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Hausdorff convergence of Einstein 4-manifolds

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§1. Introduction

Let $\mathcal{E}(D, V, R)$ denotes the class of compact 4-dimensional smooth manifolds X with Einstein metrics g (we normalized the Einstein constant k to be ± 1, or 0) which satisfy

$$\mathrm{diam}(X, g) \leq D, \ \ \mathrm{vol}(X, g) \geq V, \ \ \text{and} \ \int_X |R_g|^2 dV_g \leq R$$

($|R_g|$ denotes the norm of the curvature tensor of g). By the result of Gromov [8], if we consider $\mathcal{E}(D, V, R)$ as a class of compact metric spaces, it is precompact with respect to the Hausdorff distance (see §2 for the precise definition and some properties of Hausdorff distance). In this paper we study the boundary $\partial\mathcal{E}(D, V, R)$ of $\mathcal{E}(D, V, R)$ in the set of all compact metric spaces. We show that an element in $\partial\mathcal{E}(D, V, R)$ is a metric space (X, d) which has a structure of a smooth manifold outside a finite set S, and there is an Einstein metric g on $X \backslash S$ which is compatible with the distance d.

Before we state our main theorem let us first consider a few examples.

(1.1) *Example* (Kobayashi-Todorov [13]). Let X_∞ be a $\boldsymbol{Z}_2$-quotient of T^4, where $\boldsymbol{Z}_2$ acts on T^4 as

$$\tau(z_1, z_2) = (-z_1, -z_2) \quad \text{for} \ (z_1, z_2) \in T^4 = \boldsymbol{C}^2/(\boldsymbol{Z}+i\boldsymbol{Z})^2$$

where τ is a generator of $\boldsymbol{Z}_2$. We induce the quotient metric g_∞ on X_∞ where $\boldsymbol{C}^2$ has the standard metric. Then X_∞ is *not* a Riemannian manifold. It is an orbifold. It has sixteen singular points $x_1, \cdots, x_{16}$. Let X be the minimal resolution of X_∞. It is known to be a $K3$ surface. Let $\pi : X \longrightarrow X_\infty$ be the projection. Kobayashi-Todorov showed the existence of the following sequence of Ricci flat Kähler metrics $\{g_i\}_{i=1}^\infty$ on X.

1) $\mathrm{vol}(X, g_i) = 1$
2) $\mathrm{vol}(\pi^{-1}(x_a), g_i) = 1/i \qquad \text{for } a = 1, \cdots, 16.$

They proved that the sequence of the metric spaces (X, g_i) converges to

(X_∞, g_∞) in the Hausdorff distance, and moreover g_i converges to $\pi^* g_\infty$ in the C^∞-topology on $X - \pi^{-1}\{x_1, \cdots, x_{16}\}$. For any $p_a \in \pi^{-1}(x_a)$ $(a=1, \cdots, 16)$ there exists $r_i > 0$ such that

3) $\lim_{i \to \infty} r_i = \infty$

4) $((X, r_i g_i), p_a)$ converges to the Eguchi-Hanson space in the pointed Hausdorff distance.

The Eguchi-Hanson space is the holomorphic cotangent bundle of the Riemann sphere T^*CP^1 with complete Ricci flat Kähler metric. ([14])

(1.2) *Example* (Tsuji [21]). Let $\pi : \mathfrak{X} \longrightarrow D$ be a smooth family of compact complex surfaces of general type over the unit disk $D = \{z \in \boldsymbol{C} : |z| < 1\}$. Let $X_s = \pi^{-1}(s)$ for $s \in D$. Assume that $c_1(X_s)$ is negative if $s \neq 0$, and X_0 be a minimal surface of general type with non-empty E where E denotes the union of all (-2)-curves on X_0. It is known that X_0 can be blown down to the canonical model $\bar{X}_0$ contracting all (-2)-curves to rational double points. So $\bar{X}_0$ is a singular variety. By the result of Aubin and Yau, there exists an Einstein-Kähler metric g_s on X_s for each $s \neq 0$. Moreover Kobayashi [12] proved that $\bar{X}_0$ has an Einstein-Kähler orbifold metric $\bar{g}_0$ on $\bar{X}_0$.

Tsuji has shown that (X_s, g_s) converges to $(\bar{X}_0, \bar{g}_0)$ in the Hausdorff distance, and g_s converges to $\pi^* \bar{g}_0$ in the C^∞-topology on $X_0 \backslash E$. Let $\bar{x} \in \bar{X}_0$ be a singular point. Then a neighborhood of $\bar{x}$ looks like U/Γ where U is a open ball in $\boldsymbol{C}^2$ centered at 0, and Γ is a finite subgroup of $SU(2)$. There exist $x_s \in X_s$ and $r_s > 0$ such that

1) $\lim_{s \to 0} r_s = \infty$

2) $x_s \longrightarrow x \in \pi^{-1}(\bar{x})$

3) $((X_s, r_s g_s), x_s)$ converges in the pointed Hausdorff distance to a complete, non-compact Ricci flat Kähler surface M whose end is diffeomorphic to $\boldsymbol{R} \times (S^3/\Gamma)$.

Inspired by the above examples, we obtain the following result.

(1.3) THEOREM. *Let (X_i, g_i) be a sequence in $\mathcal{E}(D, V, R)$ with some positive constants D, V, R. Then there exist a subsequence $\{j\} \subset \{i\}$, a compact metric space (X_∞, d_∞), and a finite set $S = \{x_1, \cdots, x_k\} \subset X_\infty$ (possibly empty) such that*

1) *(X_j, g_j) converges to (X_∞, d_∞) in the Hausdorff distance,*

2) *$X_\infty \backslash S$ has a structure of C^∞-manifold and an Einstein metric g_∞ which is compatible with the distance d_∞ on $X_\infty \backslash S$,*

3) *For every compact subset $K \subset X_\infty \backslash S$, there exists an into diffeomorphism $F_j : K \longrightarrow X_j$ for each j such that $F_j^* g_j$ converges to g_∞ in the C^∞-topology on K,*

4) *For every $x_a \in S$ $(a=1, \cdots, k)$ and j, there exist $x_{a,j} \in X_j$ and positive number r_j such that*

4.1) *$B(x_{a,j}; \delta)$ converges to $B(x_a; \delta)$ in the Hausdorff distance for all $\delta > 0$,*

4.2) $\lim_{j \to \infty} r_j = \infty$

4.3) *$((X_j, r_j g_j), x_{a,j})$ converges to $((M_a, h_a), x_{a,\infty})$ in the pointed Hausdorff distance where (M_a, h_a) is a complete, non-compact, Ricci-flat, non-flat 4-manifold with*

$$0 < \int_{M_a} |R_{h_a}|^2 dV_{h_a} < \infty,$$

and the Sobolev constant $S(M_a, h_a)$ of (M_a, h_a) is positive where

$$S(M_a, h_a) = \inf \{ \|\nabla f\|_{L^1} / \|f\|_{L^{4/3}} : f \in C_0^\infty(M_a) \},$$

4.4) *For every $B(x_{a,\infty}; r)$ $(r > 0)$ there exists an into diffeomorphism $G_j : B(x_{a,\infty}; r) \longrightarrow X_j$ such that $G_j^*(r_j g_j)$ converges to h_a in the C^∞-topology on $B(x_{a,\infty}; r)$,*

5) *It holds*

$$\lim_{j \to \infty} \int_{X_j} |R_{g_j}|^2 dV_j \geq \int_{X_\infty} |R_{g_\infty}|^2 dV_\infty + \sum_a \int_{M_a} |R_{h_a}|^2 dV_{h_a}.$$

(1.4) REMARK. 1) The same phenomenon as in theorem (1.3) occurs in many other situations;

(a) a harmonic map [18],
(b) a Yang-Mills connection [22], [20], [5],
(c) a surface with constant mean curvature [2],
(d) the equation

$$\Delta u = -u^{(n+2)/(n-2)} \qquad [1].$$

The common feature is the conformal invariance of the action. But in the above cases, we know that the limiting objects ((X_∞, d_∞) in our case) can be extended smoothly. These are removable singularities theorem. But we do not have the corresponding result in our case. We conjecture that (X_∞, d_∞) is an orbifold.

2) A complete, non-compact Ricci flat manifold appeared in theorem (1.3) 4.3) corresponds to a finite action instanton on $\boldsymbol{R}^4$ (this extends to an instanton on S^4) in the Yang-Mills theory. The classifications of such manifolds must be an important problem. Kronheimer [14] have treated this problem when manifolds are Kähler and asymptotically locally Euclidean (ALE). We conjecture that the above manifolds are automatically ALE.

3) Chern-Weil theory says

$$\frac{1}{8\pi^2}\int_X |R_g|^2 dV = \text{the Euler number of } X$$

for a compact Einstein 4-manifold (X, g).

4) For an ALE Ricci-flat manifold (M, h), we have

$$\frac{1}{8\pi^2}\int_M |R_h|^2 dV = (\text{the Euler number of } M) - 1/\sharp\Gamma$$

when the end of M is diffeomorphic to $\boldsymbol{R}\times(S^3/\Gamma)$ with a finite group Γ. The right hand side of the above equality is closely related to the Chern number inequality of Miyaoka [16] and Kobayashi [12]. They have shown that for (X_0, g_0) in example (1.2), the first Chern class and the second Chern class satisfy

$$3c_2(X_0) - c_1(X_0)^2 \geq \Sigma \ (\text{the Euler number of } M_x) - 1/\sharp\Gamma_x$$

where the summation runs the all singular points x in $\bar{X}_0$, and M_x and Γ_x are the non-compact Ricci flat Kähler surface and the finite group corresponding to x.

5) Even if all manifolds X_i are diffeomorphic to each other, X_∞ is not always diffeomorphic to X_i. (See example (1.1).)

6) Our proof is restricted to 4-dimensional case, but we conjecture that the same conclusion holds in the higher dimensions if we replace "a finite set S" by "a compact subset S with the finite $(n-4)$ dimensional Hausdorff measure $H_{n-4}(S)<\infty$" where n is the dimension of manifolds. (cf. [17]) We can prove the same conclusion as the theorem under the condition

$$\int_{X_i} |R_{g_i}|^{n/2} dV_i \leq R$$

with positive constant R. But this condition may not be natural in higher dimensions.

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Added in Proof: After the completion of this paper, Bando, Kasue, and the author [23] proved that the limiting space (X_∞, d_∞) is an orbifold, and (M_a, h_a) is ALE of order 4. The same results except that (M_a, h_a) is ALE of order 4 are independently proved by M. Anderson [24].

§ 2. Hausdorff convergence

In this section we shall define the Hausdorff distance, and fix notation.

(2.1) Definition [8]. Let X and Y be compact metric spaces and $f: X \longrightarrow Y$ a map which is not necessarily continuous. We say f is an *ε-Hausdorff approximation* if

1) $|d_X(p, q) - d_Y(f(p), f(q))| < \varepsilon$ for $p, q \in X$,
2) ε-neighborhood of $f(X)$ contains Y.

The Hausdorff distance $d_H(X, Y)$ between X and Y is the infimum of all numbers ε such that there exist ε-Hausdorff approximations from X to Y and from Y to X. d_H defines a distance on the space of all compact metric spaces.

When a sequence $\{X_i\}$ of compact metric spaces converges to a compact metric space X_∞ in the Hausdorff distance, we write

$$\lim_H X_i = X_\infty.$$

We denote by $\mathcal{S}(n, D)$ the class of compact n-dimensional Riemannian manifolds which satisfy

$$\mathrm{Ric} \geq -(n-1), \qquad \mathrm{diam} \leq D.$$

The following theorem is due to Gromov [8].

(2.2) Theorem. *The closure of $\mathcal{S}(n, D)$ with respect to the Hausdorff distance is compact.*

In this paper we denote by $B(p; r)$ a ball in a metric space X centered at p with the radius r.

For noncompact spaces we define the pointed Hausdorff distance.

(2.3) DEFINITION [8]. Let X and Y be metric spaces, x and y points in X and Y respectively, and $f: X \longrightarrow Y$ a map which is not necessarily continuous. We say f is an *ε-pointed Hausdorff approximation* if

1) $f(x)=y$
2) $f(B(p; \varepsilon^{-1})) \subset B(q; \varepsilon^{-1}+\varepsilon)$
3) $f|B(p; \varepsilon^{-1}): B(p; \varepsilon^{-1}) \longrightarrow B(q; \varepsilon^{-1}+\varepsilon)$ is an ε-Hausdorff approximation.

The pointed Hausdorff distance $d_{p,H}((X, x), (Y, y))$ between pointed metric spaces (X, x) and (Y, y) is the infimum of all numbers ε such that there exist ε-pointed Hausdorff approximations from (X, x) to (Y, y) and from (Y, y) to (X, x). $d_{p,H}$ defines a distance on the space of all pointed metric spaces whose metric balls are all precompact.

When a sequence $\{(X_i, x_i)\}$ of pointed metric spaces converges to (X_∞, x_∞) in the pointed Hausdorff distance, we write

$$\lim_{p,H}(X_i, x_i) = (X_\infty, x_\infty).$$

We denote by $\mathcal{S}(n, \infty)$ the class of all complete Riemannian manifolds which satisfy

$$\mathrm{Ric} \geq -(n-1).$$

The following is also due to Gromov [8].

(2.4) THEOREM. *The closure of $\mathcal{S}(n, \infty)$ with respect to $d_{p,H}$ is compact.*

Recently many peoples study the limiting behaviors of Hausdorff convergent Riemannian manifolds. But almost all of them restrict their attention to the class of Riemannian manifolds satisfying the stronger assumptions than $\mathrm{Ric} \geq -(n-1)$, namely they treat $\mathcal{M}(n, D)$ which consists of Riemannian manifolds satisfying

$$|\text{sectional curvature}| \leq 1, \ \mathrm{diam} \leq D.$$

But in our situation this condition is too strong. Our interest is in the case that the sectional curvature becomes concentrated in small region, and its absolute value goes to infinity. On the other hand we assume the lower bound of volumes. So "collapsing" does not happen.

§ 3. Apriori estimate of curvature

In this section we derive a local estimate of curvature of Einstein metric. We treat an Einstein metric on an arbitrary dimensional manifold. The method is same as the case of a Yang-Mills connection [17]. But we must be careful since the base metric changes. The following lemma is obvious in the Yang-Mills case, if we deal with a fixed base manifold.

(3.1) LEMMA. *Let (X, g) be an n-dimensional Riemannian manifold, $B(p; r)$ a geodesic ball in X (we assume $r \leq \operatorname{diam}(X, g)/2$ if X is compact) which satisfies*

$$\text{the Sobolev constant } S(B(p; r)) \geq K,$$

for some positive constant K. Suppose a non-negative function u defined on $B(p; r)$ satisfies

$$\Delta u + au \geq 0$$

in the weak sense with some constant a. Then there exists a constant $C_1 = C_1(n, ar^2, K)$ such that

$$\sup_{B(p;r/2)} u \leq C_1 \left\{ r^{-n} \int_{B(p;r)} u^2 dV \right\}^{1/2}.$$

PROOF. Since the proof is a straightforward modification of the proof of the case that $B(p; r)$ is a ball in the Euclidean space, we shall omit it.

Using (3.1), we can prove a local curvature estimate as in [17]. The technique is originally due to Schoen (Theorem (2.2) in [19]).

(3.2) THEOREM. *Let (X, g) be an n-dimensional Riemannian manifold $(n \geq 4)$, $B(p; r)$ a geodesic ball in (X, g) $(2r \leq \operatorname{diam}(X, g)$ if X is compact) satisfying*

$$\operatorname{Ric} g = kg \qquad (k = \pm 1, \text{ or } 0),$$
$$\text{the Sobolev constant } S(B(p; r)) \geq K,$$

for some positive constant K. Then there exist positive constants $\varepsilon_1 = \varepsilon_1(n, K)$ and $C_3 = C_3(n, K)$ such that if

$$\int_{B(p;r)} |R|^{n/2} dV \leq \varepsilon_1,$$

we have

$$\sup_{B(p;r/4)} |R| \leq C_3 r^{-2} \left\{ \int_{B(p;r)} |R|^{n/2} dV \right\}^{2/n}.$$

PROOF. Since the Levi-Civita connection of Einstein metric is a Yang-Mills connection, we can apply Bochner-Weitzenböck formula, and have

$$\Delta |R| \geq -C_4 |R|^2$$

for some constant $C_4 = C_4(n)$. We take $p_0 \in \mathrm{Closure}(B(p; r/2))$ such that

$$\{r/2 - d(p, p_0)\}^2 |R(p_0)| = \sup_{B(p;r/2)} \{r/2 - d(p, *)\}^2 |R(*)|.$$

Let $\rho := 1/2\{r/2 - d(p, p_0)\}$. If $\rho = 0$, our assertion is true since $R = 0$. So we assume $\rho > 0$. Then we have

$$\sup_{B(p_0;\rho)} |R| \leq 4 |R(p_0)|.$$

We shall study two cases $|R(p_0)| \leq \rho^{-2}$ and $|R(p_0)| > \rho^{-2}$ separately. Now suppose $|R(p_0)| \leq \rho^{-2}$, then we have

$$\Delta |R| \geq -4C_4 \rho^{-2} |R| \qquad \text{on } B(p_0; \rho).$$

By (3.1) we have

$$\begin{aligned} |R(p_0)| &\leq C_5 \operatorname{vol}(B(p_0; \rho))^{-1/2} \|R\|_{L^2(B(p_0;\rho))} \\ &\leq C_5 \operatorname{vol}(B(p_0; \rho))^{-2/n} \|R\|_{L^{n/2}(B(p_0;\rho))} \end{aligned} \tag{3.3}$$

for some constant $C_5 = C_5(n, K, V)$. We have used Hölder's inequality in the second inequality above. Since the Sobolev inequality implies the isoperimetric inequality, we have

$$\operatorname{vol}(\partial B(x; \rho)) \geq K \operatorname{vol}(B(x; \rho))^{(n-1)/n}$$

for all $B(x; \rho) \subset B(p; r)$. Integrating the above inequality we have

$$\operatorname{vol}(B(p_0; \rho)) \geq C_6^{-1} \rho^n \tag{3.4}$$

where C_6 is a constant depending on n, K. On the other hand from the definition of p_0, we have

$$\sup_{B(p;r/4)} |R| \leq 2^{+6} r^{-2} \rho^2 |R(p_0)|. \tag{3.5}$$

Combining (3.3)(3.4)(3.5), we have verified our assertion in this case.

Next suppose $|R(p_0)|>\rho^{-2}$. Let $r_0:=|R(p_0)|^{-1/2}$. Then we have $r_0<\rho$. So we have got

$$\Delta|R|\geq -4C_4r_0^{-2}|R| \qquad \text{on } B(p_0;r_0).$$

By the same argument as the previous case, we have obtain

$$r_0^{-2}=|R(p_0)|\leq C_3r_0^{-2}\|R\|_{L^{n/2}(B(p;r))}.$$

But this contradicts with the assumption

$$\|R\|_{L^{n/2}(B(p;r))}\leq\varepsilon_1$$

if we take ε_1 so that $C_3\varepsilon_1<1$. So this case cannot happen.

(3.6) REMARK. In [17], for a Yang-Mills connection A, using the monotonicity formula

$$r^{4-n}\int_{B(p;r)}|R_A|^2dV\leq C_7s^{4-n}\int_{B(p;s)}|R_A|^2dV \qquad \text{for } r\leq s$$

we have shown that there exist constant ε_2 and C_8 such that if

$$r^{4-n}\int_{B(p;r)}|R_A|^2dV\leq\varepsilon_2,$$

then

$$\sup_{B(p;r/4)}|R_A|\leq C_8\left\{r^{-n}\int_{B(p;r)}|R_A|^2dV\right\}^{1/2}.$$

But we do not know that the constant C_7 depends only on n, K, V. This is the only reason why we cannot apply the proof of an apriori estimate for a Yang-Mills connection to the Einstein metric case. In the proof of (3.2) we have used the obvious inequality

$$\int_{B(p;r)}|R_A|^{n/2}dV\leq\int_{B(p;s)}|R_A|^{n/2}dV \qquad \text{for } r\leq s$$

instead of the monotonicity formula.

§ 4. Proof of Theorem

In this section we shall give a proof of Theorem (1.3). We restrict ourselves to 4-dimensional case.

By Gromov's compactness theorem, there exists a subsequence $\{j\}\subset\{i\}$ such that

$$\lim_H(X_j, g_j)=(X_\infty, d_\infty)$$

for some compact metric space (X_∞, d_∞). Taking a subsequence again (from now we shall often extract subsequences, but we shall use the same notation $\{j\}$ for all of them), we may assume that there exists $1/j$ Hausdorff approximation

$$\varphi_j:(X_j, g_j)\longrightarrow(X_\infty, d_\infty)$$

for all j. For all $p\in X_\infty$ we can find $p_j\in X_j$ such that

$$d_\infty(p, \varphi_j(p_j))<1/j.$$

Define the singular set S by

$$S:=\{p\in X_\infty: \text{for arbitrary } \{p_j\} \text{ as above and } r>0$$
$$\liminf_{j\to\infty}\int_{B(p_j;r)}|R_{g_j}|^2 dV_j\geq\varepsilon_1\},$$

where ε_1 is a constant appeared in Theorem (3.2). We remark that the Sobolev constant is uniformly bounded from below on $\mathcal{E}(D, V, R)$ by the result of Croke [4].

(4.1) CLAIM. *S is finite.*

PROOF. Fix a small number $r>0$. We take a collection of balls $\{B(x_a; r): x_a\in S\}$ such that

1) $S\subset\bigcup_a B(x_a; 2r)$
2) $B(x_a; r)\cap B(x_b; r)=\emptyset$ for $a\neq b$.

Since $x_a\in S$, for some large j, we have

$$\int_{B(x_{a,j};r/2)}|R_{g_j}|^2 dV_j\geq\varepsilon_1/2 \tag{4.2}$$

where $x_{a,j}$ is a point in X_j such that $d(x_a, \varphi_j(x_{a,j}))<1/j$. We may assume $\{B(x_{a,j}; r/2)\}_a$ are mutually disjoint. So we have

$$\text{the number of } \{x_a\}\leq 2\varepsilon_1^{-1}\sum_a\int_{B(x_{a,j};r/2)}|R_{g_j}|^2 dV_j$$
$$\leq 2\varepsilon_1^{-1}\int_{X_j}|R_{g_j}|^2 dV_j\leq 2R\varepsilon_1^{-1}.$$

Since $2R\varepsilon_1^{-1}$ is independent of r and r is arbitrary, we have verified the assertion.

First we observe that $X_\infty \backslash S$ is a C^∞-manifold, and there exists an Einstein metric g_∞ on $X_\infty \backslash S$. Let $p \in X_\infty \backslash S$. Taking a subsequence, there exist $r>0$, $p_j \in X_j$ such that

1) $d_\infty(\varphi_j(p_j), p) < 1/j$
2) $\int_{B(p_j;r)} |R_{g_j}|^2 dV_j \leq \varepsilon_1$ for all j.

By (3.1) we have

$$\sup_{B(p_j;r/4)} |R_{g_j}|^2 \leq C_3^2 r^{-4} \int_{B(p_j;r)} |R_{g_j}|^2 dV_j \leq C_3^2 r^{-4} \varepsilon_1. \tag{4.3}$$

By (3.4) we have got

$$\mathrm{vol}(B(p_j; r) \geq C_5^{-1} r^4. \tag{4.4}$$

Plug (4.3)(4.4) into a local injectivity radius estimate (Theorem (4.7) in [3]), we have got the lower bound of the injectivity radius at p_j independent of j. Applying [10], we can take for each j a harmonic coordinate system $h_j : B(p_j; r) \longrightarrow \boldsymbol{R}^4$ if we replace r smaller. Moreover by Lemma (2, 2) in [7],

$$h_j(B(p_j; r)) \supset B(0; \delta)$$

for some $\delta > 0$ independent of j.

Let $G_j := (h_j^{-1})^* g_j$ which is a Riemannian metric on $B(0; \delta)$. By [10] we have

$$\|G_j\|_{C^{1,\alpha}} \leq C_9$$

for some constant C_9 independent of j. In a harmonic coordinate system the Einstein equation turns out to be a quasi-linear elliptic system on G_j. By Schauder estimate [6], we can obtain

$$\|G_j\|_{C^k} \leq C_{10}(k)$$

for some constant $C_{10}(k)$ independent of j. Extracting a subsequence, we have got

$$G_j \longrightarrow G_\infty \quad \text{in } C^\infty \text{ topology}$$

where G_∞ is an Einstein metric on $B(0;\delta)$.

Now $\varphi_j\circ h_j^{-1}$ is a $1/j$ Hausdorff approximation from $(B(0;\delta), G_j)$ to (a neighborhood U of p, d_∞), we can show that it converges to a distance preserving map $F=F_p$ from $(B(0;\delta), G_\infty)$ to (U, d_∞). In fact, let $\{p_k\}_{k=1}^\infty$ be a countable dense subset of $B(0;\delta)$. By the diagonal argument, we can take a subsequence such that $\varphi_j\circ h_j^{-1}(p_k)$ converges for all k. We define $F(p_k)$ by this limit. Since F preserves distances, and is defined on a dense subset, we can extend F uniquely to the whole $B(0;\delta)$. Moreover F is surjective onto some neighborhood of p since $1/j$-neighborhood of $\varphi_j(X_j)$ contains X_∞. For two such maps F_p, F_q, and metrics $G_{p,\infty}, G_{q,\infty}$ for p and q respectively, $F_q^{-1}\circ F_p$ is an distance preserving map between the distances determined by $G_{p,\infty}$ and $G_{q,\infty}$. So it is an isometry between $G_{p,\infty}$ and $G_{q,\infty}$ ([9]). In particular it is differentiable. Thus $\{F_p\}_{p\in X_\infty\setminus S}$ gives a coordinate system on $X_\infty\setminus S$.

Since the proof of theorem (1.3) 2) is a straightforward modification of Green-Wu's proof of Gromov's compactness theorem [7], we shall omit it. (See also Kasue [11].)

Next we shall study the phenomena around the singular set.

PROOF OF THEOREM (1.3) 4). Fix $x_a\in S$. There exists $x_{a,j}\in X_j$ such that $d_\infty(\varphi_j(x_{a,j}), x_a)<1/j$. Take $\delta>0$ so that $(B(x;2\delta)\backslash\{x\})\cap S=\varnothing$. This can be done since S is a finite set. Define a positive number r_j by

$$r_j := \sup_{B(x_{a,j};\delta)} \sqrt{|R_{g_j}|}.$$

Since $x_a\in S$, $r_j\to\infty$ as $j\to\infty$. Moreover we may assume that $|R_{g_j}|$ take a local maximum value r_j at $x_{a,j}$. Consider a sequence of pointed Riemannian manifolds $((X_j, \tilde g_j), x_{a,j})$ where $\tilde g_j=r_j g_j$. It satisfies

$$\mathrm{Ric}\,\tilde g_j=(k/r_j)\tilde g_j$$

and

$$\sup_{\tilde B(x_{a,j};r_j\delta)} \sqrt{|R_{\tilde g_j}|}=1, \quad |R_{\tilde g_j}(x_{a,j})|=1, \tag{4.5}$$

where $\tilde B(x_{a,j};r_j\delta)$ is a geodesic ball with respect to the metric $\tilde g_j$. By Theorem (2.4) taking a subsequence we may assume

$$\lim\nolimits_{p,H}((X_j, \tilde g_j), x_{a,j})=((M_a, d_a), x_{a,\infty})$$

for some metric space (M_a, d_a) and a point $x_{a,\infty}\in M_a$. By the same argument as above we can show that M_a is a smooth manifold, and there exists a smooth Riemannian metric h_a on M_a compatible with d_a. Moreover we can construct a diffeomorphism $\Psi_j : B(x_{a,\infty}; r) \longrightarrow X_j$ so that $\Psi_j^* \tilde{g}_j$ converges to h_a in the C^∞-topology on $B(x_{a,\infty}; r)$ for all $r>0$. At last by (4.5) we have

$$|R_{h_a}(x_{a,\infty})|=1.$$

So h_a is not flat. And the Sobolev constant $S(M_a, h_a)$ of (M_a, h_a) is positive, since the Sobolev constants $S(X_j, g_j)$ are estimated from below by a positive constant independent of j.

PROOF OF THEOREM (1.3) 5). It easily follows from the lower semicontinuity of the curvature integral. We remark that we may assume that

$$\int_{X_j} |R_{g_j}|^2 dV_j$$

are the same value for all j since these are integers. Take $\rho>0$ and $r>0$. Let $S=\{x_1, \cdots, x_a, \cdots, x_k\}$. We have

$$\begin{aligned}&\int_{X_\infty - \bigcup_a B(x_a;\rho)} |R_{g_\infty}|^2 dV_\infty + \sum_a \int_{B(x_{a,\infty};r)} |R_{h_a}|^2 dV_{h_a}\\ &= \lim_{j\to\infty} \int_{F_j(X_\infty - \bigcup_a B(x_a;\rho))} |R_{g_j}|^2 dV_j + \sum_a \lim_{j\to\infty} \int_{G_j(B(x_{a,\infty};r))} |R_{g_j}|^2 dV_j.\end{aligned}$$

Since $F_j(X_\infty - \bigcup_a B(x_a; \rho)) \cap \{\bigcup_a G_j(B(x_{a,\infty}; r))\} = \emptyset$ for sufficiently large j, we have got the assertion.

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