follows $n_1' = n_1$. Therefore we have $l_1 + \cdots + l_p = m_1 + \cdots + m_p = -2p$ and $l_j \leqq -2$ $(1 \leqq j \leqq p)$, where $p = n - n_1$. Hence we must have $l_j = m_j$ $(= -2, 1 \leqq j \leqq p)$ and then $l_j = m_j$ $(1 \leqq j \leqq n)$. This shows that $V_x \simeq V_0$ for $|x| < x_1$.

REMARK 13.4. If $m_1 \leqq \cdots \leqq m_n$, $m_n - m_1 \geqq 2$, then a deformation $V_x \longrightarrow \boldsymbol{P}^1$ with $V_0 \simeq \mathcal{O}(m_1) \oplus \cdots \oplus \mathcal{O}(m_n)$ does not necessarily preserve the structure even in any small neighbourhood of $x = 0$. Indeed, in this case, the space of infinitesimal deformations at V_0, $H^1(\boldsymbol{P}^1, \mathcal{O}(\mathrm{End}(V_0)))$, which, by Serre duality, is canonically dual to $H^0(\boldsymbol{P}^1, \mathcal{O}(\mathrm{End}\,(V_0)^* \otimes T^*\boldsymbol{P}^1))$, is easily shown to be a non-zero space.

THEOREM 13.5 (see Kodaira [20]). *Let $\rho : \mathcal{M} \longrightarrow S$ be a smooth family of compact complex manifolds and let $\pi : \mathcal{B} \longrightarrow \mathcal{M}$ be a smooth family of holomorphic vector bundles over $\mathcal{M}$. Let us set $M_x = \rho^{-1}(x)$ and $B_x = \pi^{-1}(M_x)$ $(x \in S)$. If $\dim H^p(M_x, \mathcal{O}(B_x))$ is independent of $x \in S$, then a natural restriction map*:

$$r_x : H^p(\mathcal{M}, \mathcal{O}(\mathcal{B})) \longrightarrow H^p(M_x, \mathcal{O}(B_x))$$

is surjective. Here $\mathcal{O}(B_x)$ is the sheaf of germs of holomorphic sections of the holomorphic vector bundle $B_x \longrightarrow M_x$ and $\mathcal{O}(\mathcal{B})$ is the sheaf of germs of smooth sections of the smooth vector bundle $\mathcal{B} \longrightarrow \mathcal{M}$ such that their restrictions to $B_x \longrightarrow M_x$ belong to $\mathcal{O}(B_x)$.

REMARK 13.6. Kodaira [20] stated the above theorem for a smooth family. But it will turn out to be true for real analytic family, if the proof is examined carefully by the method of Kato [19].

THEOREM 13.7 (Parameter-dependence of a trivialization of bundles). *Let $\pi : \mathcal{V} \longrightarrow \boldsymbol{R}^1 \times \boldsymbol{P}^1$ be a real-analytic family of holomorphic vector bundles over $\boldsymbol{P}^1$ and let $V_x = \pi^{-1}(x)$, $(x \in \boldsymbol{R}^1)$. Suppose that V_{x_0} is a trivial* *bundle for an* $x_0 \in \boldsymbol{R}^1$. *Then Theorem* 13.3 *implies that* V_x *is pointwisely trivial whenever* x *is sufficiently near* x_0. *What we assert here is that there exists a trivialization of* V_x *depending real analytically on the parameter* x *in a neighbourhood of* x_0. *More precisely, by the definition of real analytic family, there exist a neighbourhood* $|x-x_0|<r$ *of* x_0 *in* $\boldsymbol{R}^1$, *an open covering* $\{D_0, D_\infty\}$ *of* $\boldsymbol{P}^1$, *where* D_ν *is a neighbourhood of* $k=\nu$ $(\nu=0, \infty)$, *and a transition function*

$$T(x, k) \in \Gamma((x_0-r, x_0+r) \times (D_0 \cap D_\infty),\ GL(n, \tilde{\mathcal{O}}))$$

of the vector bundle $V_x(|x-x_0|<r)$ *such that* $T(x_0, k)=\mathrm{id}$. *Then there exist* $\Phi_\nu(x, k) \in GL(n, \tilde{\mathcal{O}})((x_0-\tilde{r}, x_0+\tilde{r}) \times \tilde{D}_\nu)$, $(\nu=0, \infty)$ *such that*

$$\Phi_0(x, k)\, T(x, k) = \Phi_\infty(x, k) \quad in\ |x-x_0|<\tilde{r},\ k \in \tilde{D}_0 \cap \tilde{D}_\infty,$$

where $\{\tilde{D}_0, \tilde{D}_\infty\}$ *is an open covering of* $\boldsymbol{P}^1$ *with* $\nu \in \tilde{D}_\nu \subset D_\nu$ $(\nu=0, \infty)$ *and* $\tilde{r}$ *is a sufficiently small number such that* $0<\tilde{r}<r$.

PROOF. By Theorem 13.3, the bundle V_x is trivial for each $|x-x_0|<r$ if $r>0$ is sufficiently small. In particular the dimension of $H^0(\boldsymbol{P}^1, \mathcal{O}(V_x))$ does not depend on x. Hence Theorem 13.5 implies that the restriction map

$$r_x : \Gamma(J \times \boldsymbol{P}^1, \mathcal{O}(\mathcal{V})) \longrightarrow \Gamma(\boldsymbol{P}^1, \mathcal{O}(V_x)) \qquad (x \in J)$$

is surjective, where $J=(x_0-r, x_0+r)$. Let $\xi={}^t(\xi_1, \cdots, \xi_n)$ and $\eta={}^t(\eta_1, \cdots, \eta_n)$ be fiber coordinates of $\mathcal{V}$ over $J \times D_0$ and $J \times D_\infty$ respectively with respect to the transition function $T(x, k)$. Then, since $T(x_0, k)=\mathrm{id}$, we can take n sections of V_{x_0} whose fiber coordinates are $e_j={}^t(0, \cdots, \underset{j}{1}, \cdots, 0)$, $(j=1, \cdots, n)$ over both $J \times D_0$ and $J \times D_\infty$. Therefore it follows from the above surjectivity that there exist $\psi_{\nu,j}(x, k) \in (\tilde{\mathcal{O}}(J \times D_\nu))^n$, $(\nu=0, \infty, j=1, \cdots, n)$ such that

$$\begin{aligned} &T(x, k)\psi_{\infty,j}(x, k) = \psi_{0,j}(x, k) \qquad \text{in } J \times D_0 \cap J \times D_\infty, \\ &\psi_{\nu,j}(x_0, k) = e_j \qquad (\nu=0, \infty). \end{aligned}$$

Thus if we set $\Psi_\nu(x, k) = (\psi_{\nu,1}(x, k), \cdots, \psi_{\nu,n}(x, k))$, $(\nu=0, \infty)$, then Ψ_ν belongs to $\mathfrak{gl}(n, \tilde{\mathcal{O}})(J \times D_\nu)$ and, moreover, satisfies $\det \Psi_\nu(x_0, k) \equiv 1$ in $k \in D_\nu$. Hence, we can choose an open covering $\{\tilde{D}_0, \tilde{D}_\infty\}$ of $\boldsymbol{P}^1$ and an interval $\tilde{J}=(x_0-\tilde{r}, x_0+\tilde{r})$ such that

$$\begin{gathered} \det \psi_\nu(x, k) \neq 0 \qquad \text{in } (x, k) \in \tilde{J} \times \tilde{D}_\nu \ (\nu=0, \infty), \\ \tilde{D}_0 \subset D_0,\ \ \tilde{D}_\infty \subset D_\infty,\ \ \tilde{J} \subset J. \end{gathered}$$

Hence $\Phi_\nu(x, k)=\Psi_\nu(x, k)^{-1}\in GL(n, \tilde{\mathcal{O}})((x_0-\tilde{r}, x_0+\tilde{r})\times\tilde{D}_\nu)$, $(\nu=0, \infty)$ are desired matrices. The theorem is thus established.

§ 14. A result for the trivial bundle: $Gauge_\infty\cdot Diff^{(0)}=Diff_\infty$

In this section we shall give an affirmative answer to Problem 11.8 in the case where the vector bundle $E_{\mathfrak{m}}$ defined in § 11 is trivial, i.e. $\mathfrak{m}=(m_1, \cdots, m_n)=(0, \cdots, 0)$.

REMARK 14.1. When $\mathfrak{m}=(0, \cdots, 0)$, the pair $(\mathfrak{m}, P)$ is admissible for arbitrary $P\in\mathfrak{gl}(n, \boldsymbol{C})$. Moreover, in this case, we have

$$Diff^{(0)}=\left\{q=\begin{pmatrix} q_0 \\ q_\infty \end{pmatrix}=\begin{pmatrix} kP+q(x) \\ q(x) \end{pmatrix};\ q(x)\in\mathfrak{gl}(n, \mathfrak{a})\right\}.$$

THEOREM 14.2. *Let $P\in\mathfrak{gl}(n, \boldsymbol{C})$ be a semi-simple matrix. Then*

$$Gauge_\infty\cdot Diff^{(0)}=Diff_\infty.$$

Namely, the map

$$Diff^{(0)}\longrightarrow Stokes_\infty$$

is surjective.

IDEA OF THE PROOF. From the commutative diagram

$$\begin{tikzcd} \boldsymbol{R}^1\times\boldsymbol{P}^1_\infty \arrow[dr, "\tilde{\pi}"] & \\ \boldsymbol{R}^1\times S^1 \arrow[u, "\text{incl.}"] \arrow[r, "\pi"'] & \boldsymbol{R}^1, \end{tikzcd} \qquad S^1=p^{-1}(\infty),$$

we have a natural restriction map

$$r:\ Stokes^{(\mathfrak{m})}\longrightarrow Stokes_\infty.$$

Let $\sigma\in Stokes_\infty$ be an arbitrary element and be a germ at $x_0\in\boldsymbol{R}^1$. We ask whether σ comes from $Diff^{(\mathfrak{m})}$. The consideration will be made through the following steps.

(i) We construct an element $\tau\in Stokes^{(\mathfrak{m})}$ such that $r(\tau)=\sigma$ in a natural way.

(ii) We ask whether τ comes from $Diff^{(\mathfrak{m})}$.

(iii) Recall that τ represents some cohomology class over a set $J\times\boldsymbol{P}^1_\infty$, where J is an open interval containing x_0. By making use of Sibuya-Malgrange type theorem, we "crush" τ into a cohomology class over the space $J\times\boldsymbol{P}^1$.

(iv) By (iii) we obtain a deformation of holomorphic vector bundles

$$V_x \longrightarrow \boldsymbol{P}^1 \qquad (x \in J).$$

(v) If the above deformation is trivial, i.e. does not change the type of vector bundles, in a neighbourhood of $x=x_0$, then it turns out that τ actually comes from $Diff^{(\mathfrak{m})}$.
(vi) When $\mathfrak{m}=(0, \cdots, 0)$, we utilize Theorem 13.3 to show that the If-part of (v) is valid, whence Problem 11.8 is affirmatively solved.

PROOF. Let σ be an arbitrary element of $Stokes_\infty$ and be a germ at $x_0 \in \boldsymbol{R}^1$. Then we can choose a representative cocycle of σ:

$$\{S_{U,V} \text{ on } J\times U \cap J\times V;\ U, V \in \mathcal{U}\} \in Z^1(J\times\mathcal{U}, GL(n, \mathcal{A})_{\mathrm{id}} \cap \mathcal{K}er\, D),$$

where $J \subset \boldsymbol{R}^1$ is an open interval containing x_0 and $\mathcal{U}$ is a covering of S^1. After passing to a refinement of the covering if necessary, by Theorem 2.5, there exist $W_U \in \Gamma(J\times U, GL_0(n, \mathcal{A}))$, $(U \in \mathcal{U})$ such that

$$W_U S_{U,V} = W_V \quad \text{in} \quad J\times U \cap J\times V. \tag{14.0}$$

We define $S_U(x, k)$ in $J\times D_0 \cap J\times U$ as follows.

$$S_U(x_0, k) = W_U(x_0, k),$$
$$S_U(x, k) = \exp\{k(x-x_0)P\} S_U(x_0, k) \exp\{-k(x-x_0)P\}.$$

Then $S_U(x, k) \in \Gamma(J\times D_0 \cap J\times U,\ GL(n, \tilde{\mathcal{O}}) \cap \mathcal{K}er\, D)$ and $\{S_U\}_{U\in\mathcal{U}}$ satisfies the cocycle condition

$$S_U(x, k) S_{U,V}(x, k) = S_V(x, k) \quad \text{in} \quad J\times D_0 \cap J\times U \cap J\times V. \tag{14.1}$$

To see this, if we denote the right-hand side of (14.1) by $\tilde{S}_{U,V}(x, k)$, it is clear from the definition of $S_U(x_0, k)$ that $\tilde{S}_{U,V}(x_0, k) = S_V(x_0, k)$. Moreover, both $\tilde{S}_{U,V}$ and S_V satisfy the differential equation $DS=0$. Hence, by the uniqueness of the solution of an initial value problem, we have $\tilde{S}_{U,V} = S_V$ i.e. (14.1). Thus we have obtained a cocycle τ such that $r(\tau)=\sigma$, where

$$\tau = \{S_U, S_{U,V};\ U, V \in \mathcal{U}\} \in Z^1(J\times\mathcal{U}_0, GL(n, \tilde{\mathcal{A}})_{\mathrm{id}} \cap \mathcal{K}er\, D).$$

Let us now define $T(x, k)$ by

$$T(x, k) = S_U(x, k) W_U(x, k)^{-1} \quad \text{if} \quad (x, k) \in J\times U \cap J\times D_0.$$

It follows from (14.0) and (14.1) that

$$S_U(x,k)W_U(x,k)^{-1}=S_V(x,k)W_V(x,k)^{-1} \quad \text{in} \quad J\times U\cap J\times V\cap J\times D_0.$$

This shows that $T(x,k)$ is well-defined and satisfies

$$T(x,k)\in GL(n,\tilde{\mathcal{O}})(J\times D_0\cap J\times D_\infty), \qquad T(x_0,k)\equiv \mathrm{id}.$$

Let V_x $(x\in J)$ be holomorphic vector bundles over $\boldsymbol{P}^1$ with transition function $T(x,\cdot)$ with respect to the covering $\{D_0,D_\infty\}$. Then a deformation of holomorphic vector bundles over $\boldsymbol{P}^1$,

$$V_x \longrightarrow \boldsymbol{P}^1 \ (x\in J), \qquad V_{x_0}\simeq\mathcal{O}(0)\oplus\cdots\oplus\mathcal{O}(0) \quad (n\text{-tuple})$$

is defined. Therefore, by Theorem 13.3, this deformation is trivial in a neighbourhood of $x=x_0$. Hence Theorem 13.7 implies that, if we choose a covering $\{\tilde{D}_0,\tilde{D}_\infty\}$ of $\boldsymbol{P}^1$ such that $\nu\in\tilde{D}_\nu\subset D_\nu$ $(\nu=0,\infty)$, a sufficiently small open sub-interval $\tilde{J}$ of J containing x_0 and rewrite $\tilde{D}_\nu$, $\tilde{J}$ by D_ν, J respectively, then there exist $\Phi_\nu(x,k)\in GL(n,\tilde{\mathcal{O}})(J\times D_\nu)$, $(\nu=0,\infty)$ such that

$$\Phi_0(x,k)T(x,k)=\Phi_\infty(x,k) \quad \text{for} \quad (x,k)\in J\times D_0\in J\times D_\infty.$$

Combined with the definition of $T(x,k)$, this shows that

$$\Phi_0(x,k)S_U(x,k)=\Phi_\infty(x,k)W_U(x,k) \quad \text{in} \quad J\times D_0\cap J\times U$$

for every $U\in\mathcal{U}$. We can regard $\Phi_\infty(x,k)$ as an element of $GL(n,\mathfrak{a}\{1/k\})$. Then we may assume further that $\Phi_\infty(x,k)$ belongs to $GL_0(n,\mathfrak{a}\{1/k\})$ by replacing $\Phi_\infty(x,k)$ and $\Phi_0(x,k)$ with $\Phi_\infty(x,\infty)^{-1}\cdot\Phi_\infty(x,k)$ and $\Phi_\infty(x,\infty)^{-1}\Phi_0(x,k)$, respectively. Thus, if we set $\tilde{W}_U(x,k)=\Phi_\infty(x,k)W_U(x,k)\in GL(n,\mathcal{A})(J\times U)$, then

$$\tilde{W}_U(x,k)S_{U,V}(x,k)=\tilde{W}_V(x,k) \quad \text{for} \quad (x,k)\in J\times U\cap J\times V.$$

From this equality and the fact that $S_{U,V}\in\mathcal{K}er\,D$, we obtain

$$(D\tilde{W}_U)\tilde{W}_U^{-1}\in\mathfrak{gl}(n,\mathcal{A})(J\times U), \qquad (U\in\mathcal{U}),$$
$$(D\tilde{W}_U)\tilde{W}_U^{-1}=(D\tilde{W}_V)\tilde{W}_V^{-1} \quad \text{for} \quad (x,k)\in J\times U\cap J\times V.$$

Hence an element $q_\infty(x,k)\in\mathfrak{gl}(n,\tilde{\mathcal{O}})(J\times D_\infty)$ is (well-)defined by putting $q_\infty(x,k)=(D\tilde{W}_U)\tilde{W}_U^{-1}$ if $(x,k)\in J\times U$. Similarly

$$q_0(x,k)=(D\Phi_0(x,k)\Phi_0(x,k)^{-1}+kP\in\mathfrak{gl}(n,\tilde{\mathcal{O}})(J\times D_0)$$

is also defined. It follows from $\Phi_0S_U=\tilde{W}_U$ and $S_U\in\mathcal{K}er\,D$ that

$$q_0(x,k)=q_\infty(x,k)+kP \quad \text{for} \quad (x,k)\in J\times D_0\cap J\times D_\infty.$$

Since q_ν is holomorphic in $k\in D_\nu$, $(\nu=0, \infty)$, we see that $q_\infty(x, k)$ is independent of k. Writing $q_\infty(x, k)=q_\infty(x)$, we also have $q_0(x, k)=q_\infty(x)+kP$. Therefore,

$$q={}^t(q_\infty(x)+kP, q_\infty(x))\in \mathit{Diff}^{(0)}.$$

Now it is evident from the construction of q that a cohomology class represented by the cocycle τ is the global Stokes phenomenon $\in \mathit{Stokes}^{(0)}$ associated with $q\in \mathit{Diff}^{(0)}$. This shows that the map $\mathit{Diff}^{(0)}\to \mathit{Stokes}_\infty$ is surjective. In view of the bijection $\mathit{Gauge}_\infty\backslash \mathit{Diff}_\infty \xrightarrow{\sim} \mathit{Stokes}_\infty$ (Theorem 8.4), the above statement is nothing other than the equality $\mathit{Gauge}_\infty\cdot \mathit{Diff}^{(0)}=\mathit{Diff}_\infty$. Hence the theorem is established.

§ 15. Fundamental system of solutions for $\mathit{Diff}^{(\mathfrak{m})}_{A\text{-}cyclic}$, the sheaf $\mathit{Stokes}^{(\mathfrak{m})}_{A\text{-}cyclic}\to \boldsymbol{R}^1$ and a map $\mathit{Gauge}^{(\mathfrak{m})}_{A\text{-}cyclic}\backslash \mathit{Diff}^{(\mathfrak{m})}_{A\text{-}cyclic}\to \mathit{Stokes}^{(\mathfrak{m})}_{A\text{-}cyclic}$

In this section we shall make a similar discussion as in § 12 in the cyclic case. Thus we shall hereafter assume that

$$A^N=\mathrm{id.}, \qquad \omega P=APA^{-1}.$$

Moreover a multi-index $\mathfrak{m}=(m_1, \cdots, m_n)$ and a transition function $\Omega(k)=\mathrm{diag}(k^{m_1}, \cdots, k^{m_n})$ which determines a vector bundle $\boldsymbol{E}_\mathfrak{m}$ will be fixed. Further we put the following assumption on $\mathfrak{m}$ and A.

ASSUMPTION 15.1. If $m_i\neq m_j$, then $A_{ij}=0$, where $A=(A_{ij})$.

REMARK 15.2. The above assumption is a condition in order that the matrix $\Omega(\omega k)A\Omega(k)^{-1}$ is an entire function of k, or equivalently that the matrices A and $\Omega(k)$ are commutative.

Notation 15.3. Set

$$B=\Omega(\omega)A=A\Omega(\omega).$$

Note that Remark 15.2 implies that

$$B=\Omega(\omega k)A\Omega(k)^{-1}.$$

DEFINITION 15.4 ($\mathit{Diff}^{(\mathfrak{m})}_{A\text{-}cyclic}\to \boldsymbol{R}^1$). Let $(\mathfrak{m}, P)$ be an admissible pair and satisfies the above mentioned conditions. We define the sheaf $\mathit{Diff}^{(\mathfrak{m})}_{A\text{-}cyclic}$ by

$$\mathit{Diff}^{(\mathfrak{m})}_{A\text{-}cyclic}=\left\{q=\begin{pmatrix} q_0(x, k) \\ q_\infty(x, k)\end{pmatrix}\in \mathit{Diff}^{(\mathfrak{m})};\ \begin{matrix} q_\infty(x, \omega k)=Aq_\infty(x, k)A^{-1} \\ q_0(x, \omega k)=Bq_0(x, k)B^{-1}\end{matrix}\right\}.$$

DEFINITION 15.5. A covering $\mathcal{U}_0=\{D_0\}\cup\mathcal{U}$ of $\boldsymbol{P}^1_\infty=(\boldsymbol{P}^1\setminus\{\infty\})\cup S^1$ is said to be *N-cyclic* iff $\mathcal{U}$ is an N-cyclic covering of S^1 and D_0 satisfies $\omega D_0=D_0$.

DEFINITION 15.6 ($Stokes^{(m)}_{A\text{-}cyclic}\to\boldsymbol{R}^1$). We define a subsheaf $Stokes^{(m)}_{A\text{-}cyclic}$ of $Stokes^{(m)}$ as follows. Set

$$C(x,k)=\exp(xkP)\Omega(k)\exp(-xkP)\in\mathcal{K}er\,D.$$

Let $\sigma\in Stokes^{(m)}$ and be a germ at $x_0\in\boldsymbol{R}^1$. We say that σ belongs to $Stokes^{(m)}_{A\text{-}cyclic}$ iff there exist an open interval $J\subset\boldsymbol{R}^1$ containing x_0, a suitable N-cyclic covering $\mathcal{U}_0=\{D_0\}\cup\mathcal{U}$ of $\boldsymbol{P}^1$ and a cocycle $\{S_U,S_{U,V};\ U,V\in\mathcal{U}\}$ $\in Z^1(J\times\mathcal{U}_0,GL(n,\tilde{\mathcal{A}})_{\mathrm{id}}\cap\mathcal{K}er\,D)$ representing the cohomology class σ such that

$$\begin{aligned}
&S_U\in\Gamma(J\times D_0\cap J\times V,\ GL(n,\tilde{\mathcal{A}})_{\mathrm{id}}\cap\mathcal{K}er\,D),\\
&S_{U,V}\in\Gamma(J\times U\cap J\times V,\ GL(n,\tilde{\mathcal{A}})_{\mathrm{id}}\cap\mathcal{K}er\,D),\\
&S_{\omega U}(x,\omega k)=AC(x,k)S_U(x,k)A^{-1}\quad\text{for}\quad (x,k)\in J\times D_0\cap J\times U,\\
&S_{\omega U,\omega V}(x,\omega k)=AS_{U,V}(x,k)A^{-1}\quad\text{for}\quad (x,k)\in J\times U\in J\times V.
\end{aligned}$$

DEFINITION 15.7 ($Gauge^{(m)}_{A\text{-}cyclic}\to\boldsymbol{R}^1$). Let us set

$$Gauge^{(m)}_{A\text{-}cyclic}=\left\{g=\begin{pmatrix}g_0\\ g_\infty\end{pmatrix}\in Gauge^{(m)};\ \begin{matrix}g_\infty(x,\omega k)=Ag_\infty(x,k)A^{-1}\\ g_0(x,\omega k)=Bg_0(x,k)B^{-1}\end{matrix}\right\}.$$

We observed in §12 that a natural map $Diff^{(m)}\to Stokes^{(m)}$ is defined. Similarly we shall show that a natural map $Diff^{(m)}_{A\text{-}cyclic}\to Stokes^{(m)}_{A\text{-}cyclic}$ is also defined. To do this, we state the existence of a fundamental system of cyclic solutions for $q\in Diff^{(m)}_{A\text{-}cyclic}$.

LEMMA 15.8. *Let $q={}^t(q_0,q_\infty)$ be an element of $Diff^{(m)}_{A\text{-}cyclic}$ and be a germ at $x_0\in\boldsymbol{R}^1$. Let $\mathcal{U}_0=\{D_0\}\cup\mathcal{U}$ be an N-cyclic covering of $\boldsymbol{P}^1_\infty$. Then, for a sufficiently small open interval J containing x_0, the system of differential equations*

$$\text{(15.1)}\qquad\begin{aligned}
&DZ=\{q_0(x,k)-kP\}Z &&((x,k)\in J\times D_0),\\
&DW=q_\infty(x,k)W &&((x,k)\in J\times D_\infty),
\end{aligned}$$

has a fundamental system of solutions $Z(x,k)$, $W_U(x,k)$ such that

$$Z\in GL(n,\tilde{\mathcal{O}})(J\times D_0),\quad W_U\in GL_0(n,\tilde{\mathcal{A}})(J\times U),\quad (U\in\mathcal{U}),$$

(i) $Z(x, \omega k)=BZ(x, k)C(x, k)^{-1}A^{-1}$ *in* $J\times D_0$,
(ii) $W_{\omega U}(x, \omega k)=AW_U(x, k)A^{-1}$ *in* $J\times U$.
(iii) *There exists a* $W(x, k)\in GL_0(n, \mathfrak{a}[[1/k]])_{A\text{-}cyclic}(J)$ *such that*

$$W_U(x, k)\sim W(x, k) \quad (U\ni k\to\infty, x\in J).$$

We refer to such a system $\{Z, W_U\}$ *as a fundamental system of cyclic solutions.*

PROOF. The existence of $\{W_U\}$ is already shown in Theorem 7.1. We shall show the existence of $Z(x, k)$. The transformation

$$\Xi(x, k)=Z(x, k)\exp(xkP) \tag{15.2}$$

converts the equation (15.1) into the equation

$$\partial\Xi=q_0(x, k)\Xi. \tag{15.3}$$

Let $\Xi(x, k)$ be a solution of the equation (15.3) having an initial value $\Xi(x_0, k)$ at $x=x_0$ which is independent of k and commutative with B. For example, $\Xi(x, k)=\mathrm{id}.$ will do. By using $q_0(x, \omega k)=Bq_0(x, k)B^{-1}$, we see that $B^{-1}\Xi(x, k)B$ also satisfies the equation (15.3). Moreover, $B^{-1}\Xi(x, k)B$ has the same initial value at x_0 as $\Xi(x, k)$ does. Hence,

$$\Xi(x, \omega k)=B\Xi(x, k)B^{-1}.$$

By way of (15.2), this equality leads to the property (i) of the lemma. This shows the existence of $Z(x, k)$ and proves the lemma.

By using this lemma, a map $\mathit{Diff}^{(m)}_{A\text{-}cyclic}\to\mathit{Stokes}^{(m)}_{A\text{-}cyclic}$ is defined in a natural manner.

DEFINITION 15.9. Let $q={}^t(q_0, q_\infty)\in\mathit{Diff}^{(m)}_{A\text{-}cyclic}$. By Lemma 15.8, we can take a fundamental system of cyclic solutions Z, W_U $(U\in\mathcal{U})$ for q. Assosiated with this system, there exists a cocycle

$$\{S_U, S_{U,V};\ U, V\in\mathcal{U}\}\in Z^1(J\times\mathcal{U}_0,\ GL(n, \tilde{\mathcal{A}})_{\mathrm{id}}\in\mathcal{K}er\,D)$$

which satisfies the following conditions

$$\begin{aligned} Z(x, k)S_U(x, k)&=\Omega(k)W_U(x, k) &&\text{for}\quad (x, k)\in J\times D_0\cap J\times V,\\ W_U(x, k)S_{U,V}(x, k)&=W_V(x, k) &&\text{for}\quad (x, k)\in J\times U\cap J\times V.\end{aligned}$$

This cocycle satisfies the conditions of Definition 15.6, whence it determines an element of $\mathit{Stokes}^{(m)}_{A\text{-}cyclic}$, which will be denoted by $s(q)$ and called the *global A-cyclic Stokes phenomenon* of q. We have thus obtained a map $s: Diff^{(\mathfrak{m})}_{A\text{-}cyclic} \to Stokes^{(\mathfrak{m})}_{A\text{-}cyclic}$, $q \mapsto s(q)$. As in § 12 (see Remark 12.5), this map is well-defined, i.e. independent of a choice of a fundamental system of cyclic solutions for q.

PROPOSITION 15.10. *The map* $s: Diff^{(\mathfrak{m})}_{A\text{-}cyclic} \to Stokes^{(\mathfrak{m})}_{A\text{-}cyclic}$ *induces an injection*

$$s:\ Gauge^{(\mathfrak{m})}_{A\text{-}cyclic} \backslash Diff^{(\mathfrak{m})}_{A\text{-}cyclic} \longrightarrow Stokes^{(\mathfrak{m})}_{A\text{-}cyclic}.$$

PROOF. Proof is similar to that of Proposition 12.7 and is omitted.

§ 16. A result for the trivial bundle in the cyclic case: $Gauge_{\infty, A\text{-}cyclic} \cdot Diff^{(0)}_{A\text{-}cyclic} = Diff_{\infty, A\text{-}cyclic}$

In this section we shall answer to Problem 11.8 in the A-cyclic case. Here we assume that the bundle $\boldsymbol{E}_{\mathfrak{m}}$ is trivial, i.e. $\mathfrak{m}=(0, \cdots, 0)$. Then, note that Assumption 15.1 put no restriction on the matrices P and A.

REMARK 16.1. If $\mathfrak{m}=(0, \cdots, 0)$, then we have

$$A=B,$$
$$Diff^{(0)}_{A\text{-}cyclic}=\left\{q=\begin{pmatrix} q_\infty(x)+kP & \\ & q_\infty(x)\end{pmatrix};\ \begin{matrix} q_\infty(x) \in \mathfrak{gl}(n, \mathfrak{a}) \\ [A, q_\infty(x)]=0 \end{matrix}\right\}.$$

THEOREM 16.2. *Suppose that* $P \in \mathfrak{gl}(n, \boldsymbol{C})$ *is semi-simple*, $A \in GL(n, \boldsymbol{C})$, $A^N=\mathrm{id.}$ *and* $\omega P=APA^{-1}$. *Then*

$$Gauge_{\infty, A\text{-}cyclic} \cdot Diff^{(0)}_{A\text{-}cyclic} = Diff_{\infty, A\text{-}cyclic}, \quad i.e.$$
$$Diff^{(0)}_{A\text{-}cyclic} \longrightarrow Stokes_{\infty, A\text{-}cyclic}:\ surjective.$$

PROOF. Let σ be an arbitrary element of $Stokes_{\infty, A\text{-}cyclic}$ and be a germ at $x_0 \in \boldsymbol{R}^1$. Then there exists a representative cocycle

$$\{S_{U,V} \text{ on } J\times U \cap J\times V\} \in Z^1(J\times \mathcal{U},\ GL(n, \mathcal{A})_{\mathrm{id}} \cap \mathcal{K}er\, D)_{A\text{-}cyclic}$$

of σ, where J is an open interval containing x_0 and $\mathcal{U}$ is an N-cyclic covering of S^1. After passing to a refinement of the covering if necessary, we apply Sibuya-Malgrange theorem of cyclic version (Theorem 3.4) to this cocycle. Then there exist $W_U(x, k) \in \Gamma(J\times U,\ GL_0(n, \mathcal{A}))$, $(U \in \mathcal{U})$ such that

$$
\begin{aligned}
&W_U(x, k)S_{U,V}(x, k) = W_V(x, k), \quad (x, k) \in J\times U \cap J\times V. \\
&W_{\omega U}(x, \omega k) = A\,W_U(x, k)A^{-1}, \quad (x, k) \in J\times U.
\end{aligned} \tag{16.1}
$$

Now let us choose a neighbourhood D_0 of $0 \in \boldsymbol{P}^1$ such that $\mathcal{U}_0=\{D_0\}\cup\mathcal{U}$ becomes an N-cyclic covering of $\boldsymbol{P}^1_\infty$. As in the proof of Theorem 14.2, we define $S_U(x, k) \in \Gamma(J\times D_0 \cap J\times U,\ GL(n, \widetilde{\mathcal{O}}) \cap \mathcal{K}er\,D)$ in the following manner,

$$
\begin{aligned}
&S_U(x_0, k) = W_U(x, k_0), \\
&S_U(x, k) = \exp\{k(x-x_0)P\}S_U(x_0 k)\exp\{-k(x-x_0)P\}.
\end{aligned}
$$

It then follows from the second part of (16.1) and the assumption $\omega P = APA^{-1}$ that the relation

$$S_{\omega U}(x, \omega k) = AS_U(x, k)A^{-1}$$

holds. Hence, in a similar manner as in the proof of Theorem 14.2, we obtain a cocycle

$$\{S_U, S_{U,V}\} \in Z^1(J\times\mathcal{U}_0,\ GL(n, \widetilde{\mathcal{A}})_{\mathrm{id}} \cap \mathcal{K}er\,D)_{A\text{-}cyclic}.$$

Let τ be an element of $Stokes^{(0)}_{A\text{-}cyclic}$ represented by this cocycle. Now what we have to do is to find an element of $Diff^{(0)}_{A\text{-}cyclic}$ which gives rise to τ as its global A-cyclic Stokes phenomenon. As in § 14, we can define

$$T(x, k) \in \Gamma(J\times D_0 \cap J\times D_\infty,\ GL(n, \widetilde{\mathcal{O}})),$$

by putting $T(x, k) = S_U(x, k)\,W_U(x, k)^{-1}$ if $(x, k) \in J\times D_0 \cap J\times U$, $U \in \mathcal{U}$, where we put $D_\infty = \bigcup_{U\in\mathcal{U}} U$, a neighbourhood of ∞ in $\boldsymbol{P}^1$. By the definition, we have

$$T(x, \omega k) = A\,T(x, k)A^{-1}, \quad (x, k) \in J\times D_0 \cap J\times D_\infty. \tag{16.2}$$

We regard $T(x, k)$ as transition functions of holomorphic vector bundles over $\boldsymbol{P}^1$ parametrised by x. Since $T(x_0, k) \equiv \mathrm{id}.$ by definition, Theorem 13.3 implies that the structure of these vector bundles does not change near $x = x_0$, whence Theorem 13.7 implies the following: let $\{\tilde{D}_0, \tilde{D}_\infty\}$ be an open covering of $\boldsymbol{P}^1$ such that $\nu \in \tilde{D}_\nu \subset D_\nu$ $(\nu = 0, \infty)$, $\tilde{J}$ be a sufficiently small interval such that $x_0 \in \tilde{J} \subset J$ and rewrite $\tilde{D}_\nu$, $\tilde{J}$ as D_ν, J respectively. Then there exist $\Phi_\nu(x, k) \in \Gamma(J\times D_\nu,\ GL(n, \widetilde{\mathcal{O}}))$, $(\nu = 0, \infty)$ such that

$$\Phi_0(x, k)\,T(x, k) = \Phi_\infty(x, k), \quad (x, k) \in J\times D_0 \cap J\times D_\infty. \tag{16.3}$$

Replacing k by ωk in (16.3) and using (16.2), we obtain

$$(16.4) \qquad \Phi_0(x, \omega k) A T(x, k) A^{-1} = \Phi_\infty(x, \omega k).$$

Combining (16.3) and (16.4), we find that

$$(16.5) \qquad \Phi_0(x, \omega k) A \Phi_0(x, k)^{-1} = \Phi_\infty(x, \omega k) A \Phi_\infty(x, k)^{-1} \text{ in } J\times D_0 \cap J\times D_\infty.$$

Here we may assume that $\Phi_0(x, 0) = \mathrm{id}$, since, if necessary, we can replace $\Phi_0(x, k)$ and $\Phi_\infty(x, k)$ by $\Phi_0(x, 0)^{-1}\Phi_0(x, k)$ and $\Phi_0(x, 0)^{-1}\Phi_\infty(x, k)$ respectively. The equality (16.5) implies that the both sides of (16.5) are holomorphic in k on $\boldsymbol{P}^1$, whence they are constant in k. Thus let us denote them by $\tilde{A}(x)$. Substituting $k=0$ into (16.5) and using $\Phi_0(x, 0) = \mathrm{id}$, we obtain $\tilde{A}(x) \equiv A$. Hence we have

$$(16.6) \qquad \Phi_\nu(x, \omega k) = A \Phi_\nu(x, k) A^{-1}, \quad (x, k) \in J\times D_\nu, \quad \nu = 0, \infty.$$

Now let us put $\widetilde{W}_U(x, k) = \Phi_\infty(x, k) W_U(x, k)$. Then, as in the proof of Theorem 14.2, a function $q_\infty(x, k) = q_\infty(x) \in \mathfrak{gl}(n, \mathfrak{a})(J)$ is (well-)defined by putting

$$q_\infty(x, k) = \{D\widetilde{W}_U(x, k)\}\widetilde{W}_U^{-1} \quad \text{if } (x, k) \in J\times U, \ U \in \mathcal{U}.$$

Moreover, set $q_0(x, k) = \{D\Phi_0(x, k)\}\Phi_0(x, k)^{-1} + kP \in \mathfrak{gl}(n, \tilde{\mathcal{O}})(J\times D_0)$. Then we have

$$q_0(x, k) = q_\infty(x) + kP.$$

Taking into account (16.6) and $\widetilde{W}_{\omega U}(x, \omega k) = A\widetilde{W}_U(x, k)A^{-1}$, we have

$$q_\infty(x) = q_\infty(x, \omega k) = A\widetilde{W}_U(x, k)A^{-1} = Aq_\infty(x)A^{-1},$$

i.e. $[A, q_\infty(x)] = 0$. Hence it follows that $q = {}^t(q_0(x, k), q_\infty(x)) \in \mathit{Diff}^{(0)}_{A\text{-cyclic}}$. As in the proof of Theorem 14.2, it is now clear that q gives rise to the A-cyclic global Stokes phenomenon σ. This establishes the theorem.

§ 17. A result for the bundle $E_{\mathfrak{m}}$ with $\mathfrak{m} = (0, 1, \cdots, n-1)$ in the cyclic case: $\mathit{Gauge}_{\infty, A\text{-cyclic}} \cdot \mathit{diff} = \mathit{Diff}_{\infty, A\text{-cyclic}}$

In this section we shall solve Problem 11.8 for $\mathfrak{m} = (0, 1, \cdots, n-1)$ in the A-cyclic case. By doing this, our Main Theorem stated in § 0 will be simultaneously established. Thus we assume throughout this section that the n-tuple $\mathfrak{m}$ of Chern classes of the vector bundle $\boldsymbol{E}_{\mathfrak{m}}$ is as follows.

$$(17.1) \qquad \mathfrak{m} = (0, 1, \cdots, n-1).$$

We note that this choice of $\mathfrak{m}$ comes from the shearing transformation which we usually make in studying an asymptotic behaviour of a solution of a differential equation of the form

$$Lf=\{\partial^n+a_1(x)\partial^{n-1}+\cdots+a_n(x)\}f=k^n f,$$

as k tends to infinity. Our aim in this section is to show an equality (see Theorem 17.4) which implies

$$Gauge_{\infty,A\text{-}cyclic}\cdot Diff^{(\mathfrak{m})}_{A\text{-}cyclic}=Diff_{\infty,A\text{-}cyclic},$$

as a simple corollary.

We start with providing a typical example of the matrices $P=(P_{ij})$ and $A=(A_{ij})$ which are "compatible" with $\mathfrak{m}$ assumed in (17.1). Recall that the imposed conditions on $(\mathfrak{m}, P, A)$ are as follows.

(1) $P\in\mathfrak{gl}(n, C)$: semi-simple,
(2) $A\in GL(n, C)$: $A^N=\mathrm{id}$.
(3) $\omega P=APA^{-1}$,
(4) $(\mathfrak{m}, P)$: admissible (see Definition 11.4).
(5) $[\Omega(\omega), A]=0$ i.e. $m_i\neq m_j$ implies $A_{ij}=0$,

where $\Omega(k)=\mathrm{diag}(1, k, k^2, \cdots, k^{n-1})$. First, (5) implies that A must be diagonal. Let us now assume $N=n$. Then, by (2), the diagonal entries of A are n-th roots of unity. We thus assume that $A=\mathrm{diag}(1, \omega^{-1}, \cdots, \omega^{1-n})$, where ω is a primitive n-th root of unity. Then (3) is written as $\omega P_{ij}=\omega^{j-i}P_{ij}$, whence $P_{ij}=0$ if $j-i\not\equiv 1 \pmod n$. Thus we pick up a permutation matrix

$$P=\begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix}. \tag{17.2}$$

This matrix clearly satisfies (1). The admissibility (4) reads that $P_{ij}=0$ $(j-i>1)$, whence is satisfied by P. In this case the matrices A and B are

$$A=\mathrm{diag}\{1, \omega^{-1}, \cdots, \omega^{1-n}\}, \quad B=\mathrm{id}. \tag{17.3}$$

Throughout this section, we assume that the triple $(\mathfrak{m}, P, A)$ is defined by (17.1), (17.2) and (17.3).

In order to utilize a result in §16, we note that

$$Diff^{(0)}_{A\text{-}cyclic}=\left\{q=\begin{pmatrix} q_\infty(x)+kP \\ q_\infty(x)\end{pmatrix};\ q_\infty(x)\in \mathrm{Diag}(n,\mathfrak{a})\right\},$$

where $\mathrm{Diag}(n,\mathfrak{a})$ denotes the set of n-by-n diagonal matrices with entries in $\mathfrak{a}$. Furthermore, $Diff^{(m)}$ and $Diff^{(m)}_{A\text{-}cyclic}$ are described in the following manner. Let $\mathfrak{a}[t]\to \boldsymbol{R}^1$ be the sheaf of polynomials in t with coefficients in $\mathfrak{a}$.

LEMMA 17.1.

$$Diff^{(m)}=\left\{q=\begin{pmatrix} q_0(x,k) \\ q_\infty(x,k)\end{pmatrix}\in \mathfrak{gl}(n,\mathfrak{a}[k])\times\mathfrak{gl}(n,\mathfrak{a}[1/k]);\right.$$

$$q_0(x,k)_{ij}=\begin{cases} k^{i-j+1}P_{ij} & (i-j<0)\\ k^{i-j+1}P_{ij}+\sum_{\nu=0}^{i-j}a_{ij,\nu}(x)k^{i-j-\nu} & (i-j\geqq 0),\end{cases}$$

$$\left. q_\infty(x,k)_{ij}=\begin{cases} 0 & (i-j<0)\\ \sum_{\nu=0}^{i-j}a_{ij,\nu}(x)k^{-\nu} & (i-j\geqq 0)\end{cases}\right\},$$

$$Diff^{(m)}_{A\text{-}cyclic}=\left\{q=\begin{pmatrix} q_0(x,k) \\ q_\infty(x,k)\end{pmatrix}\in \mathfrak{gl}(n,\mathfrak{a}[k])\times\mathfrak{gl}(n,\mathfrak{a}[1/k]);\right.$$

$$q_0(x,k)_{ij}=k^{i-j+1}P_{ij}+a_{ij}(x),$$

$$q_\infty(x,k)_{ij}=a_{ij}(x)k^{j-i},$$

$$\left.\text{where}\quad a_{ij}(x)\in\mathfrak{a},\quad a_{ij}(x)\equiv 0\quad (i<j)\right\}.$$

PROOF. By the definition, $q={}^t(q_0(x,k),q_\infty(x,k))$ belongs to $Diff^{(m)}$ iff $q_0\in\mathfrak{gl}(n,\mathfrak{a}\{k\})$, $q_\infty\in\mathfrak{gl}(n,\mathfrak{a}\{1/k\})$ and $q_0(x,k)=\Omega(k)\{kP+q_\infty(x,k)\}\Omega(k)^{-1}$. An explicit calculation of this conditions gives the description of $Diff^{(m)}$ in the lemma. Secondly, $Diff^{(m)}_{A\text{-}cyclic}$ consists of such elements $q={}^t(q_0,q_\infty)\in Diff^{(m)}$ that $q_0(x,\omega k)=q_0(x,k)$ holds. Here we used $B=\mathrm{id}$. Hence $q_0(x,k)$ must belong to $\mathfrak{gl}(n,\mathfrak{a}[k^n])$. This condition, together with the first assertion of the lemma, gives the description of $Diff^{(m)}_{A\text{-}cyclic}$ in the lemma.

DEFINITION 17.2 ($diff\to\boldsymbol{R}^1$). For an open set $J\subset\boldsymbol{R}^1$, set

$$diff(J)=\{L=\partial^n+a_1(x)\partial^{n-1}+\cdots+a_n(x);\ a_j(x)\in\mathfrak{a}(J)\}.$$

Denote by $diff$ the sheaf associated with the presheaf $J\mapsto diff(J)$. Alternatively the sheaf $diff$ can be defined by

$$diff=\{{}^t(q_0(x,k;\ \boldsymbol{a}), q_0(x,k;\ \boldsymbol{a}));\ \boldsymbol{a}=(a_1(x),\cdots,a_n(x))\in\mathfrak{a}^n\},$$

where

$$q_0(x,k;\ \boldsymbol{a})=\begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ k^n-a_n(x) & -a_{n-1}(x) & \cdots & -a_2(x) & -a_1(x) \end{pmatrix},$$

$$q_\infty(x,k;\ \boldsymbol{a})=\begin{pmatrix} & & 0 & & \\ -\frac{a_n(x)}{k^{n-1}} & -\frac{a_{n-1}(x)}{k^{n-2}} & \cdots & -\frac{a_2(x)}{k} & -a_1(x) \end{pmatrix}.$$

The two sheaves, defined above and denoted by the same notation $diff$ are identified in an obvious manner.

REMARK 17.3. By considering $diff$ in the sense of the second definition, we see that

$$diff\subset Diff^{(\mathfrak{m})}_{A\text{-}cyclic}\subset Diff_{\infty,A\text{-}cyclic}.$$

THEOREM 17.4. *Let* $\mathfrak{m}=(0,1,\cdots,n-1)$ *and let the matrices* P *and* A *are defined by* (17.2) *and* (17.3). *Then we have*

$$Gauge_{\infty,A\text{-}cyclic}\cdot diff=Diff_{\infty,A\text{-}cyclic},$$
$$diff\longrightarrow Stokes_{\infty,P,A\text{-}cyclic}:\ surjective.$$

In particular, an arbitrary A-cyclic Stokes phenomenon does arise from some n-th order single ordinary differential equation of the form,

$$Lf=\{\partial^n+a_1(x)\partial^{n-1}+\cdots+a_n(x)\}f=k^nf,\quad a_j(x)\in\mathfrak{a}.$$

PROOF. From Remark 17.3 it is evident that

$$Gauge_{\infty,A\text{-}cyclic}\cdot diff\subset Diff_{\infty,A\text{-}cyclic}.$$

Conversely, if it is assumed that $Gauge_{\infty,A\text{-}cyclic}\cdot diff\supset Diff^{(0)}_{A\text{-}cyclic}$, which will be proved in the next lemma, then Theorem 16.2 implies that

$$Gauge_{\infty,A\text{-}cyclic}\cdot diff\subset Gauge_{\infty,A\text{-}cyclic}\cdot Diff^{(0)}_{A\text{-}cyclic}\subset Diff_{\infty,A\text{-}cyclic}.$$

This establishes the first equality in the theorem. The second surjection in the theorem is a result of Theorem 9.3 and the equality just established. Hence what remains to be shown is the following:

LEMMA 17.5. $Gauge_{\infty,A\text{-}cyclic}\cdot diff\supset Diff^{(0)}_{A\text{-}cyclic}$.

PROOF. Recall that any element of $Diff^{(0)}_{A\text{-}cyclic}$ is of the form ${}^t(q(x)+kP,\ q(x))$ where $q(x)$ is a diagonal matrix with entries in $\mathfrak{a}(J)$ with J an interval in $\boldsymbol{R}^1$, namely

$$q(x)=\mathrm{diag}\{b_1(x),\cdots,b_n(x)\},\quad b_j(x)\in\mathfrak{a}(J).$$

To establish the lemma, it suffices to find $g\in Gauge_{\infty,A\text{-}cyclic}$ and

$$p(x,k)=\begin{pmatrix} & & 0 & \\ -\dfrac{a_n(x)}{k^{n-1}} & \cdots & -\dfrac{a_2(x)}{k} & -a_1(x)\end{pmatrix},\qquad a_j(x)\in\mathfrak{a}(J),$$

such that $p=\rho(g)q$ holds. In order for g to be A-cyclic, let g be assumed to be of the form

$$g=(k^{j-i}g_{ij}(x))_{ij=1,\cdots,n},\quad g_{ij}(x)\in\mathfrak{a}(J).$$

Further, we assume that

$$g_{ij}(x)\equiv 0\ \ (i<j),\quad g_{jj}(x)\equiv 1\ \ (1\leqq j\leqq n). \tag{17.4}$$

Clearly such a g belongs to $Gauge_{\infty,A\text{-}cyclic}$. Secondly, we shall see that each entry $g_{ij}(x)$ of g and $a_j(x)$ of p can be determined successively with respect to an ordering introduced in these entries in a way mentioned later. To do this, we rewrite the condition $p=\rho(g)q$ as

$$Dg=pg-gq, \tag{17.5}$$

and equate the both sides of the entries of (17.5). If $i<j$ or $i=j<n$, then the (i,j)-entries of the both sides are equal to zero and there is no condition. For other four cases (i) $1\leqq i<n$, $j=1$, (ii) $1<j\leqq i<n$, (iii) $i=n$, $1<j\leqq n$, (iv) $i=n$, $j=1$, the (i,j)-entries of (17.5) give rise to

the following equalities.

(i)$_i$ $\quad g_{i+1,1}=\partial g_{i,1}+g_{i,1}b_1 \quad (1\leqq i<n)$,

(ii)$_{ij}$ $\quad g_{i+1,j}=g_{i,j-1}+\partial g_{ij}+g_{ij}b_j \quad (1\leqq i<n,\ 1<j<n)$,

(iii)$_j$ $\quad a_j=-(\partial g_{n,n+1-j}+g_{n,n+1-j}b_{n+1-j}+g_{n,n-j})-\sum_{l=1}^{j-1} a_l g_{n+1-l,n+1-j} \quad (1\leqq j<n)$,

(iv) $\quad a_n=-(\partial g_{n,1}+g_{n,1}b_1)-\sum_{l=1}^{n-1} a_l g_{n+1-l,1}$.

Let us now introduce a total ordering $\prec$ in the set $\{g_{ij};\ 1\leqq j<i<n\}$ as follows:

$$g_{ij}\prec g_{\nu,\mu} \quad \text{if} \quad j<\mu \quad \text{or} \quad j=\mu,\ i<\nu.$$

Then the recursion formulas (i)$_i$ $(1\leqq i<n-1)$, (ii)$_{ij}$ $(1<j\leqq i<n-1)$ and the assumption (17.4) allow us to determine g_{ij} $(1\leqq j<i<n)$ successively with respect to the above mentioned ordering. Then the recursion formulas (i)$_{n-1}$ and (ii)$_{n-1,j}$ $(1<j<n)$ determine $g_{n,j}$ $(1\leqq j\leqq n-1)$. Hence all g_{ij}'s are determined. Now the first terms in the right-hand sides of (iii)$_j$ and (iv) are regarded as known. Let us further introduce a total ordering in $a_1(x), \cdots, a_n(x)$ in this order. Then the recursion formulas (iii)$_j$ $(1\leqq j<n)$ and (iv) allow us to determine $a_1(x), \cdots, a_n(x)$ successively. Therefore it is shown that the desired gauge transformation g and the functions $a_1(x), \cdots, a_n(x)$ actually exist. This proves the theorem.

Note added in proof. In this paper, only the cyclic action is considered. But most results can be extended to the case where a more general group action is admitted. As a special consequence, Stokes phenomena arising from self-adjoint differential operators can be characterized completely. These results will be taken up in a separated paper.

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