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Monoidal transformations of Hirzebruch surfaces and Weyl groups of type C

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The study of the 27 lines on a nonsingular cubic surface has a long history. The configuration of these 27 lines has a high degree of symmetry. It is well known that their elegant symmetry is controlled by the Weyl group of type E_6. Furthermore the configuration of the exceptional curves of the first kind on a Del Pezzo surface V^d of degree d ($d=1,2,3$) can be controlled by the Weyl group of type E_{9-d} (see Du Val [D1] [D2], Manin [M]).

In this paper we show that there is a similar correspondence between the Hirzebruch surface X with several points blown up and the Weyl group of type C and that there is a natural realization of the root system of type C in the Picard group of X.

§ 1. Notation and statement of the results.

For each integer $e\in N$, an e-th Hirzebruch surface, or a rational ruled surface with invariant e, is a surface H_e which is isomorphic to the subvariety in $\boldsymbol{P}^2\times\boldsymbol{P}^1$

$$\{(\zeta_0:\zeta_1:\zeta_2)\times(s:t)\in\boldsymbol{P}^2\times\boldsymbol{P}^1\,|\,s^e\zeta_2-t^e\zeta_1=0\}.$$

Let F be a fiber of the projection $H_e\to\boldsymbol{P}^1$ and let S be a section:

$$F=\{(\zeta_0:\zeta_1:\zeta_2)\times(s:t)\in H_e\,|\,s=0\}$$
$$S=\{(\zeta_0:\zeta_1:\zeta_2)\times(s:t)\in H_e\,|\,\zeta_1=\zeta_2=0\}.$$

The linear system $|eF+S|$ is base-point-free (see Hartshorne [H] Chap. V, 2.17). Therefore the rational map $\Phi_{|eF+S|}:H_e\to\boldsymbol{P}^{e+1}$ turns out to be a morphism. Let Y be the image of H_e by $\Phi_{|eF+S|}$. We see easily that Y is a projective cone over a nonsingular curve C_1 which lies in a hyperplane of $\boldsymbol{P}^{e+1}$ and whose degree is exactly e, with vertex P_0.

Conversely the morphism obtained by blowing up P_0, φ: $\tilde{Y}\to Y$, is isomorphic to $\Phi_{|eF+S|}: H_e\to Y$.

Let $P_1, \cdots, P_m$ be points such that no two of them lie on a fiber and no one lies on the section S. Let $\Phi(P_i)=Q_i$. If $m=e+1$, then there exists a hyperplane H containing $Q_1, \cdots, Q_m$, not containing the vertex P_0. By Bertini's theorem, the set of hyperplanes H such that $H\cap Y$ is an irreducible nonsingular curve is an open dense subset of the complete linear system $|H|$, considered as a projective space. Therefore the set of the points $(P_1, \cdots, P_n)$ which has the following property $(*)$ is an open dense subset of the variety $H_e\times \cdots \times H_e$ (n times).

$(*)$ there exists an irreducible nonsingular curve in $|eF+S|$ passing through $P_1, \cdots, P_{e+1}$ but not $P_{e+2}, \cdots, P_n$ (passing through $P_1, \cdots, P_n$ if $e+1>n$).

We say that $P_1, \cdots, P_n$ are *in general position* if $(P_1, \cdots, P_n)$ satisfies the property $(*)$.

Let $P_1, \cdots, P_n$ be points of H_e in general position. Let $\pi: X_e(n)\to H_e$ be the morphism obtained by blowing up these n points.

PROPOSITION 1.1. *For each $e\geq 2$ there exists exactly $2n$ exceptional curves of the first kind on $X_e(n)$ if and only if $e\geq n$. They are* (1) *the exceptional curves $E_1, \cdots, E_n$* (2) *the strict transform $\tilde{F}_i$ of the fiber F_i containing P_i.*

We extend the intersection form in Picard group $\mathrm{Pic}(X_e(n))$ to $\mathrm{Pic}(X_e(n))\otimes_{\boldsymbol{Z}}\boldsymbol{R}$. We define a subspace V of $\mathrm{Pic}(X_e(n))\otimes_{\boldsymbol{Z}}\boldsymbol{R}$ and a set Δ as follows:

$$V=\{\boldsymbol{v}\in \mathrm{Pic}(X_e(n))\otimes_{\boldsymbol{Z}}\boldsymbol{R}\,|\,\boldsymbol{v}\cdot\boldsymbol{k}=\boldsymbol{v}\cdot\boldsymbol{f}=0\}$$

$$\Delta=\{\boldsymbol{l}_1-\boldsymbol{l}_2\,|\,\boldsymbol{l}_1 \text{ and } \boldsymbol{l}_2 \text{ are classes of exceptional curves of the first kind on } X_e(n),\ \boldsymbol{l}_1\neq\boldsymbol{l}_2\}$$

where $\boldsymbol{k}$ is the canonical class and $\boldsymbol{f}$ is the linear equivalence class of total transform of a fiber of the projection $H_e\to\boldsymbol{P}^1$. Then we have

THEOREM 1.2. *For each $e\geq 2$, if $e\geq n$ then Δ is a root system of type C_n of* rank n *in V. The Weyl group W corresponding to Δ can be characterized as follows*: $W=\{\sigma\in GL(\mathrm{Pic}(X_e(n)))\,|$ (1) *σ preserves intersection form,* (2) *σ fixes $\boldsymbol{k}$,* (3) *σ fixes $\boldsymbol{f}$*$\}$.

§ 2. Exceptional curves of the first kind on $X_e(n)$.

Let f and s be the linear equivalence classes of the total transforms of the fiber F and the section S on $X_e(n)$ respectively. Let $E_1, \cdots, E_n$ $(E_i=\pi^{-1}(P_i))$ be the exceptional curves and let $e_1, \cdots, e_n \in \mathrm{Pic}(X_e(n))$ be their linear equivalence classes. We know (see e.g. Hartshorne [H] Chap. V)

PROPOSITION 2.1.

(1) $\mathrm{Pic}(X_e(n)) \simeq \boldsymbol{Z}^{n+2}$, *generated by* $f, s, e_1, \cdots, e_n$.

(2) *The intersection pairing on* $X_e(n)$ *is given by* $f^2=0$, $s^2=-e$, $e_i^2=-1$ $(1\leq i\leq n)$, $f\cdot s=1$, $f\cdot e_i=0$, $s\cdot e_i=0$, $e_i\cdot e_j=0$ $(i\neq j)$.

(3) *The canonical class is* $k=-(e+2)f-2s+e_1+\cdots+e_n$.

We next prove Proposition 1.1. Let C be an exceptional curve of the first kind on $X_e(n)$ and not one of $E_i, \tilde{F}_j$. Since $C\simeq \boldsymbol{P}^1$ and $C^2=-1$, $C\cdot K=-1$ by the adjunction formula, where K is the canonical divisor on $X_e(n)$. On the other hand, C is not one of $E_i, \tilde{F}_j$ and H_e has no exceptional curve of the first kind, because $e\geq 2$, thus we have

$$C\cdot(\pi^*F)>0,\ C\cdot S\geq 0,\ C\cdot E_i\geq 0 \qquad (1\leq i\leq n), \tag{2.1}$$

where π^*F is the total transform of F. Let $c=xf+ys+\sum_{i=1}^{n} b_ie_i$ be the linear equivalence class of C. It follows from $C^2=K\cdot C=-1$ and (2.1) that

$$(2.2)\qquad \begin{aligned} &(1)\quad \sum_{i=1}^{n} b_i=(e-2)y-2x+1\\ &(2)\quad \sum_{i=1}^{n} b_i^2=2xy-ey^2+1\\ &(3)\quad y>0,\ x\geq ey,\ b_i\leq 0 \qquad (1\leq i\leq n). \end{aligned}$$

First of all we shall show that no integers satisfy (2.2) when $e\geq n$. We write $e=n+r$ $(r\geq 0)$. By Schwarz's inequality,

$$\left(\sum_{i=1}^{n} b_i\right)^2\leq n\sum_{i=1}^{n} b_i^2.$$

Thus it follows from (2.2) (1) (2) that

$$\{(e-2)y-2x+1\}^2\leq n(2xy-ey^2+1).$$

Then we have

$$4x^2-2\{2(e-2)y+ny+2\}x+\{(e-2)^2+ne\}y^2+2(e-2)y+1-n\leq 0. \tag{2.3}$$

Let us denote by $f(x, y)$ the left hand side of (2.3). Let α, β be the solutions of the quadratic equation $f(x, y)=0$ of variable x. Then we have $\alpha, \beta<ey$. Indeed assume that $ey<\alpha, \beta$. Then $2ey<\alpha+\beta=\{2(e-2)y+ny+2\}/2$. Substituting $e=n+r$ $(r\geq 0)$ from this inequality, we obtain $ny+2ry+4y-2<0$. This contradicts $n\geq 1$, $y\geq 1$. Also assume that $\alpha\leq ey\leq\beta$. Then we have $f(ey, y)\leq 0$. On the other hand we have

$$\begin{aligned} f(ey, y) &= (e^2-ne+4e+4)y^2-2(e+2)y+1-n \\ &= (r^2+nr+4n+4r+4)y^2-2(n+r+2)y+1-n>0, \quad (y\geq 1). \end{aligned}$$

This contradicts $f(ey, y)\leq 0$. Hence we have $\alpha, \beta<ey$.

Since $x\geq ey$, we have $\alpha, \beta<x$. Therefore we must have $f(x, y)>0$. This contradicts (2.3). This implies that there exists no integers satisfying (2.2).

It remains to show that there exists an exceptional curve of the first kind which is not one of $E_i, \tilde{F}_j$ if $e<n$. It is easy to check that integers $x=e$, $y=1$, $b_1=\cdots=b_{e+1}=-1$, $b_{e+2}=\cdots=b_n=0$ satisfy (2.2). The points $P_1, \cdots, P_n$ are in general position, thus we can find an irreducible nonsingular curve C on H_e in the linear system $|eF+S|$ such that C passes through $P_1, \cdots, P_{e+1}$, but not $P_{e+2}, \cdots, P_n$. Let $\tilde{C}$ be the strict transform of C on $X_e(n)$. Then the linear equivalence class of $\tilde{C}$ is $\boldsymbol{ef}+\boldsymbol{s}-\boldsymbol{e}_1-\cdots-\boldsymbol{e}_{e+1}$. Let g be the genus of $\tilde{C}$. It follows from the adjunction formula $2g-2=\tilde{C}\cdot(\tilde{C}+K)$ and Proposition 2.1 that $g=0$. Thus $\tilde{C}\simeq \boldsymbol{P}^1$. Since $\tilde{C}^2=-1$, $\tilde{C}$ is an exceptional curve of the first kind on $X_e(n)$ which is not one of $E_i, \tilde{F}_j$. Q.E.D.

§ 3. Weyl group of type C_n and $\mathrm{Pic}(X_e(n))$.

Notation is as in § 1. We shall prove Theorem 1.2. It follows from Proposition 1.1 that

$$\begin{aligned} \Delta = &\{\pm(\boldsymbol{e}_i-\boldsymbol{e}_j) \mid 1\leq i<j\leq n\} \cup \{\pm(\boldsymbol{f}-\boldsymbol{e}_i-\boldsymbol{e}_j) \mid 1\leq i\neq j\leq n\} \\ &\cup \{\pm(\boldsymbol{f}-2\boldsymbol{e}_i) \mid 1\leq i\leq n\}. \end{aligned}$$

We next observe that Δ generates V. Let $\boldsymbol{v}=x\boldsymbol{f}+y\boldsymbol{s}+\sum_{i=1}^{n} b_i\boldsymbol{e}_i \in V$. Since $\boldsymbol{v}\cdot\boldsymbol{k}=0$ and $\boldsymbol{v}\cdot\boldsymbol{f}=0$, we have $y=0$ and $2x+b_1+\cdots+b_n=0$. Hence $b_n=-2x-\sum_{i=1}^{n-1} b_i$ and $\boldsymbol{v}=x(\boldsymbol{f}-2\boldsymbol{e}_n)+\sum_{i=1}^{n-1} b_i(\boldsymbol{e}_i-\boldsymbol{e}_n)$. Thus Δ generates V. Also we can check that $2\alpha\cdot\beta/\alpha\cdot\alpha\in \boldsymbol{Z}$ for any $\alpha, \beta\in\Delta$. For $\alpha\in\Delta$, let $S_\alpha(\boldsymbol{v})=\boldsymbol{v}-2(\boldsymbol{v}\cdot\alpha/\alpha\cdot\alpha)\alpha$. We shall show that $S_\alpha(\beta)\in\Delta$ for $\beta\in\Delta$. If $\alpha=\boldsymbol{e}_i-\boldsymbol{e}_j$, then $S_\alpha(\boldsymbol{f})=\boldsymbol{f}$, $S_\alpha(\boldsymbol{e}_k)=\boldsymbol{e}_k$ $(k\neq i, j)$, $\boldsymbol{e}_i$ $(k=j)$, $\boldsymbol{e}_j$ $(k=i)$. If $\alpha=\boldsymbol{f}-\boldsymbol{e}_i-\boldsymbol{e}_j$, then $S_\alpha(f)=f$, $S_\alpha(e_k)=e_k$ $(k\neq i, j)$, $f-e_j$ $(k=i)$, $f-e_i$ $(k=j)$. If $\alpha=f-2e_i$, then $S_\alpha(f)=f$, $S_\alpha(e_j)=e_j$ $(i\neq j)$, $f-e_i$ $(i=j)$. Hence $S_\alpha(\beta)\in\Delta$. Finally if α, $c\alpha\in\Delta$ $(c\in \boldsymbol{R})$, then $c=\pm1$. Thus Δ is a reduced root system. Let $\Pi=\{e_1-e_2, \cdots, e_{n-1}-e_n, -f+2e_n\}$. Π is a base of Δ, i.e. Π is a linearly independent subset of Δ such that every element of Δ is a linear combination of elements of Π with all positive or all negative coefficients. Indeed, $e_i-e_j=(e_i-e_{i+1})+\cdots+(e_{j-1}-e_j)$, $-f+2e_i=(-f+2e_n)+2(e_i-e_n)$, $-f+e_i+e_j=(-f+2e_j)+(e_i-e_j)$.

Dynkin diagram of Π is as follows:

$\alpha_i=e_i-e_{i+1}$ $(1\leq i\leq n-1)$
$\alpha_n=-f+2e_n$

Thus Δ is a root system of type C_n.

Let $G=\{\sigma\in GL(\mathrm{Pic}(X_e(n)))\mid(1)$ σ preserves the intersection form, (2) σ fixes k, (3) σ fixes $f\}$. Let S_i be the reflection with respect to α_i. Let $W=\langle S_1, \cdots, S_n\rangle$. W is the Weyl group of type C_n. We can easily check that $S_i\in G$. Thus $W\subset G$. For $\varphi\in G$, we write $\varphi(e_i)=xf+ys+\sum_{i=1}^{n} b_ie_i$. Then $0=f\cdot e_i=f\cdot\varphi(e_i)=y$. Since $-1=e_i^2=\varphi(e_i)^2=-\sum_{i=1}^{n} b_i^2$, there exists l such that $b_l=\pm1$, $b_j=0$ $(j\neq l)$. Furthermore since $-1=e_i\cdot k=\varphi(e_i)\cdot k=-2x-b_l$, we have $x=1$ if $b_l=-1$, $x=0$ if $b_l=1$. Hence

$$\varphi(e_i)=e_l \quad\text{or}\quad f-e_l.$$

Therefore the order of G is $2^n n!$. On the other hand the order of W is also $2^n n!$. Hence we must have $G=W$. Q.E.D.

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