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Equivariant projective imbedding theorem for symplectic manifolds

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§ 1. Introduction

For Kähler manifolds, Kodaira [7] has proved that Hodge manifolds can be imbedded in a complex projective space, where a Hodge manifold is a Kähler manifold M with a Kähler form ω such that the de Rham cohomology class $[\omega]$ satisfies

$$[\omega] \in \mathrm{Image}(H^2(M, \boldsymbol{Z}) \longrightarrow H^2(M, \boldsymbol{R})). \tag{1.1}$$

Similarly, Gromov and Tischler [12] proved the following

THEOREM (Gromov and Tischler). *If (M, ω) is a closed symplectic manifold satisfying the condition* (1.1), *then for sufficiently large N there is an imbedding $f:(M, \omega) \longrightarrow (CP^N, \omega_0)$ with $f^*\omega_0 = \omega$, where ω_0 is the Kähler form of Fubini-Study metric.*

These two theorems are similar, but neither implies the other.

On the other hand, Blanchard [3] has proved the equivariant version of Kodaira's imbedding theorem as follows:

THEOREM (Blanchard). *Let M be a Hodge manifold and G a connected Lie group acting on M holomorphically. If every vector field determined by the G-action has zeros, then there is a G-equivariant projective imbedding.*

The purpose of this paper is to prove the symplectic analogues of Blanchard's equivariant projective imbedding theorem.

THEOREM 1. *(M, ω) be a closed symplectic manifold, L a complex line bundle $c_1(L^*) = [\omega]$, and G a compact Lie group acting on (M, ω) symplectically, i.e. G-action preserves ω. If the G-action on M has a lift to a G-action on L, then for sufficiently large N there is a G-equivariant imbedding $f:(M, \omega) \longrightarrow (CP^N, \omega_0)$ with $f^*\omega_0 = \omega$.*

The converse of Theorem 1 is also true. Namely, if (M, ω) is G-equivariantly imbedded in (CP^N, ω_0), there is a complex line bundle L with $c_1(L^*)=[\omega]$ such that the G-action can be lifted to L.

Theorem 1 will be proved in § 2 and § 3. There is a close relation between the existence of moment maps and the liftings of group actions to line bundles. We shall discuss that aspect in § 4. There we shall show the following symplectic analogue of Frankel's theorem concerning the existence of moment maps.

THEOREM 2. *Let* (M, ω) *be a* $2n$*-dimensional closed symplectic manifold satisfying the following condition*

$$\wedge\omega^{n-1}: H^1(M; \boldsymbol{R}) \xrightarrow{\cong} H^{2n-1}(M; \boldsymbol{R}). \tag{1.2}$$

If a compact Lie group G *acts on* (M, ω) *symplectically and every vector field determined by the* G*-action has zeros, then there exists a moment map*

$\mu: M \longrightarrow \mathfrak{g}^*$, *where* $\mathfrak{g}^*$ *is the dual space of the Lie algebra* $\mathfrak{g}$ *of* G.

Hard Lefshetz theorem asserts that every closed Kähler manifold satisfies the condition (1.2) [13].

THEOREM 3. *Let* (M, ω) *be a* $2n$*-dimensional closed symplectic manifold satisfying* (1.1) *and* (1.2). *If a compact connected Lie group* G *acts on* (M, ω) *symplectically and every vector field determined by the* G*-action has zeros, then for sufficiently large* N *there exists a* G*-equivariant imbedding* $f: (M, \omega) \longrightarrow (CP^N, \omega_0)$ *with* $f^*\omega_0 = k \cdot \omega$ *for some positive integer* k.

Theorem 1 can be also derived from a result of Schlafly on the existence of equivariant universal connections [11]. But our method is more elementary, and can be applied to a problem treated in Remark (3.9) in § 3.

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§ 2. Equivariant imbeddings into infinite dimensional projective spaces

Let (M, ω) and L be as in Theorem 1. By the theorem of Gromov and Tischler, there is an imbedding $f_0: M \longrightarrow CP^N$ such that $f_0^*\omega_0 = \omega$ and $f_0^*L_0 = L$ where L_0 is the tautological line bundle on CP^N namely we have the following diagram.

(2.1)
$$\begin{array}{ccccc} L & \xrightarrow{\tilde{f}_0} & L_0 \overset{\iota}{\subset} CP^N \times C^{N+1} & \xrightarrow{P} & C^{N+1} \\ \pi \downarrow & & \downarrow & & \\ M & \xrightarrow{f_0} & CP^N & & \end{array}$$

Then, for any $x \in M$ and $\tilde{x} \in L-M$ such that $\pi(\tilde{x})=x$, $P \circ \iota \circ \tilde{f}_0(\tilde{x})$ is homogeneous coordinate of $f_0(x)$, where $M \subset L$ is the zero section of L. We define a map $\tilde{\Phi}: L \longrightarrow Map(G, C^{N+1})$ by

$$\tilde{\Phi}(\tilde{x})(g) = P \circ \iota \circ \tilde{f}_0(g^{-1}\tilde{x}) \quad \text{for } \tilde{x} \in L \text{ and } g \in G.$$

Then it is easy to see that $\tilde{\Phi}$ is G-equivariant with respect to the action of G on $Map(G, C^{N+1})$ given by

$$(h \cdot \phi)(g) = \phi(h^{-1}g) \quad \text{for } h \in G \text{ and } \phi \in Map(G, C^{N+1}).$$

To make $Map(G, C^{N+1})$ a Hilbert space, we fix an invariant Riemannian metric on G and the standard Euclidean metric on C^{N+1}. The completion of $Map(G, C^{N+1})$ with respect to the L^2 norm is also denoted by $Map(G, C^{N+1})$. As $\tilde{\Phi}$ is C^*-equivariant with respect to the complex multiplication on these spaces, we get the following diagram and define Φ as follows:

(2.2)
$$\begin{array}{ccc} L-M & \xrightarrow{\tilde{\Phi}} & Map(G, C^{N+1})-\{0\} \\ \downarrow & & \downarrow \\ (L-M)/C^* & \xrightarrow{\Phi} & \{Map(G, C^{N+1})-\{0\}\}/C^* \\ \| & & \| \\ M & & P(Map(G, C^{N+1})) \end{array}$$

First of all, we describe the symplectic structure of $P(Map(G, C^{N+1}))$. Let B_0 be the standard symplectic form on C^{N+1}. We define the symplectic form B on $Map(G, C^{N+1})$ by

$$B(X, Y) = \int_G B_0(X(g), Y(g))dg \tag{2.3}$$

where X and Y are elements of $Map(G, C^{N+1})$ and dg is the Haar measure on G. Applying the "symplectic reduction" to $S(Map(G, C^{N+1}))$, the unit sphere in $Map(G, C^{N+1})$, and S^1-action by complex multiplication, we get a symplectic form Ω_∞ on $P(Map(G, C^{N+1}))$. Namely, for tangent vectors u and v at some point of $P(Map(G, C^{N+1}))$ we have

$\Omega_\infty(u, v)=B(\tilde{u}, \tilde{v})$, where $\tilde{u}$ and $\tilde{v}$ are lifts of u and v.

The standard symplectic structure ω_0 (determined by the Fubini-Study metric) is also described as the "symplectic reduction" of the unit sphere S^{2N+1} in C^{N+1} with the standard symplectic form B_0. Then the unit sphere S^{2N+1} is identified with the unit circle bundle $S(L_0)$ of L_0, so we get a hermitian metric on L_0 and a hermitian metric on $L=f_0^*L_0$. From now on, we denote by $S(L)$ the unit circle bundle of L with respect to this metric. By changing the G action to the new action $(g, \tilde{x})\longrightarrow \frac{\|\tilde{x}\|}{\|g\cdot\tilde{x}\|}\cdot(g\cdot\tilde{x})$, if necessary, we can assume that G acts on $S(L)$. Now we shall prove the following

PROPOSITION (2.3). *$\Phi:(M, \omega)\longrightarrow(P(Map(G, C^{N+1})), \Omega_\infty)$ is a G-equivariant imbedding such that $\Phi^*\Omega_\infty=\omega$.*

PROOF. As f_0 is an imbedding, Φ is also an imbedding. G-equivariance of Φ follows from that of $\tilde{\Phi}$, so it is enough to show that $\Phi^*\Omega_\infty=\omega$.

For $x\in M$ and $u, v\in T_xM$, let $\tilde{x}\in S(L)$ and $\tilde{u}, \tilde{v}\in T_{\tilde{x}}S(L)$ be lifts of x, u and v respectively, i.e.

$$\pi(\tilde{x})=x,\quad \pi_*(\tilde{u})=u,\ \text{and}\ \pi_*(\tilde{v})=v.$$

Then we have

$$\begin{aligned}\omega(u, v)&=\omega_0(f_{0*}u, f_{0*}v)\\ &=B_0(\tilde{f}_{0*}\tilde{u}, \tilde{f}_{0*}\tilde{v}).\end{aligned} \tag{2.4}$$

Since $\Phi_*u(g)=f_{0*}(g^{-1}{}_*u)$, $\Phi_*v(g)=f_{0*}(g^{-1}{}_*v)$, and moreover, $\tilde{\Phi}_*\tilde{u}(g)=\tilde{f}_{0*}(g^{-1}{}_*\tilde{u})$ and $\tilde{\Phi}_*\tilde{v}(g)=\tilde{f}_{0*}(g^{-1}{}_*\tilde{v})$ are lifts of Φ_*u and Φ_*v to $T_{\tilde{\Phi}(\tilde{x})}S(Map(G, C^{N+1}))$ respectively, we get

$$\begin{aligned}\Omega_\infty(\Phi_*u, \Phi_*v)&=B(\tilde{\Phi}_*\tilde{u}, \tilde{\Phi}_*\tilde{v})\\ &=\int_G B_0(\tilde{f}_{0*}(g^{-1}{}_*\tilde{u}), \tilde{f}_{0*}(g^{-1}{}_*\tilde{v}))dg\\ &=\int_G \omega(g^{-1}{}_*u, g^{-1}{}_*v)dg && \text{(by (2.4))}\\ &=\int_G \omega(u, v)dg && \text{(since } \omega \text{ is } G\text{-invariant)}\\ &=\omega(u, v).\end{aligned}$$

Hence we obtain $\Phi^*\Omega_\infty=\omega$. //

§ 3. Equivariant imbeddings into finite dimensional projective spaces

By Proposition (2.3), we have a G-equivariant symplectic imbedding $\Phi:(M,\omega)\longrightarrow(\boldsymbol{P}(Map(G,\boldsymbol{C}^{N+1})),\Omega_{\infty})$. Since unitary representations of compact Lie groups are completely reducible, we have the decomposition as follows:

$Map(G,\boldsymbol{C}^{N+1})=\Sigma^{\oplus}V_k$, where each V_k is a finite dimensional irreducible representation space.

LEMMA (3.1). *For a sufficiently large integer K, there is a G-equivariant imbedding into a finite dimensional projective space*

$$g_K: M\longrightarrow \boldsymbol{P}\Big(\bigoplus_{k=-K}^{K}V_k\Big),$$

such that

(3.2) $tg_K{}^*\Omega_K+(1-t)\omega$ *is a symplectic form for* $0\leqq t\leqq 1$, *where* Ω_K *is the standard symplectic form on* $\boldsymbol{P}\Big(\bigoplus_{k=-K}^{K}V_k\Big)$.

PROOF. As M is compact, there is an integer K_0 such that for any $K\geqq K_0$

(3.3) $p_K\circ\tilde{\Phi}(\tilde{x})\neq 0$ for any $\tilde{x}\in S(L)$, where p_K is the projection from $Map(G,\boldsymbol{C}^{N+1})$ to $\bigoplus_{k=-K}^{K}V_k$.

Thus we get

$$(3.4)\qquad \begin{array}{ccc} L-M & \xrightarrow{\tilde{g}_K} & \bigoplus_{k=-K}^{K}V_k-\{0\} \\ \downarrow & & \downarrow \\ M & \xrightarrow{g_K} & \boldsymbol{P}\big(\bigoplus_{k=-K}^{K}V_k\big) \end{array}\qquad \text{for any } K\geqq K_0.$$

Since Φ is an imbedding, there is an integer K_1 such that for any $K\geqq K_1$, g_K is an immersion. Then there is an open neighborhood U of the diagonal subset Δ_M in $M\times M$ such that $g_K(p)\neq g_K(q)$ for any $(p,q)\in U-\Delta_M$. As Φ is an imbedding, there is an integer $K(x,y)$ for any $(x,y)\notin\Delta_M$ such that $g_{K(x,y)}(x)\neq g_{K(x,y)}(y)$. Moreover for any $(x,y)\notin\Delta_M$ there exists a neighborhood $V(x,y)$ such that $g_{K(x,y)}(z)\neq g_{K(x,y)}(w)$ for any $(z,w)\in V(x,y)$. Since $M\times M-U$ is compact, $M\times M-U$ is covered by finite open subsets $V(x_j,y_j)$ and we set $K_2=\max_j\{K(x_j,y_j)\}$. Then for any $K\geqq\max(K_1,K_2)$,

g_K is an imbedding. Moreover for any $\varepsilon>0$, we can seek an integer K such that

$$\|(1-p_K)\circ\tilde{\Phi}(\tilde{x})\|<\varepsilon\cdot\|\tilde{\Phi}(\tilde{x})\| \quad \text{for } \tilde{x}\in L, \quad \text{and} \tag{3.5}$$

$$\|(1-p_K)_*\circ\tilde{\Phi}_*(\tilde{X})\|<\varepsilon\cdot\|\tilde{\Phi}_*(\tilde{X})\| \quad \text{for } \tilde{X}\in TL. \tag{3.6}$$

Then by easy calculation, condition (3.2) holds for a sufficiently large integer K. //

The following lemma is an equivariant version of Moser's result [10].

LEMMA (3.7). *Let M be a closed manifold and $\{\omega_t\}$ a 1-parameter family of symplectic forms representing the same de Rham cohomology class. If a compact Lie group G acts on M preserving each ω_t, then there exists a 1-parameter family of G-equivariant diffeomorphisms*

$$\phi_t : M \longrightarrow M \text{ such that } \phi_t{}^*\omega_t=\omega_0.$$

PROOF. At first we review Moser's proof of his result. If $\phi_t{}^*\omega_t=\omega_0$, then $\frac{d}{dt}\phi_t{}^*\omega_t=0$. Set $X_t=\phi_{-t*}\left(\frac{d}{dt}\phi_t\right)$. Then, since

$$\phi_{-t}{}^*\frac{d}{dt}\phi_t{}^*\omega_t=\mathcal{L}_{X_t}\omega_t+\frac{d}{dt}\omega_t=d\circ i(X_t)\omega_t+\frac{d}{dt}\omega_t,$$

we have

$$d\circ i(X_t)\omega_t+\frac{d}{dt}\omega_t=0. \tag{3.8}$$

With this in mind we seek a 1-parameter family of vector fields X_t satisfying (3.8). Since each ω_t represents the same de Rham cohomology class, there exists such $\{X_t\}$ and it generates a 1-parameter family of diffeomorphisms $\phi_t : M\longrightarrow M$ such that $\phi_t{}^*\omega_t=\omega_0$. In our case every ω_t is preserved by G-action, thus $\{g_*X_t\}$ also satisfies (3.8) for any $g\in G$. Hence $\bar{X}_t=\int_G g_*X_t dg$ also satisfies (3.8), and $\{\bar{X}_t\}$ generates a 1-parameter family of G-equivariant diffeomorphisms $\bar{\phi}_t : M\longrightarrow M$ such that $\bar{\phi}_t{}^*\omega_t=\omega_0$. //

Applying Lemma (3.7) to (3.2), we get a G-equivariant diffeomorphism $\phi : (M, \omega)\longrightarrow(M, g_K{}^*\Omega_K)$ such that $\phi^*g_K{}^*\Omega_K=\omega$. Thus we have proved Theorem 1 where $f=g_K\circ\phi$.

REMARK (3.9). By a similar argument we can prove the following proposition.

Let (M, ω) be a compact Kähler manifold, L a holomorphic line bundle with $c_1(L^*)=[\omega]$, and G a compact Lie group acting on (M, ω) isometrically. If the G-action on M has a lift to a G-action on L, then there is an equivariant isometric holomorphic projective imbedding

$$\Phi : (M, \omega) \longrightarrow (\boldsymbol{P}(Map(G, \boldsymbol{C}^{N+1})), \Omega_\infty) \quad \text{for some integer } N.$$

Moreover, for any $\varepsilon>0$, there is an integer K such that there is an equivariant holomorphic imbedding

$$f : (M, \omega) \longrightarrow (\boldsymbol{C}P^K, \omega_0)$$

such that

$\sup_{x \in M} \| f^*\omega_0(x) - \omega(x) \|_\omega < \varepsilon$, where $\| \quad \|_\omega$ is the norm determined by the Kähler metric with Kähler form ω.

§ 4. Existence of moment maps

For symplectic actions of semi-simple groups on symplectic manifolds there is always a unique moment map [9]. On the other hand, for toral actions it is not true that there always exists a moment map. For closed symplectic manifolds, if there exists a moment map, then the fixed point set is not empty. Conversely if a toral group acts a Kähler manifold holomorphically with non-empty fixed point set then there is a moment map [5]. We consider the following problem.

PROBLEM. *Let (M, ω) be a closed symplectic manifold and a circle group S^1 act on (M, ω) symplectically. If the fixed point set is not empty, does there exist a moment map?*

We shall answer this problem in the affirmative under condition (1.2).

THEOREM (4.1). *Let (M, ω) be a $2n$-dimensional closed symplectic manifold satisfying condition* (1.2), *and S^1 act on (M, ω) symplectically. If the fixed point set is not empty, there exist a moment map $\mu : M \longrightarrow \boldsymbol{R}$.*

It is well known that μ is a perfect Morse function with non-degenerate critical set in the sense of Bott, so we obtain the following

COROLLARY (4.2). *In the above situation, the following equality holds.*

$$H_q(M; K) \xrightarrow{\simeq} \bigoplus_i H_{q-\lambda_i}(N_i; K)$$

where $M^{S^1} = \bigcup_i N_i$ *is the decomposition of the fixed point set into connected components,* K *is an arbitrary field, and* λ_i *is the index of Hessian of* μ *along* N_i.

REMARK. 1) If M is a Kähler manifold and the S^1-action is holomorphic, then this statement is Frankel's theorem.

2) The above theorem and corollary also hold for toral actions instead of S^1-actions.

PROOF OF THEOREM (4.1). Let C be an S^1-orbit which is homeomorphic to S^1. First of all we show that the Poincaré dual of $[C]$ is represented by a constant multiple of $i(X)\omega^n$ where $i(X)$ is the interior product with a vector field X which corresponds to the generator of the Lie algebra of S^1.

We identify the tubular neighborhood $N(C)$ of C with the normal bundle of C, then the Poincaré dual of $[C]$ is the extension of the Thom class of the normal bundle by excision [4]. Let Φ be an S^1-invariant $2n$-form whose support is contained in $N(C)$, then it is easy to see that the constant multiple of $i(X)\Phi$ represents the Poincaré dual of $[C]$. On the other hand ω^n is also an S^1-invariant $2n$-form, hence $i(X)\omega^n$ also represents the Poincaré dual of $[C]$ up to a constant multiple.

By the assumption of Theorem the fixed point set is non-empty so the homology class $[C]$ is zero in $H_1(M; \boldsymbol{R})$. This means the Poincaré dual of $[C]$ is also zero in $H^{2n-1}(M; \boldsymbol{R})$, then the condition (1.2) implies $i(X)\omega$ is cohomologous to zero. It asserts the existence of a moment map. //

For a compact Lie group G, there is a covering group $\bar{G}$ which is isomorphic to the product of a compact semi-simple Lie group H and a toral group T. If there exists a moment map for $\bar{G}$-action, so does for G-action. Therefore using the existence of moment maps for semi-simple group actions [9] together with Theorem (4.1) and Remark above, we obtain Theorem 2.

The following fact is well known [8].

PROPOSITION (4.3). *Let* (M, ω) *be a symplectic manifold satisfying*

(1.1) *and H a complex line bundle with* $c_1(H)=[\omega]$. *If a Lie group G acts on M preserving* ω *and the action is lifted to H, then there exists a moment map. Converse is also true if G is* 1-*connected.*

In case G is compact this proposition can be strengthened in the following way. First we recall a theorem on liftings of group actions to line bundles, and a theorem of existence of moment maps.

PROPOSITION (4.4) ([6]). *Let G be a compact Lie group acting on a connected locally finite CW-complex X such that* $X_G := BG \times_G X$ *has the homotopy type of a CW-complex. Then a complex line bundle H over X admits a lifting of the group action of G on X if and only if the following condition holds*

$$c_1(H) \in \mathrm{Image}(H^2(X_G; \boldsymbol{Z}) \longrightarrow H^2(X; \boldsymbol{Z})). \tag{4.5}$$

PROPOSITION (4.6) ([1]). *Let G be a compact Lie group acting on a symplectic manifold* (M, ω). *Then there exists a moment map* $\mu : M \longrightarrow \mathfrak{g}^*$ *if and only if the following condition holds*

$$[\omega] \in \mathrm{Image}(H^2(M_G; \boldsymbol{R}) \longrightarrow H^2(M; \boldsymbol{R})). \tag{4.7}$$

Using (4.4) and (4.6) we obtain

LEMMA (4.8). *Let G be a compact Lie group acting on a symplectic manifold* (M, ω) *satisfies the condition* (1.1) *and if there exists a moment map, then for a complex line bundle L with* $c_1(L^*)=[\omega]$, *there is an integer k such that the G-action can be lifted to* L^k. *Moreover if G is a product of a compact* 1-*connected semi-simple Lie group H and a toral group T, then the G-action can be lifted to L.*

For the proof of the second part, we remark the fact that conditions (4.5) and (4.7) are equivalent to (4.9) and (4.10) respectively [6].

$$d_2^{\boldsymbol{Z}}(c_1(H))=0 \quad \text{and} \quad d_3^{\boldsymbol{Z}}(c_1(H))=0 \tag{4.9}$$

$$d_2^{\boldsymbol{R}}([\omega])=0 \quad \text{and} \quad d_3^{\boldsymbol{R}}([\omega])=0 \tag{4.10}$$

where
$$\begin{aligned} d_2^{\boldsymbol{Z}} &: E_2^{0,2}=H^2(X; \boldsymbol{Z}) \longrightarrow E_2^{2,1}=H^2(BG; H^1(X; \boldsymbol{Z})) \\ d_2^{\boldsymbol{R}} &: E_2^{0,2}=H^2(M; \boldsymbol{R}) \longrightarrow E_2^{2,1}=H^2(BG; H^1(M; \boldsymbol{R})) \\ d_3^{\boldsymbol{Z}} &: E_3^{0,2} \longrightarrow E_3^{3,0} \\ d_3^{\boldsymbol{R}} &: E_2^{0,2} \longrightarrow E_3^{3,0} \end{aligned}$$

are differentials of Serre's spectral sequence. By the assumption $d_2^R([\omega])=0$. As $H^1(M;Z)$ is free, $d_2^Z(c_1(L))=0$ if $c_1(L)$ coincides with $-[\omega]$ in de Rham cohomology of M. If G is a product of a 1-connected semi-simple Lie group and a toral group, then $E_3^{3,0}=0$ since $E_3^{3,0}$ is a quotient of $H^3(BG;Z)=0$. Thus we have proved the second part of Lemma (4.8).

We shall prove Lemma (4.8) in smooth category for the sake of completeness, since [6] treats the lifting problem in topological category.

PROOF OF LEMMA (4.8) in smooth category. First of all we review the following result due to Berline and Vergne [2].

LEMMA (4.9) ([2]). *Let (P,α) be a principal H-bundle over M with connection α and G a simply connected group of automorphisms of M. If there exists a map*

$J:\mathfrak{g}\longrightarrow\Gamma(P(\mathfrak{h}))$ *where $\Gamma(P(\mathfrak{h}))$ is the space of sections of $P(\mathfrak{h})=P\times_{ad}\mathfrak{h}$, such that*

1) $DJ_X+i(X_M^*)\Omega=0$ *for any $X\in\mathfrak{g}$ where D is the covariant differentiation, Ω is the curvature form of α, J_X is $J(X)$, and X_M^* is the vector field determined by the action of* $\exp(tX)$ *on M.*

2) $$\Omega(X_M^*,Y_M^*)=[J_X,J_Y]-J_{[X,Y]}.$$

Then the G-action can be lifted to P as automorphisms on P, i.e. the G-action commutes with H-action, such that α is G-invariant and $J_X=\alpha(X_P^*)$.

(1) The case of $G=S^1$.

Let P be the associated principal S^1-bundle of L, and α a connection on P with curvature form ω. The moment map μ satisfies conditions 1) and 2) in Lemma (4.9). Thus there is an $\boldsymbol{R}$-action $\{\phi_t\}$ on P which covers the S^1-action on M, where S^1 is identified with $\boldsymbol{R}/\boldsymbol{Z}$. Then ϕ_1 is a gauge transformation on the principal S^1-bundle P, so it can be identified with $\phi:M\longrightarrow S^1$. If ϕ is null homotopic, it can be lifted to $\bar{\phi}:M\longrightarrow\boldsymbol{R}$ and the vector field $X_P^*-\bar{\phi}\cdot v$ defines an S^1-action on P which covers S^1-action on M, where v is the fundamental vector field of the principal S^1-bundle P. Therefore it suffices to show that ϕ is null homotopic. As S^1 is a $K(\boldsymbol{Z};1)$-space, it suffices to show that $\phi^*:H^1(S^1;\boldsymbol{Z})\longrightarrow H^1(M;\boldsymbol{Z})$ is a zero map. Since there is a moment map μ, there is a connected components F of fixed point set M^{S^1} such that $H^1(M;\boldsymbol{Z})\xrightarrow{\iota^*}H^1(F;\boldsymbol{Z})$ is an isomorphism where ι is the inclusion map. It is obvious that $F \longrightarrow S^1$, the composition of ι and ϕ, is a constant map. Therefore Lemma (4.9) is proved in the case of $G=S^1$.

(2) The general case.

It suffices to prove in the case of $G=G' \times S^1$ where G' is a simply connected Lie group. By Lemma (4.9), there is a $\bar{G}$-action on P which covers G-action on M where $\bar{G}$ is the universal covering group of G. We modify the vector field corresponding to S^1-component as (1), then we get G'-action and S^1-action. It suffices to show the commutativity of these actions, and it is clear by the construction in (1). //

Finally Theorem 3 is a direct consequence of Theorem 1, Theorem 2, and Lemma (4.8).

REMARK. We can show the existence of moment maps in the following situation, in the spirit of the above proof of Lemma (4.8).

PROPOSITION (4.10). *Let* (M, ω) *be a closed symplectic manifold. If* S^1 *acts on* (M, ω) *symplectically and*

$$H^1(M; \boldsymbol{R}) \longrightarrow H^1(M^{S^1}; \boldsymbol{R}) \quad \textit{is injective,} \tag{4.11}$$

then there exists a moment map.

PROOF. The obstruction to existence of a moment map is the cohomology class $[i(X)\omega] \in H^1(M; \boldsymbol{R})$ where X is the vector field determined by the S^1-action on M. Since X vanishes on the fixed set M^{S^1}, (4.11) asserts that $[i(X)\omega]$ is zero. //

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