J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
35 (1988), 1-26.

On Hartogs type continuation theorem for regular solutions of linear partial differential equations with constant coefficients

Dedicated to Professor Seizô ITÔ, my first instructor
in advanced mathematics,
on the celebration of his sixtieth birthday.

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This is a continuation of our series of study on the extension of regular (especially real analytic) solutions of linear partial differential equations. This time we improve the result of [7], which may be considered to be part I of our present article, on continuation of real analytic solutions of a linear partial differential equation with constant coefficients to a thick obstacle K which is open to one side. Namely, let K denote the intersection of a compact set and the half space $\langle\vartheta, x\rangle<0$; we shall discuss when the following quotient space becomes trivial:

$$\mathscr{A}_P(U\setminus K)/\mathscr{A}_P(U). \tag{0.1}$$

Here in general $\mathscr{A}_P(U)$ denotes the totality of real analytic solutions of the equation $P(D)u=0$ on U, and U denotes a (or any) neighborhood of K, i.e. an open set containing K as a relatively closed subset. By a due reason we assume that $U\setminus K$ is connected. In the sequel we shall call such a pair $K\subset U$ a *Hartogs pair*. Also, without special mention we set $\vartheta=(0, \cdots, 0, 1)$, hence $\langle\vartheta, x\rangle=x_n$, in the sequel.

We first review the known results briefly: When K is compact we could give a necessary and sufficient condition for the triviality of (0.1), namely that P has no elliptic factor. (See [6], [12]. Recently Abramczuk [1] showed that this condition is common to the extension of any quasi-analytic class solutions, as long as U is convex.) When K is itself not compact but is open to one side as above, we only gave in [7] the following sufficient condition:

THEOREM 0.1. *Let $P(D)$ be a linear partial differential operator with constant coefficients, and let $\vartheta=(0, \cdots, 0, 1)$. We have the continuation of real analytic solutions for a fixed Hartogs pair, if each ir-*

reducible component q of P satisfies either of the following conditions in relation to the given Hartogs pair:

1) *There exists a sequence of directions ϑ_k tending to ϑ such that q is hyperbolic to each direction ϑ_k (in the hyperfunction sense, i.e. in the sense that the principal symbol q^0 of q satisfies $q^0(\vartheta_k)\neq 0$ and $q^0(\xi+\tau\vartheta_k)\neq 0$ for $\xi\in\boldsymbol{R}^n$,* $\operatorname{Im}\tau\neq 0$.

2) *By a suitable linear coordinate transformation fixing the hyperplane $x_n=0$, dx_1 becomes a non-characteristic direction and the following inequalities hold on the variety $N(q)=\{\zeta\in\boldsymbol{C}^n;\ q(\zeta)=0\}$: There exists k $(2\leqq k\leqq n-1)$ such that for any $\varepsilon>0$ we have, with some C_ε, on the region* $\operatorname{Im}\zeta_n>0$

$$(0.2)\qquad |\operatorname{Im}\zeta_1|\leqq\varepsilon\sum_{j=2}^{n}|\zeta_j|+A\sum_{j=2}^{k}|\operatorname{Re}\zeta_j|+B\sum_{j=2}^{n-1}|\operatorname{Im}\zeta_j|+C_\varepsilon,$$

$$(0.3)\qquad \sum_{j=2}^{k}|\operatorname{Re}\zeta_j|\leqq\varepsilon|\zeta|+B\sum_{j=1}^{n-1}|\operatorname{Im}\zeta_j|+C_\varepsilon.$$

3) *By a suitable linear coordinate transformation fixing the hyperplane $x_n=0$, $x_1=0$ becomes a non-characteristic hyperplane containing K and the following inequality holds on $N(q)$: For any $\varepsilon>0$ we have, with some C_ε, on the region* $\operatorname{Im}\zeta_n>0$

$$(0.4)\qquad |\operatorname{Im}\zeta_1|\leqq\varepsilon|\zeta_n|+b|\operatorname{Im}\zeta_n|+C_{\varepsilon,\zeta_2,\cdots,\zeta_{n-1}}.$$

Note that the linear coordinate transformations in 2) or 3) may depend on the irreducible components. The more P contains irreducible components of type 3), the thinner the obstacle K must be.

Among these, the third one, i.e. the result for a thin K contained in a non-characteristic hyperplane, is rather sharp and also it was later interpreted from a micro-local viewpoint, by means of the hyperfunctional boundary value problem, and thus generalized to the case of operators with real analytic coefficients. (See [11], [14].) As for those for general thick K, however, the first one is rather a trivial consequence from the hyperbolicity, whereas the second one is not so simple and we could give only the Heat equation as a meaningful example. Therefore it is now time to restart the study for general K. Here in this article we shall give an abstract necessary and sufficient condition for the continuation of real analytic solutions of the above type, based on a refined analysis of the Grušin transform by means of the Fourier hyperfunctions. Employing this, we deduce the following more simple and ample sufficient condition:

THEOREM 0.2. *Assume that the principal part P_m of $P(D)$ does not contain D_n. Assume further that it satisfies either of the following*:

1) *$P_m(D)$ is elliptic with respect to the variables actually appearing in it.*

2) *$P_m(D)$ is of second order and with real coefficients.*

3) *$P(D)$ itself does not contain D_n.*

Then for any Hartogs pair (0.1) *becomes trivial. That is, any real analytic solution defined in $U\setminus K$ can be continued as a real analytic solution to the whole U.*

More precisely speaking, we can add irreducible factors satisfying the assumption of this theorem among the list in Theorem 0.1.

We shall also give some necessary conditions.

Recall here the main difficulty arizing for general K in comparison with a thin K: It lies in the fact that the boundary values to a suitably chosen non-characteristic hyperplane, which were hyperfunctions for a thin K, now become complex analytic functionals. Thus we need an argument equivalent with the reduction of support of an analytic functional, which is in general rather delicate. The general project of our future study is given in [15].

In the appendix we discuss the case of parabolic equation with real analytic coefficients by a functional-analytic method, expressing our respects to Prof. S. Itô.

§ 1. Various spaces representing the obstruction for the continuation of real analytic solutions

As the problem of continuation of real analytic solutions from $U\setminus K$ to U, we seek parallelly two kinds of conditions to be posed on the operator $P(D)$, namely, one for any K and the other for a fixed K. Concerning the latter, however, as is seen below it is stiff to consider only the quotient space (0.1) for a given fixed pair. Therefore, letting $U^\varepsilon = U\cap\{x_n<-\varepsilon\}$, we rather seek below the condition for the triviality of

$$\varinjlim_{\varepsilon\to+0} \mathscr{A}_P(U^\varepsilon\setminus K)/\mathscr{A}_P(U^\varepsilon), \tag{1.1}$$

or more strongly, the triviality of each quotient space

$$\mathscr{A}_P(U^\varepsilon\setminus K)/\mathscr{A}_P(U^\varepsilon), \quad \varepsilon>0. \tag{1.2}$$

The reason for such modification is that in the discussion below we often have to solve the equation $P(D)u=f$ in the real analytic category, and this is possible in general only on a neighborhood of a little shrunk set $K^{\varepsilon}=K\cap\{x_n<-\varepsilon\}$. Note that there is a natural injective mapping from the quotient space (0.1) to the space (1.1), hence the vanishing of (1.1) trivially implies that of (0.1). Conversely, we see that the vanishing of (0.1) implies even the vanishing of (1.2) if K satisfies the following:

$$K+\varepsilon\vartheta\cap\{x_n<0\}\subset K \quad \text{for any} \quad \varepsilon>0. \tag{1.3}$$

(See Remark 1.3 below.) Hence we need not worry too much about such delicate distinction, except when we consider very special K, e.g. a bulged K placed in such a way as $\bar{K}\cap\{x_n=0\}=\{0\}$. Indeed, all known sufficient conditions ensure not only the vanishing of (0.1) but also of (1.2).

Now as for the triviality of (1.1) we know the following. (Similar assertion concerning the triviality of (1.2) was given in [7] for convex K and in [10] for general K.)

PROPOSITION 1.1. 1) *The space* (1.1) *depends only on* K *but not on a special choice of the neighborhood* U. *Hence in studying the triviality of* (1.1) *we can always choose, if required, a convex neighborhood* U *or even* $U=\{x_n<0\}$.

2) (1.1) *is trivial if and only if for every irreducible factor* q *of* P *the space corresponding to* (1.1) *is trivial. Hence when we study continuation of real analytic solutions in that form, we can assume without loss of generality that* P *is irreducible.*

PROOF. 1) Let $U\supset V$ be two neighborhoods of a fixed K. For $\varepsilon'<\varepsilon$, consider the following diagram consisting of natural mappings.

$$\begin{array}{ccc} \mathscr{A}_P(U^{\varepsilon'}\setminus K)/\mathscr{A}_P(U^{\varepsilon'}) & \xrightarrow{\rho^{\varepsilon'}_{VU}} & \mathscr{A}_P(V^{\varepsilon'}\setminus K)/\mathscr{A}_P(V^{\varepsilon'}) \\ \downarrow{\scriptstyle \rho^U_{\varepsilon\varepsilon'}} & & \downarrow{\scriptstyle \rho^V_{\varepsilon\varepsilon'}} \\ \mathscr{A}_P(U^{\varepsilon}\setminus K)/\mathscr{A}_P(U^{\varepsilon}) & \xrightarrow{\rho^{\varepsilon}_{VU}} & \mathscr{A}_P(V^{\varepsilon}\setminus K)/\mathscr{A}_P(V^{\varepsilon}). \end{array} \tag{1.4}$$

The horizontal arrows $\rho^{\varepsilon'}_{VU}$, ρ^{ε}_{VU} are clearly injective. Hence the corresponding mapping between the projective limits is injective. On the other hand, by the same argument as in [10], of which the outline we shall review below, we can see that the image of $\rho^V_{\varepsilon\varepsilon'}$ is contained in the image of ρ^{ε}_{VU}, whence we see that the mapping between the projective limits is surjective.

The essence of the argument of [10] is as follows: Let $v_{\varepsilon'} \in \mathscr{A}_P(V^{\varepsilon'}\setminus K)$. Decompose it as $v_{\varepsilon'}=g_{\varepsilon'}-h_{\varepsilon'}$, where $h_{\varepsilon'} \in \mathscr{A}(V^{\varepsilon'})$ and $g_{\varepsilon'} \in \tilde{\mathscr{O}}(\boldsymbol{D}^n\setminus\bar{K})$. (Here $\tilde{\mathscr{O}}$ is the sheaf of slowly increasing holomorphic functions on $\boldsymbol{D}^n+i\boldsymbol{R}^n$. See [16]. The above decomposition is assured by the cohomology vanishing theorem for this sheaf corresponding to Malgrange's theorem for $\mathscr{A}$, applied to the pair of "real" open sets $V^{\varepsilon'}$, $\boldsymbol{D}^n\setminus\bar{K}$ whose intersection equals $V^{\varepsilon'}\setminus K$. See ibid., Theorem 2.1.6. We use this sheaf because we must choose a convex complex domain containing U^{ε} in the domain of $f_{\varepsilon'}$ appearing below; If U is bounded, it suffices to choose $g_{\varepsilon'} \in \mathscr{A}(\{x_n<-\varepsilon'\}\setminus K)$ using the ordinary Malgrange theorem.) Then $f_{\varepsilon'}=Pg_{\varepsilon'}=Ph_{\varepsilon'}$ defines an element of $\tilde{\mathscr{O}}((\boldsymbol{D}^n\setminus\bar{K})\cup U^{\varepsilon'})$. Recall that an element of this space is holomorphic on a convex complex neighborhood of $x_n<-\varepsilon''$ for any $\varepsilon''>\varepsilon'$. Hence by the existence theorem of global holomorphic solution we can choose $h_{\varepsilon} \in \mathscr{A}(\{x_n<-\varepsilon\})\subset\mathscr{A}(U^{\varepsilon})$ such that $Ph_{\varepsilon}=f_{\varepsilon'}|_{U^{\varepsilon}}$. Put $u_{\varepsilon}=g_{\varepsilon'}-h_{\varepsilon}$. Then $u_{\varepsilon} \in \mathscr{A}_P(U^{\varepsilon}\setminus K)$ and $u_{\varepsilon}=v_{\varepsilon'} \bmod \mathscr{A}_P(U^{\varepsilon})$.

2) In view of 1) we can assume that U is a bounded convex neighborhood. Then the proof is just the same as Lemma 2.4 in [7].

Q.E.D.

Note that in view of the above assertion 1) we can assume without loss of generality that K is connected.

REMARK 1.2. In view of the uniqueness of analytic continuation the projective limit space (1.1) is isomorphic to the following:

$$\varprojlim_{\varepsilon\to+0} \mathscr{A}_P(U^{\varepsilon}\setminus\bar{K}_{\varepsilon})/\mathscr{A}_P(U^{\varepsilon}). \tag{1.5}$$

Here $\bar{K}_{\varepsilon}$ denotes the closed ε-neighborhood of $\bar{K}$. (We distinguish this from $K^{\varepsilon}=K\cap\{x_n<-\varepsilon\}$ byp lacing the suffix ε to the below.) Indeed, the natural mapping from (1.1) to (1.5) is obviously injective, and it is also surjective because of the relation

$$\mathscr{A}_P(U^{\varepsilon}\setminus\bar{K}_{\varepsilon})\cap\bigcap_{0<\varepsilon'<\varepsilon}(\mathscr{A}_P(U^{\varepsilon}\setminus\bar{K}_{\varepsilon'})+\mathscr{A}_P(U^{\varepsilon}))\subset\mathscr{A}_P(U^{\varepsilon}\setminus K).$$

Moreover, the above proof shows in fact that (1.1) is also isomorphic to the following:

$$(1.6)\qquad \varprojlim_{\varepsilon\to+0}\varinjlim_{\gamma\to+0}\mathcal{O}_P((\{x_n<-\varepsilon\}\setminus\bar{K}_\varepsilon)+i\{|y|<\gamma\})/\mathcal{O}_P(\{x_n<-\varepsilon\}+i\{|y|<\gamma\}).$$

Here $\mathcal{O}_P$ denotes the holomorphic solution of $Pu=0$. In fact, put $U=\{x_n<0\}$, and let B_γ denote $\{|y|<\gamma\}$ for the sake of simplicity. For $\varepsilon'<\varepsilon$ consider the following diagram corresponding to (1.4):

$$\begin{array}{ccc}
\varinjlim_{\gamma\to+0}\mathcal{O}_P((U^{\varepsilon'}\setminus\bar{K}_{\varepsilon'})+iB_\gamma)/\mathcal{O}_P(U^{\varepsilon'}+iB_\gamma) & \xrightarrow{\rho^{\varepsilon'}_{VU}} & \mathcal{A}_P(V^{\varepsilon'}\setminus K)/\mathcal{A}_P(V^{\varepsilon'}) \\
\big\downarrow \rho^U_{\varepsilon\varepsilon'} & & \big\downarrow \rho^V_{\varepsilon\varepsilon'} \\
\varinjlim_{\gamma\to+0}\mathcal{O}_P((U^{\varepsilon}\setminus\bar{K}_{\varepsilon})+iB_\gamma)/\mathcal{O}_P(U^{\varepsilon}+iB_\gamma) & \xrightarrow{\rho^{\varepsilon}_{VU}} & \mathcal{A}_P(V^{\varepsilon}\setminus K)/\mathcal{A}_P(V^{\varepsilon}).
\end{array}$$

This time the horizontal arrow $\rho^{\varepsilon'}_{VU}$ is not injective in itself, because an element of $\mathcal{A}(U^{\varepsilon'})$ is not necessarily holomorphic on a domain of the form $U^{\varepsilon'}+i\{|y|<\gamma\}$. However, an element of its kernel becomes zero when it goes down by the restriction $\rho^U_{\varepsilon\varepsilon'}$. Hence the natural mapping from (1.6) to (1.5) is injective. On the other hand, it is clear from the above proof that Image $\rho^V_{\varepsilon\varepsilon'}\subset$Image ρ^ε_{VU}, hence our mapping is surjective.

REMARK 1.3. We don't know if the assertion 1) holds as well concerning (0.1) for any fixed K. (If $\mathcal{A}_P(U\setminus K)/\mathcal{A}_P(U)=H^1_K(U,\mathcal{A}_P)$ holds, then this will be trivial in view of the excision theorem. But this isomorphism is not at all obvious, because $\mathcal{A}_P$ has in general the non-trivial first cohomology group even on a convex open set. See [10] in this respect.) Here we only prove that for K satisfying (1.3) the triviality of (0.1) for some fixed U implies the triviality of (1.2) for any U. Let $u\in\mathcal{A}(U^\varepsilon\setminus K)$. We decompose it as $u=v-w$, with

$$v\in\tilde{\mathcal{O}}(\boldsymbol{D}^n\setminus\bar{K}),\qquad w\in\mathcal{A}(U^\varepsilon),$$

as above. Then $f=Pv=Pw$ defines an element of $\tilde{\mathcal{O}}((\boldsymbol{D}^n\setminus\bar{K})\cup U^\varepsilon)$. For any $\varepsilon'>\varepsilon$ choose a solution g of $Pg=f$ in $\mathcal{A}(\{x_n<-\varepsilon'\})$. Then $v-g\in\mathcal{A}_P(\{x_n<-\varepsilon'\}\setminus K)$, and by the assumption (1.3) for K the translated one $v(x',x_n+\varepsilon')-g(x',x_n+\varepsilon')$ belongs to $\mathcal{A}_P(\{x_n<0\}\setminus K)$. The vanishing of (0.1) for some U implies that this element belongs to $\mathcal{A}_P(\{x_n<0\})$. After the backward translation, this implies that $u(x)$ belongs to $\mathcal{A}_P(U^{\varepsilon'})$. Since ε' is arbitrary, we conclude that $u(x)\in\mathcal{A}_P(U^\varepsilon)$, that is, the triviality of (1.2).

REMARK 1.4. We have seen that the main obstruction to prove an assertion like Proposition 1.1 was the problem of the global existence of real analytic solutions. Another way to avoid this is to consider the following problem instead of the triviality of (0.1): Find the condition for

$$u \in \mathscr{A}(U \setminus K),\ P(D)u \in \mathscr{A}(U) \Longrightarrow u \in \mathscr{A}(U). \tag{1.7}$$

This is equivalent to the triviality of the following space:

$$\begin{aligned} \operatorname{Ker}\{P(D)\colon & \mathscr{A}(U \setminus K)/(\mathscr{A}(U)+\mathscr{B}[K]) \\ & \longrightarrow \mathscr{A}(U \setminus K)/(\mathscr{A}(U)+\mathscr{B}[K])\}. \end{aligned} \tag{1.8}$$

For this assertion, the analogy of Proposition 1.1 nicely holds. The assertion (1.7) obviously implies the triviality of (0.1). On the other hand, the triviality of (1.1) implies (1.7). In fact, let $u \in \mathscr{A}(U \setminus K)$ be such that $P(D)u = f \in \mathscr{A}(U)$. By solving the cohomology with coefficients in $\mathscr{A}$ as in the proof of Proposition 1.1, we can assume that $u \in \tilde{\mathscr{O}}(\boldsymbol{D}^n \setminus \bar{K})$, hence that $f \in \tilde{\mathscr{O}}(\boldsymbol{D}^n \setminus (\bar{K} \setminus K))$. For each $\varepsilon > 0$ let $v_\varepsilon \in \mathscr{A}(U^\varepsilon)$ be a solution of $P(D)v = f$ on U^ε, and put $u_\varepsilon = u - v_\varepsilon$. Then $u_\varepsilon \bmod \mathscr{A}_P(U^\varepsilon)$ define an element of the space (1.1). Its triviality obviously implies the analytic extendability of the original u to U. This way of formulation was suggested by Prof. G. Bratti.

§2. An abstract necessary and sufficient condition

In this section we present an abstract necessary and sufficient condition for the continuation of real analytic solutions, or more precisely the vanishing of the obstruction space (1.1), expressed in terms of the Grušin representation for that space. For this purpose, we first review the Grušin representation for the quotient space (0.1) given in [7], and then modify it for the projective limit space (1.1). In the sequel we employ the fundamental principle of Ehrenpreis-Palamodov. The reader, bothered with the notion of Noether operator d, may simply assume that P is irreducible, and hence d is simply the restriction to the algebraic variety $N(P)$. As shown in Proposition 1.1, we do not lose so much by assuming this. Also we assume that K is convex from now on. Since $\complement K$ is connected by assumption, we have

$$\mathscr{A}_P(U \setminus K) \hookrightarrow \mathscr{A}_P(U \setminus \operatorname{ch} K)$$

for a convex neighborhood U. Therefore this is neither much restrictive in view of the same proposition.

By a little abuse of the symbol, we shall denote by ∂K the thin compact set $\bar{K}\setminus K$ contained in $x_n=0$. Now for $u \in \mathscr{B}_P(U\setminus K)$ (that is, for a hyperfunction solution of $Pu=0$ on $U\setminus K$), we extend it as a hyperfunction to an element $[u] \in \mathscr{B}(U)$. Then $P(D)[u]$ will have support in K. Let us denote by $[[P(D)[u]]] \in \mathscr{B}[\bar{K}]$ an extension with minimal support, and by $\widehat{[[P(D)[u]]]}$ its Fourier transform. By a Paley-Wiener type theorem this becomes an entire function $F(\zeta)$ with the following growth condition: for any $\varepsilon>0$ we can find $C_\varepsilon>0$ such that

$$|F(\zeta)| \leqq C_\varepsilon e^{\varepsilon|\zeta|+H_{\bar{K}}(\mathrm{Im}\,\zeta)}, \tag{2.1}$$

where $H_L(\eta)=\sup_{\eta\in L}\langle x, \eta\rangle$ denotes the supporting function of the set L. The *Grušin transform* $\tilde{d}u$ of u, introduced in [7] (and thus named in [8]), is the image of this by the *Noether operator* d. The meaning of the latter for the present case is as follows: For a germ of holomorphic function $F(\zeta)$ on $\boldsymbol{C}^n$, $d\cdot F(\zeta)$ denotes the following vector of germs of holomorphic functions on the varieties $N(q)=\{\zeta \in \boldsymbol{C}^n;\, q(\zeta)=0\}$:

$$d\cdot F(\zeta)=\left\{\left(\frac{\partial}{\partial\nu}\right)^j F|_{N(q)}\right\}_{q,j},$$

where q runs over irreducible factors of P and $0\leqq j\leqq m(q)-1$, $m(q)$ being the multiplicity of q, and ν is a fixed vector transversal to all $N(q)$.

PROPOSITION 2.1 ([7], Proposition 1.4). *The above construction defines the following isomorphism.*

$$\begin{array}{ccc} \tilde{d}\colon \mathscr{B}_P(U\setminus K)/\mathscr{B}_P(U) & \xrightarrow{\sim} & \widehat{\mathscr{B}[\bar{K}]}\{P, d\}/\widehat{\mathscr{B}[\partial K]}\{P, d\} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ u & \longmapsto & d\cdot\widehat{[[P(D)[u]]]}. \end{array} \tag{2.2}$$

Here in general $\widehat{\mathscr{B}[L]}\{P, d\}$ denotes the space of families of global holomorphic functions on the algebraic varieties $N(q)$, locally in the image of d and satisfying the same growth condition as $\widehat{\mathscr{B}[L]}$. By Kawai's theorem on the propagation of regularity ([16], Theorem 5.1.1) we have the canonical injective mapping

$$\mathscr{A}_P(U\setminus K)/\mathscr{A}_P(U) \hookrightarrow \mathscr{B}_P(U\setminus K)/\mathscr{B}_P(U).$$

In order to study real analytic solutions we must describe the image of this left-hand side in the above representation. In our former work [7] we did this employing the local operators with constant coefficients. Since that method is not convenient to obtain an abstract necessary and sufficient condition, we introduce here another idea, more adapted to the space (1.1).

Adopt therefore, in place of a hyperfunction extension $[u]$ the cut-off $\chi_\varepsilon P(D)(\psi_\varepsilon u)$, where $\psi_\varepsilon \in C^\infty(U)$ is such that $\psi_\varepsilon(x)\equiv 1$ outside the ε-neighborhood of K, $\equiv 0$ in the $\varepsilon/2$-neighborhood of K, and $\chi_\varepsilon \in C^\infty(\boldsymbol{R}^n)$ is such that $\chi_\varepsilon(x)\equiv 1$ on $x_n \leqq -\varepsilon$, $\equiv 0$ on $x_n \geqq -(\varepsilon/2)$, each with values in the interval $[0, 1]$. Here $P(D)(\psi_\varepsilon u)$ of course denotes the element of $C^\infty(U)$ defined as zero on K. (Hence in this argument u may be a distribution solution of $Pu=0$ in $U\setminus K$ as we shall assume it for a while.) Then we will have

$$|\widehat{\chi_\varepsilon P(D)(\psi_\varepsilon u)}| \leqq C(1+|\zeta|)^M \exp(H_{\bar{K}_\varepsilon}(\mathrm{Im}\,\zeta)),$$

where now $\bar{K}_\varepsilon$ denotes somewhat abusingly the part in $x_n \leqq 0$ of the closed ε-neighborhood of K, because the part in $x_n>0$ has no use for us. (Of course, when u is a real analytic solution the factor $(1+|\zeta|)^M$ is unnecessary, or may be replaced by $C_N(1+|\zeta|)^{-N}$ for every $N>0$.) If we consider the difference of this with $\widehat{[[P(D)[u]]]}$ as in [7], we cannot avoid to treat an infra-exponential growth order to the real direction (even if we are treating a distribution solution). Therefore we consider instead the difference of two such elements for different ε. Then,

$$\begin{aligned}
&\widehat{\chi_\varepsilon P(D)(\psi_\varepsilon u)} - \widehat{\chi_{\varepsilon'} P(D)(\psi_{\varepsilon'} u)} \\
&\quad = \widehat{\chi_\varepsilon P(D)(\psi_\varepsilon u - \psi_{\varepsilon'} u)} + \widehat{(\chi_\varepsilon - \chi_{\varepsilon'}) P(D)(\psi_{\varepsilon'} u)} \\
&\quad = P(\zeta)\cdot \widehat{\chi_\varepsilon(\psi_\varepsilon u - \psi_{\varepsilon'} u)} + \widehat{[\chi_\varepsilon, P(D)](\psi_\varepsilon u - \psi_{\varepsilon'} u)} + \widehat{(\chi_\varepsilon - \chi_{\varepsilon'}) P(D)(\psi_\varepsilon u)}.
\end{aligned}$$

Here the first term in the last side, i.e. the main ambiguity coming from the choice of ψ_ε, vanishes after the application of d, and for the remaining terms we obtain an estimate of the form

$$\leqq C(1+|\zeta|)^M \exp(H_{\partial K_\varepsilon}(\mathrm{Im}\,\zeta)),$$

with the larger ε among ε, ε', where ∂K_ε now denotes the part in $x_n \leqq 0$ of the closed ε-neighborhood of ∂K. Thus it is reasonable to take

$$\lim_{\substack{\longleftarrow \\ \varepsilon\to+0}} \widehat{\mathscr{E}'(\bar{K}_\varepsilon)}\{P, d\}/\widehat{\mathscr{E}'(\partial K_\varepsilon)}\{P, d\} \tag{2.3}$$

as our new representation space. Here as above $\widehat{\mathscr{E}'(L)}\{P, d\}$ denotes in general the space of vectors of global holomorphic functions on $N(q)$ locally belonging to the image of d, and satisfying the same growth condition as $\widehat{\mathscr{E}'(L)}$. This projective system is not so beautiful one compared to $\widehat{\mathscr{B}[\bar{K}]}\{P, d\}/\widehat{\mathscr{B}[\partial K]}\{P, d\}$ as the representation space. But it is more adapted to the space (1.1) and also provides an estimate easier to treat for us. The above construction gives rise to the following natural mapping:

$$\mathscr{D}'_P(U\setminus K)/\mathscr{D}'_P(U) \hookrightarrow \lim_{\substack{\longleftarrow \\ \varepsilon\to+0}} \widehat{\mathscr{E}'(\bar{K}_\varepsilon)}\{P, d\}/\widehat{\mathscr{E}'(\partial K_\varepsilon)}\{P, d\}.$$

We shall show that this is generalized to an isomorphic representation of the space

$$\lim_{\substack{\longleftarrow \\ \varepsilon\to+0}} \mathscr{D}'_P(U^\varepsilon\setminus\bar{K}_\varepsilon)/\mathscr{D}'_P(U^\varepsilon). \tag{2.4}$$

PROPOSITION 2.2. *The construction explained above gives rise to the following isomorphism*

$$\tilde{d}\colon \lim_{\substack{\longleftarrow \\ \varepsilon\to+0}} \mathscr{D}'_P(U^\varepsilon\setminus\bar{K}_\varepsilon)/\mathscr{D}'_P(U^\varepsilon) \xrightarrow{\sim} \lim_{\substack{\longleftarrow \\ \varepsilon\to+0}} \widehat{\mathscr{E}'(\bar{K}_\varepsilon)}\{P, d\}/\widehat{\mathscr{E}'(\partial K_\varepsilon)}\{P, d\}. \tag{2.5}$$

PROOF. For an element $u_\varepsilon \in \mathscr{D}'_P(U^{\varepsilon/2}\setminus\bar{K}_{\varepsilon/2})$ representing an element of the quotient space $\mathscr{D}'_P(U^{\varepsilon/2}\setminus\bar{K}_{\varepsilon/2})/\mathscr{D}'_P(U^{\varepsilon/2})$, we construct the element $\tilde{d}u_\varepsilon \in \widehat{\mathscr{E}'(\bar{K}_\varepsilon)}\{P, d\}/\widehat{\mathscr{E}'(\partial K_\varepsilon)}\{P, d\}$, regarding $U^{\varepsilon/2}$ as U. It is clear that the elements thus obtained are compatible with the mapping of the projective system (2.3). The verification of the isomorphy (2.5) is just parallel to the proof of Proposition 2.1: We only have to replace the corresponding fundamental principle from the one for hyperfunctions to the one for distributions. (Note that each quotient space $\widehat{\mathscr{E}'(\bar{K}_\varepsilon)}\{P, d\}/\widehat{\mathscr{E}'(\partial K_\varepsilon)}\{P, d\}$ represents isomorphically the space of distribution solutions on $U\setminus\bar{K}_\varepsilon$ which are prolongeable to $\bar{K}_\varepsilon$ modulo those extendable as distribution solutions on $U\setminus\partial K_\varepsilon$. However, the prolongeability becomes redundant after the projective limit is taken.)

Now we study the image of (1.1). First recall that in place of (1.1) we can consider the space (1.5) which resembles more to (2.4). (For the case of distribution or C^∞ solutions the corresponding relation is more complicated because it depends on the unique continuation property of the solutions in relation to P and K.) Note further that (1.5) is injectively contained in (2.4) because of Kawai's theorem on the propagation of analyticity (or rather by Boman's theorem, in this case of distribution solutions).

THEOREM 2.3. *The image of the space* (1.1) *by the Grušin transform is characterized by the following condition: An element of* (2.3) *is in the image of* (1.1) *if and only if for any* $\varepsilon>0$ *we can find* δ_ε, $C_\varepsilon>0$ *and a representative* $\{F_\varepsilon^{q,k}(\zeta)\}\in\widehat{\mathscr{E}'(\bar{K}_\varepsilon)}\{P, d\}$ *such that*

$$(2.6)\qquad \begin{aligned}&|F_\varepsilon^{q,k}(\zeta)|\\ &\leqq C_\varepsilon \exp(-\delta_\varepsilon|\zeta|+H_{\bar{K}_\varepsilon}(\operatorname{Im}\zeta))+C_\varepsilon(1+|\zeta|)^m\exp(H_{\partial K_\varepsilon}(\operatorname{Im}\zeta)).\end{aligned}$$

PROOF. Fix $\varepsilon>0$ and let $u_\varepsilon\in\mathscr{A}_P(U^{\varepsilon/2}\setminus K)$ be a representative of an element of $\mathscr{A}_P(U^{\varepsilon/2}\setminus K)/\mathscr{A}_P(U^{\varepsilon/2})$ which in turn represents an element of the projective limit (1.1). We construct $\tilde{d}u_\varepsilon$ as above, but now using a sequence of approximately real analytic functions $\{\psi_{\varepsilon,k}\}_{k=0}^\infty$, $\{\chi_{\varepsilon,k}\}_{k=0}^\infty$ as cut-off functions. That is, $\psi_{\varepsilon,k}$, $\chi_{\varepsilon,k}$ satisfy, besides the above listed properties, the following estimate

$$|D^\alpha\psi_{\varepsilon,k}(x)|,\ |D^\alpha\chi_{\varepsilon,k}(x)|\leqq C_\varepsilon B_\varepsilon^k|\alpha|^{|\alpha|},\quad\text{for}\quad|\alpha|\leqq k.$$

Then, by applying this estimate and the Paley-Wiener theorem to $D^\alpha(\chi_{\varepsilon,k}(x)P(D)(\psi_{\varepsilon,k}u))$, $|\alpha|\leqq k$, we find that $F_{\varepsilon,k}(\zeta)=\widehat{\chi_{\varepsilon,k}(x)P(D)(\psi_{\varepsilon,k}u)}$ will satisfy,

$$(2.7)\qquad |F_{\varepsilon,k}(\zeta)|\leqq C_\varepsilon B_\varepsilon^k k^k(1+|\zeta|)^{-k}\exp(H_{\bar{K}_\varepsilon}(\operatorname{Im}\zeta)),$$

with other constants B_ε, C_ε independent of k. Also, a closer look at the construction of $\psi_{\varepsilon,k}$, $\chi_{\varepsilon,k}$ shows that we can employ those satisfying further

$$|D^\alpha\psi_{\varepsilon,k}(x)|,\ |D^\alpha\chi_{\varepsilon,k}(x)|\leqq C_\varepsilon k^m\quad\text{for}\quad|\alpha|\leqq m.$$

Thus we see by the same calculus as before that the difference of thus constructed functions for different k, after the application of d, is estimated by

$$C_\varepsilon k^m\exp(H_{\partial K_\varepsilon}(\operatorname{Im}\zeta)).$$

Note that the constant C_ε here does not depend on k. Hence for given ζ with $|\zeta|\gg1$, estimating (2.7) with the representative such that $k\sim$

$a_\varepsilon|\zeta|$, $a_\varepsilon \ll 1$, we obtain

$$\begin{aligned}|d\cdot F_{\varepsilon,k_0}(\zeta)| &\leqq |d\cdot F_{\varepsilon,k}(\zeta)| + |d\cdot F_{\varepsilon,k_0}(\zeta) - d\cdot F_{\varepsilon,k}(\zeta)| \\ &\leqq C_\varepsilon \exp(a_\varepsilon(\log(B_\varepsilon a_\varepsilon))|\zeta| + H_{\bar{K}_\varepsilon}(\operatorname{Im}\zeta)) + C_\varepsilon(1+|\zeta|)^m \exp(H_{\partial K_\varepsilon}(\operatorname{Im}\zeta)),\end{aligned}$$

hence (2.6), with $\delta_\varepsilon = -a_\varepsilon|\log(B_\varepsilon a_\varepsilon)|$.

Conversely assume that there exists an element of (2.3) satisfying (2.6). Fix $\varepsilon_0 > 0$ and for an arbitrarily chosen $\varepsilon < \varepsilon_0$ take a representative $\{F_\varepsilon^{q,j}(\zeta)\} \in \widehat{\mathscr{E}'(\bar{K}_\varepsilon)}\{P, d\}$ of this element. By the fundamental principle for $\mathscr{E}'$, such $\{F_\varepsilon^{q,j}\}$ can first of all be represented as $d\cdot\hat{f}_\varepsilon(\zeta)$, with a distribution f_ε with support in $\bar{K}_\varepsilon$. On the other hand, if we choose $A>0$ large so that $\bar{K} \subset \{|x|<A\}$, then for $|\operatorname{Im}\zeta| \leqq (\delta_\varepsilon/A)|\zeta| + C$ the first term of the right-hand side of (2.3) can be neglected and hence

$$|F_\varepsilon^{q,j}(\zeta)| \leqq C_\varepsilon(1+|\zeta|)^m \exp(H_{\partial K_\varepsilon}(\operatorname{Im}\zeta)) \quad \text{on} \quad N(P) \cap \left\{|\operatorname{Im}\zeta| \leqq \frac{\delta_\varepsilon}{A}|\zeta| + C\right\}.$$

Replacing this by the following weaker estimate

$$|F_\varepsilon^{q,j}(\zeta)| \leqq C_{\varepsilon,\gamma} \exp(\gamma|\zeta| + H_{\partial K_\varepsilon}(\operatorname{Im}\zeta)) \tag{2.8}$$

on the following conic neighborhood of the real axis

$$|\operatorname{Im}\zeta| < \frac{\delta_\varepsilon}{A}|\operatorname{Re}\zeta| + C, \tag{2.9}$$

we can apply the fundamental principle for this growth order to obtain a holomorphic function $F_\varepsilon(\zeta)$ with the same growth condition (2.8) on the whole domain (2.9) but shrunk a little by diminishing the constant C. (The fundamental principle for the growth order (2.8) is proved in [6], II, §3 only on the whole space $\boldsymbol{C}^n$. But the proof goes just similarly on the domain (2.9), or even easier, if we admit to shrink it as above.) By a theorem of Kawai ([16], Lemma 5.1.2; see also [9], Remark after Lemma 2.3), such an element is the Fourier image of a rapidly decreasing modified Fourier hyperfunction $g_\varepsilon(x)$ with the singular support contained in ∂K_ε. That is, $g_\varepsilon(x)$ is a section of the sheaf of rapidly decreasing holomorphic functions $\underset{\approx}{\mathscr{O}}$ outside ∂K_ε, or it can be extended to an exponentially decreasing holomorphic function on a conical neighborhood of $\boldsymbol{D}^n \setminus \partial K_\varepsilon$. The difference $\hat{f}_\varepsilon(\zeta) - \hat{g}_\varepsilon(\zeta)$ obviously satisfies the estimate (2.8) with ∂K_ε replaced by $\bar{K}_\varepsilon$ and vanishes by d. Hence it is divisible by $P(\zeta)$, and in view of Malgrange's inequality combined with the above cited theorem of Kawai, we can find another modified Fourier hyperfunction $v_\varepsilon(x)$ with the singular support contained in $\bar{K}_\varepsilon$ such that

$$f_\varepsilon(x)-g_\varepsilon(x)=P(D)v_\varepsilon(x).$$

Thus, solving $Ph_\varepsilon=g_\varepsilon\in\tilde{\tilde{\mathcal{O}}}(\{x_n<\varepsilon\})$ for $h_\varepsilon\in\mathscr{A}(U^{\varepsilon_0})$, we obtain an element $u_{\varepsilon_0}=v_\varepsilon+h_\varepsilon$ of $\mathscr{A}_P(U^{\varepsilon_0}\setminus\bar{K}_{\varepsilon_0})$.

Let us show that the elements $\{u_\varepsilon\}_{\varepsilon>0}$ thus constructed are compatible with the morphisms of the projective system (1.5). Let u_{ε_0}, u_{ε_1} be two such elements for $\varepsilon_1<\varepsilon_0$. Let us denote the elements used for the construction of u_{ε_1} by the same symbols as above with the suffix ε' in place of ε (ε' is chosen so that $\varepsilon'<\min\{\varepsilon_1,\varepsilon\}$). Since we started from the same element of the space (2.3), we have

$$\hat{f}_\varepsilon-\hat{f}_{\varepsilon'}\in\widehat{\mathscr{E}'(\partial K_\varepsilon)}\{P,d\},$$

hence, again by the fundamental principle for $\mathscr{E}'$ we have, with some $w_\varepsilon\in\mathscr{E}'(\partial K_\varepsilon)$,

$$(Pv_\varepsilon+g_\varepsilon)-(Pv_{\varepsilon'}+g_{\varepsilon'})=f_\varepsilon-f_{\varepsilon'}=Pw_\varepsilon,$$

hence

$$P(v_\varepsilon-v_{\varepsilon'}-w_\varepsilon)=-(g_\varepsilon-g_{\varepsilon'}).$$

Therefore $v_\varepsilon-v_{\varepsilon'}-w_\varepsilon$ is a hyperfunction solution of the equation $Pu=0$ in U^ε, and by Kawai's theorem on the propagation of analyticity, it is in fact an element of $\mathscr{A}_P(U^\varepsilon)$. Thus we conclude that

$$\begin{aligned}u_{\varepsilon_0}-u_{\varepsilon_1}&=v_\varepsilon+h_\varepsilon-(v_{\varepsilon'}+h_{\varepsilon'})\\&=(v_\varepsilon-v_{\varepsilon'}-w_\varepsilon)+w_\varepsilon+h_\varepsilon-h_{\varepsilon'}\in\mathscr{A}_P(U^{\varepsilon_0}).\end{aligned}$$

Note that the same argument also shows that our construction does not depend on the special choice of the intermediate elements.

REMARK 2.4. The above proof shows that the factor $C_\varepsilon(1+|\zeta|)^m$ in (2.6) can be equivalently replaced by the tempered one $C_\varepsilon(1+|\zeta|)^{M_\varepsilon}$ which is more consistent with our representation space (2.3). Further, it can also be replaced by the infra-exponential one: $C_{\varepsilon,\gamma}e^{\gamma|\zeta|}$ for any $\gamma>0$. This can be interpreted as follows: As the space containing (1.1) we can employ, instead of (2.4), the space

$$\varprojlim_{\varepsilon\to+0}\mathscr{B}_P(U^\varepsilon\setminus\bar{K}_\varepsilon)/\mathscr{B}_P(U^\varepsilon), \tag{2.10}$$

or equivalently, instead of (2.3) the representation space

$$\varprojlim_{\varepsilon\to+0} \widehat{\mathscr{B}[\bar{K}_\varepsilon]}\{P, d\}/\widehat{\mathscr{B}[\partial K_\varepsilon]}\{P, d\}, \tag{2.11}$$

which is isomorphic to (2.10) in view of Proposition 1.1. Indeed the above proof implicitly employs this fact in the last stage. The relation between the spaces of solutions we have introduced until now is as follows:

$$\begin{array}{ccccc}
\mathscr{A}_P(U\setminus K)/\mathscr{A}_P(U) & \hookrightarrow & \mathscr{D}'_P(U\setminus K)/\mathscr{D}'_P(U) & \longrightarrow & \mathscr{B}_P(U\setminus K)/\mathscr{B}_P(U) \\
\cap & & \cap & & \cap \\
\varprojlim_{\varepsilon\to+0} \mathscr{A}_P(U^\varepsilon\setminus\bar{K}_\varepsilon)/\mathscr{A}_P(U^\varepsilon) & \hookrightarrow & \varprojlim_{\varepsilon\to+0} \mathscr{D}'_P(U^\varepsilon\setminus\bar{K}_\varepsilon)/\mathscr{D}'_P(U^\varepsilon) & \longrightarrow & \varprojlim_{\varepsilon\to+0} \mathscr{B}_P(U^\varepsilon\setminus\bar{K}_\varepsilon)/\mathscr{B}_P(U^\varepsilon).
\end{array}$$

Note that the horizontal mappings between $\mathscr{D}'$ and $\mathscr{B}$ are not necessarily injective, because $\mathscr{D}'_P(U\setminus K)\cap\mathscr{B}_P(U)\supsetneqq\mathscr{D}'_P(U)$ in general.

COROLLARY 2.5. (1.1) *is trivial if and only if for each irreducible component q of P we have the following: Let $\{F_\varepsilon(\zeta)\}_{\varepsilon>0}$ be a projective system of holomorphic functions on $N(q)$ satisfying*

$$|F_\varepsilon(\zeta)|\leqq C_\varepsilon \exp(-\delta_\varepsilon|\zeta|+H_{\bar{K}_\varepsilon}(\operatorname{Im}\zeta))+C_\varepsilon(1+|\zeta|)^m \exp(H_{\partial K_\varepsilon}(\operatorname{Im}\zeta)), \tag{2.6$'$}$$

and for any pair $\varepsilon'<\varepsilon$

$$|F_\varepsilon(\zeta)-F_{\varepsilon'}(\zeta)|\leqq C_\varepsilon(1+|\zeta|)^m \exp(H_{\partial K_\varepsilon}(\operatorname{Im}\zeta)).$$

Then F_ε satisfy indeed

$$|F_\varepsilon(\zeta)|\leqq C_\varepsilon(1+|\zeta|)^m \exp(H_{\partial K_\varepsilon}(\operatorname{Im}\zeta)). \tag{2.12}$$

Here the factors C_ε may be changed by the general tempered one or by the infra-exponential one as indicated in Remark 2.4. (As a sufficient condition we can do this replacement only in (2.12) to obtain the best result.)

PROOF. In view of Proposition 1.1 we can verify the triviality by each irreducible factor q of P. Then the condition for the triviality of (1.1) is clear from the established representation theorem. The fact that we can modify the estimates as above follows from the above Remark. Q.E.D.

The above necessary and sufficient condition for the triviality of

(1.1) is not so transparent. The condition for the triviality of (1.2) may be expressed in a little more refined way:

THEOREM 2.6. *In order that* (1.2) *becomes trivial for a fixed* K, *it is necessary and sufficient that the following Phragmén-Lindelöf type principle holds on* $N(q)$ *for every irreducible factor* q *of* P *and for any* $\varepsilon>0$: *If a holomorphic function* $F(\zeta)$ *on* $N(q)$ *satisfies, for any positive* $\varepsilon'<\varepsilon$

$$(2.13)\qquad |F(\zeta)|\leqq C_{\varepsilon'}\exp(-\delta_{\varepsilon'}|\zeta|+H_{\bar{K}_{\varepsilon'}}(\operatorname{Im}\zeta))+C_{\varepsilon'}(1+|\zeta|)^m\exp(H_{\partial K_\varepsilon}(\operatorname{Im}\zeta)),$$

then it satisfies indeed, for any $\gamma>0$,

$$(2.14)\qquad |F(\zeta)|\leqq C_\gamma(1+|\zeta|)^m\exp(\gamma|\operatorname{Im}\zeta|+H_{\partial K_\varepsilon}(\operatorname{Im}\zeta)).$$

As above, as a necessary condition we can equivalently replace the factors $C_{\varepsilon'}(1+|\zeta|)^m$, $C_\gamma(1+|\zeta|)^m e^{\gamma|\operatorname{Im}\zeta|}$ in (2.13)–(2.14) all by infra-exponential ones, and only the factor in (2.14) as a sufficient condition.

The proof is similar. What is not so trivial is the necessity part. This time, by repeating the last part of the proof of Theorem 2.3 we can see that $F(\zeta)$ corresponds to an element of

$$\varprojlim_{\gamma\to+0}\mathscr{A}_P(U^{\varepsilon+\gamma}\setminus\bar{K}_\gamma)/\mathscr{A}_P(U^{\varepsilon+\gamma})\cong\varprojlim_{\gamma\to+0}\mathscr{A}_P(U^{\varepsilon+\gamma}\setminus K)/\mathscr{A}_P(U^{\varepsilon+\gamma}).$$

Thus in view of Corollary 2.5 the triviality of (1.2) will imply that $F(\zeta)$ satisfies (2.14).

The condition for the extendability of real analytic solutions for any K (for fixed ϑ) may be expressed far more simply:

THEOREM 2.7. *Let* $\vartheta=(0,\cdots,0,1)$. *Then we have the triviality of* (0.1) *for any Hartogs pair if and only if the following Phragmén-Lindelöf type principle holds on* $N(q)$ *for each irreducible factor* q *of* P: *If a holomorphic function* $F(\zeta)$ *on* $N(q)$ *satisfies, for any* $\varepsilon>0$,

$$(2.15)\qquad |F(\zeta)|\leqq C_\varepsilon(1+|\zeta|)^m\exp(A|\operatorname{Im}\zeta'|+\varepsilon(\operatorname{Im}\zeta_n)_+ \\ +\max\{-\delta|\zeta'|-\delta_\varepsilon|\zeta_n|+a(\operatorname{Im}\zeta_n)_+,0\}),$$

for some constants A, a, $\delta>0$ *independent of* ε *and* δ_ε, $C_\varepsilon>0$ *depending on* ε, *then with another constant* $B>0$ *it satisfies indeed, for any* $\gamma>0$,

$$(2.16)\qquad |F(\zeta)|\leqq C_\gamma\exp(\gamma|\zeta|+B|\operatorname{Im}\zeta'|).$$

Here $\zeta'=(\zeta_1, \cdots, \zeta_{n-1})$ *and* $(\mathrm{Im}\,\zeta_n)_+=\max\{\mathrm{Im}\,\zeta_n, 0\}$.

PROOF. The triviality of (0.1) for any K is equivalent to the triviality of (1.2) for any K. Hence applying Proposition 1.1 we can assume that P is irreducible, $U=\{x_n<0\}$ and even $K=\{|x'|\leqq A, -a\leqq x_n<0\}$. Note that for this K we have

$$H_{\bar K}(\eta)=A|\eta'|+a(\eta_n)_+. \tag{2.17}$$

Then the necessity of the above Phragmén-Lindelöf type principle for the triviality of (1.2) is evident from the preceding theorem, because (2.15) is stronger than (2.13). In fact, a function $F(\zeta)$ satisfying (2.15) will satisfy (2.13) for any $\varepsilon>0$, hence (2.14) for any $\varepsilon>0$. (Therefore as a matter of fact we can obtain the conclusion (2.16) always with $B=A$. As a sufficient condition, however, it is convenient to employ a different constant B as above.)

To show the sufficiency, we must repeat the corresponding part of Theorem 2.3 a little. Take $u\in\mathscr{A}_P(U\setminus K)$. Since K is arbitrary now we can assume that u is in fact real analytic also on a neighborhood of the boundary of K:

$$\{|x'|=A, -a\leqq x_n\leqq 0\}\cup\{|x'|<A, x_n=-a\}.$$

Therefore choosing a fixed C^∞-function $\psi(x)$ such that $\psi\equiv 1$ on $\complement K$, and the characteristic function $\chi(x)=\chi(x_n)$ of $\{x_n\leqq 0\}$, we can consider the function

$$F(\zeta)=\widehat{\chi(x)P(D)(\psi(x)u)}|_{N(P)},$$

which satisfies

$$|F(\zeta)|\leqq C\exp(H_{\bar K}(\mathrm{Im}\,\zeta)).$$

Now, for any $\varepsilon>0$ we construct the functions $F_\varepsilon(\zeta)$ as in the proof of Theorem 2.3. Note that this time, in view of the assumed regularity of u at the boundary of K, we can use $\psi_{\varepsilon,k}(x)=\psi_k(x)$ independent of ε such that $\psi_k(x)=1$ on $\complement K$, and vary only $\chi_{\varepsilon,k}(x)=\chi_{\varepsilon,k}(x_n)$ with ε. Then from the proof of Theorem 2.3 we see that

$$\begin{aligned}|F_\varepsilon(\zeta)|\leqq C_\varepsilon\exp(-\delta|\zeta'|-\delta_\varepsilon|\zeta_n|+A|\mathrm{Im}\,\zeta'|+(a+\varepsilon)(\mathrm{Im}\,\zeta_n)_+)\\ +C_\varepsilon(1+|\zeta|)^m\exp(A|\mathrm{Im}\,\zeta'|+\varepsilon(\mathrm{Im}\,\zeta_n)_+),\end{aligned}$$

$$|F(\zeta)-F_\varepsilon(\zeta)|\leqq C_\varepsilon(1+|\zeta|)^m\exp(A|\mathrm{Im}\,\zeta'|+\varepsilon(\mathrm{Im}\,\zeta_n)_+).$$

Thus the fixed function $F(\zeta)$ on $N(P)$ satisfies, for any $\varepsilon>0$,

$$\begin{aligned}|F(\zeta)|&\leqq|F_\varepsilon(\zeta)|+|F(\zeta)-F_\varepsilon(\zeta)|\\&\leqq C_\varepsilon(1+|\zeta|)^m\exp(A|\mathrm{Im}\,\zeta'|+\varepsilon(\mathrm{Im}\,\zeta_n)_++\max\{-\delta|\zeta'|-\delta_\varepsilon|\zeta_n|+a(\mathrm{Im}\,\zeta_n)_+,0\}).\end{aligned}$$

Thus $F(\zeta)$ satisfies (2.15), hence (2.16). Again with a little abuse of the notation, put

$$L=\mathrm{ch}(K\cup\partial L)\cap U,\quad\text{where}\quad\partial L=\{|x'|\leqq B,\ x_n=0\}.$$

(Of course we are assuming that $B\geqq A$.) Then by the fundamental principle for hyperfunctions we can choose $v\in\mathscr{B}[\partial L]$, $w\in\mathscr{B}[\bar{L}]$ such that

$$F(\zeta)=\hat{v}(\zeta)+P(\zeta)\hat{w}(\zeta),$$

that is, $\chi P(D)((\psi u)-w)=v$. This implies that after modified on L, u can be continued to a hyperfunction solution on U. Thus u can actually be continued to an element of $\mathscr{A}_P(U)$. Q.E.D.

REMARK 2.8. As before, the condition of theorem 2.7 can be replaced by the following equivalent one which apparently improves the necessity part: If a holomorphic function $F(\zeta)$ on $N(P)$ satisfies the estimate, for any $\gamma>0$,

$$\begin{aligned}|F(\zeta)|\leqq C\exp(-\delta|\zeta|+A|\mathrm{Im}\,\zeta'|+a|\mathrm{Im}\,\zeta_n|)&\\+C_\gamma\exp(\gamma|\zeta|+A|\mathrm{Im}\,\zeta'|+b|\mathrm{Im}\,\zeta_n|),&\end{aligned}\tag{2.18}$$

with constants $a>b\geqq 0$, $\delta>0$, then it satisfies, for any $\gamma>0$,

$$|F(\zeta)|\leqq C_\gamma\exp(\gamma|\zeta|+A|\mathrm{Im}\,\zeta'|+b|\mathrm{Im}\,\zeta_n|).\tag{2.19}$$

The proof is quite similar.

A similar Phragmén-Lindelöf type condition was employed in a different problem of finding a global real analytic solution on a convex open set. In this respect, it is a very interesting question whether our conditions can be equivalently paraphrased to the corresponding ones on the principal part P_m as in [5], or they can be localized with respect to ξ (or at least with respect to ξ'), as in [21].

§ 3. A new sufficient condition for general K

In this section we seek a concrete class of operators P which satisfy the abstract conditions obtained in the preceding section. First we give

LEMMA 3.1. *Assume that the principal part of P does not contain D_n. Then every holomorphic function $F(\zeta)$ on $N(P)$ satisfying the condition* (2.15) *automatically satisfies*

$$|F(\zeta)|\leqq C_\varepsilon(1+|\zeta|)^m \exp(A|\mathrm{Im}\,\zeta''|+bA|\zeta''|+\lambda A|\zeta_n|^q+\varepsilon(\mathrm{Im}\,\zeta_n)_+), \tag{3.1}$$

with some constants b, $\lambda>0$, $q<1$ depending only on P, where $\zeta''=(\zeta_2, \cdots, \zeta_{n-1})$.

PROOF. Choose dx_1 as a non-characteristic direction. Let m be the order of P. Then from the algebraic assumption on P, we can see that the roots $\zeta_1=\tau_j(\zeta'', \zeta_n)$, $j=1, \cdots, m$ of the equation $P(\zeta)=0$ for ζ_1 all satisfy

$$|\tau_j(\zeta'', \zeta_n)|\leqq\lambda|\zeta_n|^q+b|\zeta''|+C, \quad j=1, \cdots, m, \tag{3.2}$$

with some constants λ, b, $C>0$ and $q\leqq 1-1/m$. (For the proof see e.g. [13], Remark 2.1.) Substituting this to a given $F(\zeta)$ satisfying (2.15), we find that for any $\varepsilon>0$,

$$\begin{aligned}&|F(\tau_j(\zeta'', \zeta_n), \zeta'', \zeta_n)|\\ &\leqq C_\varepsilon(1+|\zeta|)^m \exp(A|\mathrm{Im}\,\zeta''|+bA|\zeta''|+\lambda A|\zeta_n|^q+\varepsilon(\mathrm{Im}\,\zeta_n)_+\\ &\qquad +\max\{-\delta_\varepsilon|\zeta_n|+a(\mathrm{Im}\,\zeta_n)_+, 0\}).\end{aligned} \tag{3.3}$$

(Here we use estimate (3.2) only for $|\mathrm{Im}\,\tau_j(\zeta'', \zeta_n)|$. This remark, however, does not serve to weaken the assumption on P because of the homogeneity of this estimate with respect to the factor i.) Let us consider the fundamental symmetric polynomials

$$F_k(\zeta'', \zeta_n)=\sigma_k(F(\tau_1(\zeta'', \zeta_n), \zeta'', \zeta_n), \cdots, F(\tau_n(\zeta'', \zeta_n), \zeta'', \zeta_n)),\quad k=1, \cdots, m,$$

to obtain entire functions of (ζ'', ζ_n) satisfying (3.3) with the exponents multiplied by k and the factor $(1+|\zeta|)$ raised to mk. Consider them as functions of one variable ζ_n, and apply the Phragmén-Lindelöf principle. For fixed $\varepsilon>0$, we have first on the sides $\delta_\varepsilon|\zeta_n|=a\,\mathrm{Im}\,\zeta_n$

$$\begin{aligned}&|F_k(\zeta'', \zeta_n)|\\ &\leqq C_\varepsilon(1+|\zeta|)^m \exp(kA|\mathrm{Im}\,\zeta''|+kbA|\zeta''|+k\lambda A|\zeta_n|^q+k\varepsilon(\mathrm{Im}\,\zeta_n)_+).\end{aligned} \tag{3.4}$$

Hence applying the Phragmén-Lindelöf principle on this sector of opening $<\pi$ (after dividing by some holomorphic function to cancel the last two factors and the polynomial-like factor from the above estimate), we conclude that (3.4) holds on $\delta_\varepsilon|\zeta_n|\leqq a\operatorname{Im}\zeta_n$ with the same constants. Joining this with the trivial estimate for $0\leqq a\operatorname{Im}\zeta_n\leqq\delta_\varepsilon|\zeta_n|$, we conclude that (3.4) holds everywhere on $\operatorname{Im}\zeta_n\geqq 0$, for any fixed ζ'', hence on C^{n-1}.

The value of the original function $F(\zeta)$ can be obtained as the roots of the algebraic equation for t:

$$t^m - F_1(\zeta'', \zeta_n)t^{m-1} + \cdots + (-1)^m F_m(\zeta'', \zeta_n) = 0,$$

hence satisfies (3.1). Q.E.D.

REMARK 3.2. Assume that $P_m(D)$ actually contains $D_1, \cdots, D_l$ only, for some $l<n$. Then writing $\zeta'''=(\zeta_2, \cdots, \zeta_l)$, $\zeta^*=(\zeta_{l+1}, \cdots, \zeta_n)$ we obtain the following strengthened inequality:

$$(3.1)' \qquad |F(\zeta)|\leqq C_\varepsilon(1+|\zeta|)^m \exp(A|\operatorname{Im}\zeta''|+bA|\zeta'''|+\lambda A|\zeta^*|^q+\varepsilon(\operatorname{Im}\zeta_n)_+).$$

In fact, in this case (3.2) can be replaced by

$$(3.2)' \qquad |\tau_j(\zeta''', \zeta^*)|\leqq\lambda|\zeta^*|^q+b|\zeta'''|+C.$$

Hence the above proof applies without modification.

If $F(\zeta)$ were entire, the estimate (3.1) would imply that $F(\zeta)$ is the Fourier image of an analytic functional with porter in

$$(3.5) \qquad \{|z''|\leqq B,\ x_n=0\},$$

with $B=A(b+1)$. Hence together with the original estimate (2.15) which implies that $F(\zeta)$ is the Fourier image of a hyperfunction with support in K, we would conclude (by the uniqueness of the support of a real analytic functional; see e.g. [19]) that $F(\zeta)$ is the Fourier image of a hyperfunction with support in the intersection of (3.5) and K, i.e. $F(\zeta)$ would satisfy

$$|F(\zeta)|\leqq C_\varepsilon \exp(\varepsilon|\zeta|+H_{\partial K}(\operatorname{Im}\zeta)).$$

But for a holomorphic function $F(\zeta)$ on the variety $N(P)$, the validity of this deduction depends on P. We infer that the assumption for P having the principal part independent of D_n would suffice for this. (However, the "proof" given in [15] was wrong.) For the moment we will content ourselves with partial results by introducing some additional assumptions on P below.

THEOREM 3.3. *Assume that the principal part P_m of P contains only $D_1, \cdots, D_l$, $l<n$, and it is elliptic in these variables. Then for any Hartogs pair* (0.1) *becomes trivial.*

PROOF. We employ the same notation as in Remark 3.2. Thus we have (3.1)$'$. The ellipticity of P_m implies the following inequality for the point $(\zeta_1, \zeta''', \zeta^*)$ on $N(P_m)$:

$$|\mathrm{Im}\,\zeta_1| \geqq \delta|\mathrm{Re}\,\zeta'''| - |\mathrm{Im}\,\zeta'''|.$$

Hence by the continuity of the roots the point $(\zeta_1, \zeta''', \zeta^*)$ on $N(P)$ satisfy

$$|\mathrm{Im}\,\zeta_1| \geqq \delta|\mathrm{Re}\,\zeta'''| - |\mathrm{Im}\,\zeta'''| - \lambda'|\zeta^*|^q - C,$$

hence in particular

$$|\mathrm{Re}\,\zeta'''| \leqq c(|\mathrm{Im}\,\zeta'''| + |\zeta^*|^q + 1).$$

Substituting this into (3.1)$'$ we obtain

$$|F(\zeta)| \leqq C_\varepsilon(1+|\zeta|)^m \exp(B|\mathrm{Im}\zeta'''| + \mu|\zeta^*|^q + \varepsilon(\mathrm{Im}\,\zeta_n)_+).$$

That is, (2.16). Q.E.D.

THEOREM 3.4. *Assume that $P_m(D')$ is of second order and with real coefficients. Then for any Hartogs pair* (0.1) *becomes trivial.*

PROOF. By a linear coordinate transformation fixing $x_n=0$, we can assume that

$$P_m(D') = D_1^2 + \cdots + D_p^2 - D_{p+1}^2 - \cdots - D_{p+q}^2,$$

with $p+q<n$. Note that the roots $\tau_j(\zeta'', \zeta_n)$, $j=1, 2$ satisfy, besides (3.1), the following

$$\begin{aligned} (3.6)\qquad |\mathrm{Im}\,\tau_j(\zeta'', 0)| - \lambda|\zeta_n|^q - c &\leqq |\mathrm{Im}\,\tau_j(\zeta'', \zeta_n)| \\ &\leqq |\mathrm{Im}\,\tau_j(\zeta'', 0)| + \lambda|\zeta_n|^q + c, \quad j=1, 2, \end{aligned}$$

where $q=1/2$, and

$$(3.7)\qquad |\mathrm{Im}\,\tau_1(\zeta'', \zeta_n)| = |\mathrm{Im}\,\tau_2(\zeta'', \zeta_n)|.$$

Returning a little to the proof of Lemma 3.1, we have now

$$
\begin{aligned}
(3.3)' \qquad & |F(\tau_j(\zeta'', \zeta_n), \zeta'', \zeta_n)| \\
& \leqq C_\varepsilon \exp(A|\mathrm{Im}\, \tau_j(\zeta'', 0)| + A|\mathrm{Im}\, \zeta''| + \lambda A|\zeta_n|^q + \varepsilon(\mathrm{Im}\, \zeta_n)_+ \\
& \qquad + \max\{-\delta_\varepsilon|\zeta_n| + a(\mathrm{Im}\, \zeta_n)_+, 0\}).
\end{aligned}
$$

Hence by the Phragmén-Lindelöf principle we now obtain

$$
\begin{aligned}
(3.4)' \qquad & |F_k(\zeta'', \zeta_n)| \\
& \leqq C_\varepsilon(1+|\zeta|)^m \exp(kA|\mathrm{Im}\, \zeta''| + kA|\mathrm{Im}\, \tau_1(\zeta'', 0)| \\
& \qquad + k\lambda A|\zeta_n|^q + k\varepsilon(\mathrm{Im}\, \zeta_n)_+).
\end{aligned}
$$

Thus $F(\tau_j(\zeta'', \zeta_n), \zeta'', \zeta_n)$ satisfies the same inequality with $k=1$. In view of the remaining side of (3.6) and (3.7), we can replace $|\mathrm{Im}\, \tau_1(\zeta'', 0)|$ by $|\mathrm{Im}\, \zeta_1| + \lambda|\zeta_n|^q + c$ in this estimate. Thus we obtain (2.16) for $F(\zeta)$.
Q.E.D.

THEOREM 3.5. *Assume that $P(D)$ does not contain D_n. Then for any Hartogs pair* (0.1) *is trivial.*

PROOF. This time the variety $N(P)$ have the fiber structure

$$N(P) = \bigcup_{P(\zeta')=0} \{\zeta'\} \times \boldsymbol{C}_{\zeta_n}.$$

Therefore we can consider the holomorphic function $F(\zeta)$ on $N(P)$ as an entire function of ζ_n simply by fixing $\zeta' \in N(P)$. Thus without mixing the estimates by use of the symmetric polynomials we can conclude that

$$|F(\zeta)| \leqq C_\varepsilon(1+|\zeta|)^m \exp(A|\mathrm{Im}\, \zeta'| + \varepsilon(\mathrm{Im}\, \zeta_n)_+),$$

that is, (2.16). Q.E.D.

§ 4. Some necessary conditions

Now we show some necessary conditions in order to have the continuation of real analytic solutions. The following theorem asserts the necessity of condition 1) in Theorem 0.1 in some sense:

THEOREM 4.1. *Assume that $\mathscr{A}_P(U \setminus K)/\mathscr{A}_P(U) = 0$ for some fixed Hartogs pair such that K contains a half-ball with center on $\langle x, \vartheta\rangle = 0$. Assume further that ϑ is a non-characteristic direction for P. Then P is hyperbolic to the direction ϑ.*

PROOF. Choose a non-characteristic real analytic hypersurface $S=\{x_n=\varphi(x')\}$ such that the domain surrounded by S and the hyperplane $x_n=0$ is non-void and contained in K: $\phi\neq\{\varphi(x')\leqq x_n<0\}\subset K$. In the sequel we simply write S in place of $S\cap\{x_n<0\}$, and use $x'=(x_1, \cdots, x_{n-1})$ as the coordinates on S. We shall first give a proof assuming the triviality of (1.1), hence assuming Proposition 1.1. Let $u_0(x'), \cdots, u_{m-1}(x')\in{}'\mathscr{A}(S)$ be an arbitrarily given set of initial data on S, where $'\mathscr{A}$ denotes the real analytic functions of $n-1$ variables x'. By Cauchy-Kowalevsky's theorem there exists a solution $u(x)$ to the equation $P(D)u=0$ with these initial data on a neighborhood of S. By Malgrange's cohomology vanishing theorem for $\mathscr{A}$, we can choose a decomposition

$$u(x)=v(x)-w(x),$$
$$v\in\mathscr{A}(\{x_n<0,\ x_n\leqq\varphi(x')\}),\qquad w\in\mathscr{A}(\{\varphi(x_n)\leqq x_n<0\}).$$

On the common domain of definition we have

$$0=Pu=Pv-Pw.$$

Hence $f=Pv=Pw$ is an element of $\mathscr{A}(\{x_n<0\})$. Therefore if we take a global real analytic solution g of $Pg=f$ on U^ε (by shrinking U if necessary), we have $v-g\in\mathscr{A}_P(U^\varepsilon\setminus K)$. By the assumption this element can be continued to the whole U^ε. This means that v, hence u itself can be continued up to $\varphi(x')\leqq x_n<-\varepsilon$. Since ε is arbitrary, this implies that the Cauchy problem for real analytic data admits a solution on a fixed real domain $\varphi(x')\leqq x_n<0$ independent of the initial data. Thus e.g. by [20] we conclude that P is hyperbolic to the direction ϑ.

The modification when we only assume the triviality of (0.1) is as follows: Consider $u_\varepsilon(x)=u(x', x_n-\varepsilon)$. Since our operator is translation invariant, u_ε is a solution of $Pu=0$, defined on a neighborhood of $S_\varepsilon=\{x_n=\varphi(x')+\varepsilon,\ x_n<\varepsilon\}$. Employing now the sheaf $\tilde{\mathscr{O}}$ of slowly increasing holomorphic functions on $\boldsymbol{D}^n+i\boldsymbol{R}^n$, we can decompose u_ε as above, now with

$$v\in\tilde{\mathscr{O}}(\{x\in\boldsymbol{D}^n;\ x_n<\varepsilon,\ x_n<\varphi(x')+\varepsilon\}),\qquad w\in\mathscr{A}(\{\varphi(x')+\varepsilon\leqq x_n<\varepsilon\}).$$

Let g be a global holomorphic solution of $Pg=f$ on a convex complex neighborhood of $x_n\leqq0$. Then $v-g\in\mathscr{A}_P(U\setminus K)$, and the proof from now on is just the same as above. Q.E.D.

Remark that the non-characteristic directions of $P(D)$ are Zariski open. Thus the above theorem means that if the direction ϑ is an accumulation point (in the usual topology) of a Zariski dense set of directions ν such that the quotient space (0.1) concerning the half space $\langle x, \nu\rangle<0$ is trivial for general K, then the case 1) of Theorem 0.1 covers all possible situations. This supports in a sense the fact that we have treated mainly operators which are degenerate to the direction ϑ in the study of sufficient conditions in this article.

Appendix. Functional-analytic approach for the parabolic equation

We now consider a class of parabolic equations with variable coefficients which contain the heat equation:

$$Pu=\left(\frac{\partial}{\partial t}+A(x, t, D_x)\right)u(x, t)=0. \tag{A.1}$$

Here A is a strongly elliptic differential operator in x of order $2m$ containing t as a parameter. Namely, we assume that

$$\operatorname{Re} A_{2m}(x, t, \xi) > 0 \quad \text{for} \quad \xi \in \boldsymbol{R}^n\setminus\{0\}. \quad (\text{Recall that } D_x=-i\partial/\partial x.)$$

We further assume that the coefficients of A are real analytic in the joint variables (x, t) in the considered region. The variables x, t here are to correspond to x', x_n respectively in the argument hitherto. We shall show

THEOREM A.1. *Let $K\subset U$ be any Hartogs pair. Then every real analytic solution of the equation* (A.1) *on $U\setminus K$ can be continued to U.*

PROOF. Obviously it suffices to show the continuation of the solution to a neighborhood of the "bottom" of K. Thus without loss of generality we can assume that $U=\Omega\times]0, T[$ with the real analytic boundary $\partial\Omega$, $K\subset\Omega\times[a, T[$, and $u(x, t)$ is real analytic on a neighborhood of the part of the boundary

$$\partial\Omega\times[0, T]\cup\bar{\Omega}\times\{0\}. \tag{A.2}$$

Choose a Stein complex neighborhood W of $\bar{\Omega}\times[0, T]$. By Cartan's theorem B we can find a holomorphic function v on W such that

$$\begin{aligned} &v(x, 0)=u(x, 0) && \text{for} \quad x\in\bar{\Omega}, \\ &\partial_\nu^k v(x, t)|_{\partial\Omega}=\partial_\nu^k u(x, t)|_{\partial\Omega}, && \text{for} \quad 0\leqq t\leqq T \ (k=0, \cdots, m-1), \end{aligned}$$

where ∂_ν denotes a real analytic vector field in x-variables defined on a neighborhood of $\partial\Omega$ and transversal to it. Then $w=u-v$ satisfies the initial-boundary condition

$$w|_{\Omega\times\{0\}}=0; \qquad \partial_\nu^k w|_{\partial\Omega\times[0,T]}=0 \quad (k=0, \cdots, m-1),$$

and $f=Pw=-Pv$ can be naturally considered as a real analytic function on $\bar{\Omega}\times[0, T]$. Here we have the following

LEMMA A.2. *For any $f(x, t)\in\mathscr{A}(\bar{\Omega}\times[0, T])$ we have a unique solution $w(x, t)\in C^0([0, T], L_2(\Omega))$ of the mixed problem*

$$\text{(A.3)} \qquad \begin{cases} \left(\dfrac{\partial}{\partial t}+A(x, t, D_x)\right)w(x, t)=f(x, t), \\ w|_{t=0}=0; \qquad \partial_\nu^k w|_{\partial\Omega}=0 \quad (k=0, \cdots, m-1), \end{cases}$$

which is real analytic in $\Omega\times]0, T[$.

PROOF. According to [17], the operator A with the Dirichlet boundary condition generates an evolution operator $U(t, s)$ on $L_2(\Omega)$ which is holomorphic in t, s when $t, s\in\Delta$ and $|\mathrm{Im}(t-s)|<\lambda\,\mathrm{Re}(t-s)$, where Δ is a fixed complex neighborhood of $[0, T]$ depending on the domain of holomorphy of the coefficients of $A(x, t, D_x)$. Thus (A.3), viewed as an abstract evolution equation on $L_2(\Omega)$ admits a unique solution $w(x, t)\in C^1([0, T], L_2(\Omega))$. Thanks to Theorem 3 loc. cit., it is holomorphic in t in a complex neighborhood of $]0, T[$ depending on Δ and the domain of holomorphy of $f(x, t)$. On the other hand, by [4], Theorem 6.2 the operator (A.1) has partial analytic hypo-ellipticity in x. Namely, any solution of (A.3) can be continued with respect to each space variable x_j to a fixed complex neighborhood when the other variables are fixed on the real domain. Thus by the Malgrange-Zerner theorem (see e.g. [18], Theorem 3), $w(x, t)$ becomes jointly holomorphic in a complex neighborhood of $\Omega\times]0, T[$. Q.E.D.

END OF PROOF OF THEOREM A.1 is now immediate: By Harnack's theorem the original solution $u(x, t)$ agrees with $v(x, t)+w(x, t)$ on $\Omega\times]0, a[$. Hence the latter gives the analytic continuation of the former to $\Omega\times]0, T[$. Q.E.D.

As we saw in §3, when $A(x, t, D_x)$ has constant coefficients the continuation of real analytic solutions takes place even if we replace $\partial/\partial t$ in (A.1) by $-\partial/\partial t$, $i\partial/\partial t$, or by $(\partial/\partial t)^\mu$, $\mu<2m$. It seems very interesting to explain this by functional-analytic method and thereby extend the result to the case of real analytic coefficients.

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(Received June 3, 1987)

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