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A note on Dehn surgery on links and divisors of varieties of group representations

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§1. Introduction

Let M be a compact connected, irreducible, orientable 3-manifold such that ∂M consists of n tori: $\partial M=\bigcup_{i=1}^{n}\partial_i M$, $\partial_i M\approx S^1\times S^1$ $(n\geqq 1)$. We assume that the interior of M has a hyperbolic metric of finite volume. Then $\pi_1(M)$ has, up to conjugacy, a canonical representation in $PSL_2(\boldsymbol{C})$, which then admits a lifting $\rho_0:\pi_1(M)\to SL_2(\boldsymbol{C})$. Let R_0 be an irreducible component of the complex affine algebraic set of all representations of $\pi_1(M)$ in $SL_2(\boldsymbol{C})$ that passes through ρ_0. The set of the characters of representations in R_0 is parametrized by a complex affine variety X_0 of dimension n. [2], [3].

Let α_i be slopes in $\partial_i M$ $(1\leqq i\leqq n)$, i.e. unoriented isotopy classes of essential simple closed curves in $\partial_i M$. Then α_i determine the well-defined regular functions f_{α_i} on X_0 by $f_{\alpha_i}(\chi_\rho)=\chi_\rho(a_i)^2-4$, where χ_ρ is the character of a representation ρ and a_i is any element of $\pi_1(M)$ whose conjugacy class is α_i. Note that $f_{\alpha_i}(\chi_\rho)$ is equal to the square of the difference of the two eigenvalues of $\rho(a_i)$, and $f_{\alpha_i}(\chi_\rho)=0$ if and only if $\bar{\rho}(a_i)\in PSL_2(\boldsymbol{C})$ is a parabolic element.

In this note we consider, as a step toward understanding Dehn surgery on links, certain techniques of showing the nontriviality of the fundamental group of the closed manifold $M_{\alpha_1,\dots,\alpha_n}$ obtained by Dehn surgery on M along the slopes α_i, under the assumption that we are given some informations on the intersections of the divisors of the functions f_{α_i}.

The first result deals with the case where certain intersection occurs in the Zariski-open subset $X_{0,\mathrm{irr}}$ of the characters of irreducible representations in X_0.

THEOREM 1.1. *Suppose that there are irreducible components Z_i of the divisors of zeroes of f_{α_i} $(1\leqq i\leqq n)$ with nonempty intersection in $X_{0,\mathrm{irr}}$, such that the order of zero of f_{α_i} is strictly larger than that of f_{β_i} along*

Z_i for some slope β_i in $\partial_i M$ ($1 \leqq i \leqq n$). Then for each $x \in X_{0,\mathrm{irr}} \cap \left(\bigcap_{i=1}^{n} Z_i\right)$, each representation $\rho \in R_0$ with $\chi_\rho = x$ satisfies $\rho(a_i) = \pm I$ ($[a_i] = \alpha_i$) for $1 \leqq i \leqq n$ and induces a nontrivial representation of $\pi_1(M_{\alpha_1,\dots,\alpha_n})$ in $PSL_2(\boldsymbol{C})$.

Recall that the order of zero of a regular function f along Z in a variety X is, by definition, $\mathrm{ord}_Z(f) = l_{\mathcal{O}_{Z,X}}(\mathcal{O}_{Z,X}/(\bar{f}))$, the length of the $\mathcal{O}_{Z,X}$-module in parentheses. Note that the closed subset $X_{0,\mathrm{red}}$ of reducible characters consists of the characters in X_0 which take the value 2 on the commutator subgroup of $\pi_1(M)$. Hence, for example, any nontrivial perfect (i.e. the 1st homology group$=0$) subgroup of $SL_2(\boldsymbol{C})$ is irreducible.

For the next result we assume $n=2$, i.e. ∂M consists of two tori, and (α_1, α_2) is a non-boundary slope pair, which means the following: if S is a properly embedded compact orientable surface in M with nonempty boundary such that the slope of $S \cap \partial_i M$ is, if nonempty, equal to α_i for $i=1,2$, then S has a boundary parallel component or S compresses in M. Let $\tilde{X}_0$ be a smooth complex projective surface with a birational regular map onto a projective closure $\bar{X}_0$ of X_0, $p: \tilde{X}_0 \to \bar{X}_0$. We call $\tilde{x} \in \tilde{X}_0$ a point at infinity if $p(\tilde{x}) \in \bar{X}_0 - X_0$. Under these additional hypotheses the next result deals with the case where certain intersection occurs at infinity.

THEOREM 1.2. *Suppose that there is an irreducible component Z of the divisor of zero of f_{α_1} in $\tilde{X}_0$ such that the order of zero of f_{α_1} is strictly larger than that of f_{β_1} along Z for some slope β_1 in $\partial_1 M$. Suppose that the divisors (f_{α_2}) and (f_{β_2}) intersect Z properly at a point $\tilde{x} \in \tilde{X}_0$ at infinity with the intersection multiplicities satisfying $i_{\tilde{x}}(Z \cdot (f_{\alpha_2})) > i_{\tilde{x}}(Z \cdot (f_{\beta_2}))$ for some slope β_2 in $\partial_2 M$. Then the closed manifold M_{α_1,α_2} contains a closed incompressible surface of positive genus. In particular $\pi_1(M_{\alpha_1,\alpha_2})$ is nontrivial.*

This note is motivated by the work in [1] by Culler-Gordon-Luecke-Shalen on the Property P Conjecture for knots. The results stated above are partial generalizations of some of the arguments in [1] to higher dimensions, obtained through our attempt to see if one can apply the techniques of that paper to Dehn surgery on hyperbolic links. Our proofs, thus, depend heavily on that paper and related parts in the papers [2], [3], and [4].

§ 2. Intersections in $X_{0,\mathrm{irr}}$

In this section we give a proof of Theorem 1.1 stated in § 1. Let $\nu: X_0^\nu \to X_0$ be the normalization of X_0. For a nonzero function f in the function field $C(X_0) \simeq C(X_0^\nu)$, the order of f along a prime divisor Z of X_0 is given by $\mathrm{ord}_Z(f) = \sum_{\tilde{Z}} \mathrm{ord}_{\tilde{Z}}(f)[C(\tilde{Z}) : C(Z)]$, where $\tilde{Z}$ runs through all prime divisors of X_0^ν which map onto Z, and $[C(\tilde{Z}) : C(Z)]$ denotes the degree of the field extension. Hence, in particular, there is a prime divisor $\tilde{Z}_i$ of X_0^ν such that the order of zero of f_{α_i} is strictly larger than that of f_{β_i} along $\tilde{Z}_i$ for $1 \leqq i \leqq n$.

Let $P: \pi_1(M) \to SL_2(C(R_0))$ be the tautological representation defined by $P(g)(\rho) = \rho(g)$ for $g \in \pi_1(M)$, i.e. the value at ρ of the (i, j)-entry of $P(g)$ is the (i, j)-entry of $\rho(g)$. Fix k, any countable subfield of C such that R_0 is defined over k and $P(\pi_1(M)) \subset SL_2(k[R_0])$.

Theorem 1.1 is a consequence of the following lemma.

LEMMA 2.1. *Let α and β be (conjugacy classes of) commuting elements of $\pi_1(M)$, $\tilde{Z}$ a prime divisor of X_0^ν such that $v(f_\alpha) > v(f_\beta)$ for the valuation v of the function field $K = k(X_0) \simeq k(X_0^\nu)$ determined by $\tilde{Z}$ in the normal variety X_0^ν. Then for each irreducible representation ρ with its character contained in $Z = \nu(\tilde{Z})$, we have $\rho(a) = \pm I$ ($[a] = \alpha$).*

For the proof of Lemma 2.1 we consider the extension field $L = k(R_0)$ over $K = k(X_0)$, and an appropriate valuation on it. For this purpose, we use k-valuating sequences in X_0 and R_0 (instead of using the normalization of R_0 explicitly). A k-valuating sequence, as defined in I.2 of [4], is a sequence $(x_j)_{j=1}^\infty$ of k-generic points of a complex variety V, such that $\lim_{j \to \infty} f(x_j)$ exists in $C \cup \{\infty\}$ for each $f \in k(V)$. Here, a k-generic point of V is a (closed) point in V which is not contained in any proper subvariety defined over k. These points form a dense subset of V (in the usual topology). A k-valuating sequence (x_j) defines a valuation v on $k(V)$ whose valuation ring and maximal ideal are $\mathcal{O}_v = \{f \in k(V) \mid \lim_{j \to \infty} f(x_j)$ is finite.$\}$, and $\mathfrak{m}_v = \{f \in k(V) \mid \lim_{j \to \infty} f(x_j) = 0.\}$.

Choose a k-valuating sequence (x_j) in X_0 which defines the valuation v of $K = k(X_0)$ determined by $\tilde{Z}$. For example, we may take the image in X_0 of a sequence of k-generic points of X_0^ν converging to a k-generic point of $\tilde{Z}$. Since Z contains an irreducible character by hypothesis, and $X_{0,\mathrm{red}}$ is an algebraic set defined over k, the limit $x_\infty = \lim_{j \to \infty} x_j \in Z$ is an irreducible character. Hence, over an open neighborhood of x_∞, the surjective regular map $t: R_0 \to X_0$ defined by $t(\rho)=\chi_\rho$ is the projection of representations to their equivalence classes with the fibers $t^{-1}(x)$ irreducible algebraic sets obtained by conjugation by elements of $SL_2(C)$. Here, we used the facts that the irreducible variety R_0 is a union of the equivalence classes of representations in $SL_2(C)$, and that two irreducible representations are equivalent if their characters are the same (cf. Propositions 1.1.1, 1.5.2 and 1.4.4 of [2]). Then, choose a sequence of k-generic points $(\rho_j)_{j=1}^\infty$ of R_0 with $t(\rho_j)=x_j$ which converges to a k-generic point of the irreducible component Q of the algebraic set $t^{-1}(Z)$ which contains $t^{-1}(x)$ (A slight modification of the proof of Proposition 1.5.2 of [2] shows that $\mathrm{dist}(t^{-1}(x_j), \rho)\to 0$ as $j\to\infty$ for any $\rho\in t^{-1}(x_\infty)$.).

Since $L=k(R_0)$ is a countable field, we may assume, after passing to subsequences of (x_j) and (ρ_j), that for each $f\in L$, $\lim_{j\to\infty} f(\rho_j)$ exists in $C\cup\{\infty\}$, i.e. (ρ_j) is a k-valuating sequence. Let u be the valuation of L defined by this k-valuating sequence. Regarding K as a subfield of L, we have $f(\rho_j)=f(x_j)$ for $f\in K$ and $j\geqq 1$. Hence the valuation ring $\mathcal{O}_v$ consists of the functions $f\in K$ such that $\lim_{j\to\infty} f(\rho_j)$ is finite, i.e. $\mathcal{O}_v=\mathcal{O}_u\cap K$, which then implies that the valuation u is an extension of v. Note that $u(f_\alpha)>u(f_\beta)$ from the assumption $v(f_\alpha)>v(f_\beta)$. Since the sequence (ρ_j) is bounded, $P(a)$ and $P(b)$ are commuting elements of $SL_2(\mathcal{O}_u)$, where P: $\pi_1(M)\to SL_2(L)$ is the tautological representation and $[a]=\alpha$, $[b]=\beta$. Then by Lemma 1.5.7 of [1] (which is shown by explicitly writing down the commutativity condition of the matrices), we have $P(a)\equiv\pm I \bmod \mathfrak{m}_u$. From the choice of Q and the proof of Proposition 1.4.4 of [2], Q contains $t^{-1}(x)$ for each irreducible character x in Z. Note that for $f\in k[R_0]$, $f\in\mathfrak{m}_u$ is equivalent to $f\,|\,Q=0$, since $\lim_{j\to\infty}\rho_j$ is a k-generic point of Q. Hence $\rho(a)=P(a)(\rho)=\pm I$, for each irreducible representation ρ in $t^{-1}(Z)$, which completes the proof of Lemma 2.1.

To complete the proof of Theorem 1.1, we only need to check that the image of an irreducible representation in $PSL_2(C)$ is nontrivial. This follows from a general argument that representations with noncyclic image in $PSL_2(C)$ are at least dense in $t^{-1}(x)$ for every $x\in X_0$ (Proposition 1.5.5 of [1]). In our case, $t^{-1}(x)$ consists of a single equivalence class, and any representation in it has noncyclic image in $PSL_2(C)$.

§ 3. Intersections at infinity

In this section we give a proof of Theorem 1.2 stated in § 1. The point is to observe that we can choose an affine curve C in X_0, and a discrete rank one valuation v of the function field $F=\boldsymbol{C}(C)$ of C such that the valuation v is supported at infinity, i.e. $\mathcal{O}_v \not\supset \boldsymbol{C}[C]$, and $v(\bar{f}_{\alpha_i})>v(\bar{f}_{\beta_i})$ for both $i=1,2$, where $\bar{f}_{\alpha_i}$ and $\bar{f}_{\beta_i}$ are the restrictions of f_{α_i} and f_{β_i} to C.

To see this, fix a local equation h of Z in a neighborhood of the ideal point $\tilde{x}$, and let C' be an irreducible curve whose local equation h' may be written as $h'(z,w)=h(z,w-z^N)$ for some local parameters z,w around $\tilde{x}$, and for sufficiently large N, so that to each branch of Z at $\tilde{x}$ there corresponds a branch of C' intersecting the branch of Z with sufficiently large multiplicity compared with other components of the divisors (f_{α_i}) and (f_{β_i}). Let $\nu:\tilde{C}\to C'$ be the normalization of C', and $\{y_1,\cdots,y_r\}=\nu^{-1}(\tilde{x})$ be the inverse image of $\tilde{x}\in C'$. If $N\geqq 1$ is sufficiently large, (using local uniformization parameters for corresponding branches of Z and C' at $\tilde{x}$) one can check that (i) $v_j(\bar{f}_{\alpha_1})\geqq v_j(\bar{f}_{\beta_1})$ for each valuation v_j of the function field $\boldsymbol{C}(C')\simeq\boldsymbol{C}(\tilde{C})$ determined by the point $y_j\in\tilde{C}$ $(1\leqq j\leqq r)$, and (ii) $v_{j_0}(\bar{f}_{\alpha_2})>v_{j_0}(\bar{f}_{\beta_2})$ for some $1\leqq j_0\leqq r$. In fact, since $\bar{h}(x)$ goes to zero rapidly as x approaches $\tilde{x}$ in C', $f_{\alpha_1}(\nu(y))/f_{\beta_1}(\nu(y))$ goes to zero as y approaches one of y_j, which implies (i). If N is sufficiently large, $i_{\tilde{x}}(C'\cdot(f_{\alpha_2}))=i_{\tilde{x}}(Z\cdot(f_{\alpha_2}))>i_{\tilde{x}}(Z\cdot(f_{\beta_2}))=i_{\tilde{x}}(C'\cdot(f_{\beta_2}))$, and (ii) follows from the fact that these intersection multiplicities are given by adding the valuation of the respective functions through $1\leqq j\leqq r$.

For a generic choice of C', C' intersects a Zariski-open subset of $\tilde{X}_0$ which maps isomorphically onto a Zariski-open subset of X_0. Then the image C of such a C' is birationally equivalent to C, and y_{j_0} determines a valuation $v=v_{j_0}$ of $F=\boldsymbol{C}(C)$, such that $v(\bar{f}_{\alpha_i})>v(\bar{f}_{\beta_i})$ for both $i=1,2$. Also, since $\tilde{x}$ is a point at infinity, we have $\mathcal{O}_v\not\supset\boldsymbol{C}[C]$.

Let E be the function field $\boldsymbol{C}(D)$ of an irreducible component D of $t^{-1}(C)$ such that the restriction of t to D, $t:D\to C$ is dominating. Let u be a valuation on E (abstractly) extending the valuation v on F as a subfield of E.

As in [5] let $\mathcal{T}_E$ be the combinatorial tree whose vertices are the E^*-orbits of finitely generated $\mathcal{O}_u$-submodules of E^2 which span the E-vector space E^2, and each of whose edges connects two vertices which are represented by submodules L and L' such that $L\subset L'$ and the length of the quotient L'/L is 1. As in § 2 of [2] (See § 1.3 of [1].) we can construct a surface S in M which is associated to the action of $\Pi=\pi_1(M)$

on the tree $\mathcal{T}_E$ induced by the tautological representation $P: \Pi \rightarrow SL_2(E)$. Let $\mathcal{K}$ be the quotient of the product of the universal covering of M and the tree $\mathcal{T}_E$ under the diagonal action of $\Pi = \pi_1(M)$: $\mathcal{K} = \tilde{M} \underset{\Pi}{\times} \mathcal{T}_E$, and $\mathcal{E}$ be the subcomplex $\mathcal{E} = \tilde{M} \underset{\Pi}{\times} \{\text{midpoints of edges of } \mathcal{T}_E\} \subset \mathcal{K}$. Then S may be written as the inverse image $\varphi^{-1}(\mathcal{E})$ of $\mathcal{E}$ for some continuous map $\varphi: M \rightarrow \mathcal{K}$ which is transverse to $\mathcal{E}$ and induces the isomorphism $\pi_1(M) \simeq \pi_1(\mathcal{K})$ (Proposition 1.3.4 of [1]). Since the valuation v is supported at infinity, such an S is always nonempty. We may assume that S is incompressible in M, since we can modify φ appropriately tracing the compressions of S. Also, the condition $f_{\alpha_i} \in \mathcal{O}_u$ $(i=1, 2)$ enables us to choose S so that ∂S is disjoint from some fixed simple closed curves in $\partial_i M$ with slopes α_i $(i=1, 2)$. Then such an S is closed, since (α_1, α_2) is a non-boundary slope pair by assumption.

The rest of the proof parallels the arguments in § 1.6 of [1]. Only slightest modifications will be necessary to deal with links in place of knots. The arguments are valid for links with any number of components.

For closed surfaces $S = S_1 \cup \cdots \cup S_p$, and $S' = S'_1 \cup \cdots \cup S'_q$ (S_j, S'_j the connected components), we say $S < S'$ if (i) $\chi_-(S) < \chi_-(S')$ or (ii) $\chi_-(S) = \chi_-(S')$ and $p < q$, where $\chi_-(S) = \sum_{j=1}^{p} (-1) \min(\text{euler } \# \text{ of } S_j, 0)$. If S_0 is a minimal element with respect to this ordering among all closed surfaces associated to the action of $\pi_1(M)$ on $\mathcal{T}_E$, then S_0 remains incompressible in the closed manifold M_{α_1, α_2}.

To show the incompressibility of S_0, suppose S_0 compressed in M_{α_1, α_2}. Then there would be a compressing disk D for S_0 in M_{α_1, α_2}. We may assume that D intersects J_i in a family of parallel meridian disks $d_i^1, \cdots, d_i^{k_i}$ $(i=1, 2)$, where J_i is the solid torus in M_{α_1, α_2} attached to M along $\partial_i M$. Let $I \subset \{1, 2\}$ be the set of indices i with $D \cap J_i \neq \emptyset$, N be the closure of the connected component of $M - S_0$ which contains $(\operatorname{Int} D) \cap M$. Then, since $S_0 \cap \partial_i M = \emptyset$, $\partial_i M \subset N$ for each $i \in I$. The connected component S_0^1 of S_0 which contains ∂D maps into $\mathcal{K}_e$ for some edge e, where $\mathcal{K}_e$ is the quotient $\tilde{M} \underset{\Pi_e}{\times} (\text{midpoint of } e)$, $\Pi_e =$ the stabilizer of $e \subset \pi_1(M)$. Then for some vertex v of $\mathcal{T}_E$ incident to e, N maps into $\mathcal{K}_v$, where $\mathcal{K}_v$ is the quotient $\tilde{M} \underset{\Pi_v}{\times} \operatorname{Star}(v)$, $\Pi_v =$ the stabilizer of $v \subset \pi_1(M)$, $\operatorname{Star}(v) =$ the closed star of v in the barycentric subdivision of $\mathcal{T}_E$. Fixing a base point b of N on ∂D, consider the induced homomorphism $\varphi_*: \pi_1(N, b) \rightarrow \pi_1(\mathcal{K}_v, \varphi(b))$. Joining a base point of $\partial_i M$ to b by an arc in N for each $i \in I$, we regard

$\pi_1(\partial_i M)\subset\pi_1(N)$. Then, since $u(\bar{f}_{\alpha_i})>u(\bar{f}_{\beta_i})$, conjugates of $\varphi_*(\alpha_i)$ in $\pi_1(N)$ act trivially on the link of v in $\mathcal{T}_E$ by Lemma 1.5.7 of [1] for each $i\in I$.

Let P be the punctured disk $P=D-\mathrm{Int}(\bigcup_{i\in I,1\leqq j\leqq k_i} d_i^j)$ with the base point b. Then $\varphi_*(\pi_1(P))$ is contained in the subgroup of $\pi_1(\mathcal{K}_v, \varphi(b))$, normally generated by $\varphi_*(\alpha_i)$, $i\in I$. Hence $\varphi_*(\pi_1(P))$ acts trivially on the link of v in $\mathcal{T}_E$. In particular, it is contained in the stabilizer of the edge $e:\varphi_*(\pi_1(P))\subset\Pi_e\simeq\pi_1(\mathcal{K}_e, \varphi(b))$.

Using the asphericity of $\mathcal{K}_v$, we can homotope P into $\mathcal{K}_e$ fixing the outer boundary ∂D. Then we can construct a map $\psi: M\to\mathcal{K}$ such that ψ is homotopic to φ and if $\bar{S}_0$ denotes the surface obtained by compressing S_0 along D, $\psi^{-1}(\mathcal{E})=M\cap\bar{S}_0$. Applying all possible compressions to the surface $\psi^{-1}(\mathcal{E})$ in M and deleting all 2-sphere components and all boundary parallel components, we get an incompressible closed surface S_1 in M such that S_1 is associated to the action of $\pi_1(M)$ on $\mathcal{T}_E$ and $S_1<S_0$. This contradicts the minimality of S_0 and proves the incompressibility of S_0. This completes the proof of Theorem 1.2.

References

[1] Culler, M., Gordon, C. McA., Luecke, J. and P. B. Shalen, Dehn surgery on knots, Ann. of Math. **125** (1987), 237-300.

[2] Culler, M. and P. B. Shalen, Varieties of group representations and splittings of 3-manifolds, Ann. of Math. **117** (1983), 109-146.

[3] Culler, M. and P. B. Shalen, Bounded, separating surfaces in knot manifolds, Invent. Math. **75** (1984), 537-545.

[4] Morgan, J. W. and P. B. Shalen, Valuations, trees, and degenerations of hyperbolic structures, I, Ann. of Math. **120** (1984), 401-476.

[5] Serre, J.-P. (with H. Bass), Trees, Springer-Verlag, New York, 1980.

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