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On removability of a singular submanifold for weakly harmonic maps

By D. COSTA and G. LIAO

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Introduction

In the past few years, the regularity problem for weakly harmonic maps between Riemannian manifolds has become an active research area: J. Eells and J. C. Polking proved in 1982 that a weakly harmonic map

$$u \in W^{1,2}_{\text{loc}}(M^n, N^m)$$

on M/A is weakly harmonic on M if A is a polar set of M [1, 2]; R. Schoen and K. Uhlenbeck developed a regularity theory for energy minimizing harmonic maps [3]; R. Hardt et al. proved partial regularity for minimizing maps that are similar to harmonic maps from $\boldsymbol{R}^3$ into S^2 [4].

The regularity problem for weakly harmonic maps is rather challenging. One intermediate step is to study stationary harmonic maps, which are defined to be harmonic maps that are also critical with respect to domain diffeomorphisms: R. Schoen [5] proved that for $n=2$ a stationary harmonic map cannot have interior singularities; J. Sacks and K. Uhlenbeck proved for $n=2$ the removability of isolated singularities for weakly harmonic maps [6]. Based on a result of S. Hildebrandt et al. [7], the second author proved that weakly harmonic maps with small energy cannot have isolated singularities for $n\geq 3$ [8]. For stationary harmonic maps whose singular set is contained in a smooth submanifold Σ^d with $n-d>2$, it was also proved by G. Liao in [9] that the maps are smooth if the energy is small. In physics harmonic maps correspond to nonlinear σ-models which are field models for systems with low energy. Thus it is natural to study the regularity problem under the assumption that the energy is small. Other conditions are also being considered for regularity problems: S. Takakuwa proved the following theorem [10];

Suppose that u is a stationary harmonic map satisfying

$$\int_D |\nabla u|^n dv < \infty$$

for any compact subset D of M. Then u cannot have isolated singular points.

The purpose of this paper is to extend the result of [9] to weakly harmonic maps. The stationary assumption is shown to be unnecessary if the apparent singular set has bigger codimension. We recall that a weakly harmonic map u is a map in $W^{1,2}(M^n, N^m)$ which is a weak solution of the Euler-Lagrange equations of the energy functional

$$E(u) = \int_M |\nabla u|^2 dV,$$

where M^n is taken to be the ball B_2 in $\boldsymbol{R}^n$ equipped with a metric g such that $a^{-1}(\delta_{ij}) \leq (g_{ij}) \leq a(\delta_{ij})$ for some $a \geq 1$, and N^m is a compact submanifold isometrically embedded in $\boldsymbol{R}^k$ (cf. [8] for notations). Our main results are

THEOREM 1. *Suppose $u: M^n \to N^m$ is a weakly harmonic map such that*

$$\text{(i)} \qquad \int_M |\nabla u|^p dV < \infty, \quad p = \frac{2(n-d)}{n-1-d}$$

and

$$\text{(ii)} \qquad u \in C^\infty(M \backslash \Sigma, N),$$

where Σ is the graph of a $C^{1,\alpha}$ function f with $d = \dim \Sigma < n-2$. There exists $\varepsilon > 0$, depending only on n, a and N, such that

$$u \in C^\infty(B_1, N) \ \ \textit{if} \ \int_M |\nabla u|^2 dv \leq \varepsilon.$$

THEOREM 2. *Suppose $u: M^n \to N^m$ is a weakly harmonic map such that $u \in C^\infty(M \backslash \Sigma, N)$ where Σ is the graph of a $C^{1,\alpha}$ function with $d = \dim \Sigma < n-3$. There exists $\varepsilon > 0$, depending only on n, a, N, such that $u \in C^\infty(B_1, N)$ if*

$$\int_M |\nabla u|^2 dV \leq \varepsilon.$$

REMARK. In Theorem 2, condition (i) in Theorem 1 is removed. An additional condition is that $d = \dim \Sigma$ is $< n-3$. A special case is $d = 1$,

$n=5$. Theorem 2 asserts the removability of a singular line in the interior of a manifold M^n with $n\geq 5$, provided that the total energy is small.

In the last section, some monotonicity inequalities for Yang-Mills Fields are derived. These inequalities might be useful in the study of regularity problem for Yang-Mills Fields.

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Section 1. The gradient estimate

The following estimate is our basic analytic lemma. We let $\lambda=\mathrm{dist}(x, \Sigma)$.

LEMMA 1.1. *Suppose that* $u: M\rightarrow N$ *is a weakly harmonic map with* $u\in C^2(B_{2R}/\Sigma^d, N)$ *where* $B_{2R}=\{x\in \boldsymbol{R}^n \mid |x|<2R\}$, $n-d\geq 3$, *and the Riemannian metric* g *satisfies, for some* $a\geq 1$,

$$a^{-1}(\delta_{ij})\leq(g_{ij})\leq a(\delta_{ij})$$
$$|\partial_k g_{ij}|\leq a|x|^{-1}.$$

There exists $\varepsilon_0>0$ *depending only on* n, a, N *such that if*

$$\int_M|\nabla u|^p dV<\infty \quad \text{for} \quad p=\frac{2(n-d)}{n-1-d} \quad \text{and} \quad \int_M|\nabla u|^2 dV\leq\varepsilon_0,$$

then u *satisfies*

$$|\nabla u|^2(x)\leq C\lambda^{-n}\int_{B_{2\lambda}(x)}|\nabla u|^2 dV \quad \text{for} \quad x\in B_R/\Sigma,\ 2\lambda<R.$$

We first prove a monotonicity formula, namely

PROPOSITION 1.2. *Suppose that* u *satisfies conditions* (i) *and* (ii) *in Theorem* 1. *In particular,* $n-d>2$ *and*

$$\int_M|\nabla u|^p dV<\infty \quad \text{for} \quad p=\frac{2(n-d)}{n-1-d}.$$

Then for $0<\rho_0<\rho_1\leq 1$ *we have*

$$\rho_0^{2-n}e^{c\rho_0\Lambda}\int_{B_{\rho_0}(x_0)}|\nabla u|^2\leq\rho_1^{2-n}e^{c\rho_1\Lambda}\int_{B_{\rho_1}(x_0)}|\nabla u|^2,$$

where $x_0\in\Sigma\cap B_{1/2}(0)$ *and* $\Lambda, c\geq 0$.

PROOF OF PROPOSITION 1.2. Take $0=x_0$. Denote, for $x\in B_1(x_0), r=|x|$, $\lambda=\text{dist}(x,\Sigma)$. Let $X(x)=\xi_\sigma(\lambda)\eta_\tau(r)\partial_r(x)r$ and define for small $t\in[0,1]$ a map $u_{t,\sigma}$ by

$$u_{t,\sigma}(x)=u(x+t\exp X(x)).$$

The cut-off functions η_τ and ξ_σ are defined in the following way.

Let $\phi\in C^\infty(\boldsymbol{R}^+,\boldsymbol{R})$ be so that, for $\sigma_1>0, \phi(\lambda)=1$ for $0\leq\lambda\leq 1, 0\leq\phi(\lambda)\leq 1$ for $1\leq\lambda\leq 1+\sigma_1, \phi(\lambda)=0$ for $\lambda\geq 1+\sigma_1$ and $\phi'(\lambda)=0$. Define $\eta_\tau\in C^\infty(\boldsymbol{R}^+,\boldsymbol{R})$ by $\eta_\tau(r)=\phi_{\sigma_1}\left(\frac{r}{\tau}\right)$, for $\tau\in\left(0,\frac{1}{2}\right)$. Define $\xi_\sigma\in C^\infty(\boldsymbol{R}^+,\boldsymbol{R})$ so that $\xi_\sigma(\lambda)=0$ if $0\leq\lambda\leq\sigma$, $0\leq\xi_\sigma(\lambda)\leq 1$ for $\sigma\leq\lambda\leq 2\sigma$, $\xi_\sigma(\lambda)=1$ if $\lambda\geq 2\sigma$ and $|\xi'_\sigma|\leq 2\sigma^{-1}$, where "$'$" means $\dfrac{\partial}{\partial r}$.

We denote $x=x(r,\theta)$ and let e_α $(\alpha=1,\cdots,n-1)$, ∂_r be an orthonormal frame on $M=B_2$. Also, for $\alpha,\beta=1,\cdots,n-1$, we set

$$\varepsilon_{\alpha\beta}(x)=\frac{\partial}{\partial r}(\langle\nabla_{re_\alpha}\partial_r, e_\beta\rangle)(x)$$

and let $\Lambda>0$ be such that

$$|\varepsilon_{\alpha\beta}(x)|\leq\Lambda\qquad\text{for all } x\in B_1.$$

Then we have

$$\langle\nabla_{e_\alpha}X, e_\beta\rangle=\eta\xi\langle\nabla_{re_\alpha}\partial_r, e_\beta\rangle=\eta\xi\delta_{\alpha\beta}+\eta\xi\int_0^r\varepsilon_{\alpha\beta}(r',\theta)dr'$$

and, therefore,

$$\begin{aligned}\operatorname{div}X=\sum_{i=1}^n\langle\nabla_{e_i}X, e_i\rangle&=(\eta\xi r)'+\sum_{\alpha=1}^{n-1}\langle\nabla_{e_\alpha}X, e_\alpha\rangle\\&\geq\eta\xi+\eta\xi'r+\eta'\xi r+(n-1)\eta\xi-(n-1)\eta\xi\Lambda r\end{aligned}$$

and

$$\begin{aligned}&2\sum_{i=1}^n\langle du(\nabla_{e_i}X),\ du(e_i)\rangle=2\langle du(\nabla_{\partial_r}X),\ \partial_r u\rangle+2\sum_{\alpha=1}^{n-1}\langle du(\nabla_{e_\alpha}X),\ \partial_\alpha u\rangle\\&\leq 2(\eta\xi r)'|\partial_r u|^2+2\sum_{\alpha=1}^{n-1}r\eta(\nabla_{e_\alpha}\xi)\langle\partial_r u, \partial_\alpha u\rangle+2\sum_{\alpha=1}^{n-1}\eta\xi|\partial_\alpha u|^2+2\sum_{\alpha=1}^{n-1}\eta\xi\Lambda r|\nabla u|^2.\end{aligned}$$

Hence, using the first variation formula

$$0=\frac{d}{dt}E(u_t)|_{t=0}=-\int_M[|\nabla u|^2\operatorname{div}X-2\sum_{i=1}^{n}\langle du(\nabla_{e_i}X),\ du(e_i)\rangle]dV,$$

we obtain

$$\begin{aligned}0\geq&\int_{B_1}\eta\xi' r|\nabla u|^2dV+\int_{B_1}\eta'\xi r|\nabla u|^2dV+n\int_{B_1}\eta\xi|\nabla u|^2dV\\&-(n-1)\int_{B_1}\eta\xi\Lambda r|\nabla u|^2dV-2\int_{B_1}(\eta\xi r)'|\partial_r u|^2dV\\&-2(n-1)\int_{B_1}r\eta|d\xi||\nabla u|^2dV-2\sum_{\alpha=1}^{n-1}\int_{B_1}\eta\xi|\partial_\alpha u|^2dV\\&-2(n-1)\Lambda\int_{B_1}\eta\xi r|\nabla u|^2dV.\end{aligned}$$

Now, we observe that

$$\int_{B_1}|d\xi||\nabla u|^2dV\leq C\sigma^{-1}\int_{C_{2\sigma}}|\nabla u|^2dV\longrightarrow 0$$

as $\sigma\to 0$, in view of Hölder's inequality

$$\int_{C_{2\sigma}}|\nabla u|^2dV\leq C\sigma^{(n-d)(1-2/p)}\Big(\int_{C_{2\sigma}}|\nabla u|^p dV\Big)^{2/p}$$

and the choice of $p=2(n-d)/(n-1-d)$. Therefore, the terms involving ξ', $|d\xi|$ in the previous inequality tend to zero as $\sigma\to 0$ and we obtain

$$\begin{aligned}0\geq&\int_{B_1}\eta' r|\nabla u|^2dV+(n-2)\int_{B_1}\eta|\nabla u|^2dV-3(n-1)\Lambda\int_{B_1}\eta r|\nabla u|^2dV\\&-2\int_{B_1}\eta' r|\partial_r u|^2dV.\end{aligned}$$

Recalling the definition of the cut-off function η and noticing that

$$\eta' r=-\tau\Big(\frac{\partial}{\partial\tau}\eta\Big),$$

we rewrite the above inequality as

$$\begin{aligned}0\geq&-\tau\frac{\partial}{\partial\tau}\int_{B_1}\eta|\nabla u|^2+(n-2)\int_{B_1}\eta|\nabla u|^2-3(n-1)\tau(1+\sigma_1)\Lambda\int_{B_1}\eta|\nabla u|^2\\&-2\int_{B_1}\eta' r|\partial_r u|^2\\\geq&-\tau\frac{\partial}{\partial\tau}\int_{B_1}\eta|\nabla u|^2+(n-2)\int_{B_1}\eta|\nabla u|^2-3(n-1)\tau(1+\sigma_1)\Lambda\int_{B_1}\eta|\nabla u|^2.\end{aligned}$$

Finally, multiplying the last inequality by $\tau^{1-n}e^{c\tau\Lambda}$ (where $c=3(n-1)$), integrating over $\tau\in[\rho_0, \rho_1]$ and letting $\sigma_1\to 0$, we obtain the monotonicity formula of Proposition 1.2. Q.E.D.

Now, Lemma 1.1 follows from Proposition 1.2 and from Theorem 2.2 in [5]. We observe that the gradient estimate of Lemma 1.1 enables us to bound $\lambda^2|\nabla u(x)|^2$ (where $\lambda=\mathrm{dist}(x, \Sigma)$) by the energy $E(u)$. Namely, we have the following

COROLLARY 1.3. *Assume that a weakly harmonic map $u: M\to N$ satisfies*

$$\int_M |\nabla u|^p dV<\infty, \quad p=\frac{2(n-d)}{n-1-d}.$$

Then, there is $\varepsilon_0>0$ such that if $\int_M |\nabla u|^2 dV\leq\varepsilon_0$ we have

$$\lambda^2|\nabla u(x)|^2\leq C\int_M |\nabla u|^2 dV, \quad \lambda=\mathrm{dist}(x, \Sigma),$$

for $x\notin\Sigma$, where C depends only on n, a, N and the Ricci curvature of (M, g).

PROOF OF COROLLARY 1.3. Take $\rho_0=2\lambda$, $\rho_1=1$ in Proposition 1.2 to get

$$\lambda^{2-n}\int_{B_\lambda(x)} |\nabla u|^2 dV\leq C_0(2\lambda)^{2-n}\int_{B_{2\lambda}(x_0)} |\nabla u|^2 dV\leq C\int_M |\nabla u|^2 dV,$$

where x_0 is the orthogonal projection of x onto Σ. By Lemma 1.1 there exists $\varepsilon_0>0$ such that if $\int_M |\nabla u|^2 dV\leq\varepsilon_0$ then $|\nabla u(x)|^2\leq C\lambda^{-2}\int_M |\nabla u|^2 dV$. Q.E.D.

Section 2. Proof of Theorem 1

First, we prepare two analysis lemmas which will be used in the proof of our main theorems.

LEMMA 2.1. *Let $\Omega\subset R^n$ be a bounded smooth domain. Given a subdomain $\Omega_0\subset\Omega$ with*

$$\Omega_0\subset\bar{\Omega}_0\subset\Omega,$$

there exists a constant $C=C(\Omega_0, \Omega)$ such that

$$\sup_{\Omega_0} |v| \leq C$$

for every $v \in C(\Omega)$ *solution of* $\Delta v = 0$ *in* Ω *with* $\int_\Omega v = 0$, $\int_\Omega |\nabla v|^2 \leq 1$.

PROOF. Suppose, by contradiction, that there exists $v_i \in C(\Omega)$, $i \in N$, with

$$\Delta v_i = 0 \text{ in } \Omega, \quad \int_\Omega v_i = 0, \quad \int_\Omega |\nabla v_i|^2 \leq 1$$

and

$$\sup_{\Omega_0} |v_i| = \rho_i \longrightarrow \infty.$$

Then, defining $w_i = v_i/\rho_i$, we have

$$\Delta w_i = 0 \text{ in } \Omega, \quad \int_\Omega w_i = 0, \quad \int_\Omega |\nabla w_i|^2 \leq \frac{1}{\rho_i^2} \longrightarrow 0, \tag{1}$$

and

$$\sup_{\Omega_0} |w_i| = 1.$$

Now, let

$$d_0 = \operatorname{dist}(\bar{\Omega}_0, \partial\Omega), \quad r = d_0/2,$$

and let $x_i \in \partial\Omega_0$ be such that

$$|w_i(x_i)| = \sup_{\Omega_0} |w_i| = 1.$$

By the mean value theorem we have

$$w_i(x_i) = \frac{1}{\operatorname{vol} B_r(x_i)} \int_{B_r(x_i)} w_i(x)\,dx,$$

hence (with $c_n = (\operatorname{vol} B_1)^{1/2}$)

$$1 = |w_i(x_i)| \leq \frac{1}{\operatorname{vol} B_r(x_i)} \left(\int_{B_r(x_i)} w_i^2 dx \right)^{1/2} (\operatorname{vol} B_r(x_i))^{1/2}$$

$$= \frac{1}{C_n r^{n/2}} \left(\int_{B_r(x_i)} w_i^2 dx \right)^{1/2} \leq \frac{1}{C_n r^{n/2}} \left(\int_\Omega w_i^2 dx \right)^{1/2}.$$

Since $\int_\Omega w_i = 0$ we can use Poincare's inequality to obtain

$$1 \leq \frac{\lambda_1^{1/2}}{C_n r^{n/2}} \left(\int_\Omega |\nabla w_i|^2 dx \right)^{1/2}.$$

But this contradicts the fact that $\int_\Omega |\nabla w_i|^2 \to 0$ (cf. (1)) Q.E.D.

LEMMA 2.2. *Let Ω_0, Ω be as in Lemma 2.1 with $\partial\Omega_0$ piecewise smooth. Suppose that*

$$v_i \in C^\infty(\bar{\Omega}_0) \cap W^{1,2}(\Omega)$$

is such that

$$-\Delta v_i = h_i \longrightarrow 0 \quad in \quad L^1(\Omega), \tag{2}$$
$$\int_\Omega v_i = 0, \int_\Omega |\nabla v_i|^2 = 1, \ |\nabla v_i| \leq C_0 \ in \ \Omega_0,$$

hence $v_i \to v_\infty$ weakly in $W^{1,2}(\Omega)$ [by passing, if necessary, to a subsequence]. Then

$$\int_{\Omega_0} |\nabla v_i|^2 \longrightarrow \int_{\Omega_0} |\nabla v_\infty|^2. \tag{3}$$

PROOF. By passing to a subsequence, if necessary, we may assume that

$$\begin{aligned} v_i &\longrightarrow v_\infty && \text{in } L^2(\Omega) \\ v_i &\longrightarrow v_\infty \text{ a.e.} && \text{in } \Omega. \end{aligned} \tag{4}$$

In fact, by compactness of the inclusions $W^{1,2}(\Omega) \subset W^{s,2}(\Omega)$, $0 \leq s < 1$, we have that

$$v_i \longrightarrow v_\infty \text{ in } W^{s,2}(\Omega), \qquad 0 \leq s < 1.$$

Therefore, fixing $1/2 < s < 1$ and using the fact that the trace map

$$\tau : W^{s,2}(\Omega) \longrightarrow W^{s-\frac{1}{2},2}(\partial\Omega_0)$$

is continuous, we obtain, in particular, that

$$v_i \to v_\infty \quad \text{in } L^2(\partial\Omega_0), \tag{5}$$

hence

$$\int_{\partial\Omega_0} v_i b \to \int_{\partial\Omega_0} v_\infty b \qquad \text{for all } b \in C^\infty(\partial\Omega_0). \tag{6}$$

Next, we claim that

$$\int_{\partial\Omega_0}\frac{\partial v_i}{\partial n}\chi \longrightarrow \int_{\partial\Omega_0}\frac{\partial v_\infty}{\partial n}\chi \qquad \text{for all } \chi\in C^\infty(\partial\Omega_0). \tag{7}$$

Indeed, given $\chi\in C^\infty(\partial\Omega_0)$ and letting

$$\hat{\chi}\in C_0^\infty(\Omega)$$

denote any extension of X, we obtain from (2), (5), (6) and Green's second identity that

$$\begin{aligned}\int_{\partial\Omega_0}\frac{\partial v_i}{\partial n}\chi &= \int_{\partial\Omega_0}\frac{\partial \chi}{\partial n}v_i+\int_{\Omega_0}(\Delta v_i)\hat{\chi}-\int_{\Omega_0}(\Delta\hat{\chi})v_i \\ \longrightarrow \int_{\partial\Omega_0}\frac{\partial \chi}{\partial n}v_\infty-\int_{\Omega_0}(\Delta\hat{\chi})v_\infty &= \int_{\partial\Omega_0}\frac{\partial v_\infty}{\partial n}\chi-\int_{\Omega_0}(\Delta v_\infty)\hat{\chi}.\end{aligned}$$

Therefore, (7) holds since v_∞ is a weak solution (hence, by Weyl's lemma, a C^∞ classical solution) of $\Delta v=0$ in Ω by (2). Now, in view of (2), (4) and Green's identity, we have

$$\int_{\Omega_0}|\nabla v_i|^2-\int_{\partial\Omega_0}\frac{\partial v_i}{\partial n}v_i=\int_{\Omega_0}(-\Delta v_i)v_i=\int_{\Omega_0}h_iv_i\longrightarrow 0, \tag{8}$$

where the convergence to zero follows from the fact that the v_i are uniformly bounded in Ω_0, since $v_i\to v_\infty$ a.e. and $|\nabla v_i|\le c_0$ by (2), (4). On the other hand, since (5), (7) and (again) Green's identity imply

$$\int_{\partial\Omega_0}\frac{\partial v_i}{\partial n}v_i\longrightarrow\int_{\partial\Omega_0}\frac{\partial v_\infty}{\partial n}v_\infty=\int_{\Omega_0}(\Delta v_\infty)v_\infty+\int_{\Omega_0}|\nabla v_\infty|^2 \tag{9}$$

and since v_∞ is a solution of $\Delta v=0$ in Ω, we finally obtain (3) from (8) and (9). Q.E.D.

PROOF OF THEOREM 1. By Proposition 1.2 we have the scaling inequality, for $0<\rho<1$,

$$\rho^{2-n}\int_{B_\rho}|\nabla u|^2dV\le c\int_M|\nabla u|^2dV. \tag{10}$$

Combining this with the gradient estimate, we obtain $\varepsilon_0>0$ such that

$$|\nabla u(x)|^2\le C\lambda^{-2}E(u) \qquad \text{for } x\notin\Sigma, \tag{11}$$

if $E(u)\le\varepsilon_0$. In view of (10) and (11), the argument in [9] for stationary harmonic maps can be carried over. For completeness of this paper, we include a detailed proof based on the inequalities (10), (11) and Lemmas 2.1 and 2.2.

First we show that, under the assumptions of Theorem 1, there exist numbers $\varepsilon_0>0$ and $\sigma\in(0,1)$ depending only on the metric g, $K=\|f\|_{1,\alpha}$ and N such that, if $E(u)\leq\varepsilon_0$ then

$$\sigma^{2-n}E_\sigma(u)\leq\frac{1}{2}E(u), \tag{12}$$

where

$$E_\sigma(u)=\int_{B_\sigma}|\nabla u|^2dV$$

and B_σ is the geodesic ball of radius σ.

We prove this by contradiction. So, assume that there is a sequence of weakly harmonic maps u_i satisfying the assumptions of Theorem 1, with Σ_i being the graph of a function f_i, $\|f_i\|_{1,\alpha}\leq K$, such that

$$E(u_i)\leq\frac{1}{i}$$

but

$$\sigma^{2-n}E_\sigma(u_i)\geq\frac{1}{2}E(u_i). \tag{13}$$

Define v_i by

$$v_i=\frac{1}{E(u_i)^{1/2}}(u_i-\bar{u}_i),$$

where

$$\bar{u}_i=\frac{1}{\operatorname{vol}B_1}\int_{B_1}u_i dV.$$

Then $E(v_i)=1$,

$$\int_{B_1}v_i=0$$

and, by passing to a subsequence, if necessary, we may assume that $v_i\rightarrow v_\infty$ weakly in $W^{1,2}(M,\boldsymbol{R}^k)$ and $v_i\rightarrow v_\infty$ strongly in $L^p(M,\boldsymbol{R})^k$ for $p\in[1,2n/(n-2))$. And dividing (13) by $E(u_i)$ yields

(14) $$\sigma^{2-n}E_\sigma(v_i)\geq\frac{1}{2}.$$

Now, by the harmonic map equations $\Delta u_i=g^{\alpha\beta}A(\partial_\alpha u, \partial_\beta u)$ we obtain

(15) $$\Delta v_i=h_i,$$

where $h_i=E(u_i)^{1/2}g^{\alpha\beta}A(\partial_\alpha v, \partial_\beta v)\to 0$ in L^1 since $E(u_i)\to 0$. Therefore v_∞ satisfies

$$\Delta v_\infty=0$$

in the weak sense and, by Weyl's lemma, v_∞ is a classical smooth harmonic function from M to $\boldsymbol{R}^k$.

Next, we claim that

(16) $$E_\sigma(v_i)\to E_\sigma(v_\infty) \qquad \text{as } i\to\infty.$$

To see this, let $\eta>0$ be an arbitrary number. Since $\|f_i\|_{1,\alpha}\leq K$ there is a subsequence of f_i (again denoted by f_i) such that $f_i\to f$ uniformly in the c^1 norm. Let Σ_i, Σ denote the graphs of f_i, f, respectively. Let Σ_λ denote the cylinder

$$\Sigma_\lambda=\{x\in B_1 \mid \mathrm{dist}(x, \Sigma)\leq\lambda\}$$

and choose $\lambda_0>0$ such that, for $\lambda\leq\lambda_0$,

(17) $$\int_{B_\sigma\cap\Sigma_\lambda}|\nabla v_\infty|^2 dV\leq\frac{1}{3}\eta.$$

Cover $B_\sigma\cap\Sigma_\lambda$ by balls centered at $x_j\in\Sigma$, $j=1, 2, \cdots, N(\lambda)$, with radii $\mu=c_1\lambda$. Because the metric g on B is close to the Euclidean metric, we can arrange so that $N(\lambda)\leq c_2\lambda^{-d}$. And, by the inequality (10), we have

$$\mu^{2-n}\int_{B_\mu(x_j)}|\nabla u_i|^2 dV\leq CE(u_i),$$

or, in terms of the v_i,

$$\int_{B_\mu(x_j)}|\nabla v_i|^2 dV\leq C\mu^{n-2}.$$

Therefore, we obtain

$$\int_{B_\sigma\cap\Sigma_\lambda}|\nabla v_i|^2 dV\leq\sum_{j=1}^{N(\lambda)}\int_{B_\mu(x_j)}|\nabla v_i|^2 dV\leq C\lambda^{-d}\lambda^{n-2},$$

and, since $n-d>2$, we can choose $\lambda_1>0$ so that

(18) $$\int_{B_\sigma\cap\Sigma_\lambda}|\nabla v_i|^2dV\leq\frac{1}{3}\eta \quad i=1,2,3,\cdots$$

for $\lambda\leq\lambda_1$. Fix $\lambda>0$ with $\lambda\leq\min\{\lambda_0,\lambda_1\}$. There is an integer I_1, such that $B_\sigma\cap\Sigma_i\subset B_\sigma\cap\Sigma_\lambda$ for $i\geq I_1$. By Lemma 2.2 with $\Omega_0=B_\sigma\backslash\Sigma_\lambda$, $\Omega=B_1$, we obtain that

$$\int_{\Omega_0}|\nabla v_i|^2dV\longrightarrow\int_{\Omega_0}|\nabla v_\infty|^2dV \qquad \text{as } i\to\infty.$$

Thus, we can choose $I_2\geq I_1$, such that

(19) $$\left|\int_{B_\sigma\backslash\Sigma_\lambda}(|\nabla v_i|^2-|\nabla v_\infty|^2)dV\right|\leq\frac{1}{3}\eta$$

for $i\geq I_2$. From (17), (18) and (19) we obtain

$$\left|\int_{B_r}(|\nabla v_i|^2-|\nabla v_\infty|^2)dV\right|\leq\frac{1}{3}\eta+\frac{1}{3}\eta+\frac{1}{3}\eta=\eta$$

for all $i\geq I_2$ and, hence, we have proved our claim (16):

$$E_\sigma(v_i)\longrightarrow E_\sigma(v_\infty) \qquad \text{as } i\to\infty.$$

Next, letting $i\to\infty$ in (14) gives

(20) $$\sigma^{2-n}E_\sigma(v_\infty)\geq\frac{1}{2}.$$

And by Lemma 2.1 with $\Omega_0=B_{1/2}$, there exists a constant $c>0$ such that

(21) $$\sup_{B_{1/2}}|v_\infty|\leq C.$$

Therefore, combining (20), (21) with the "interior gradient estimates" for harmonic functions yields

$$\frac{1}{2}\leq\sigma^{2-n}E_\sigma(v_\infty)\leq C\sigma^{2-n}\sigma^n\sup_{B_{1/2}}|v_\infty|^2\leq C\sigma^2,$$

which is clearly a contradiction for σ small. Thus, we have showed our initial claim that (12) holds true if $E(u)$ is small.

Finally, by a simple iteration argument as that in [9] we get that

$$r^{2-n}E_r(u)\leq Cr^{2\beta}E(u)$$

for any $r\in(0,\sigma)$ and some $\beta\in(0,1)$. Hence, by a theorem of C. B. Morrey [12], u is regular on a small ball centered at $0\in\Sigma$. Applying the above argument to every point of Σ we obtain the desired result that u is regular in the interior of M if its total energy is small. Q.E.D.

REMARK. Theorem 1 is primarily interesting for $d\geq 1$. If $d=0$ the theorem asserts the removability of isolated singularities under the assumptions that $\nabla u\in L^{p}(B_{2}, N)$ for $p=2n/(n-1)$ and the total energy

$$E(u)=\int_{B_2}|\nabla u|^{2}$$

is small. Note that for $n=3$, one has $p=3=n$ and, in this case, the smallness assumption on the energy is implied by the fact that

$$\int_{B_2}|\nabla u|^{n}<\infty.$$

Indeed, one can rescale u to get $u_{\delta}: B_{2}\rightarrow N$ defined by $u_{\delta}(x)=u(\delta x)$, $\delta>0$. Then, using the scaling invariance of

$$\|\nabla u\|_{L^{n}}$$

and choosing $\delta>0$ small we get that

$$E(u_{\delta})=\int_{B_2}|\nabla u_{\delta}|^{2}\leq C\left(\int_{B_2}|\nabla u_{\delta}|^{n}\right)^{2/n}=C\left(\int_{B_{2\delta}}|\nabla u|^{n}\right)^{2/n}$$

is also small and, hence, we can apply Theorem 1 to u_{δ} to get $u_{\delta}\in C^{\infty}(B_{1}, N)$, that is, the isolated singularity $x=0$ is removable. In particular, the main theorem of Takakuwa [10] is recovered in this case. On the other hand, note that $p=2n/(n-1)$ decreases toward 2 as n increases.

Section 3. Inequalities of monotonicity type

One of the properties that make stationary harmonic maps easier to work with is that they satisfy a monotonicity inequality. Proposition 1.2 is a typical monotonicity inequality. In this section, a similar inequality is obtained. Then Theorem 2 is proved by making use of this inequality. We let f denote the function

$$f(\rho)=\begin{cases} \ln\rho & \text{if } n=d+4\\ \rho^{d+4-n}/(d+4-n) & \text{if } n\neq d+4,\end{cases}$$

and C_ρ, D_ρ denote the cylinders

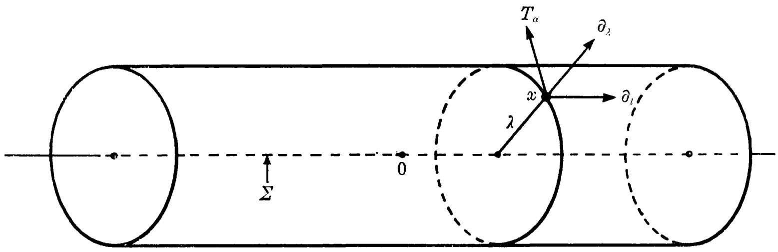
$$C_\rho=\{x|\operatorname{dist}(x,\Sigma)\le\rho\}\cap B_2(0),$$
$$D_\rho=C_\rho\cap B_{3/2}(0).$$

PROPOSITION 3.1. *Suppose that a weakly harmonic map $u\in W^{1,2}(M,N)$ satisfies condition* (ii) *in Theorem* 1 *with $d<n-2$. Then, there exist constants $c, K, \Lambda\ge 0$ such that, for $0<\rho_1<\rho_2\le 1$, we have*

$$e^{c\rho_1\Lambda}\rho_1^{d+2-n}\int_{D_{\rho_1}}|\nabla u|^2dV\le e^{c\rho_2\Lambda}\rho_2^{d+2-n}\int_{C_{\rho_2}}|\nabla u|^2dV$$
$$+K(f(\rho_2)-f(\rho_1))\Big(\int_{C_{\rho_2}}|\nabla u|^2dV\Big).$$

PROOF. For simplicity, we now assume that the metric is Euclidean and that Σ lies on a straight line, which is taken as the ρ-coordinate axis. We denote as before $\lambda=\operatorname{dist}(x,\Sigma)$ and now choose the orthonormal frame of TM

$$\partial_\lambda,\partial_l, T_\alpha\ (\alpha=1,\cdots,n-2).$$



We define cut-off functions η_τ, ξ_σ, and ζ as follows. As before, let $\varphi\in C^\infty(\boldsymbol{R}^+,\boldsymbol{R})$ be such that, for $\sigma_1>0$, $\varphi(\lambda)=1$ for $0\le\lambda\le 1$, $0\le\varphi(\lambda)\le 1$ for $1\le\lambda\le 1+\sigma_1$, $\varphi(\lambda)=0$ for $\lambda\ge 1+\sigma_1$ and $\varphi'(\lambda)\le 0$. This time, we define $\eta_\tau\in C^\infty(\boldsymbol{R}^+,\boldsymbol{R})$ by

$$\eta_\tau(\lambda)=\varphi\Big(\frac{\lambda}{\tau}\Big),$$

for $\tau\in[\rho_1,\rho_2]$, and introduce $\zeta\in C^\infty([-2,2],\boldsymbol{R})$ so that $\zeta(l)=1$ if $|l|\le 3/2$, $0\le\zeta(l)\le 1$ if $3/2\le|l|\le 7/4$ and $\zeta(l)=0$ if $|l|\ge 7/4$. The cut-off function $\xi_\sigma\in C^\infty(\boldsymbol{R}^+,\boldsymbol{R})$, $\sigma>0$, is chosen as before, that is, $\xi_\sigma(\lambda)=0$ if $0\le\lambda\le\sigma$, $0\le\xi_\sigma(\lambda)\le 1$ if $\sigma\le\lambda\le 2\sigma$, $\xi_\sigma(\lambda)=1$ if $\lambda\ge 2\sigma$ and $|\xi_\sigma'|\le 2\sigma^{-1}$, where now "′" means

$$\frac{\partial}{\partial \lambda}.$$

We will let " $\cdot$ " mean

$$\frac{\partial}{\partial l}.$$

As a variation vector field we now take $X(x)=\eta_\tau(\lambda)\xi_\sigma(\lambda)\zeta(l)\lambda\partial_\lambda(x)$ and, for small $t\geq 0$, let $u_t(x)=u(x+t\exp X(x))$. Then we compute

$$\begin{aligned}
&\nabla_{\partial_\lambda}X=(\eta\xi\lambda)'\zeta\partial_\lambda,\\
&\nabla_{\partial_l}X=\eta\xi\lambda\dot{\zeta}\partial_\lambda,\\
&\nabla_{T_\alpha}X=\nabla_{\lambda T_\alpha}(\eta\xi\zeta\partial_\lambda)=\eta\xi\zeta T_\alpha \qquad (\alpha=1,\cdots,n-2),
\end{aligned}$$

$$\operatorname{div} X=\sum_{i=1}^n\langle\nabla_{e_i}X, e_i\rangle=(\eta\xi\lambda)'\zeta+(n-2)\eta\xi\zeta=(\eta\xi)'\zeta\lambda+(n-1)\eta\xi\zeta,$$

$$\begin{aligned}
\sum_{i=1}^n\langle\nabla u(\nabla_{e_i}X), \nabla u(e_i)\rangle&=\langle\nabla u[(\eta\xi\lambda)'\zeta\partial_\lambda], \nabla u(\partial_\lambda)\rangle+\langle\nabla u(\eta\xi\lambda\dot{\zeta}\partial_\lambda), \nabla u(\partial_l)\rangle\\
&\quad+\sum_{\alpha=1}^{n-2}\langle\nabla u(\eta\xi\zeta T_\alpha), \nabla u(T_\alpha)\rangle\\
&=(\eta\xi\lambda)'\zeta|\partial_\lambda u|^2+\eta\xi\lambda\dot{\zeta}\langle\partial_\lambda u, \partial_l u\rangle+\sum_{\alpha=1}^{n-2}\eta\xi\zeta|\partial_\alpha u|^2\\
&=(\eta\xi)'\zeta\lambda|\partial_\lambda u|^2+\eta\xi\zeta|\nabla u|^2-\eta\xi\zeta|\partial_l u|^2+\eta\xi\lambda\dot{\zeta}\langle\partial_\lambda u, \partial_l u\rangle.
\end{aligned}$$

Hence, by the first variation formula for u_t we obtain

$$\begin{aligned}
0=&\int_M[(\eta\xi)'\zeta\lambda+(n-1)\eta\xi\zeta]|\nabla u|^2dV\\
&-2\int_M[(\eta\xi)'\zeta\lambda|\partial_\lambda u|^2+\eta\xi\zeta|\nabla u|^2-\eta\xi\zeta|\partial_l u|^2+\eta\xi\lambda\dot{\zeta}\langle\partial_\lambda u, \partial_l u\rangle]dV,
\end{aligned}$$

so that

$$\begin{aligned}
-2\int_M\eta'\xi\zeta\lambda|\partial_\lambda u|^2dV=&2\int_M\eta\xi'\zeta\lambda|\partial_\lambda u|^2dV+(3-n)\int_M\eta\xi\zeta|\nabla u|^2dV\\
&-\int_M\eta'\lambda\xi\zeta|\nabla u|^2dV-\int_M\eta\xi'\zeta\lambda|\nabla u|^2dV\\
&-2\int_M\eta\xi\zeta|\partial_l u|^2dV+2\int_M\eta\xi\lambda\dot{\zeta}\langle\partial_\lambda u, \partial_l u\rangle dV\\
=2\int_M\eta\xi'\zeta\lambda|\partial_\lambda u|^2dV&+(3-n)\int_M\eta\xi\zeta|\nabla u|^2dV+\tau\frac{\partial}{\partial\tau}\int_M\eta\xi\zeta|\nabla u|^2dV\\
-\int_M\eta\xi'\zeta\lambda|\nabla u|^2dV&-2\int_M\eta\xi\zeta|\partial_l u|^2dV+2\int_M\eta\xi\lambda\dot{\zeta}\langle\partial_\lambda u, \partial_l u\rangle dV,
\end{aligned}$$

where we used the fact that

$$\eta'\lambda = -\tau\left(\frac{\partial}{\partial\tau}\eta\right)$$

in the last equality.

Now, replacing ζ by ζ^2 and using the inequality $2|A||B| \leq |A|^2 + |B|^2$ with

$$|A| = (2\eta\xi)^{1/2}\zeta|\partial_t u|,$$
$$|B| = (2\eta\xi)^{1/2}\lambda|\dot{\zeta}||\partial_\lambda u|,$$

in the last term of the previous equality, we obtain the inequality

$$0 \leq 2\int_M \eta\xi'\lambda\zeta^2|\partial_\lambda u|^2 dV + (3-n)S_\sigma + \tau\frac{\partial}{\partial\tau}S_\sigma + R$$

where

$$S_\sigma = \int_M \eta\xi\zeta^2|\nabla u|^2 dV$$

and

$$R = 2\int_M \eta\xi\lambda^2|\dot{\zeta}|^2|\partial_\lambda u|^2 dV.$$

Since we have that

$$\left|\int_M \eta\xi'\lambda\zeta^2|\partial_\lambda u|^2 dV\right| \leq C\int_{C_{2\sigma}/C_\sigma} |\xi'|\lambda|\nabla u|^2 dV \longrightarrow 0$$

as $\sigma \to 0$, and R can be estimated by

$$R \leq C\int_{C_{\tau(1+\sigma_1)}} \lambda^2|\nabla u|^2 dV,$$

we obtain the inequality

$$0 \leq (3-n)S + \tau\frac{\partial}{\partial\tau}S + 2C\int_{C_{\tau(1+\sigma_1)}} \lambda^2|\nabla u|^2 dV,$$

hence

$$0 \leq (3-n)S + \tau\frac{\partial}{\partial\tau}S + 2C\tau^2(1+\sigma_1)^2\int_{C_{\rho_2}} |\nabla u|^2 dV + R_1$$

for all $\tau \leq \rho_2$, where $R_1 \to 0$ as $\sigma_1 \to 0$. Finally, multiplying the last inequality by τ^{2-n}, integrating over $\tau \in [\rho_1, \rho_2]$ and letting $\sigma_1 \to 0$, we obtain

$$0\leq\int_{\rho_1}^{\rho_2}\frac{\partial}{\partial\tau}(\tau^{3-n}S)d\tau+2C\Big(\int_{C_{\rho_2}}|\nabla u|^2dV\Big)\int_{\rho_1}^{\rho_2}\tau^{4-n}d\tau,$$

which implies the monotonicity formula of Proposition 3.1. Q.E.D.

PROOF OF THEOREM 2. Suppose $n-d\geq 4$. Recall that, for the cut-off function $\xi=\xi_\sigma$ in Proposition 1.2, we have

$$\int_{B_1}|d\xi||\nabla u|^2dV\leq C(2\sigma)^{-1}\int_{C_{2\sigma}}|\nabla u|^2dV.$$

And Proposition 3.1 with $\rho_1=2\sigma<1=\rho_2$ implies

$$(2\sigma)^{d+2-n}\int_{C_{2\sigma}}|\nabla u|^2dV\leq C(1+|f(2\sigma)|)\int_M|\nabla u|^2dV,$$

where

$$f(2\sigma)=\begin{cases}ln(2\sigma) & \text{if } n=d+4\\ (2\sigma)^{d+4-n}/(d+4-n) & \text{if } n>d+4.\end{cases}$$

Therefore, if $n>d+4$ we obtain

$$(2\sigma)^{-1}\int_{C_{2\sigma}}|\nabla u|^2dV\leq C\sigma^{-1}\sigma^{n-d-2}(1+\sigma^{d+4-n})\int_M|\nabla u|^2dV\longrightarrow 0$$

as $\sigma\to 0$. Similarly, if $n=d+4$, we also have

$$(2\sigma)^{-1}\int_{C_{2\sigma}}|\nabla u|^2dV\leq C\sigma^{-1}\sigma^{n-d-2}ln(\sigma^{-1})\int_M|\nabla u|^2dV\longrightarrow 0$$

as $\sigma\to 0$. Thus

$$\lim_{\sigma\to 0}\int_{B_1}|d\xi||\nabla u|^2dV=0$$

if $n-d\geq 4$, and we can follow the argument in Proposition 1.2 to prove all the results in Sections 1 and 2 without the assumption (i) of Theorem 1. In particular, Theorem 2 holds true. Q.E.D.

Section 4. Inequalities of monotonicity type for Yang-Mills fields

In this section, we derive analogous formulas (Proposition 1.2 and Proposition 3.1) for Yang-Mills fields.

Let $P=P(M,G)$ be a principal bundle with compact structure group

G over a Riemannian manifold M. Suppose that an Ad_G invariant inner product has been put on the Lie algebra $\mathfrak{G}$ of the group G. Let $\mathrm{Ad}P = P \times_{\mathrm{Ad}G} \mathfrak{G}$. The inner product on $\mathfrak{G}$ induces a fibre metric on $\mathrm{Ad}P$, making all tensor product bundles $\wedge^k T^*M \otimes \mathrm{Ad}P$ into Riemannian vector bundles over M.

The Sobolev spaces of connections on P are defined by

$$W^{k,q}(P) = \{w_0 + B \mid B \in W^{k,q}(T^*M \otimes \mathrm{Ad}P)\},$$

where w_0 is a fixed smooth connection on P. The space of smooth connections on P is an affine space. As a consequence, the definition of Sobolev space $W^{k,q}(P)$ is independent of the choice of w_0. For a smooth connection w on P, its curvature $\Omega \in \wedge^2 T^*M \otimes \mathrm{Ad}P$ is given by

$$\Omega = \Omega(w) = D \circ \nabla,$$

where ∇ is the full covariant differential on $\wedge^k T^*M \otimes \mathrm{Ad}P$ induced by w. The exterior covariant derivative D is the projection by exterior product of ∇, D and ∇ agree on 0-forms. Let $\tilde{w} = w + B \in W^{1,2}(P) \cap W^{0,4}(P)$, then

$$\Omega(\tilde{w}) = \Omega(w) + DB + [B, B]$$

and

$$\Omega(\tilde{w}) \in W^{0,2}(\wedge^2 T^*M \otimes \mathrm{Ad}P).$$

The Yang-Mills functional, for $w \in W^{1,2}(P) \cap W^{0,4}(P)$, is given by

$$F(w) = \int_M |\Omega|^2 dV.$$

Let

$$YM(w) = \{w' = w + B \in W^{1,2}(P) \cap W^{0,4}(P) \mid B \in W_0^{1,2} \cap W_0^{0,4}(T^*M) \otimes \mathrm{Ad}P)\}.$$

DEFINITION. W is called a Yang-Mills connection if for all 1-parameter families $w^t \in \mathrm{YM}(w)$ such that $w^0 = w$ and $F(w^t)$ is differentiable in t, $\left.\frac{d}{dt}\right|_{t=0} F(w^t) = 0$.

In [10], it was shown that, for any 1-parameter family $w^t \in \mathrm{YM}(w)$ that is the variation of w generated by a 1-parameter family ϕ^t of diffeomorphisms of M with $\phi^0 = \mathrm{Identity}$, the first variation formula holds. We state this result here as a lemma.

LEMMA 4.1. *Let* $X=\frac{d}{dt}\phi^t|_{t=0}$. *Let* $\{e_i\}$ *be an orthonormal frame field for TM. Then*

$$\frac{d}{dt}F(w^t)|_{t=0}=-\int_M\left[(\operatorname{div} x)|\Omega|^2-4\langle\Omega(\nabla_{e_i}X, e_j),\ \Omega(e_i, e_j)\rangle\right]dV,$$

where $|\Omega|^2=\sum_{i,j=1}^{n}\langle\Omega(e_i, e_j),\ \Omega(e_i, e_j)\rangle$, $n=\dim M$.

Based on this formula, Price obtained a monotonicity inequality for *r-stationary* Yang-Mills fields. The r-stationary means that $\frac{d}{dt}F(w^t)|_{t=0}=0$ in Lemma 4.1. He also derived a similar formula assuming that, instead of r-stationarity, the generalized quark current density is bounded in a suitable sense.

In this section, we derive inequalities of monotonicity type for Yang-Mills fields without the assumption of r-stationarity. For the rest of the section, take $M=B_2(0)$ as before. Let Σ be the graph of a $c^{1,\alpha}$ function with $d=\dim\Sigma$. Suppose that

$$w\in W^{1,q}(P)\cap W^{0,4}(P) \qquad (\text{for } q\geq 2)$$

is a Yang-Mills connection such that its induced connection on $P(M\backslash\Sigma, G)$ by the inclusion mapping

$$i:\ M\backslash\Sigma\longrightarrow M$$

is smooth.

As a counterpart of Proposition 1.2, we have

PROPOSITION 4.2. *If* $d=\dim\Sigma<n-3$ *and* $q=\frac{2(n-d)}{n-1-d}$, *then there exists a constant* $C, \Lambda\geq 0$ *such that, for* $0<\rho_1<\rho_2\leq 1$, $x_0\in\Sigma\cap B_{1/2}(0)$,

$$e^{c\Lambda\rho_1}\rho_1^{4-n}\int_{B_{\rho_1}(x_0)}|\Omega|^2dV\leq e^{c\Lambda\rho_2}\rho_2^{4-n}\int_{B_{\rho_2}(x_0)}|\Omega|^2dV.$$

PROOF. Take exactly the same frame field $\{e_i\}$ and the same 1-parameter family ϕ^t of diffeomorphisms of M as in the proof of Proposition 1.2. Recall that for $\alpha=1, 2, \cdots, n-1$ $\nabla_{e_\alpha}X=\eta\xi r(\nabla_{e_\alpha}\partial_r)+\eta r(\nabla_{e_\alpha}\xi)\partial_r$. We compute

$$\begin{aligned}
&\sum_{i,j=1}^{n}\langle\Omega(\nabla_{e_i}X, e_j),\ \Omega(e_i, e_j)\rangle\\
&=\sum_{\alpha,\beta=1}^{n-1}\langle\Omega(\nabla_{e_\alpha}X, e_\beta),\ \Omega(e_\alpha, e_\beta)\rangle+\sum_{\alpha=1}^{n-1}\langle\Omega(\nabla_{\partial_r}X, e_\alpha),\ \Omega(\partial_r, e_\alpha)\rangle\\
&\le\sum_{\alpha,\beta=1}^{n-1}\langle\Omega(r\eta\xi(\nabla_{e_\alpha}\partial_r)+\eta r(\nabla_{e_\alpha}\xi)\partial_r, e_\beta),\ \Omega(e_\alpha, e_\beta)\rangle\\
&\quad+\sum_{\alpha=1}^{n-1}\langle\Omega((\eta\xi r)'\partial_r, e_\alpha),\ \Omega(\partial_r, e_\alpha)\rangle\\
&\le\sum_{\alpha,\beta=1}^{n-1}\langle\Omega(\eta\xi(\nabla_{re_\alpha}\partial_r), e_\beta),\ \Omega(e_\alpha, e_\beta)\rangle\\
&\quad+\sum_{\alpha=1}^{n-1}\langle\Omega((\eta\xi+\xi\eta' r+\xi'\eta r)\partial_r, e_\alpha),\ \Omega(\partial_r, e_\alpha)\rangle\\
&\quad+\sum_{\alpha,\beta=1}^{n-1}\langle\Omega(\eta r(\nabla_{e_\alpha}\xi)\partial_r, e_\beta),\ \Omega(e_\alpha, e_\beta)\rangle\\
&\le\sum_{\alpha,\beta=1}^{n-1}\left\langle\Omega\left(\sum_{\mu=1}^{n-1}\xi\eta\left(\delta_{\alpha\mu}+\int_0^r\varepsilon_{\alpha\mu}ds\right)e_\mu, e_\beta\right),\ \Omega(e_\alpha, e_\beta)\right\rangle\\
&\quad+(\eta\xi+\xi\eta' r+\xi'\eta r)|\partial_r\lrcorner\Omega|^2\\
&\quad+(\eta\zeta+\zeta\eta' r+\zeta'\eta r)|\partial_r\lrcorner\Omega|^2\\
&\quad+(n-1)\eta r|\nabla_{e_\alpha}\xi||\Omega|^2\\
&\le\xi\eta\sum_{\alpha,\beta=1}^{n-1}|\Omega(e_\alpha, e_\beta)|^2+(n-1)\xi\eta r\cdot\Lambda\cdot|\Omega|^2\\
&\quad+\eta\xi|\partial_r\lrcorner\Omega|^2+\xi\left(-\tau\frac{\partial}{\partial\tau}\eta\right)|\partial_r\lrcorner\Omega|^2\\
&\quad+|\xi'|\eta r|\partial_r\lrcorner\Omega|^2+(n-1)\eta r|\nabla_{e_\alpha}\xi||\Omega|^2.
\end{aligned}$$

The integral of the last two terms converges to zero as σ (the parameter in the definition of cut-off function ξ) goes to zero, since we have $|d\xi|\le C\cdot\sigma^{-1}$ and by assumption

$$\int_M|\Omega|^q dV<\infty \qquad \text{for } q=\frac{2(n-d)}{n-1-d}.$$

The fact that w induces a smooth connection on $P(M\backslash\Sigma, G)$ implies that $F(w^t)$ is differentiable in t, where w^t is the variation of w generated by ϕ^t. The connection w is Yang-Mills then implies that

$$\frac{d}{dt}F(w^t)|_{t=0}=0.$$

By Lemma 4.1 we get

$$0=\int_M(\operatorname{div}X)|\Omega|^2dV-4\int_M\langle\Omega(\nabla_{e_i}X, e_j),\ \Omega(e_i, e_j)\rangle dV.$$

Letting $\sigma \to 0$, we get

$$0 \geq n\int_M \eta|\Omega|^2 dV - (n-1)\int_M \eta\Lambda r|\Omega|^2 dV$$
$$+\int_M \eta' r|\Omega|^2 dV - 4\int_M \eta|\Omega|^2 dV - 4(n-1)\int_M \eta\Lambda r|\Omega|^2 dV$$
$$-4\int_M \eta' r|\partial_r \lrcorner \Omega|^2 dV.$$

Note that $\eta' r = -\tau \frac{\partial}{\partial \tau}\eta$. We get

$$0 \geq (n-4)\int_M \eta|\Omega|^2 dV - 5(n-1)\int_M \eta r\Lambda|\Omega|^2 dV$$
$$-\tau\frac{\partial}{\partial \tau}\int_M \eta|\Omega|^2 dV + 4\tau\frac{\partial}{\partial \tau}\int_M \eta|\partial_r \lrcorner \Omega|^2 dV.$$

The conclusion follows from this differential inequality by the same argument as in the proof of Proposition 1.2. Q.E.D.

Next, we derive an inequality for a Yang-Mills connection $w \in W^{1,2}(P) \cap W^{0,4}(P)$. This inequality is the counterpart of Proposition 3.1.

PROPOSITION 4.3. *Suppose that* $d = \dim \Sigma < n-4$, $q=2$. *There exist constants* c, K, $\Lambda \geq 0$ *such that for* $0 < \rho_1 < \rho_2 \leq 1$

$$e^{c\rho_1\Lambda}\rho_1^{d+4-n}\int_{D_{\rho_1}} |\Omega|^2 dV \leq e^{c\rho_2\Lambda}\rho_2^{d+4-n}\int_{C_{\rho_2}} |\Omega|^2 dV + K(f(\rho_2) - f(\rho_1))\int_{C_{\rho_2}} |\Omega|^2 dV,$$

where

$$f(\rho) = \begin{cases} \ln \rho & \text{if } n = d+6 \\ \rho^{d+6-n}/(d+6-n) & \text{if } n \neq d+6, \end{cases}$$
$$C_\rho = \{x:\ \mathrm{dist}(x, \Sigma) \leq \rho\} \cap B_2(0),$$
$$D_\rho = C_\rho \cap B_{3/2}(0).$$

PROOF. Again, for simplicity of the computation, we assume that the metric M is Euclidean and that $d = \dim \Sigma = 1$. We also take the same local frame field $\{e_i\}$ $i = 1, 2, \cdots, n$, where $e_\alpha = T_\alpha$, $1 \leq \alpha \leq n-2$, $e_{n-1} = \partial_\lambda$, $e_n = \partial_l$, and the same variation field X of M as in the proof of Proposition 2.1.

Recall that

$$X=\eta_\tau(\lambda)\xi_\sigma(\lambda)\zeta(l)\lambda\partial_\lambda(x),$$
$$\operatorname{div} X=(\eta\xi)'\zeta+(n-1)\eta\xi\zeta.$$

Compute

$$\begin{aligned}
&\sum_{i,j=1}^{n}\langle\Omega(\nabla_{e_i}X, e_j),\ \Omega(e_i, e_j)\rangle\\
&=\sum_{j=1}^{n}\langle\Omega(\nabla_{\partial_\lambda}X, e_j),\ \Omega(\partial_\lambda, e_j)\rangle+\sum_{j=1}^{n}\langle\Omega(\nabla_{\partial_l}X, e_j),\ \Omega(\partial_l, e_j)\rangle\\
&\quad+\sum_{\alpha=1}^{n-2}\sum_{j=1}^{n}\langle\Omega(\nabla_{T_\alpha}X, e_j),\ \Omega(T_\alpha, e_j)\rangle\\
&=\sum_{j=1}^{n}\langle\Omega((\eta\xi\lambda)'\zeta\partial_\lambda, e_j),\ \Omega(\partial_\lambda, e_j)\rangle\\
&\quad+\sum_{j=1}^{n}\langle\Omega(\eta\xi\lambda\dot\zeta\partial_\lambda, e_j),\ \Omega(\partial_l, e_j)\rangle\\
&\quad+\sum_{\alpha=1}^{n-2}\sum_{j=1}^{n}\langle\Omega(\eta\xi\zeta T_\alpha, e_j),\ \Omega(T_\alpha, e_j)\rangle\\
&=(\eta\xi\lambda)'\zeta|\partial_\lambda\lrcorner\Omega|^2+\eta\xi\lambda\dot\zeta\sum_{j=1}^{n}\langle\Omega(\partial_\lambda, e_j),\ \Omega(\partial_l, e_j)\rangle\\
&\quad+\eta\xi\zeta\sum_{\alpha=1}^{n-2}|T_\alpha\lrcorner\Omega|^2.
\end{aligned}$$

By adding and subtracting $\eta\xi\zeta|\partial_l\lrcorner\Omega|^2$, we get

$$\begin{aligned}
&\sum_{i,j=1}^{n}\langle\Omega(\nabla_{e_i}X, e_j),\ \Omega(e_i, e_j)\rangle\le\eta\xi\zeta|\Omega|^2\\
&\quad+(\eta\xi)'\lambda\zeta|\partial_\lambda\lrcorner\Omega|^2+\eta\xi\lambda\dot\zeta\sum_{\alpha=1}^{n-2}\langle\Omega(\partial_\lambda, T_\alpha),\ \Omega(\partial_l, T_\alpha)\rangle-\eta\xi\zeta|\partial_l\lrcorner\Omega|^2.
\end{aligned}$$

Replacing ζ by ζ^2 throughout, we get

$$\begin{aligned}
&\sum_{i,j=1}^{n}\langle\Omega_{e_i}(X, e_j),\ \Omega(e_i, e_j)\rangle\le\eta\xi\zeta^2|\Omega|^2+(\eta\xi)'\lambda\zeta^2|\partial_l\lrcorner\Omega|^2\\
&\quad+2\eta\xi\lambda\zeta\dot\zeta\sum_{\alpha=1}^{n-2}|\Omega(\partial_\lambda, T_\alpha)||\Omega(\partial_l, T_\alpha)|-\eta\xi\zeta^2|\partial_l\lrcorner\Omega|^2\\
&\le\eta'\xi\lambda\zeta^2|\partial_\lambda\lrcorner\Omega|^2+\eta\xi\zeta^2|\Omega|^2+\eta\xi'\lambda\zeta^2|\partial_l\lrcorner\Omega|^2+\eta\xi\lambda^2\dot\zeta^2\sum_{\alpha=1}^{n-2}|\Omega(\partial_\lambda, T_\alpha)|^2.
\end{aligned}$$

By Lemma 3.1,

$$0=\int_M|\Omega|^2\operatorname{div} X\,dV-4\int_M\langle\Omega(\nabla_{e_i}X, e_j),\ \Omega(e_i, e_j)\rangle dV.$$

It follows that

$$0\ge\int_M|\Omega|^2(\lambda(\eta\xi)'\zeta^2+(n-1)\eta\xi\zeta^2)dV$$

$$-4\int_M \eta'\xi\lambda\zeta^2|\partial_\lambda \lrcorner \Omega|^2 dV - 4\int_M \eta\xi'\lambda\zeta^2|\partial_\lambda \lrcorner \Omega|^2 dV$$
$$-4\int_M \eta\xi\lambda^2\dot{\zeta}^2 \sum_{\alpha=1}^{n-2} |\Omega(\partial_\lambda, T_\alpha)|^2 dV.$$

Noticing that $\tau\frac{\partial}{\partial\tau}\eta = -\eta'\lambda$, we get

$$4\tau\frac{\partial}{\partial\tau}\int_M \eta\xi\zeta^2|\partial_\lambda \lrcorner \Omega|^2 dV \leq (5-n)\int_M \eta\xi\zeta^2|\Omega|^2 dV$$
$$+\tau\frac{\partial}{\partial\tau}\int_M \eta\xi\zeta^2|\Omega|^2 dV + 4\sum_{\alpha=1}^{n-2}\int_M \eta\xi\lambda^2\dot{\zeta}^2|\Omega(\partial_\lambda, T_\alpha)|^2 dV$$
$$+4\int_M |\Omega|^2\lambda\eta|\xi'|\zeta^2 dV + 4\int_M \eta|\xi'|\lambda\zeta^2|\partial_\lambda \lrcorner \Omega|^2 dV.$$

The last two terms converge to zero as $\sigma \to 0$ (σ is the parameter in the cut-off function ξ). Letting $\sigma \to 0$, we have at the limit

$$0 \leq 4\tau\frac{\partial}{\partial\tau}\int_M \eta\zeta^2|\partial_\lambda \lrcorner \Omega|^2 dV \leq (5-n)\int_M \eta\zeta^2|\Omega|^2 dV + \tau\frac{\partial}{\partial\tau}\int_M \eta\zeta^2|\Omega|^2 dV + R,$$

where

$$R = 4\sum_{\alpha=1}^{n-2}\int_M \eta\lambda^2\dot{\zeta}^2|\Omega(\partial_\lambda, T_\alpha)|^2 dV.$$

Note that

$$0 \leq R \leq 4C\int_M \eta\lambda^2|\Omega|^2 dV \leq 4C\int_{C_{\tau(1+\sigma_1)}} \lambda^2|\Omega|^2 dV,$$

where C=constant. Multiply the differential inequality by τ^{4-n}, integrate from $\tau=\rho_1$ to ρ_2, take the limit $\sigma_1 \to 0$ (σ_1 and τ are the parameters in the definition of cut-off function η). We have

$$0 \leq \rho_2^{5-n}\int_{C_{\rho_2}} |\Omega|^2 dV - \rho_1^{5-n}\int_{D_{\rho_1}} |\Omega|^2 dV + \lim_{\sigma_1\to 0}\int_{\rho_1}^{\rho_2} \tau^{4-n} R\, d\tau.$$
$$\lim_{\sigma_1\to 0}\int_{\rho_1}^{\rho_2} \tau^{4-n} R\, d\tau \leq 4C\int_{\rho_1}^{\rho_2} \tau^{4-n}\left(\int_{C_\tau} \lambda^2|\Omega|^2 dV\right) d\tau \leq 4C\left(\int_{\rho_1}^{\rho_2} \tau^{6-n} d\tau\right)\int_{C_{\rho_2}} |\Omega|^2 dV,$$
$$= 4C(f(\rho_2) - f(\rho_1))\int_{C_{\rho_2}} |\Omega|^2 dV,$$

where

$$f(\rho)=\begin{cases} ln\,\rho & \text{if } n=7 \\ \dfrac{\rho^{7-n}}{7-n} & \text{if } n\neq 7. \end{cases}$$

Q.E.D.

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D. Costa
Departamento de Matematica
Universidade de Brasilia
70.910 Brasilia, DF
Brazil
and
University of Utah
Salt Lake City, UT 84112
U. S. A.
(Visiting 1986/87)

G. Liao
Department of Mathematics
University of Utah
Salt Lake City
UT 84112
U. S. A.