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Singular perturbation of domains and semilinear elliptic equation

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Contents

§1. Introduction.

We consider the following semilinear elliptic boundary value problem:

$$(1.1)\qquad \begin{cases} \Delta v+f(v)=0 & \text{in } \Omega \\ \dfrac{\partial v}{\partial \nu}=0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\boldsymbol{R}^n$ with smooth boundary $\partial\Omega$ and ν denotes the unit outer normal vector on $\partial\Omega$. $\Delta=\sum_{j=1}^{n}\partial^2/\partial x_j^2$ is the Laplace operator and f is a real valued smooth function on $\boldsymbol{R}$.

The structure of the solutions of (1.1) and their stability largely depend upon the geometrical property of the domain Ω. Therefore, if the shape of Ω is deformed, then the structure of solutions changes accordingly, and it is usually the case that this change is continuous in some sense, so far as Ω is deformed smoothly. On the other hand, when the domain is perturbed "singularly", it is in general much more obscure how the structure of solutions of (1.1) varies, and not much study has been done on this subject except for a few pioneering works including the beautiful papers of Hale and Vegas [10] and Vegas [22]. The subject of the present paper is to deal with such a singular perturbation of Ω in a setting similar to but more general than that of [10] and [22], and to study in detail the structure of solutions of

(1.1) at its "singular limit". Among other things we shall show that, when a portion of the region Ω becomes thinner and degenerates into a one-dimensional line segment in the limit process, the limit problem for (1.1) will be described in part by an ordinary differential equation on this line segment, which is so to speak an "infinitesimal remnant" of a portion of the original region. One of the contributions of our study is to emphasize the role of such an infinitesimal remnant, the importance of which has hardly been recognized before.

Before describing our result more precisely, let us first specify the type of domain that we deal with in this paper. Roughly speaking, it is a domain with parameter $\zeta>0$ and is decomposed as $\Omega(\zeta)=D_1\cup D_2\cup Q(\zeta)$ where D_1 and D_2 are mutually disjoint regions and $Q(\zeta)$ is a changing part which approaches a line segment as $\zeta\downarrow 0$. (See Figure 1.) In particular, the volume of $Q(\zeta)$ decreases to zero as $\zeta\downarrow 0$.

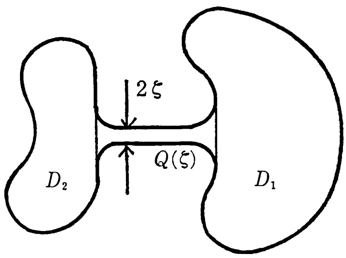


Figure 1

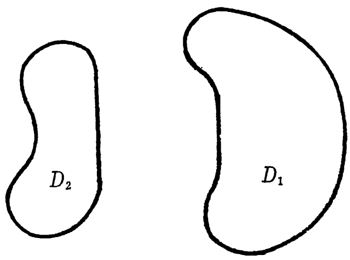
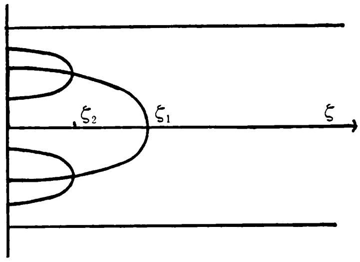


Figure 2

One of the important questions to arise in this situation is whether or not the influence of $Q(\zeta)$ over (1.1) for $\Omega=\Omega(\zeta)$ vanishes as $\zeta\downarrow 0$, i.e., whether the structure of the solutions of (1.1) for $\Omega=\Omega(\zeta)$ (for small $\zeta>0$) is equivalent to that of (1.1) for $\Omega=\Omega_0\equiv D_1\cup D_2$ (Figure 2) or not. Closely related to this question is an observation by Vegas [22] and Hale-Vegas [10], who have considered (1.1) for $f(\lambda, u)=\lambda u-u^p$ on the same domain as that in Figure 1 and have analyzed the bifurcation phenomenon for the bifurcation parameter ζ (when $\lambda>0$ is a sufficiently small constant). Their bifurcation diagram in the case where p is an odd natural number and the domain $\Omega(\zeta)$ is symmetric, is as in Figure 3.

In their situation, when ζ is very small (i.e. $0<\zeta<\zeta_2$ in Figure 3) there are exactly nine solutions and each of them takes values near one of the values $\{0, \lambda^{1/(p-1)}, -\lambda^{1/(p-1)}\}$ in D_i $(i=1, 2)$. Similarly, (1.1) for $\Omega=\Omega_0$ has exactly nine solutions, each of which is equal to one of the values $\{0, \lambda^{1/(p-1)}, -\lambda^{1/(p-1)}\}$ in D_i for each i. Moreover each solution of

(1.1) for $\Omega=\Omega(\zeta)$ converges as $\zeta\rightarrow 0$ to a corresponding solution for $\Omega=\Omega_0$, with its stability properties unchanged in the limit process. Thus the structure of the solutions for $\Omega(\zeta)$ $(0<\zeta<\zeta_2)$ is equivalent to that for $\Omega=\Omega_0$, which is a non-connected open set.



Fifiure 3 (Bifurcation Diagram)

These results may seem to suggest that problem (1.1) for $\Omega=\Omega(\zeta)$ can be regarded as a perturbation of that for $\Omega=\Omega_0$. With all these observations, it is, however, our conclusion in this paper that, so far as the structure of the solution set of (1.1) is concerned, it is more natural to regard $\Omega(\zeta)$ as a perturbation of the set $\Omega_*=D_1\cup D_2\cup L$ (exhibited in Figure 4), where $L=\bigcap_{\zeta>0}Q(\zeta)$, rather than of $\Omega_0=D_1\cup D_2$.

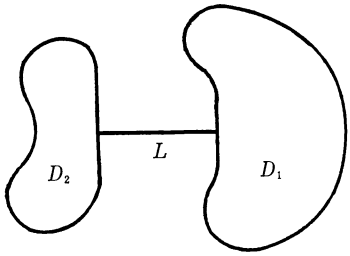


Figure 4

In other words, the influence of the varying portion $Q(\zeta)$ over (1.1) does not vanish as $\zeta\rightarrow 0$ but in fact is asymptotically equal to that of the line segment L. The reason why the role of L did not appear in the results of Hale and Vegas [10] and Vegas [22] is the following: In their situation, $\partial f/\partial u$ is small around the solutions of (1.1) because of the smallness of $\lambda>0$. In such a case, as one can easily show, if v is a solution of (1.1) for $\Omega=\Omega_*$ then its behavior on the whole region Ω_* is automatically determined by its behavior on $D_1\cup D_2$, therefore the structure of the solution set of (1.1) for $\Omega=\Omega_*$ is equivalent to that for $\Omega=\Omega_0$; In other words, the role of L does not appear explicitly in the limit problem. This, however, is not always the case if we consider a more general situation in which $\partial f/\partial u$ is not necessarily small, and the segment L may often play an important role as we shall see in Sections 3 and 4 of this paper.

Solutions of the boundary value problem (1.1) can be regarded as "equilibrium"— or time-independent — solutions of the parabolic initial boundary value problem

$$(1.2)\qquad \begin{cases} \dfrac{\partial u}{\partial t}=\Delta u+f(u) & \text{in} \quad (0,\infty)\times\Omega, \\ \dfrac{\partial u}{\partial \nu}=0 & \text{on} \quad (0,\infty)\times\partial\Omega, \\ u(0,x)=u_0(x) & \text{in} \quad \Omega. \end{cases}$$

When we speak of the stability of the solutions of (1.1), we mean the stability of those solutions as equilibrium solutions of (1.2). More precisely, we define the stability as follows:

DEFINITION 1. The equilibrium solution v of (1.2) is said to be *stable* if given any $\varepsilon>0$, there exists a $\delta>0$ such that $\|u(t,\cdot)-v(\cdot)\|_{L^\infty(\Omega)}\leqq\varepsilon$ $(0<t<\infty)$ for any $w\in C^0(\bar{\Omega})$ satisfying $\|v-w\|_{L^\infty(\Omega)}\leqq\delta$, where u is a solution of (1.2) with the initial condition $u(0,x)=w(x)$. We say that v is *unstable* if v is not stable.
For details, see, for example, Matano [15].

It has been observed by several authors that the stability and the structure of the equilibrium solutions of (1.2), together with their stability properties, is closely related to the geometry of the domain Ω. It is known that any non-constant equilibrium solution is unstable if Ω is a bounded convex domain in $\boldsymbol{R}^n$. (See N. Chafee [4] for $n=1$ and see H. Matano [15] and Casten-Holland [3] for general n.) More generally, the same result holds in the case where Ω is a Riemannian manifold with non-negative Ricci curvature and $\partial\Omega$ has non-positive definite second fundamental form with respect to the unit outer normal vector ν on $\partial\Omega$ (S. Jimbo [11]). On the other hand, Matano [15] has constructed a non-constant stable equilibrium solution on the same type of domain as $\Omega(\zeta)$ in Figure 1. We shall give in Section 2 an improved version of a result of Matano [15]. For related topics concerning systems of reaction-diffusion equations, see K. Kishimoto and H. F.

Weinberger [13], H. Matano and M. Mimura [17].

The contents of this paper are as follows:

In Section 2, we shall establish a theorem on the existence of stable solutions of (1.1) for domains of the form $\Omega(\zeta)=\bigcup_{i=1}^{N} D_i \cup Q(\zeta)$ where D_i $(i=1, 2, \cdots, N)$ are mutually disjoint regions and $Q(\zeta)$ is a channel connecting these regions such that the measure of $Q(\zeta)$ tends to zero as $\zeta\to 0$. More precisely, given an arbitrary sequence of (not necessarily distinct) values $a_1, a_2, \cdots, a_N$ satisfying $f(a_i)=0$ and $f'(a_i)<0$ $(i=1, 2, \cdots, N)$, we shall show for every small $\zeta>0$ the existence of a stable solution v_ζ of (1.1) with $\Omega=\Omega(\zeta)$ such that v_ζ takes values very close to a_i on D_i for each $i=1, 2, \cdots, N$ (Theorem 1). In this theorem we shall impose only very weak conditions on $Q(\zeta)$, thereby allowing $\Omega(\zeta)$ to be possibly a very wild perturbation of the region $\bigcup_{i=1}^{N} D_i$. Theorem 1 is a much improved version of the results obtained by Matano [15; Theorem 6.2 and Corollary 6.3].

In Section 3, in order to obtain a more detailed information about the singular limit of (1.1), we shall consider the case $N=2$ as in Figure 1 and impose additional conditions on the shape of $Q(\zeta)$, so that we may carry out a more delicate argument. The main result in this section (Theorem 3) states that if v_ζ is a solution of (1.1) for $\Omega=\Omega(\zeta)$ with $\zeta>0$ sufficiently small (which means that the channel region $Q(\zeta)$ is thin enough) and $\|v_\zeta-a_i\|_{L^2(D_i)}$ is sufficiently small for $i=1, 2$, where a_1, a_2 are constants satisfying $f(a_i)=0$, $f'(a_i)<0$, then $v_\zeta|_{Q(\zeta)}$ is closely approximated by a solution V of the two point boundary value problem

$$(1.3)\qquad \begin{cases} \dfrac{d^2V}{dz^2}+f(V)=0 & \text{in} \quad L, \\ V=a_i & \text{on} \quad \bar{D}_i\cap\bar{L} \quad (i=1, 2), \end{cases}$$

(see Figure 4), and that the stability property of v_ζ coincides with that of V (Theorem 3).

In Section 4, we shall somewhat show the converse of Theorem 3: We start from solutions of (1.3) and then construct solutions of (1.1) approximate them on $Q(\zeta)$. Our study will be confined to a specific example of f, for which we have $a_1=a_2$ and (1.3) has three distinct $a_1\equiv V^{(0)}<V^{(1)}<V^{(2)}$, with $V^{(1)}$ unstable and $V^{(0)}$, $V^{(2)}$ both stable. We shall then construct, for each small $\zeta>0$, three distinct solutions $a_1\equiv v_\zeta^{(0)}<v_\zeta^{(1)}<v_\zeta^{(2)}$ of (1.1) for $\Omega=\Omega(\zeta)$, such that

$$(1.4)\qquad \lim_{\zeta\to 0}\sup_{x\in D_1\cup D_2}|v_\zeta^{(i)}(x)-a_1|=0 \quad (i=0, 1, 2),$$

(1.5) $$\lim_{\zeta\to 0}\sup_{x\in Q(\zeta)}|v_\zeta^{(i)}(x_1, x_2, \cdots, x_n)-V^{(i)}(x_1)|=0 \quad (i=0, 1, 2),$$

and that $v_\zeta^{(1)}$ is unstable while the rest is stable (Theorem 4). What is particularly interesting about this result is that, although the behaviors of the solutions $v_\zeta^{(i)}$ $(i=0, 1, 2)$ on $\Omega(\zeta)$ are almost indistinguishable except on the extremely thin portion $Q(\zeta)$, their stability properties are quite different from one another. Theorem 4, together with Theorem 3, indicates that much of the information about the structure of the solution set of (1.1) for $\Omega=\Omega(\zeta)$—with ζ sufficiently small—is contained in the ordinary differential equation (1.3).

For technical reasons we assumed that the space dimension n is larger than or equal to 3 throughout Sections 3 and 4, while we only assume $n\geqq 2$ in Section 2. All the functions that we consider in this paper are real valued.

Acknowledgement

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§2. Existence of Stable Solutions.

Let $D_1, D_2, \cdots, D_N$ be bounded domains in $\boldsymbol{R}^n$ $(n\geqq 2)$ such that each D_j has a smooth boundary ∂D_j and that $D_i\cap D_j=\emptyset$ holds for any i and j with $i>j$. We specify the situation as follows:

(II-1) $\{\Omega(\zeta)\}_{\zeta>0}$ is a family of bounded domains in $\boldsymbol{R}^n$ which satisfies the following conditions (1) and (2):

(1) Each $\Omega(\zeta)$ has a smooth boundary and $\Omega(\zeta_1)\supset\Omega(\zeta_2)\supset\bigcup_{i=1}^N D_i$ holds for any ζ_1 and ζ_2 such that $\zeta_1>\zeta_2>0$.

(2) $$\lim_{\zeta\to 0}\mathrm{Vol}\left(\Omega(\zeta)-\bigcup_{i=1}^N D_i\right)=0,$$

where $\mathrm{Vol}(S)$ denotes the Lebesgue measure of a set $S\subset\boldsymbol{R}^n$.

(II-2) Let f be a real valued smooth function on $\boldsymbol{R}$ such that the set $\Pi\equiv\{\xi\in\boldsymbol{R}\mid f(\xi)=0, f'(\xi)<0\}$ is not empty.

Under the above conditions (II-1) and (II-2), we consider solutions of (1.1), or, equivalently, equilibrium solutions of the following semilinear diffusion equation (2.1):

$$(2.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t}=\Delta u+f(u) & \text{in} \quad (0,\ \infty)\times\Omega(\zeta), \\ \dfrac{\partial u}{\partial \nu}=0 & \text{on} \quad (0,\ \infty)\times\partial\Omega(\zeta), \\ u(0,\ x)=u_0(x) & \text{in} \quad \Omega(\zeta). \end{cases}$$

We present the main theorem of this section, which is in some sense a much improved version of a theorem of Matano [15].

THEOREM 1. *For any sequence of (not necessarily distinct) values* $\{a_i\}_{i=1}^N$ *contained in the set* $\Pi=\{\xi\in\boldsymbol{R}\mid f(\xi)=0,\ f'(\xi)<0\}$ *and for any sufficiently small* $\zeta>0$, *problem* (2.1) *has at least one stable equilibrium solution* v_ζ *which satisfies the following condition*:

$$(2.2)\qquad \begin{cases} \lim\limits_{\zeta\to 0}\|v_\zeta-a_i\|_{L^2(D_i)}=0 \quad (1\leqq i\leqq N) \\ \lim\limits_{\zeta\to 0} v_\zeta=a_i \ \textit{in}\ C^\infty(\overline{D_i(\eta)}) \ \textit{for any}\ \eta>0, \quad (1\leqq i\leqq N) \end{cases}$$

where $D_i(\eta)\equiv\{x\in D_i\mid \mathrm{dis}(x,\ \Omega(\eta)-D_i)>\eta\}$ *for* $\eta>0$.

REMARK. It is Matano [15; Theorem 6.2 and Corollary 6.3] who has first obtained a result on the existence of nonconstant stable equilibrium solutions of (2.1), though he has not considered the singular limit as $\zeta\to 0$. Hale and Vegas [10] have studied this singular limit and have obtained a result similar to our Theorem 1 by using the implicit function theorem. Their results require additional conditions on the shape of $Q(\zeta)$ and also have to assume that $\partial f/\partial u$ is small, in order to ensure the uniqueness of the solution v_ζ satisfying (2.2). Our theorem, on the other hand, does not impose any conditions on the bound of $\partial f/\partial u$, nor on the shape of $Q(\zeta)$. Note that the method in [10] or [22], which is based on the implicit function theorem, does not apply in such a situation; in fact, as we shall see in Section 4, one cannot in general expect the uniqueness of a solution v_ζ that satisfies (2.2). We therefore take a different approach, in which a result of Matano [15; Theorem 4.2] plays an essential role. We write down his theorem in a slightly modified form (Proposition 1 below).

DEFINITION 2. A closed set $Y\subset C^1(\bar{\Omega})\cap C^2(\Omega)$ is said to be *positively invariant* under the semiflow defined by (1.2) (or simply "under (1.2)" if, given any $w\in Y$, the solution $u(t,\ x)$ of (1.2) with initial data $u_0=w$ is defined globally on $[0,\ \infty)\times\Omega$ and satisfies $u(t,\ \cdot)\in Y$ for all $t\geqq 0$.

PROPOSITION 1 (Matano [15]). *Let Ω be a bounded domain in $\boldsymbol{R}^n$ with smooth boundary and let f be a smooth function. Let $Y, Y_1, Y_2, Y_3, \cdots$ be a family of non-empty sets in $C^1(\bar{\Omega})\cap C^2(\Omega)$ such that*

(a) *$Y_1\supset Y_2\supset Y_3\supset\cdots$ and $\cap_{m=1}^{\infty} Y_m=Y$;*

(b) *each Y_m is closed in $C^1(\bar{\Omega})\cap C^2(\Omega)$ and bounded in $L^\infty(\Omega)$; moreover, for each m, Y_{m+1} is contained in the interior of Y_m with respect to the topology of $C^1(\bar{\Omega})\cap C^2(\Omega)$;*

(c) *each Y_m is positively invariant under the semiflow defined by* (1.2).

Then Y contains at least one stable equilibrium solution of (1.2).

For the proof of this proposition, see [15]. The following well-known identity will be useful later in this section:

PROPOSITION 2. *Let D be a bounded domain in $\boldsymbol{R}^n$ with smooth boundary ∂D. Let $\lambda_1<\lambda_2\leqq\lambda_3\leqq\cdots$ be the eigenvalues of $-\Delta$ on D under the Neumann boundary conditions and $\psi_1, \psi_2, \psi_3, \cdots$ be the corresponding eigenfunctions that are orthonormalized with respect to the $L^2(D)$ inner product. Then*

$$\int_D |\mathrm{grad}\,\psi|^2 dx=\sum_{k=1}^{\infty}\lambda_k\left(\int_D \psi\psi_k dx\right)^2=\sum_{k=2}^{\infty}\lambda_k\left(\int_D \psi\psi_k dx\right)^2$$

for any $\psi\in H^1(D)$.

We omit the proof. (Note that $\lambda_1=0$.) We denote by $\{\lambda_{i,q}\}_{q=1}^{\infty}$ and $\{\psi_{i,q}\}_{q=1}^{\infty}$, respectively, the sequence of eigenvalues arranged in increasing order and the system of corresponding orthonormalized eigenfunctions associated with the operator $-\Delta$ on D_1 with the Neumann boundary conditions. We need some notations. Set

$$Q(\zeta)\equiv\Omega(\zeta)-\bigcup_{i=1}^{N} D_i$$

$$a^*\equiv\max_{1\leq i\leq N} a_i,\quad a_*\equiv\min_{1\leq i\leq N} a_i,$$

where $a_1, a_2, \cdots, a_N$ are the constants that appear in Theorem 1. Let $A(x)$ be a smooth function on $\boldsymbol{R}^n$ satisfying

$$\text{(2.3)}\qquad \begin{cases} A(x)=a_i \text{ for any } x\in D_i\ (1\leqq i\leqq N);\\ a_*\leqq A(x)\leqq a^* \text{ for } x\in\boldsymbol{R}^n \text{ and } \mathrm{grad}\,A(x) \text{ has compact support in } \boldsymbol{R}^n. \end{cases}$$

We define, for $w\in H^1(\Omega(\zeta))\cap L^\infty(\Omega(\zeta))$, a functional

$$J_\zeta(w) \equiv \int_{\Omega(\zeta)} \left(\frac{1}{2} |\text{grad}\, w|^2 - \int_{A(x)}^{w(x)} f(\xi) d\xi \right) dx \tag{2.4}$$

and also we define, for $\zeta > 0$ and $\delta > 0$,

$$\tilde{E}(\delta, \zeta) \equiv \{ w \in C^1(\overline{\Omega(\zeta)}) \cap C^2(\Omega(\zeta)) \mid a_* - \delta \leqq w(x) \leqq a^* + \delta \text{ in } \Omega(\zeta), \\ J_\zeta(w) \leqq J_\zeta(A) + \delta^3 \}, \tag{2.5}$$

$$E(\delta, \zeta) \equiv \{ w \in \tilde{E}(\delta, \zeta) \mid \|w - a_i\|_{L^2(D_i)} \leqq \delta, \ i = 1, 2, \cdots, N \}. \tag{2.6}$$

LEMMA 2.1. *Let $\delta_0 > 0$ be such that $f'(\xi) < 0$ for any $\xi \in [a_* - \delta_0, a_*) \cup (a^*, a^* + \delta_0]$. Then for any $\zeta > 0$ and $\delta \in (0, \delta_0)$, the set $\tilde{E}(\delta, \zeta)$ is positively invariant under the semiflow defined by* (2.1) (*see Definition* 2).

PROOF. Given $\zeta > 0$ and $\delta > 0$, let $u(t, x)$ be a solution of (2.1) with initial data $u(0, \cdot) \in \tilde{E}(\delta, \zeta)$. Since $f(a_* - \delta) > 0$ and $f(a^* + \delta) < 0$, the constant function $a_* - \delta$ is a time-independent subsolution of (2.1) while $a^* + \delta$ is a supersolution. In view of this, and applying the comparison theorem together with the local existence theorem for (2.1), we easily find that the solution $u(t, x)$ exists globally on $[0, \infty) \times \Omega(\zeta)$ and satisfies the inequality $a_* - \delta \leqq u(t, x) \leqq a^* + \delta$ there. To complete the proof of the lemma, we have only to show that $J_\zeta(u(t, \cdot)) \leqq J_\zeta(A) + \delta^3$ for all $t \geqq 0$. But this inequality is obvious since $J_\zeta(u(t, \cdot))$ is monotone non-increasing in $t \geqq 0$, which follows from the well-known formula

$$\frac{d}{dt} J_\zeta(u(t, \cdot)) = -\int_{\Omega(\zeta)} \left| \frac{\partial}{\partial t} u(t, x) \right|^2 dx \leqq 0 \quad (t > 0)$$

and the continuity of $J_\zeta(u(t, \cdot))$ at $t = 0$ (see, for example, [15]). Thus Lemma 2.1 is proved.

To prove the positive invariance of $E(\delta, \zeta)$, however, requires a more delicate argument as well additional assumptions on δ and ζ. Once $E(\delta, \zeta)$ is proved to be positively invariant, we would be in a stage to apply Proposition 1, from which the conclusion of Theorem 1 will follow. Much of the rest of this section will be devoted to finding conditions on δ and ζ for $E(\delta, \zeta)$ to be positively invariant under (2.1).

REMARK. As we are concerned with only those solutions of (2.1) satisfying $a_* - \delta_0 \leqq u \leqq a^* + \delta_0$, their behaviors remain unchanged if we replace f by another function $\bar{f}$ satisfying $f = \bar{f}$ on $[a_* - \delta_0, a^* + \delta_0]$ and having a compact support in $\boldsymbol{R}$.

To make the calculation easier, we divide the functional J_ζ as follows:

$$J_\zeta(w)=J_{Q(\zeta)}(w)+\sum_{i=1}^{N} J_{D_i}(w),$$

where $J_{Q(\zeta)}$ and J_{D_i} are obtained by replacing $\Omega(\zeta)$ in (2.4) by $Q(\zeta)$ and D_i respectively.

LEMMA 2.2. *There exists a positive integer q_0 such that*

$$J_{D_i}(w)\geqq\int_{D_i}(w(x)-w_{i,q}(x))^2dx+\int_{D_i}\int_{w_{i,q}(x)}^{a_i} f(\xi)d\xi dx \tag{2.7}$$

for any $q\geqq q_0$, $i=1, 2, \cdots, N$ and $w\in C^1(\bar{D}_i)$, where

$$w_{i,q}(x)=\sum_{k=1}^{q}\psi_{i,k}(x)\int_{D_i}\psi_{i,k}(y)w(y)dy.$$

PROOF. Set

$$\nu_{i,k}=\int_{D_i}\psi_{i,k}(y)w(y)dy.$$

By Proposition 2, we have

$$\begin{aligned}\int_{D_i}|\mathrm{grad}\, w|^2dx&=\sum_{k=2}^{\infty}\lambda_{i,k}(\nu_{i,k})^2\\ &\geqq\lambda_{i,q+1}\sum_{k=q+1}^{\infty}(\nu_{i,k})^2+\lambda_{i,2}\sum_{k=2}^{q}(\nu_{i,k})^2\\ &=\lambda_{i,q+1}\int_{D_i}(w-w_{i,q})^2dx+\lambda_{i,2}\int_{D_i}(w_{i,q}-w_{i,1})^2dx.\end{aligned} \tag{2.8}$$

Next put

$$c_1\equiv\max_{\xi\in R}|f'(\xi)|.$$

c_1 is well-defined since we have assumed (without loss of generality) that f has compact support. By Taylor's formula,

$$\begin{aligned}\int_{w_{i,q}}^{w} f(\xi)d\xi&\leqq f(w_{i,q})(w-w_{i,q})+\frac{c_1}{2}(w-w_{i,q})^2\\ &\leqq f(w_{i,1})(w-w_{i,q})+c_1|w_{i,q}-w_{i,1}|\,|w-w_{i,q}|+\frac{c_1}{2}(w-w_{i,q})^2.\end{aligned}$$

Integrating this inequality over D_i and using the fact that $w-w_{i,q}$ is orthogonal to $f(w_{i,1})$, which is a constant function (hence parallel to $\psi_{i,1}$), we see that

$$\text{(2.9)} \quad \int_{D_i}\int_{w_{i,q}(x)}^{w(x)} f(\xi)d\xi dx$$
$$\leqq c_1\int_{D_i}|w_{i,q}-w_{i,1}|\,|w-w_{i,q}|\,dx+\frac{c_1}{2}\int_{D_i}(w-w_{i,q})^2dx$$
$$\leqq \frac{1}{2}c_1\alpha\int_{D_i}(w_{i,q}-w_{i,1})^2dx+c_1\Big(\frac{1}{2}+\frac{1}{2\alpha}\Big)\int_{D_i}(w-w_{i,q})^2dx$$

for any constant $\alpha>0$. Set $\alpha=\alpha_i=\lambda_{i,2}/c_1$ and choose q_0 sufficiently large so that

$$\text{(2.10)} \quad \frac{1}{2}\lambda_{i,q+1}-c_1\Big(\frac{1}{2}+\frac{1}{2\alpha_i}\Big)\geqq 1$$

for all $q\geqq q_0$ and $i=1, 2, \cdots, N$. Subtracting (2.9) from (2.8) multiplied by 1/2, and using (2.10) and the fact that $A(x)=a_i$ on D_i, we easily find that (2.7) holds. This completes the proof of Lemma 2.2.

We also need the following lemma, which is an easy consequence of Cauchy-Schwarz's inequality:

LEMMA 2.3. *Let q be a positive integer and put*

$$\text{(2.11)} \quad c_2\equiv\max_{1\leqq i\leqq N,1\leqq k\leqq q}\|\psi_{i,k}\|_{L^\infty(D_i)}.$$

Then for any $i=1, 2, \cdots, N$ and $w\in L^\infty(D_i)$, we have

$$\text{(2.12)} \quad \|w_{i,q}-a_i\|_{L^\infty(D_i)}\leqq c_2q^{1/2}\|w_{i,q}-a_i\|_{L^2(D_i)}$$
$$\leqq c_2q^{1/2}\|w-a_i\|_{L^2(D_i)},$$

where $w_{i,q}$ is as in Lemma 2.2.

Now let

$$\text{(2.13)} \quad \mu\equiv\min_{1\leqq i\leqq N}(-f'(a_i))$$

and let σ_* be a positive constant with $0<\sigma_*\leqq\delta_0$ such that

$$\text{(2.14)} \quad -f'(\xi)\geqq\mu/2 \quad \text{for any} \quad \xi\in(a_i-\sigma_*, a_i+\sigma_*), \quad i=1, 2, \cdots, N.$$

By the assumption on a_i, we have $\mu>0$. We also use the notation

(2.15)
$$c_3 \equiv \max_{x \in R^n} |\mathrm{grad}\, A(x)| + \int_R |f(\xi)| d\xi.$$

LEMMA 2.4. *Let $q = q_0$ be as in* (2.10) *and let the constants c_2, μ, σ_*, c_3 be as in* (2.11), (2.13), (2.14), (2.15), *respectively. Suppose that $\delta > 0$ and $\zeta > 0$ satisfy the inequalities*

(2.16a)
$$0 < \delta \leqq \min\left\{\delta_0, \frac{\sigma_*}{c_2 q^{1/2}}\right\},$$

(2.16b)
$$c_3 \mathrm{Vol}(Q(\zeta)) + \delta^3 < \frac{1}{2} \min\{1, \mu/4\} \delta^2.$$

Then $E(\delta, \zeta)$ is positively invariant under (2.1).

PROOF. Let $u_\zeta(t, x)$ be a solution of (2.1) satisfying $u_\zeta(0, \cdot) \in E(\delta, \zeta)$. Set

$$T^* = \sup\{T \geqq 0 \,|\, u_\zeta(t, \cdot) \in E(\delta, \zeta) \text{ for } 0 \leqq t \leqq T\}.$$

Assuming $T^* < \infty$, we shall derive a contradiction. By Lemma 2.1, $\tilde{E}(\delta, \zeta)$ is positively invariant under (2.1). In view of this and the fact that $\|u_\zeta(t, \cdot) - a_i\|_{L^2(D_i)}$ is continuous in t and by the definition of T^*, we see that

(2.17)
$$\|u_\zeta(T^*, \cdot) - a_i\|_{L^2(D_i)} \leqq \delta \quad \text{for} \quad i = 1, 2, \cdots, N.$$

(2.18)
$$\|u_\zeta(T^*, \cdot) - a_j\|_{L^2(D_j)} = \delta \quad \text{for some} \quad j,\ 1 \leqq j \leqq N.$$

Fix such an integer j as in (2.18). By (2.12), (2.16a) and (2.17), we have

$$\|u_{i,q}^\zeta(T^*, \cdot) - a_i\|_{L^\infty(D_i)} \leqq c_2 q^{1/2} \delta \leqq \sigma_*$$

for $i = 1, 2, \cdots, N$, where

$$u_{i,q}^\zeta(t, x) = \sum_{k=1}^{q} \psi_{i,k}(x) \int_{D_i} \psi_{i,k}(y) u_\zeta(t, y) dy.$$

It follows from this and (2.14) that

(2.19)
$$\int_{D_i} \int_{u_{i,q}^\zeta}^{a_i} f(\xi) d\xi dx \geqq \frac{\mu}{4} \int_{D_i} (u_{i,q}^\zeta(T^*, x) - a_i)^2 dx.$$

Combining (2.7), (2.19) and the inequality $J_\zeta(u_\zeta(T^*, \cdot)) \leqq J_\zeta(A) + \delta^3$, we obtain

$$J_\zeta(A)+\delta^3 \geqq J_{Q(\zeta)}(u_\zeta(T^*,\ \cdot))+\sum_{i=1}^{N}\left\{\int_{D_i}(u_\zeta-u_{i,q}^\zeta)^2dx+\frac{\mu}{4}\int_{D_i}(u_{i,q}^\zeta-a_i)^2dx\right\}$$

$$\geqq J_{Q(\zeta)}(u_\zeta(T^*,\ \cdot))+\frac{1}{2}\min\left\{1,\ \frac{\mu}{4}\right\}\sum_{i=1}^{N}\int_{D_i}(u_\zeta(T^*,\ \cdot)-a_i)^2dx.$$

A simple calculation shows that

$$J_\zeta(A)-J_{Q(\zeta)}(u_\zeta(T^*,\ \cdot))\leqq c_3\operatorname{Vol}(Q(\zeta)).$$

It follows that

$$c_3\operatorname{Vol}(Q(\zeta))+\delta^3\geqq\frac{1}{2}\min\left\{1,\ \frac{\mu}{4}\right\}\int_{D_j}(u_\zeta(T^*,\ x)-a_j)^2dx,$$

hence, by (2.16b),

$$\{\|u_\zeta(T^*,\ \cdot)-a_j\|_{L^2(D_j)}\}^2<\delta^2.$$

But this contradicts (2.18), thus the lemma is proved.

PROOF OF THEOREM 1. Put

$$\delta_1=\min\{\delta_0,\ \sigma_*/(c_2q^{1/2})\},$$
$$\mu_1=\frac{1}{2}\min\{1,\ \mu/4\}.$$

By Lemma 2.4, $E\delta(,\ \zeta)$ is positively invariant under (2.1) if $0<\delta<\delta_1$ and

$$c_3\operatorname{Vol}(Q(\zeta))<\mu_1\delta^2-\delta^3. \tag{2.20}$$

By the assumption on $Q(\zeta)$, the left-hand side of (2.20) is monotone increasing in $\zeta>0$ and tends to 0 as $\zeta\to 0$, while the right-hand side is strictly monotone increasing in $\delta\in(0,\ \delta_2]$, where

$$\delta_2=\min\left\{\delta_1,\ \frac{2\mu_1}{3}\right\}.$$

In view of this, one can easily construct a monotone increasing function $\delta=\delta(\zeta)$ defined on some interval $0<\zeta\leqq\zeta_2$ such that

$$0<\delta(\zeta)<\delta_2,\ c_3\operatorname{Vol}(Q(\zeta))<\mu_1\delta(\zeta)^2-\delta(\zeta)^3\quad\text{for}\quad\zeta\in(0,\ \zeta_2],\ \lim_{\zeta\to 0}\delta(\zeta)=0.$$

It is clear that (2.20) holds for any $\zeta\in(0,\ \zeta_2]$ and any $\delta\in[\delta(\zeta),\ \delta_2]$.

Now, for each $\zeta\in(0,\ \zeta_2]$, choose a sequence $\{\delta^{(m)}\}_{m=1}^{\infty}$ satisfying

$$\delta_2=\delta^{(1)}>\delta^{(2)}>\delta^{(3)}>\cdots\to\delta(\zeta),$$

and set

$$Y=E(\delta(\zeta),\zeta),\quad Y_m=E(\delta^{(m)},\zeta)$$

for $m=1, 2, 3, \cdots$. Clearly we have $Y_1\supset Y_2\supset Y_3\supset\cdots$ and $\cap_{m=1}^{\infty}Y_m=Y$. Y is not empty since it contains the function A. And it isn't difficult to check that the assumptions in Proposition 1 are all satisfied. Applying this proposition, we see that $Y=E(\delta(\zeta),\zeta)$ contains at least one stable equilibrium solution of (2.1), say v_ζ.

Next we examine the behavior of v_ζ as $\zeta\to 0$. Since $v_\zeta\in E(\delta(\zeta),\zeta)$, we have

$$\|v_\zeta-a_i\|_{L^2(D_i)}\leqq\delta(\zeta),$$

hence

$$\lim_{\zeta\to 0}v_\zeta=a_i\quad\text{in}\quad L^2(D_i) \tag{2.21}$$

for $i=1, 2, \cdots, N$. (Here and in what follows we simply write v_ζ instead of writing $v_\zeta|_{D_i}$ or $v_\zeta|_{D_i(\eta)}$, as there will be no confusion in using such an abbreviation.) For each i $(1\leqq i\leqq N)$, v_ζ satisfies

$$\Delta v_\zeta+f(v_\zeta)=0\quad\text{in}\quad D_i \tag{2.22}$$

$$a_*-\delta(\zeta)\leqq v_\zeta(x)\leqq a^*+\delta(\zeta)\quad\text{in}\quad D_i \tag{2.23}$$

$$\frac{\partial v_\zeta}{\partial\nu}=0\quad\text{on}\quad\partial\Omega(\zeta)\cap\partial D_i. \tag{2.24}$$

Fix $\eta>0$ arbitrarily. In view of (2.22)$\sim$(2.24), and using the L^p estimate of Agmon, Douglis and Nirenberg [1] together with Sobolev imbedding theorem on the domain $D_i(\eta/2)$, we obtain the boundedness of $\{v_\zeta\}_{\zeta>0}$ in $C^{1+\beta}(\overline{D_i((1-(1/2)^2)\eta)})$ for some $\beta\in(0,1)$ and also the boundedness of $\{f(v_\zeta)\}_{\zeta>0}$ in $C^{1+\beta}(\overline{D_i((1-(1/2)^2)\eta)})$. Applying the Schauder estimate to the domain $\overline{D_i((1-(1/2)^2)\eta)}$, we obtain the boundedness of $\{v_\zeta\}_{\zeta>0}$ in $C^{3+\beta}(\overline{D_i((1-(1/2)^3)\eta)})$. Repeating this bootstrap argument, we obtain the boundedness of $\{v_\zeta\}_{\zeta>0}$ in $C^\infty(\overline{D_i(\eta)})$, hence its relative compactness. Combining this and (2.21) yields

$$\lim_{\zeta\to 0}v_\zeta=a_i\text{ in }C^\infty(\overline{D_i(\eta)})\quad\text{for any}\quad\eta>0,\quad(1\leqq i\leqq N).$$

This completes the proof of Theorem 1.

§3. Asymptotic Behavior on the Thin Part.

In this section we further investigate the behavior of solutions, especially their behavior on the perturbation part $Q(\zeta)$. As the condition (II-1) on the family of domains $\{\Omega(\zeta)\}_{\zeta>0}$ introduced in Section 2 is very weak, it allows the perturbation $Q(\zeta)$ to be extremely wild. In order to carry out a more delicate argument about the behavior of solutions, we have to consider a more limited class of perturbations, as specified below.

We set the domain $\Omega(\zeta)$ in the form

$$\Omega(\zeta)=D_1\cup D_2\cup Q(\zeta),$$

where the parameter ζ varies in some interval $(0, \zeta_*)$ with $0<\zeta_*<1/2$, and D_i $(i=1, 2)$ and $Q(\zeta)$ are domains satisfying the conditions (III-1) and (III-2) below, in which we use the notation $x'=(x_2, x_3, \cdots, x_n)\in \boldsymbol{R}^{n-1}$:

(III-1) D_1 and D_2 are mutually disjoint bounded domains in $\boldsymbol{R}^n$ with smooth boundaries and satisfy the following conditions for some constant ζ_*.

$$\begin{aligned}&\bar{D}_1\cap\{x=(x_1, x')\in \boldsymbol{R}^n \mid x_1\leqq 1,\ |x'|<3\zeta_*\}\\ &=\{(1, x')\in \boldsymbol{R}^n \mid |x'|<3\zeta_*\}\\ &\bar{D}_2\cap\{x=(x_1, x')\in \boldsymbol{R}^n \mid x_1\geqq -1,\ |x'|<3\zeta_*\}\\ &=\{(-1, x')\in \boldsymbol{R}^n \mid |x'|<3\zeta_*\},\end{aligned}$$

(III-2) $Q(\zeta)=R_1(\zeta)\cup R_2(\zeta)\cup \Gamma(\zeta)$

$$\begin{aligned}&R_1(\zeta)=\{(x_1, x')\in \boldsymbol{R}^n \mid 1-2\zeta<x_1\leqq 1,\ |x'|<\zeta\rho((x_1-1)/\zeta)\}\\ &R_2(\zeta)=\{(x_1, x')\in \boldsymbol{R}^n \mid -1\leqq x_1<-1+2\zeta,\ |x'|<\zeta\rho((-1-x_1)/\zeta)\}\\ &\Gamma(\zeta)=\{(x_1, x')\in \boldsymbol{R}^n \mid -1+2\zeta\leqq x_1\leqq 1-2\zeta,\ |x'|<\zeta\},\end{aligned}$$

where $\rho\in C^0((-2, 0])\cap C^\infty((-2, 0))$ is a positive valued monotone increasing function such that $\rho(0)=2$, $\rho(s)=1$ for $s\in(-2, -1)$ and its inverse function ρ^{-1}: $(1, 2)\to(-1, 0)$ satisfies

$$\lim_{r\uparrow 2-0}\frac{d^k\rho^{-1}(r)}{dr^k}=0 \text{ holds for any positive integer } k.$$

We also assume that

(III-3) $\overline{\lim_{\xi\to\infty}} f(\xi)<0,\ \underline{\lim_{\xi\to-\infty}} f(\xi)>0.$

REMARK. The domain characterized by (III-1) and (III-2) above clearly satisfies (II-1), therefore it is a special case of that discussed in Section 2; so there will be no confusion in using the same notation $\Omega(\zeta)$. Note that the last condition in (III-2) implies that $\Omega(\zeta)$ has a C^∞-boundary. Such a smoothness condition, as one easily sees, is not essential in our analysis and can be relaxed considerably.

Under the conditions (II-2), (III-1), (III-2) and (III-3), we analyze the asymptotic behavior of a class of certain solutions (which will be characterized by (III-4)) of the following semilinear elliptic boundary value problem:

$$(3.1)\qquad \begin{cases} \Delta v+f(v)=0 & \text{in } \Omega(\zeta), \\ \dfrac{\partial v}{\partial \nu}=0 & \text{on } \partial\Omega(\zeta). \end{cases}$$

(III-4) $\{v_\zeta\}_{0<\zeta<\zeta_*}$ is a family of functions such that for each $\zeta\in(0,\zeta_*)$, v_ζ is a solution of (3.1) and that

$$\lim_{\zeta\to 0}\|v_\zeta-a_i\|_{L^2(D_i)}=0 \quad (i=1, 2)$$

for some constants a_1, a_2 with $f(a_i)=0$ and $f'(a_i)<0$ $(i=1, 2)$. (See (II-2).)

Notation 1. Let $\mu_1(\zeta)$ be the first eigenvalue of the following eigenvalue problem:

$$(3.2)\qquad \begin{cases} \Delta\psi+f'(v_\zeta)\psi+\mu\psi=0 & \text{in } \Omega(\zeta), \\ \dfrac{\partial\psi}{\partial \nu}=0 & \text{on } \partial\Omega(\zeta). \end{cases}$$

REMARK. It is well-known that if $\mu_1(\zeta)>0$ (resp. $\mu_1(\zeta)<0$) then v_ζ is stable (resp. unstable) as an equilibrium solution of (1.2) for $\Omega=\Omega(\zeta)$.

REMARK. The two values a_1 and a_2 are not necessarily distinct.

We set $M_*=\inf\{\xi\in\boldsymbol{R}\mid f(\xi)=0\}$ and $M^*=\sup\{\xi\in\boldsymbol{R}\mid f(\xi)=0\}$. It is easily seen by (II-2) and (III-3) that M_* and M^* are well defined and that

$$(3.3)\qquad M_*\leqq v_\zeta(x)\leqq M^* \quad \text{for} \quad x\in\Omega(\zeta).$$

As mentioned before, the aim of this section is to investigate the behavior of v_ζ on $Q(\zeta)$ as $\zeta\to 0$. The first theorem in this section concerns its behavior on the region

$$D_1 \cup D_2 \cup R_1(\zeta) \cup R_2(\zeta) = \Omega(\zeta) - \Gamma(\zeta):$$

THEOREM 2. *Assume $n \geqq 3$, then we have, for $i=1, 2$,*

$$\lim_{\zeta \to 0} \sup_{x \in D_i \cup R_i(\zeta)} |v_\zeta(x) - a_i| = 0.$$

Next we discuss the behavior of v_ζ on the region $Q(\zeta)$. The following ordinary differential equation plays a key role in describing the asymptotic behavior of v_ζ on $Q(\zeta)$ as $\zeta \to 0$:

$$\text{(3.4)} \qquad \begin{cases} \dfrac{d^2 V}{dz^2} + f(V) = 0 \quad \text{in} \quad -1 < z < 1, \\ V(1) = a_1, \quad V(-1) = a_2. \end{cases}$$

Notation 2. Given a solution V of (3.4), let λ_V and Φ_V, respectively, be the first eigenvalue and the first eigenfunction of the following eigenvalue problem:

$$\text{(3.5)} \qquad \begin{cases} \dfrac{d^2 \Phi}{dz^2} + f'(V(z))\Phi + \lambda \Phi = 0 \quad \text{in} \quad -1 < z < 1, \\ \Phi(1) = 0, \quad \Phi(-1) = 0. \end{cases}$$

Now we present one of the main results of this paper.

THEOREM 3. *Assume $n \geqq 3$. Then for any sequence of positive values $\{\zeta_m\}_{m=1}^\infty$ such that $\lim_{m \to \infty} \zeta_m = 0$, there exist a subsequence $\{\kappa_m\}_{m=1}^\infty \subset \{\zeta_m\}_{m=1}^\infty$ and a solution V of* (3.4) *with the following asymptotic property*:

$$\text{(3.6)} \qquad \lim_{m \to \infty} \sup_{x \in Q(\kappa_m)} |v_{\kappa_m}(x_1, x') - V(x_1)| = 0.$$

Furthermore, if $\lambda_V > 0$ (resp. $\lambda_V < 0$), then

$$\varliminf_{m \to \infty} \mu_1(\kappa_m) > 0 \ (\textit{resp.}\ \varlimsup_{m \to \infty} \mu_1(\kappa_m) < 0)$$

holds.

Before starting the proof we introduce some more notations:

$$\begin{aligned} &p_1 = (1, 0, \cdots, 0), \quad p_2 = (-1, 0, \cdots, 0), \\ &\Sigma_1(\eta) = \{(x_1, x') \in \boldsymbol{R}^n \mid x_1 > 1, \ |x - p_1| < \eta\}, \\ &\Sigma_2(\eta) = \{(x_1, x') \in \boldsymbol{R}^n \mid x_1 < -1, \ |x - p_2| < \eta\}. \end{aligned}$$

It can be easily seen by the last part of the proof of Theorem 1 and the condition (III-4) that

$$\lim_{\zeta\to 0} v_\zeta = a_i \quad \text{in} \quad C^\infty(\overline{D_i - \Sigma_i(\eta)}) \tag{3.7}$$

for any small positive constant η ($i=1, 2$).

For $\varepsilon>0$ and $0<\zeta<\zeta_*$, we set

$$K(\varepsilon, \zeta) \equiv \{x \in D_1 \mid |v_\zeta(x) - a_1| \geqq \varepsilon\},$$
$$\eta(\varepsilon, \zeta) \equiv \inf\{\eta>0 \mid \Sigma_1(\eta) \supset K(\varepsilon, \zeta)\}.$$

Then it follows from (3.7) that

$$\lim_{\zeta\to 0} \eta(\varepsilon, \zeta) = 0 \quad \text{for any} \quad \varepsilon>0. \tag{3.8}$$

The convergence rate of (3.8) is estimated as follows:

LEMMA 3.1. *For any* $\varepsilon>0$ *we have*

$$\overline{\lim_{\zeta\to 0}} \frac{\eta(\varepsilon, \zeta)}{\zeta} < \infty.$$

PROOF. If we assume the contrary, there exist $\varepsilon_1>0$ and a sequence of positive values $\{\zeta_m\}_{m=1}^\infty$ such that

$$\lim_{m\to\infty} \zeta_m = 0 \quad \text{and} \quad \lim_{m\to\infty} \frac{\eta(\varepsilon_1, \zeta_m)}{\zeta_m} = \infty. \tag{3.9}$$

Since $\eta(\varepsilon, \zeta)$ increases as ε decreases, this last equality also holds if ε_1 is replaced by a positive constant which is smaller than ε_1. Therefore we assume without loss of generality that ε_1 is sufficiently small so that $f'(\xi)<0$ holds for any $\xi \in (a_1-\varepsilon_1, a_1+\varepsilon_1)$. We hereafter denote $\eta(\varepsilon_1, \zeta_m)$ by η_m for simplicity.

For the analysis of the behavior of v_ζ on the small part $\Sigma_1(\zeta)$, we change the scale of the variable x around the point p_1 as follows:

$$\begin{cases} x - p_1 = \eta_m \cdot (y - p_1) \\ U_m(y) = v_{\zeta_m}(\eta_m \cdot (y-p_1) + p_1). \end{cases} \tag{3.10}$$

The equation (3.1) is then transformed into the following in some neighborhood of p_1:

$$\begin{cases} \Delta_y U_m + \eta_m^2 f(U_m) = 0 \quad \text{in} \quad \Sigma_1(3\zeta_*/\eta_m) \\ \dfrac{\partial U_m}{\partial y_1}(1, y') = 0 \ \text{ for } y' \text{ such that } \dfrac{2\zeta_m}{\eta_m} < |y'| < \dfrac{3\zeta_*}{\eta_m}. \end{cases} \tag{3.11}$$

We put

$$\gamma_m \equiv \max_{y_1 \geqq 1, |y-p_1|=3\zeta_*/\eta_m} |U_m(y)-a_1| \equiv \max_{x \in D_1, |x-p_1|=3\zeta_*} |v_{\zeta_m}(x)-a_1|.$$

Then it is easy to see that

$$\lim_{m\to\infty} \gamma_m = 0. \tag{3.12}$$

By the definition of $\eta_m = \eta(\varepsilon_1, \zeta_m)$ and U_m, we have

$$\max_{y_1 \geqq 1, |y-p_1|=1} |U_m(y)-a_1| = \max_{x \in D_1, |x-p_1|=\eta_m} |v_{\zeta_m}(x)-a_1| = \varepsilon_1, \tag{3.13}$$

$$|U_m(y)-a_1| \leqq \varepsilon_1 \quad \text{in} \quad \overline{\Sigma_1(3\zeta_*/\eta_m)-\Sigma_1(1)}, \tag{3.14}$$

$$M_* \leqq U_m(y) \leqq M^* \quad \text{in} \quad \Sigma_1(3\zeta_*/\eta_m). \tag{3.15}$$

Now we define a comparison function G_m which will estimate U_m for large y:

$$G_m(y) = \frac{\varepsilon_1}{|y-p_1|^{n-2}} + \gamma_m.$$

It can be easily seen by (3.14) and the assumption on ε_1 that

$$f(U_m(y))<0 \quad \text{for any} \quad y \in (\Sigma_1(3\zeta_*/\eta_m)-\Sigma_1(1)) \cap \{y \mid U_m(y)>a_1\}$$
$$f(U_m(y))>0 \quad \text{for any} \quad y \in (\Sigma_1(3\zeta_*/\eta_m)-\Sigma_1(1)) \cap \{y \mid U_m(y)<a_1\}$$

and that G_m is a harmonic function in $\Sigma_1(3\zeta_*/\eta_m)-\Sigma_1(1)$ with the boundary condition $(\partial G_m/\partial y_1)(1, y')=0$ $(1<|y'|<3\zeta_*/\eta_m)$. In view of this, (3.11), (3.13) and the definition of γ_m, and applying the maximum principle to the function $U_m - a_1$ in the domain $\Sigma_1(3\zeta_*/\eta_m)-\Sigma_1(1)$, we obtain the following estimate (3.16) for sufficiently large m. (Recall $\lim_{m\to\infty} \zeta_m/\eta_m = 0$.)

$$|U_m(y)-a_1| \leqq G_m(y) \quad \text{for} \quad y \in \Sigma_1(3\zeta_*/\eta_m)-\Sigma_1(1). \tag{3.16}$$

Applying a bootstrap argument (similar to that in last part of the proof of Theorem 1) to the family $\{U_m\}_{m=1}^{\infty}$ satisfying (3.11) and the uniform bound (3.15), and then using the diagonal argument, we can choose a convergent subsequence $\{U_{m_j}\}_{j=1}^{\infty}$. By virtue of (3.9), U belongs to $C^{\infty}(\{(y_1, y') \in \mathbf{R}^n \mid y_1 \geqq 1\}-\{p_1\})$ and satisfies the following:

$$M_* \leqq U(y) \leqq M^* \quad \text{in} \quad \{(y_1, y') \mid y_1 \geqq 1\}-\{p_1\} \tag{3.17}$$

(3.18) $$\Delta_y U=0 \quad \text{in} \quad \{(y_1, y')\in \boldsymbol{R}^n \mid y_1>1\}$$

(3.19) $$\frac{\partial U}{\partial y_1}(1, y')=0 \quad \text{for } y'\in \boldsymbol{R}^{n-1} \text{ such that } y'\neq 0$$

(3.20) $$\lim_{j\to\infty} U_{m_j}=U$$
$$\text{in } C^\infty\left(\left\{(y_1, y') \mid y_1\geqq 1,\ \eta\leqq |y-p_1|\leqq\frac{1}{\eta}\right\}\right) \text{ for any } \eta>0.$$

On the other hand, by virtue of the estimate (3.16), the convergence (3.20) and (3.12), U satisfies

(3.21) $$|U(y)-a_1|\leqq\frac{\varepsilon_1}{|y-p_1|^{n-2}}$$
$$\text{in } \{(y_1, y')\in \boldsymbol{R}^n \mid y_1\geqq 1,\ |y-p_1|\geqq 1\}$$

(3.22) $$M_*\leqq U(y)\leqq M^* \quad \text{in} \quad \{(y_1, y')\in \boldsymbol{R}^n \mid y_1>1\}.$$

From (3.13), (3.20) and the compactness of the set $\{(y_1, y')\in \boldsymbol{R}^n \mid y_1\geqq 1, |y-p_1|=1\}$, it follows that

(3.23) $$\max_{y_1\geqq 1, |y-p_1|=1} |U(y)-a_1|=\varepsilon_1.$$

Now define a function $\bar{U}\in C^\infty(\boldsymbol{R}^n-\{p_1\})$ by

$$\bar{U}(y_1, y')=\begin{cases} U(y) & \text{for } y_1\geqq 1,\ y\neq p_1 \\ U(2-y_1, y') & \text{for } y_1<1. \end{cases}$$

By (3.18), (3.19) and (3.22), we have

$$\begin{cases} \Delta_y U=0 & \text{in } \boldsymbol{R}^n-\{p_1\} \\ M_*\leqq U(y)\leqq M^* & \text{in } \boldsymbol{R}^n-\{p_1\}. \end{cases}$$

Therefore, applying the removable singularity theorem, we can extend $\bar{U}$ on $\boldsymbol{R}^n$ as a bounded harmonic function. We denote it also by $\bar{U}$. Thus $\bar{U}$ must be a constant function by the Harnack Theorem. But it is impossible by (3.21) and (3.23). This is a contradiction and we complete the proof of Lemma 3.1.

PROOF OF THEOREM 2. We change the variable x into y around p_1 by the following:

(3.24) $$\begin{cases} x-p_1=\zeta\cdot(y-p_1), \\ U_\zeta(y)=v_\zeta(\zeta(y-p_1)+p_1). \end{cases}$$

Then the equation (3.1) is transformed into the following:

$$\Delta_y U_\zeta + \zeta^2 f(U_\zeta) = 0 \quad \text{in} \quad H_\zeta, \tag{3.25}$$

$$\frac{\partial U_\zeta}{\partial \nu} = 0 \quad \text{on} \quad \partial H_\zeta \cap \partial H. \tag{3.26}$$

Here we have put

$$H \equiv \hat{D} \cup \hat{R} \cup \hat{\Gamma}$$

$$H_\zeta \equiv \left\{ y \in \hat{D} \,|\, |y - p_1| < \frac{3\zeta_*}{\zeta} \right\} \cup \hat{R} \cup \left\{ (y_1, y') \in \hat{\Gamma} \,|\, y_1 \geqq 3 - \frac{2}{\zeta} \right\}$$

for $0 < \zeta < 1/2$, where

$$\hat{D} \equiv \{(y_1, y') \in \boldsymbol{R}^n \,|\, y_1 > 1\},$$
$$\hat{R} \equiv \{(y_1, y') \in \boldsymbol{R}^n \,|\, -1 < y_1 \leqq 1,\ |y'| < \rho(y_1 - 1)\},$$
$$\hat{\Gamma} \equiv \{(y_1, y') \in \boldsymbol{R}^n \,|\, y_1 \leqq -1,\ |y'| < 1\},$$

and ν denotes the unit outer normal vector on ∂H. Set

$$\tau_\zeta = \max_{y_1 \geqq 1,\, |y - p_1| = 3\zeta_*/\zeta} |U_\zeta(y) - a_1| = \max_{x \in D_1,\, |x - p_1| = 3\zeta_*} |v_\zeta(x) - a_1|. \tag{3.27}$$

It is easily seen by (3.7) that

$$\lim_{\zeta \to 0} \tau_\zeta = 0. \tag{3.28}$$

Suppose now that the conclusion of Theorem 2 does not hold, say for $i=1$. Then there exist a constant $\varepsilon_0 > 0$ and a sequence of positive numbers $\zeta_1 > \zeta_2 > \zeta_3 > \cdots \to 0$ such that

$$\sup_{x \in D_1 \cup R_1(\zeta_m)} |v_{\zeta_m}(x) - a_1| > \varepsilon_0 \quad (m = 1, 2, \cdots). \tag{3.29}$$

By Lemma 3.1, there exists a constant $\beta > 0$ such that

$$\eta(\varepsilon_0, \zeta) < \zeta/\beta \quad \text{for sufficiently small } \zeta > 0, \tag{3.30}$$

which implies

$$\sup_{x \in D_1 - \Sigma_1(\zeta/\beta)} |v_\zeta(x) - a_1| \leqq \varepsilon_0.$$

Combining this and (3.29), we see that

$$|U_{\zeta_m}(x)-a_1|\leqq\varepsilon_0 \quad \text{in} \quad \Sigma_1(3\zeta_*/\zeta_m)-\Sigma_1(1/\beta), \tag{3.31}$$

$$\sup_{y\in\Sigma_1(1/\beta)\cup\hat{R}} |U_{\zeta_m}(y)-a_1|>\varepsilon_0 \tag{3.32}$$

for sufficiently large m. It is also clear that

$$M_*\leqq U_{\zeta_m}(y)\leqq M^* \quad \text{in} \quad H_{\zeta_m} \tag{3.33}$$

for $m=1, 2, 3, \cdots$. (3.27), (3.31) and the comparison theorem imply

$$|U_{\zeta_m}(y)-a_1|\leqq\frac{\varepsilon_0}{\beta^{n-2}|y-p_1|^{n-2}}+\tau_{\zeta_m} \quad \text{in} \quad \Sigma_1(3\zeta_*/\zeta_m)-\Sigma_1(1/\beta). \tag{3.34}$$

See (3.25), (3.26) and (3.33), and using the same argument as in the proof of Lemma 3.1, we can choose a convergent subsequence of $\{U_{\zeta_m}\}_{m=1}^{\infty}$, again denoted by $\{U_{\zeta_m}\}_{m=1}^{\infty}$ for simplicity, and it is clear from (3.28) and (3.34) that the limit function $U\in C^{\infty}(\bar{H})$ satisfies the following:

$$\Delta_y U=0 \quad \text{in} \quad H, \tag{3.35}$$

$$\frac{\partial U}{\partial\nu}=0 \quad \text{on} \quad \partial H, \tag{3.36}$$

$$\lim_{m\to 0} U_{\zeta_m}=U \text{ in } C^{\infty}(\bar{H}_\eta) \quad \text{for any} \quad \eta>0, \tag{3.37}$$

$$|U(y)-a_1|\leqq\frac{\varepsilon_0}{\beta^{n-2}|y-p_1|^{n-2}} \quad \text{in} \quad \{(y_1, y')\in \boldsymbol{R}^n \mid y_1\geqq 1,\ |y-p_1|\geqq 1/\beta\}, \tag{3.38}$$

$$M_*\leqq U(y)\leqq M^* \quad \text{in} \quad H. \tag{3.39}$$

On the other hand, from (3.32), (3.37) and the compactness of the set $\overline{\Sigma_1(1/\beta)\cup\hat{R}}$, we see that

$$\sup_{y\in\Sigma_1(1/\beta)\cup\hat{R}} |U(y)-a_1|\geqq\varepsilon_0. \tag{3.40}$$

Thus (3.38) and (3.40) imply that U is a non-constant function in H. But this is impossible by (3.35), (3.36), (3.39) and Lemma 3.2 below. This completes the proof of Theorem 2.

LEMMA 3.2. *Let ψ be a bounded function which belongs to $C^{\infty}(\bar{H})$ and satisfy*

(3.41) $$\Delta_y \psi = 0 \quad in \quad H,$$

(3.42) $$\frac{\partial \psi}{\partial \nu} = 0 \quad on \quad \partial H,$$

(3.43) $$\lim_{y_1 \geqq 1, |y| \to \infty} |\psi(y) - a| = 0.$$

Then $\psi \equiv a$ *in* H.

PROOF OF LEMMA 3.2. We assume the contrary. Without loss of generality, we may assume

(3.44) $$\sup_{y \in H} \psi(y) = M > a.$$

We choose a sequence of points $\{r_m\}_{m=1}^{\infty} \subset H$ such that $\lim_{m \to \infty} \psi(r_m) = a$. Using the strong maximum principle, the Hopf lemma (See [19]) and the equation (3.41)-(3.42), we can easily see that ψ cannot attain its maximum on $\bar{H}$, because ψ is a non-constant function. Consequently $\{r_m\}_{m=1}^{\infty}$ does not have an accumulation point on $\bar{H}$. In view of this and (3.43), we see that

$$\lim_{m \to \infty} r_{m,1} = -\infty,$$

where $r_{m,i}$ denotes the i-th coordinate of the point r_m. Without loss of generality we may assume that $r_{m,1} < -2$ for all m. We define a family of functions $\{\psi_m\}_{m=1}^{\infty}$ as follows:

$$\psi_m(y_1, y') = \psi(y_1 + r_{m,1} + 2, y').$$

Each ψ_m satisfies

(3.45) $$\Delta_y \psi_m = 0 \quad \text{in} \quad H \cap \{y_1 < 0\},$$

(3.46) $$\frac{\partial \psi_m}{\partial \nu} = 0 \quad \text{on} \quad \partial H \cap \{y_1 < 0\},$$

(3.47) $$\psi_m(y) \leqq M \quad \text{in} \quad H,$$

(3.48) $$\lim_{m \to \infty} \max_{H \cap \{y_1 = -2\}} \psi_m(y) = M.$$

By a standard compactness argument and the maximum principle, we get

(3.49) $$\lim_{m \to \infty} \psi_m = M \quad \text{in} \quad C^{\infty}(\overline{H \cap \{-3 < y_1 < -1\}}).$$

On the other hand, integrating the equation (3.41) in y' on $\{|y'|<1\}$ and using the Neumann boundary condition, we have

$$\frac{d^2}{dy_1^2}\int_{|y'|<1}\psi(y_1, y')dy'=0 \quad \text{for} \quad y_1\leqq 0.$$

Since the boundedness of ψ implies the boundedness of

$$\int_{|y'|<1}\psi(y_1, y')dy' \quad \text{in} \quad -\infty<y_1\leqq 0,$$

we see that $\int_{|y'|<1}\psi(y_1, y')dy'$ is independent of y_1 when y_1 is negative. We denote this value by K. Then

$$\int_{|y'|<1}\psi_m(-2, y')dy'=\int_{|y'|<1}\psi(r_{m,1}, y')dy'\equiv K. \tag{3.50}$$

We remark that the left hand side of (3.50) tends to the value $M\int_{|y'|<1}1dy'$ when m tends to ∞. Consequently,

$$\int_{|y'|<1}\psi(r_{m,1}, y')dy'=\int_{|y'|<1}Mdy' \quad \text{for any } m.$$

By virtue of (3.44), the above equality implies $\psi(r_{m,1}, y')=M$ for y' such that $|y'|<1$. But this contradicts to the fact that ψ cannot attain its maximum on $\bar{H}$. This completes the proof of Lemma 3.2 (hence that of Theorem 2).

PROOF OF THE FORMER HALF OF THEOREM 3

To analyze the asymptotic behavior of v_ζ in the thin part $Q(\zeta)$, we change the variable $x=(x_1, x')$ into $y=(y_1, y')$ as follows:

$$\begin{cases} y_1=x_1, \\ \zeta y'=x', \\ U_\zeta(y)=v_\zeta(y_1, \zeta y'). \end{cases} \tag{3.51}$$

We set

$$\iota(\zeta)\equiv\sum_{i=1}^{2}\sup_{x\in R_i(\zeta)}|v_\zeta(x)-a_i|.$$

By Theorem 2, we have $\lim_{\zeta\to 0}\iota(\zeta)=0$. We put

$$\omega=\max_{M_*\leqq\xi\leqq M^*}|f(\xi)|.$$

By (3.51), the equation (3.1) is transformed into the following equation in the part corresponding to $Q(\zeta)$.

(3.52) $$\left(\frac{\partial^2}{\partial y_1^2}+\frac{1}{\zeta^2}\sum_{j=2}^{n}\frac{\partial^2}{\partial y_j^2}\right)U_\zeta+f(U_\zeta)=0 \quad \text{in} \quad G(\zeta),$$

(3.53) $$\frac{\partial U_\zeta}{\partial \nu}=0 \quad \text{on} \quad \partial G\cap\{-1+\zeta<y_1<1-\zeta\},$$

where

$$G=\{(y_1, y')\in \boldsymbol{R}^n \mid |y'|<1, \ -\infty<y_1<\infty\}$$
$$G(\zeta)=G\cap\{-1+\zeta<y_1<1-\zeta\}$$

and ν denotes the unit outer normal vector on ∂G. We decompose U_ζ as $U_\zeta=U_{1,\zeta}+U_{2,\zeta}$ where $U_{1,\zeta}$ and $U_{2,\zeta}$ are uniquely determined by equations (3.54)~(3.57) below.

(3.54) $$\left(\frac{\partial^2}{\partial y_1^2}+\frac{1}{\zeta^2}\sum_{j=2}^{n}\frac{\partial^2}{\partial y_j^2}\right)U_{1,\zeta}(y)=0 \quad \text{in} \quad G(\zeta),$$

(3.55) $$\begin{cases} U_{1,\zeta}(y)=U_\zeta(y) & \text{on} \quad G\cap\{y_1=1-\zeta\}, \\ U_{1,\zeta}(y)=U_\zeta(y) & \text{on} \quad G\cap\{y_1=-1+\zeta\}, \end{cases}$$

(3.56) $$\frac{\partial U_{1,\zeta}}{\partial \nu}(y)=0 \quad \text{on} \quad \partial G\cap\{-1+\zeta<y_1<1-\zeta\},$$

(3.57) $$U_{2,\zeta}=U_\zeta-U_{1,\zeta}.$$

By the above definition, $U_{2,\zeta}$ satisfies the following equation:

(3.58) $$\left(\frac{\partial^2}{\partial y_1^2}+\frac{1}{\zeta^2}\sum_{j=2}^{n}\frac{\partial^2}{\partial y_j^2}\right)U_{2,\zeta}+f(U_\zeta)=0 \quad \text{in} \quad G(\zeta),$$

(3.59) $$U_{2,\zeta}(1-\zeta, y')=U_{2,\zeta}(-1+\zeta, y')=0 \quad (|y'|<1),$$

(3.60) $$\frac{\partial U_{2,\zeta}}{\partial \nu}=0 \quad \text{on} \quad \partial G\cap\{-1+\zeta\leqq y_1\leqq 1-\zeta\}.$$

Hereafter we denote by P_ζ the following differential operator:

$$P_\zeta=\frac{\partial^2}{\partial y_1^2}+\frac{1}{\zeta^2}\sum_{j=2}^{n}\frac{\partial^2}{\partial y_j^2}.$$

Since $U_{1,\zeta}$ satisfies (3.54)~(3.56), the maximum principle yields the following:

LEMMA 3.3. *For any* $\zeta \in (0, \zeta_*)$, *we have*

$$\sup_{y \in G(\zeta)} \left| U_{1,\zeta}(y) - \frac{1-\zeta-y_1}{2-2\zeta} a_1 - \frac{1-\zeta+y_1}{2-2\zeta} a_2 \right| \leqq \iota(\zeta). \tag{3.61}$$

We define functions Φ_+, Φ_- on $G(\zeta)$ which will estimate U_ζ roughly:

$$\Phi_{\pm,\zeta}(y_1, y') = \frac{y_1+1-\zeta}{2-2\zeta} a_1 + \frac{1-\zeta-y_1}{2-2\zeta} a_2$$
$$\pm \frac{\omega}{2}(y_1+1-\zeta)(1-\zeta-y_1) \pm \iota(\zeta).$$

LEMMA 3.4. *For any* $\zeta \in (0, \zeta_*)$, *we have*

$$\Phi_{-,\zeta}(y) \leqq U_\zeta(y) \leqq \Phi_{+,\zeta}(y) \quad in \quad G(\zeta). \tag{3.62}$$

PROOF. By an easy calculation, we have

$$P_\zeta \Phi_\pm = \mp \omega \quad \text{in} \quad G(\zeta),$$
$$\frac{\partial \Phi_\pm}{\partial \nu} = 0 \quad \text{on} \quad \partial G \cap \{-1+\zeta < y_1 < 1-\zeta\},$$

and by the definition of $\iota(\zeta)$, we also have

$$a_1 - \iota(\zeta) = \Phi_{-,\zeta}(1-\zeta, y') \leqq U_\zeta(1-\zeta, y') \leqq \Phi_{+,\zeta}(1-\zeta, y') = a_1 + \iota(\zeta),$$
$$a_2 - \iota(\zeta) = \Phi_{-,\zeta}(-1+\zeta, y') \leqq U_\zeta(-1+\zeta, y') \leqq \Phi_{+,\zeta}(-1+\zeta, y') = a_2 + \iota(\zeta).$$

Applying the comparison theorem, we obtain the desired inequalities.

LEMMA 3.5. *There exists a positive constant* c_1 *such that*

$$\int_{G(\zeta)} \left| \frac{\partial U_{2,\zeta}}{\partial y_1} \right|^2 dy + \frac{1}{\zeta^2} \sum_{j=2}^{n} \int_{G(\zeta)} \left| \frac{\partial U_{2,\zeta}}{\partial y'} \right|^2 dy \leqq c_1 \tag{3.63}$$

for any $\zeta \in (0, \zeta_*)$.

We can deduce this inequality by integrating the equation (3.58) over $G(\zeta)$ after multiplying $U_{2,\zeta}$ and using the estimates (3.3) and (3.61). Next we define a function which bounds $U_{2,\zeta}$ in $G(\zeta)$:

$$\Psi_\zeta(y_1, y') = \frac{\omega}{2}(1-\zeta-y_1)(1-\zeta+y_1).$$

LEMMA 3.6. *There exists a positive constant* c_2 *such that*

$$|U_{2,\zeta}(y)| \leqq \Psi_\zeta(y) \quad in \quad G(\zeta), \tag{3.64}$$

$$\left|\frac{\partial U_{2,\zeta}}{\partial y_1}(1-\zeta, y')\right| \leqq c_2 \tag{3.65}$$

$$(|y'| \leqq 1).$$

$$\left|\frac{\partial U_{2,\zeta}}{\partial y_1}(-1+\zeta, y')\right| \leqq c_2 \tag{3.66}$$

PROOF. Ψ_ζ satisfies the following:

$$P_\zeta \Psi_\zeta + \omega = 0 \quad \text{in} \quad G(\zeta), \tag{3.67}$$

$$\frac{\partial \Psi_\zeta}{\partial \nu} = 0 \quad \text{on} \quad \partial G \cap \{-1+\zeta \leqq y_1 \leqq 1-\zeta\}, \tag{3.68}$$

$$\Psi_\zeta(-1+\zeta, y') = \Psi_\zeta(1-\zeta, y') = 0 \quad |y'|<1. \tag{3.69}$$

Applying the comparison theorem to (3.58)-(3.60) and (3.67)-(3.69), we see that

$$-\Psi_\zeta(y) \leqq U_{2,\zeta}(y) \leqq \Psi_\zeta(y) \quad \text{in} \quad G(\zeta). \tag{3.70}$$

Then taking into account the boundary conditions (3.59) and (3.69), we have

$$\left|\frac{\partial U_{2,\zeta}}{\partial y_1}(1-\zeta, y')\right| \leqq \left|\frac{\partial \Psi_\zeta}{\partial y_1}(1-\zeta, y')\right| = \omega(1-\zeta) \leqq \omega.$$

By the same argument, we have $|(\partial U_{2,\zeta}/\partial y_1)(-1+\zeta, y')| \leqq \omega$. Thus Lemma 3.6 is proved.

LEMMA 3.7. *For any $\delta \in (0, 1)$, there exists a constant $c_{3,\delta}>0$ such that*

$$\left|\frac{\partial U_\zeta}{\partial y_1}(y)\right| \leqq c_{3,\delta} \quad in \quad G(\delta) \quad (0<\zeta \leqq \delta/2) \tag{3.71}$$

$$\left|\frac{\partial U_{1,\zeta}}{\partial y_1}(y)\right| \leqq c_{3,\delta} \quad in \quad G(\delta) \quad (0<\zeta \leqq \delta/2) \tag{3.72}$$

$$\left|\frac{\partial U_{2,\zeta}}{\partial y_1}(y)\right| \leqq c_{3,\delta} \quad in \quad G(\delta) \quad (0<\zeta \leqq \delta/2). \tag{3.73}$$

PROOF. We shall first prove (3.71). For each $y_* \in [0, 1-\delta]$, we define a function W_1 as follows:

$$W_1(y_1, y') = (U_\zeta(y) - U_\zeta(2y_* - y_1, y'))/2 \quad \text{in} \quad G \cap \{2y_* - 1 + \zeta \leqq y_1 \leqq y_*\}.$$

W_1 satisfies the following equations:

$$\frac{\partial W_1}{\partial y_1}(y_*, y') = \frac{\partial U_\zeta}{\partial y_1}(y_*, y') \quad \text{for} \quad |y'|<1, \tag{3.74}$$

$$P_\zeta W_1 + \frac{1}{2}(f(U_\zeta(y)) - f(U_\zeta(2y_* - y_1, y'))) = 0 \quad \text{in} \quad G \cap \{2y_* - 1 + \zeta < y_1 < y_*\}, \tag{3.75}$$

$$\frac{\partial W_1}{\partial \nu} = 0 \quad \text{on} \quad \partial G \cap \{2y_* - 1 + \zeta \leqq y_1 \leqq y_*\}, \tag{3.76}$$

$$W_1(y_*, y') = 0 \quad \text{for} \quad |y'|<1. \tag{3.77}$$

Next we define a comparison function Θ_1 as follows:

$$\Theta_1(y_1, y') = \frac{\omega}{2}(y_* - y_1)(y_1 - 2y_* + 1 - \zeta) + \frac{\bar{M}}{1 - y_* - \zeta}(y_* - y_1),$$

where $\bar{M} = \max(|M_*|, |M^*|)$. Θ_1 satisfies

$$P_\zeta \Theta_1 + \omega = 0 \quad \text{in } G \cap \{2y_* - 1 + \zeta < y_1 < y_*\}, \tag{3.78}$$

$$\frac{\partial \Theta_1}{\partial \nu} = 0 \quad \text{on} \quad \partial G \cap \{2y_* - 1 + \zeta \leqq y_1 \leqq y_*\}, \tag{3.79}$$

$$\Theta_1(2y_* - 1 + \zeta, y') = \bar{M} \quad \text{for} \quad |y'|<1, \tag{3.80}$$

$$\Theta_1(y_*, y') = 0 \quad \text{for} \quad |y'|<1. \tag{3.81}$$

Applying the comparison theorem to (3.74)-(3.76) and (3.78)-(3.80) (notice $P_\zeta(\Theta_1 - W_1)(y) \leqq 0$), we obtain

$$-\Theta_1(y) \leqq W_1(y) \leqq \Theta_1(y) \quad \text{in} \quad G \cap \{2y_* - 1 + \zeta \leqq y_1 \leqq y_*\}. \tag{3.82}$$

Taking notice of the boundary conditions (3.77) and (3.81), we deduce from (3.82) and (3.74) that

$$\begin{aligned} \left|\frac{\partial U_\zeta}{\partial y_1}(y_*, y')\right| &= \left|\frac{\partial W_1}{\partial y_1}(y_*, y')\right| \leqq \left|\frac{\partial \Theta_1}{\partial y_1}(y_*, y')\right| \\ &= \frac{\omega}{2}(1 - y_* - \zeta) + \frac{\bar{M}}{1 - y_* - \zeta} \leqq \frac{\omega}{2} + \frac{2\bar{M}}{\delta} \end{aligned} \tag{3.83}$$

for any $\zeta \in (0, \delta/2]$. The above estimate holds uniformly in $y_* \in [0, 1-\delta]$. Thus (3.71) is proved in the case $y_* \in [0, 1-\delta]$. The case $y_* \in [-1+\delta, 0]$

can be treated in quite the same manner. Using a similar reflection technique, one can easily verify (3.72) and (3.73). This completes the proof of Lemma 3.7.

LEMMA 3.8. *For any* $\delta \in (0, \zeta_*)$, *there exists a positive constant* $c_{4,\delta}$ *such that*

$$(3.84) \qquad \sum_{j=2}^{n} \left| \frac{\partial U_\zeta}{\partial y_j}(y) \right|^2 \leqq c_{4,\delta}\zeta^4 \quad on \quad \partial G \cap \{-1+\delta \leqq y_1 \leqq 1-\delta\},$$

for any $\zeta \in (0, \delta/2]$.

PROOF. For the sake of constructing a comparison function, we take a function $h \in C^\infty([0, \infty))$ which satisfies

(i) $h(0)=0$, $h(1)=1$

(ii) $\frac{d^k h}{d\xi^k}(0)=0$ for any natural number k.

$$\frac{dh}{d\xi}(\xi)>0 \quad \text{for any} \quad \xi \in (0, 1).$$

Take an arbitrary hyperplane π in $\boldsymbol{R}^n$ which contains the y_1-axis. By an appropriate orthogonal transformation of coordinates in $(y_2, \cdots, y_n)$, we can assume without loss of generality that π is expressed by the equation $y_2=0$. Note that the equation (3.52) is invariant under the above transformation.

Now we define a domain $G_+(\zeta)$ and a function $W_2(y)$ in $G_+(\zeta)$ as follows:

$$(3.85) \qquad G_+(\zeta)=G(\zeta) \cap \{y_2>0\},$$

$$(3.86) \qquad W_2(y)=\frac{1}{2}(U_\zeta(y_1, y_2, \cdots, y_n)-U_\zeta(y_1, -y_2, y_3, \cdots, y_n)).$$

It is easily seen that W_2 satisfies the following:

$$(3.87) \qquad \frac{\partial W_2}{\partial y_2}(y)=\frac{\partial U_\zeta}{\partial y_2}(y) \quad \text{on} \quad \pi \cap \partial G_+(\zeta),$$

$$(3.88) \qquad P_\zeta W_2+\frac{1}{2}(f(U_\zeta)-f(U_\zeta(y_1, -y_2, y_3, \cdots, y_n)))=0 \quad \text{in} \quad G_+(\zeta),$$

$$(3.89) \qquad W_2(y)=0 \quad \text{on} \quad \pi \cap \partial G_+(\zeta),$$

(3.90) $$\frac{\partial W_2}{\partial \nu}(y)=0 \quad \text{on} \quad \partial G_+(\zeta)\cap\partial G(\zeta).$$

We define a comparison function $\Theta_2(y)$ as follows:

(3.91) $$\Theta_2(y)=\begin{cases} e(\delta)\zeta^2 y_2(3-y_2)+\bar{M}h\left(\dfrac{y_1-1+\delta}{\delta-\zeta}\right) & (y_1>1-\delta) \\ e(\delta)\zeta^2 y_2(3-y_2) \quad (-1+\delta\leqq y_1\leqq 1-\delta) & \\ e(\delta)\zeta^2 y_2(3-y_2)+\bar{M}h\left(\dfrac{-y_1-1+\delta}{\delta-\zeta}\right) & (y_1<-1+\delta), \end{cases}$$

where

$$e(\delta)=1+\omega+\frac{2\bar{M}}{\delta^2}\sup_{\xi\in[0,1]}|h''(\xi)|.$$

By a simple calculation, we obtain

(3.92) $$P_\zeta\Theta_2(y)=\begin{cases} -2e(\delta)+\dfrac{\bar{M}}{(\delta-\zeta)^2}h''\left(\dfrac{y_1-1+\delta}{\delta-\zeta}\right) & (y_1>1-\delta) \\ -2e(\delta) \quad (-1+\delta\leqq y_1\leqq 1-\delta) & \\ -2e(\delta)+\dfrac{\bar{M}}{(\delta-\zeta)^2}h''\left(\dfrac{-y_1-1+\delta}{\delta-\zeta}\right) & (y_1<-1+\delta), \end{cases}$$

(3.93) $$\frac{\partial\Theta_2}{\partial\nu}(y)>0 \quad \text{on} \quad \partial G_+(\zeta)\cap\partial G,$$

(3.94) $$\Theta_2(y)\geqq\bar{M} \quad \text{on} \quad (\partial G_+(\zeta)\cap\{y_1=1-\zeta\})\cup(\partial G_+(\zeta)\cap\{y_1=-1+\zeta\}),$$

(3.95) $$\Theta_2(y)=0 \quad \text{on} \quad \partial G_+(\delta)\cap\pi.$$

By using $0<\zeta\leqq\delta/2$ and the definition of $e(\delta)$, we obtain from (3.88)$\sim$(3.90) and (3.92)$\sim$(3.94) that

(3.96) $$P_\zeta(\Theta_2-W_2)(y)<0 \quad \text{in} \quad G_+(\zeta),$$

(3.97) $$\frac{\partial}{\partial\nu}(\Theta_2-W_2)>0 \quad \text{on} \quad \partial G_+(\zeta)\cap\partial G,$$

(3.98) $$\Theta_2(y)-W_2(y)\geqq 0 \quad \text{on} \quad \partial G_+(\zeta)\cap\pi,$$

(3.99) $$\Theta_2(y)-W_2(y)\geqq 0 \quad \text{on} \quad (\partial G_+(\zeta)\cap\{y_1=1-\zeta\})\cup(\partial G_+(\zeta)\cap\{y_1=-1+\zeta\}),$$

(3.100) $$\Theta_2(y)-W_2(y)=0 \quad \text{on} \quad \partial G_+(\delta)\cap\pi.$$

Applying the maximum principle to (3.96)~(3.99), we obtain

$$\Theta_2(y)-W_2(y)\geqq 0 \quad \text{in} \quad G_+(\zeta).$$

By a similar argument, we have $-\Theta_2(y)\leqq W_2(y)$ in $G_+(\zeta)$, hence,

(3.101) $$|W_2(y)|\leqq\Theta_2(y) \quad \text{in} \quad G_+(\zeta).$$

Combining (3.101) and (3.100) yields

$$\left|\frac{\partial W_2}{\partial y_2}(y)\right|\leqq\frac{\partial\Theta_2}{\partial y_2}(y) \quad \text{on} \quad \partial G_+(\delta)\cap\pi.$$

In view of this and (3.87), we see that

$$\left|\frac{\partial U_\zeta}{\partial y_2}(y)|_{\partial G_+(\delta)\cap\pi}\right|\leqq\left|\frac{\partial\Theta_2}{\partial y_2}(y)|_{\partial G_+(\delta)\cap\pi}\right|=3e(\delta)\zeta^2,$$

hence

$$\left|\frac{\partial U_\zeta}{\partial y_2}(y)|_{\partial G_+(\delta)\cap\pi\cap\partial G}\right|\leqq 3e(\delta)\zeta^2.$$

Since $\pi=\{y\in\boldsymbol{R}^n\,|\,y_2=0\}$, this last inequality can be written as

(3.102) $$\left|\frac{\partial U_\zeta}{\partial n}(y)\right|\leqq 3e(\delta)\zeta^2 \quad \text{on} \quad \pi\cap\partial G\cap\{-1+\delta\leqq y_1\leqq 1-\delta\},$$

where $\partial/\partial n$ denotes the normal derivative on π. It is clear that (3.102) holds for any hyperplane π containing the y_1-axis. It follows that

(3.103) $$|\vec{p}\cdot\operatorname{grad}U_\zeta(y)|\leqq 3e(\delta)\zeta^2$$

for any $y\in\partial G\cap\{-1+\delta\leqq y_1\leqq 1-\delta\}$ and any unit vector $\vec{p}$ that is orthogonal to the y_1-axis and parallel to the tangent hyperplane of ∂G at y. As is easily seen, (3.103) and the Neumann boundary condition (3.53) imply

$$\sum_{j=2}^{n}\left|\frac{\partial U_\zeta}{\partial y_j}(y)\right|^2\leqq 9e(\delta)^2\zeta^4 \quad \text{on} \quad \partial G\cap\{-1+\delta\leqq y_1\leqq 1-\delta\}.$$

We complete the proof of Lemma 3.8 by putting $c_{4,\delta}=9e(\delta)^2$.

Now we proceed with the proof of the former half of Theorem 3. Len $\{\zeta_m\}_{m=1}^{\infty}$ be an arbitrary sequence of positive numbers satisfying $0<\zeta_m<\zeta_*$ and $\lim_{m\to\infty}\zeta_m=0$. For each $\delta\in(0, 1/2)$, $\{U_{2,\zeta_m}\}_{m=1}^{\infty}$ is bounded in the Sobolev space $H^1(G(\delta))$ by virtue of Lemma 3.5, hence relatively compact in $H^{1/2}(G(\delta))$. In view of this, and using the diagonal argument, we see that there exists a subsequence $\{\kappa_m\}_{m=1}^\infty \subset \{\zeta_m\}_{m=1}^\infty$ and a function V_2 on $G(0)=G\cap\{-1\leqq y_1\leqq 1\}$ such that

(3.104a) $$\lim_{m\to\infty} U_{2,\kappa_m} = V_2 \quad \text{in} \quad H^{1/2}(G(\delta))$$

for any $\delta\in(0, 1/2)$. It follows from (3.103) and the trace theorem (Taylor [21; Chap. I]) that

(3.104b) $$\lim_{m\to\infty} U_{2,\kappa_m}|_{\partial G(\delta)\cap\partial G} = V_2|_{\partial G(\delta)\cap\partial G} \quad \text{in} \quad L^2(\partial G(\delta)\cap\partial G)$$

for any $\delta\in(0, 1/2)$. Next, by Lemma 3.3 and $\lim_{m\to\infty}\iota(\kappa_m)=0$, we have

(3.105) $$\lim_{m\to\infty} U_{1,\kappa_m}(y) = \frac{1-y_1}{2}a_1 + \frac{1+y_1}{2}a_2 \quad \text{uniformly on} \quad \overline{G(\delta)}$$

for any $\delta\in(0, 1/2)$. Combining (3.103)~(3.105) and letting

$$V(y) = V_2(y) + \frac{1-y_1}{2}a_1 + \frac{1+y_1}{2}a_2,$$

we obtain

(3.106) $$\lim_{m\to\infty} U_{\kappa_m} = V \quad \text{in} \quad L^2(G(\delta)),$$

(3.107) $$\lim_{m\to\infty} U_{\kappa_m}|_{\partial G(\delta)\cap\partial G} = V|_{\partial G(\delta)\cap\partial G} \quad \text{in} \quad L^2(\partial G(\delta)\cap\partial G),$$

for any $\delta\in(0, 1/2)$. By the estimate (3.63), V_2 is independent of the variables $y_2, \cdots, y_n$ hence so is the function V. We can therefore write $V=V(y_1)$. Hereafter we shall show that V satisfies (3.4) and that the convergence in (3.106) occurs in the topology of $L^\infty(G(0))$.

By virtue of Theorem 2, Lemma 3.4 and $\lim_{m\to\infty}\iota(\kappa_m)=0$, there exist, for any $\varepsilon>0$, a constant $\bar{\delta}=\bar{\delta}(\varepsilon)>0$ and a positive integer $\bar{m}=\bar{m}(\varepsilon)$ such that

(3.108) $$\sup_{-1\leqq y_1\leqq -1+\bar{\delta},\,|y'|\leqq 1} |U_{\kappa_m}(y)-a_2| + \sup_{1-\bar{\delta}\leqq y_1\leqq 1,\,|y'|\leqq 1} |U_{\kappa_m}(y)-a_1| \leqq \varepsilon$$

for any $m\geqq\bar{m}(\varepsilon)$. By (3.3), we have

(3.109) $$M_*\leqq U_{\kappa_m}(y)\leqq M^* \quad \text{in} \quad G(0)=G\cap\{-1\leqq y_1\leqq 1\}$$

for $m=1, 2, 3, \cdots$. (3.109) implies that $M_*\leqq V(y_1)\leqq M^*$ for $-1\leqq y_1\leqq 1$.

Moreover it follows from (3.108) that $V(y_1)$ is continuous at $y_1=\pm 1$, and that

$$V(1)=a_1, \quad V(-1)=a_2. \tag{3.110}$$

In order to see that V satisfies the ordinary differential equation in (3.4), we take an arbitrary $\phi \in C_0^\infty((-1, 1))$ and integrate the equation

$$\left(\frac{\partial^2}{\partial y_1^2}+\frac{1}{\kappa_m^2}\sum_{j=2}^{n}\frac{\partial^2}{\partial y_j^2}\right)U_{\kappa_m}+f(U_{\kappa_m})=0$$

over $G(\kappa_m)$ after multiplying it by $\phi(y_1, y')=\phi(y_1)$. Then we have for sufficiently large m so that $\operatorname{supp}\phi\subset(-1+\kappa_m, 1-\kappa_m)$,

$$\int_{G(\kappa_m)} U_{\kappa_m}(y)P_{\kappa_m}\phi dy+\int_{G(\kappa_m)}\phi f(U_{\kappa_m})dy=0.$$

(Notice that $P_{\kappa_m}\phi(y)=(\partial^2\phi/\partial y_1^2)(y_1)$.) Letting $m\to\infty$ and using (3.106) and (3.109), we get

$$\int_{|y'|\leq 1}dy'\int_{-1}^{1}\left(V(y_1)\frac{\partial^2}{\partial y_1^2}\phi(y_1)+\phi(y_1)f(V(y_1))\right)dy_1=0.$$

By the arbitrariness of ϕ, we have

$$\frac{d^2}{dy_1^2}V(y_1)+f(V(y_1))=0 \quad \text{in} \quad (-1, 1).$$

This together with (3.10), implies that V satisfies (3.4), as claimed.

LEMMA 3.9.

$$\lim_{m\to\infty}\sup_{y\in G(\kappa_m)}|U_{\kappa_m}(y_1, y')-V(y_1)|=0.$$

PROOF. First we show that $U_{\kappa_m}|_{\partial G(0)\cap\partial G}$ converges to V uniformly on $\partial G(0)\cap\partial G=\partial G\cap\{-1\leq y_1\leq 1\}$. By Lemma 3.7, Lemma 3.8, (3.109) and Ascoli-Arzerà theorem, $\{U_{\kappa_m}|_{\partial G(\delta)\cap\partial G}\}$ is relatively compact in $C^0(\partial G(\delta)\cap\partial G)$ for any $\delta\in(0, 1/2)$. In view of this and (3.107), we see that

$$\lim_{m\to\infty}U_{\kappa_m}|_{\partial G(\delta)\cap\partial G}=V|_{\partial G(\delta)\cap\partial G} \quad \text{in} \quad C^0(\partial G(\delta)\cap\partial G),$$

for any $\delta\in(0, 1/2)$. Combining this and (3.108), we easily find that

$$\lim_{m\to\infty}U_{\kappa_m}|_{\partial G(0)\cap\partial G}=V|_{\partial G(0)\cap\partial G} \quad \text{in} \quad C^0(\partial G(0)\cap\partial G). \tag{3.111}$$

Now we define a comparison function $\Theta_{+,m}$, $\Theta_{-,m}$ by

$$\Theta_{\pm,m}(y)=V(y_1)\pm\frac{\omega}{n-1}(1-|y'|^2)\kappa_m^2 \pm \sup_{y\in\partial G(\kappa_m)}|U_{\kappa_m}(y_1,y')-V(y_1)|.$$

$\Theta_{+,m}$, $\Theta_{-,m}$ satisfy the following:

$$P_{\kappa_m}(\Theta_{\pm,m}-U_{\kappa_m})=-f(V)\mp 2\omega+f(U_{\kappa_m})\gtreqless 0 \quad \text{in} \quad G(\kappa_m),$$
$$\Theta_{\pm,m}(y)-U_{\kappa_m}(y)\gtreqless 0 \quad \text{on} \quad \partial G(\kappa_m).$$

Applying the maximum principle, we have

$$\Theta_{\pm,m}(y)-U_{\kappa_m}(y)\gtreqless 0 \quad \text{in} \quad G(\kappa_m),$$

or, equivalently, $\Theta_{-,m}(y)\leqq U_{\kappa_m}(y)\leqq\Theta_{+,m}(y)$ in $G(\kappa_m)$. By the definition of $\Theta_{\pm,m}$ and by (3.111) we conclude that

$$\lim_{m\to\infty}\sup_{y\in G(\kappa_m)}|U_{\kappa_m}(y)-V(y_1)|=0$$

and complete the proof of Lemma 3.9.

Expressing the equality in Lemma 3.9 in the original variable x, we complete the proof of the former assertion of Theorem 3.

PROOF OF THE LATTER HALF OF THEOREM 3

(1) *The case* $\lambda_V<0$.

We shall show that

$$\overline{\lim_{m\to\infty}}\,\mu_1(\kappa_m)\leqq\lambda_V. \tag{3.112}$$

The following well-known variational characterization of μ_1 will be useful:

$$\mu_1(\kappa_m)=\inf_{\psi\in H^1(\Omega(\kappa_m))}\frac{\int_{\Omega(\kappa_m)}(|\nabla\psi|^2-f'(v_{\kappa_m})\psi^2)dx}{\int_{\Omega(\kappa_m)}|\psi|^2dx}. \tag{3.113}$$

Since the first eigenfunction Φ_V in (3.5) does not change sign by the Krein-Rutman theory, we may assume without loss of generality that $\Phi_V(z)>0$ for $-1<z<1$. Let

$$\gamma_m=\max\{\Phi_V(1-2\kappa_m),\ \Phi_V(-1+2\kappa_m)\},$$
$$\alpha_m=\min\{z\geqq-1+2\kappa_m\,|\,\Phi_V(z)=\gamma_m\},$$
$$\beta_m=\max\{z\leqq 1-2\kappa_m\,|\,\Phi_V(z)=\gamma_m\}.$$

Clearly $-1+2\kappa_m \leqq \alpha_m \leqq \beta_m \leqq 1-2\kappa_m$ $(m=1, 2, \cdots)$ and, as is easily seen from the boundary condition $\Phi_V(-1)=\Phi_V(1)=0$, we have

$$\lim_{m\to\infty} \gamma_m = 0, \tag{3.114}$$

$$\lim_{m\to\infty} \alpha_m = -1, \quad \lim_{m\to\infty} \beta_m = 1. \tag{3.115}$$

Now we define a function $\psi_m(x)$ on $\Omega(\kappa_m)$ by

$$\psi_m(x_1, x') = \begin{cases} \Phi_V(x_1)-\gamma_m, & x \in \Gamma(\kappa_m) \cap \{\alpha_m \leqq x_1 \leqq \beta_m\} \\ 0, & x \in D_1 \cup D_2 \cup R_1(\kappa_m) \cup R_2(\kappa_m) \cup (\Gamma(\kappa_m) \cap \{x_1 < \alpha_m \text{ or } x_1 > \beta_m\}). \end{cases}$$

It is clear that $\psi_m \in H^1(\Omega(\kappa_m))$. In order to estimate the right-hand side of (3.113), we substitute $\psi = \psi_m$:

$$\begin{aligned} &\int_{\Omega(\kappa_m)} (|\nabla \psi_m|^2 - f'(v_{\kappa_m})\psi_m{}^2)dx \\ &= \int_{|x'|\leqq \kappa_m} dx' \int_{\alpha_m}^{\beta_m} \left(\left| \frac{\partial \Phi_V}{\partial x_1} \right|^2 - f'(v_{\kappa_m})|\Phi_V(x_1)-\gamma_m|^2 \right) dx_1 \\ &= -\iint \left\{ \frac{d^2\Phi_V}{dx_1{}^2} + f'(v_{\kappa_m})(\Phi_V(x_1)-\gamma_m) \right\} (\Phi_V(x_1)-\gamma_m) dx' dx_1 \\ &= \int_{|x'|\leqq \kappa_m} \left\{ \int_{\alpha_m}^{\beta_m} \{\lambda_V + f'(V(x_1)) - f'(v_{\kappa_m}(x_1, x'))\} \Phi_V(x_1)^2 dx_1 \right. \\ &\quad \left. + \int_{\alpha_m}^{\beta_m} \{-\lambda_V \Phi_V(x_1) + (2f'(v_{\kappa_m}) - f'(V))\Phi_V(x_1) - f'(v_{\kappa_m})\gamma_m\} \gamma_m dx_1 \right\} dx'. \end{aligned} \tag{3.116}$$

By the former assertion of Theorem 3 (namely (3.6)) already proved above, we have

$$\lim_{m\to\infty} \sup_{x \in \Gamma(\kappa_m)} |f'(V(x_1)) - f'(v_{\kappa_m}(x_1, x'))| = 0. \tag{3.1Ɪ7}$$

Letting $m \to \infty$ in (3.116), using (3.114), (3.115) and (3.17), we see that

$$\lim_{m\to\infty} \int_{\Omega(\kappa_m)} (|\nabla \psi_m|^2 - f'(v_{\kappa_m})\psi_m{}^2)dx \Big/ \iint_{\Gamma(\kappa_m)} \Phi_V(x_1)^2 dx_1 dx' = \lambda_V. \tag{3.118}$$

On the other hand, it is clear from (3.114) and (3.115) that

$$\lim_{m\to\infty} \frac{\displaystyle\iint_{\Gamma(\kappa_m)} \Phi_V(x_1)^2 dx_1 dx'}{\displaystyle\int_{\Omega(\kappa_m)} \psi_m(x)^2 dx} = 1.$$

Combining this, (3.118) and (3.113), we obtain (3.112). In particular we have $\lambda_V<0$, as claimed.

(2) *The case* $\lambda_V>0$.

From now on we shall prove that $\mu_1(\kappa_m)$ is bounded from below by a positive constant for sufficiently large m. To prove this, we assume that there exists a subsequence $\{m(j)\}_{j=1}^{\infty}$ such that

$$(*)\qquad \lim_{j\to\infty} m(j)=\infty, \quad \lim_{j\to\infty} \mu_1(\kappa_{m(j)})\leqq 0$$

and shall derive a contradiction. Let ψ_j be the eigenfunction of (3.2) corresponding to the eigenvalue $\mu_1(\kappa_{m(j)})$ such that

$$(3.119)\qquad \|\psi_j\|_{L^2(\Omega(\kappa_{m(j)}))}=1 \quad (j\geqq 1).$$

LEMMA 3.10. *Under the condition* $(*)$,

$$\lim_{j\to\infty}\psi_j=0 \text{ in } C^\infty((\overline{D_1-\Sigma_1(\eta)})\cup(\overline{D_2-\Sigma_2(\eta)}) \text{ for any } \eta>0.$$

For the proof of this lemma, we need the following simple lemma, the proof of which is omitted:

LEMMA 3.11. *Let A be a self-adjoint operator in a Hilbert space X with norm $\|\cdot\|$, and let $\sigma(A)$ denote the spectrum of A. Suppose there exists a constant $0<\theta\leqq 1$ and an element $\varphi\in X$ such that $\|A\varphi\|\leqq \theta\|\varphi\|$. Then*

$$\sigma(A)\cap I_\theta\neq\emptyset, \quad \|E(I_\theta)\varphi-\varphi\|\leqq\theta^{1/2}\|\varphi\|,$$

where $I_\theta=[-\theta^{1/2},\theta^{1/2}]\subset \boldsymbol{R}$ and $E(\cdot)$ is the spectral measure associated with the operator A.

PROOF. Using the a priori estimates of Agmon, Douglis and Nirenberg [1], and applying the bootstrap argument, we see that

$$(3.120)\qquad \{\psi_j\}_{j=1}^{\infty} \text{ is compact in } C^\infty((\overline{D_1-\Sigma_1(\eta)})\cup(\overline{D_2-\Sigma_2(\eta)}) \text{ for any } \eta>0.$$

Next we take two functions $\phi_1, \phi_2\in C^\infty(\boldsymbol{R}^n)$ such that

$$\phi_1(x)=1 \text{ in } D_1,\ \phi_1(x)=0 \text{ in } D_2,\ \phi_2(x)=0 \text{ in } D_1,$$
$$\phi_2(x)=1 \text{ in } D_2,\ \operatorname{supp}\phi_1\cap\operatorname{supp}\phi_2=\emptyset.$$

We put, for $i=1, 2$ and $j=1, 2, 3, \cdots,$

$$\theta_j^{(i)} \equiv \|(\Delta + f'(v_{\kappa_{m(j)}}))\phi_i - f'(a_i)\phi_i\|_{L^2(\Omega(\kappa_{m(j)}))} / \|\phi_i\|_{L^2(\Omega(\kappa_{m(j)}))}.$$

We can easily check that $\lim_{j\to\infty}\theta_j^{(i)}=0$ $(i=1, 2)$ by using Theorem 2 and a simple calculation. By Lemma 3.11, the eigenvalue problem (3.2) for $\zeta=\kappa_{m(j)}$ has eigenvalues $\mu^{(1)}(j)$ and $\mu^{(2)}(j)$ for large j such that

$$\mu^{(i)}(j) \in [-f'(a_i)-\theta_j^{(i)1/2},\ -f'(a_i)+\theta_j^{(i)1/2}] \equiv I_j^{(i)}, \tag{3.121}$$

$$\|P_{I_j^{(i)}}\phi_i - \phi_i\|_{L^2(\Omega(\kappa_{m(j)}))} / \|\phi_i\|_{L^2(\Omega(\kappa_{m(j)}))} \leqq \theta_j^{(i)1/2} \tag{3.122}$$

for $i=1, 2$, where $P_{I_j^{(i)}}$ is the eigenprojection (associated with the self-adjoint operator $-\Delta - f'(v_{\kappa_{m(j)}})$ onto the subspace of $L^2(\Omega(\kappa_{m(j)}))$ corresponding to the interval $I_j^{(i)}$. We have $\mu_1(\kappa_{m(j)}) \notin \cup_{i=1}^2 [-f'(a_i)-\theta_j^{(i)}, -f'(a_i)+\theta_j^{(i)}]$ for large j by (*), hence $(\psi_j, P_{I_j^{(i)}}\phi_i)_{L^2(\Omega(\kappa_{m(j)}))}=0$ for large j and $i=1, 2$. In view of this and (3.122), we have, for $i=1, 2$,

$$|(\psi_j, \phi_i)_{L^2(\Omega(\kappa_{m(j)}))}| / \|\phi_i\|_{L^2(\Omega(\kappa_{m(j)}))} \leqq \theta_j^{(i)1/2}$$

for large j and so we can easily deduce $\lim_{j\to\infty}\int_{D_i}\psi_j dx=0$ $(i=1, 2)$. Considering this and the fact that $\psi_j(x)>0$ in $\overline{\Omega(\kappa_{m(j)})}$, we have

$$\lim_{j\to\infty}\psi_j(x)=0 \quad \text{in} \quad L^1(D_1\cup D_2).$$

The conclusion of Lemma 3.10 now follows from this and (3.120).

By using Lemma 3.10, we can choose a monotone sequence of positive values $\{l_j\}_{j=1}^\infty$ such that

$$\begin{cases} \lim_{j\to\infty} l_j=0, \quad l_j>\kappa_{m(j)}, \\ \lim_{j\to\infty} K(j)=0, \end{cases} \tag{3.123}$$

where

$$K(j)=\sup_{x\in(D_1-\tilde{\Sigma}_1(2l_j))\cup(D_2-\tilde{\Sigma}_2(2l_j))}|\psi_j(x)|>0,$$
$$\tilde{\Sigma}_1(2l_j)=\{(x_1, x')\in \boldsymbol{R}^n \mid 1<x_1<1+2l_j,\ |x'|<2l_j\},$$
$$\tilde{\Sigma}_2(2l_j)=\{(x_1, x')\in \boldsymbol{R}^n \mid -1-2l_j<x_1<-1,\ |x'|<2l_j\}.$$

Here we define two sets:

$$S_j=(Q(\kappa_{m(j)})\cup\tilde{\Sigma}_1(2l_j)\cup\tilde{\Sigma}_2(2l_j)),$$
$$T_j=\{(x_1, x')\in \boldsymbol{R}^n \mid |x'|<\kappa_{m(j)},\ |x_1|\leqq 1+2l_j\}.$$

Now we decompose eigenfunction ψ_j as

$$\psi_j(x)=\psi_j^{(1)}(x)+\psi_j^{(2)}(x) \quad \text{in} \quad S_j,$$

where $\psi_j^{(1)}$, $\psi_j^{(2)}$ are determined uniquely by the following condition:

$$\left\{\begin{array}{l}\Delta\psi_j^{(1)}=0 \quad \text{in} \quad S_j, \\ \psi_j^{(1)}(x)=\psi_j(x) \quad \text{on} \quad \partial S_j-\partial\Omega(\kappa_{m(j)}), \\ \dfrac{\partial\psi_j^{(1)}}{\partial\nu}(x)=0 \quad \text{on} \quad \partial S_j\cap\partial\Omega(\kappa_{m(j)}),\end{array}\right. \tag{3.124}$$

$$\psi_j^{(2)}(x)=\psi_j(x)-\psi_j^{(1)}(x) \quad \text{in} \quad S_j. \tag{3.125}$$

Applying the maximum principle to (3.124), we obtain the inequality

$$0<\psi_j^{(1)}(x)\leqq K_j \quad \text{in} \quad S_j. \tag{3.126}$$

Now we calculate as follows:

$$\begin{aligned}\mu_1(\kappa_{m(j)})&=\int_{\Omega(\kappa_{m(j)})}(|\nabla\psi_j|^2-f'(v_{\kappa_{m(j)}})\psi_j^2)dx \\ &=\int_{\Omega(\kappa_{m(j)})-S_j}(|\nabla\psi_j|^2-f'(v_{\kappa_{m(j)}})\psi_j^2)dx+\int_{S_j}|\nabla\psi_j^{(1)}|^2dx \\ &\quad+\int_{S_j}(|\nabla\psi_j^{(2)}|^2-f'(v_{\kappa_{m(j)}})|\psi_j^{(2)}|^2)dx-\int_{S_j}f'(v_{\kappa_{m(j)}})(2\psi_j^{(2)}+\psi_j^{(1)})\psi_j^{(1)}dx \\ &\equiv B_1(j)+B_2(j)+B_3(j)+B_4(j).\end{aligned} \tag{3.127}$$

We have used $\int_{S_j}\nabla\psi_j^{(1)}\nabla\psi_j^{(2)}dx=0$ in the above. By Theorem 2, $-f'(v_{\kappa_{m(j)}})\geqq\beta_*/2$ in $\Omega(\kappa_{m(j)})-S_j$ for large j, where $\beta_*=\min(-f'(a_1), -f'(a_2))$. Consequently,

$$B_1(j)\geqq\min(1, \beta_*/2)(\|\psi_j\|_{L^2(\Omega(\kappa_{m(j)})-S_j)})^2 \tag{3.128}$$

for large j. By (3.123), (3.126) and the boundedness of $\|\psi_j^{(2)}\|_{L^2(S_j)}$ $(j=1, 2, 3, \cdots)$,

$$\lim_{j\to\infty}B_4(j)=0. \tag{3.129}$$

Hereafter we estimate $B_3(j)$ from below.

$$\begin{aligned}B_3(j)=&\int_{T_j}(|\nabla\psi_j^{(2)}|^2-f'(v_{\kappa_{m(j)}})|\psi_j^{(2)}|^2)dx \\ &+\int_{S_j-T_j}(|\nabla\psi_j^{(2)}|^2-f'(v_{\kappa_{m(j)}})|\psi_j^{(2)}|^2)dx.\end{aligned}$$

Again by Theorem 2, the second term of $B_3(j)$ is bounded from below by

$$\min(1, \beta_*/2)(\|\psi_j^{(2)}\|_{H^1(S_j-T_j)})^2.$$

To estimate the first term of $B_3(j)$, we change the variable x into y in T_j as follows:

$$\begin{cases} x_1=(1+\sigma_j)y_1 \\ x'=y' \\ \Xi_j(y_1, y')=\psi_j^{(2)}(\sigma_j y_1, y') \quad \text{for} \quad |y_1|\leqq 1,\ |y'|<\kappa_{m(j)} \end{cases}$$

where $\sigma_j=1+2l_j$. Note that

$$\Xi_j(y)=0 \quad \text{for} \quad y_1=\pm 1,\ |y'|<\kappa_{m(j)}. \tag{3.130}$$

By this change of variables, we have

$$\begin{aligned}
&\int_{T_j}(|\nabla\psi_j^{(2)}|^2-f'(v_{\kappa_{m(j)}})|\psi_j^{(2)}|^2)dx \\
&\geqq \int_{|y'|\leqq\kappa_{m(j)}} dy' \int_{-1}^{1} \left\{\frac{1}{\sigma_j^2}\left|\frac{\partial \Xi_j}{\partial y_1}(y_1, y')\right|^2 - f'(v_{\kappa_{m(j)}}(\sigma_j y_1, y'))\Xi_j^2\right\}\sigma_j dy_1 \\
&= \frac{1}{\sigma_j}\int_{|y'|\leqq\kappa_{m(j)}} dy' \int_{-1}^{1}\left(\left|\frac{\partial \Xi_j(y_1, y')}{\partial y_1}\right|^2 - f'(V(y_1))|\Xi_j(y_1, y')|^2\right)dy_1 \\
&\quad + \int_{|y'|\leqq\kappa_{m(j)}} dy' \int_{-1}^{1}\left(\frac{1}{\sigma_j^2}f'(V(y_1)) - f'(v_{\kappa_{m(j)}}(\sigma_j y_1, y'))\right)\Xi_j^2\sigma_j dy_1.
\end{aligned}$$

By Theorems 2 and 3 and the fact that $\lim_{j\to\infty}\sigma_j=0$, the second term of the right-hand side tends to 0 as $j\to\infty$. Using the variational characterization of the eigenvalue λ_V and the boundary condition (3.130), we easily find that the first term is bounded from below by

$$\frac{1}{\sigma_j}\int_{|y'|\leqq\kappa_{m(j)}} dy' \int_{-1}^{1} \lambda_V \Xi_j(y)^2 dy_1 \left(=\frac{\lambda_V}{\sigma_j^2}(\|\psi_j^{(2)}\|_{L^2(T_j)})^2\right).$$

Consequently we have the following inequality for j:

$$B_3(j)\geqq\frac{\lambda_V}{2}(\|\psi_j^{(2)}\|_{L^2(T_j)})^2. \tag{3.131}$$

From the inequalities (3.127), (3.128) and (3.130), follows

$$\mu_1(\kappa_{m(j)})-B_4(j)\geqq\min(1,\ \beta_*/2)(\|\psi_j\|_{L^2(\Omega(\kappa_{m(j)})-S_j)})^2$$
$$+\int_{S_j}|\nabla\psi_j^{(1)}|^2dx+\min(1,\ \beta_*/2)(\|\psi_j^{(2)}\|_{H^1(S_j-T_j)})^2$$
$$+\frac{\lambda_V}{2}(\|\psi_j^{(2)}\|_{L^2(T_j)}{}^2).$$

Let j tend to ∞ and using $\lim_{j\to\infty}\mu_1(\kappa_{m(j)})\leqq 0$ and (3.129), we have $\lim_{j\to\infty}\|\psi_j\|_{L^2(\Omega(\kappa_{m(j)}))}=0$. But this contradicts the fact that $\|\psi_j\|_{L^2(\kappa_{m(j)})}=1$ for $j\geqq 1$ (see (3.119)). This contradiction proves $\underline{\lim}_{m\to\infty}\mu_1(\kappa_m)>0$ in the case $\lambda_V>0$. Therefore we have completed the proof of Theorem 3.

§ 4. Construction of Unstable Solution.

In this section, we shall consider the equation (3.1) on the domain $\Omega(\zeta)$ established in Section 3 where we choose f in (3.1) as specified below. We shall construct a family of solutions $\{v_\zeta\}_{\zeta>0}$ in (III-4) where v_ζ is an unstable solution of (3.1) under the condition $a_1=a_2=b_1$ for small $\zeta>0$.

We set the nonlinear term f in the following form:

(4.1) $$f(\xi)=\vartheta g(\xi) \qquad (\vartheta>0),$$

where $g\in C^\infty(\boldsymbol{R})$ satisfies conditions (IV-1) and (IV-2) below and the parameter ϑ will be chosen later.

(IV-1) There exist three points $b_1<b_2<b_3$ such that

$$g(b_i)=0 \quad (1\leqq i\leqq 3), \quad g'(b_1)<0, \quad g'(b_3)<0,$$
$$g(\xi)>0 \quad \text{in} \quad (-\infty,\ b_1)\cup(b_2,\ b_3),$$
$$g(\xi)<0 \quad \text{in} \quad (b_1,\ b_2)\cup(b_3,\ \infty).$$

(IV-2) $$\int_{b_1}^{b_3}g(\xi)d\xi>0.$$

By (IV-1)$\sim$(IV-2), there exists a unique $d\in(b_2,\ b_3)$ such that $\int_{b_1}^{d}g(\xi)d\xi=0$.

With f as above, the corresponding ordinary differential equation (3.4) can be written in the form

(4.2) $$\begin{cases}\dfrac{d^2V}{dz^2}+f(V)=0 \quad \text{in} \quad -1<z<1\\ V(1)=b_1, \quad V(-1)=b_1.\end{cases}$$

PROPOSITION 3. *There exists a positive value* ϑ_0 *such that for any* $\vartheta \geqq \vartheta_0$, (4.2) *has exactly three solutions*

$$V^{(0)}(z)(\equiv b_1) < V^{(1)}(z) < V^{(2)}(z) \quad (-1<z<1)$$

with the following stability properties:

$$\lambda_{V^{(0)}}>0, \quad \lambda_{V^{(1)}}<0, \quad \lambda_{V^{(2)}}>0.$$

(See Notation 2 in Section 3 as for the definition $\lambda_{V^{(0)}}, \lambda_{V^{(1)}}, \lambda_{V^{(2)}}$.)

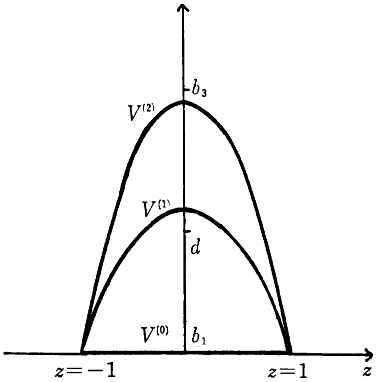

Figure 5

PROOF. First of all, as one can easily see, any solution $V(z)$ of (4.2) satisfies $b_1 \leqq V(z) < b_3$ $(-1 \leqq z \leqq 1)$. Furthermore, if $V(z)$ is a non-trivial solution of (4.2) (i.e., $V(z) \not\equiv b_1$), then

$$\text{(4.3)} \quad \begin{cases} V(z)=V(-z) & (-1\leqq z\leqq 1), \\ \dfrac{dV}{dz}(z)>0 \quad (-1\leqq z<0), & \dfrac{dV}{dz}(0)=0. \end{cases}$$

Integrating the equation (4.2) after multiplying it by dV/dz and using (4.3), we see that

$$\text{(4.4)} \quad \frac{dV}{dz}(z)=\left(2\int_{V(z)}^{V(0)} f(\rho)d\rho\right)^{1/2} \quad (-1\leqq z<0).$$

In particular we have $\int_{V(z)}^{V(0)} f(\rho)d\rho>0$ for $-1\leqq z<0$. It also follows from (4.4) that

$$\text{(4.5)} \quad \int_{b_1}^{V(z)} \left(2\int_{\sigma}^{V(0)} f(\rho)d\rho\right)^{-1/2} d\sigma=1+z.$$

Combining the above observations, we obtain the following:

(4.6) $$d<\xi<b_3,$$

(4.7) $$\int_{b_1}^{\xi}\left(2\int_{\sigma}^{\xi}f(\rho)d\rho\right)^{-1/2}d\sigma=1,$$

where $\xi=V(0)$. Conversely, if ξ satisfies (4.6) and (4.7), then there exists a solution $V(z)$ of (4.2) such that $V(0)=\xi$. Moreover it is clear from (4.5) that the value $V(0)$ determines the solution $V(z)$ uniquely. Therefore the problem of finding nontrivial solution of (4.2) reduces to that of finding ξ satisfying (4.6) and (4.7). (See also Maginu [14].)

To examine the left-hand side of (4.7) as a function of ξ, we define $s(\xi)$ on (d, b_3) as follows:

$$s(\xi)=\int_{b_1}^{\xi}\left(2\int_{\sigma}^{\xi}g(\rho)d\rho\right)^{-1/2}d\sigma.$$

$s(\xi)$ is well-defined by (IV-1) and (IV-2) and moreover we have:

LEMMA 4.1. *$s(\xi)$ is a positive differentiable function on (d, b_3) with the following asymptotic conditions*:

$$\begin{cases}\lim_{\xi\uparrow b_3}\dfrac{s(\xi)}{(-1/g'(b_3))^{1/2}\log\dfrac{1}{b_3-\xi}}=1\\[2ex] \lim_{\xi\downarrow d}\dfrac{s(\xi)}{(-1/4g'(b_1))^{1/2}\log\dfrac{1}{\xi-d}}=1\\[2ex] \lim_{\xi\uparrow b_3}\dfrac{d}{d\xi}s(\xi)=+\infty,\ \lim_{\xi\downarrow d}\dfrac{d}{d\xi}s(\xi)=-\infty.\end{cases}$$

PROOF. First we deal with the case where ξ is near b_3, i.e., $d'<(d'+b_3)/2\leqq\xi<b_3$, where d' is a point in (b_2, b_3) which will be determined later.

(4.8) $$s(\xi)=\int_{b_1}^{d'}\left(2\int_{\sigma}^{\xi}g(\rho)d\rho\right)^{-1/2}d\sigma+\int_{d'}^{\xi}\left(2\int_{\sigma}^{\xi}g(\rho)d\rho\right)^{-1/2}d\sigma.$$

It is easily seen that the first term belongs to $C^{\infty}([(d'+b_3)/2, b_3])$. Therefore the second term is essential to the asymptotic behavior of $s(\xi)$ as $\xi\uparrow b_3$. Expand $g(\rho)$ around $\rho=b_3$ as follows:

$$g(\rho)=g'(b_3)(\rho-b_3)+r_1(\rho)(\rho-b_3)^2\equiv g_1(\rho)+g_2(\rho).$$

By a simple calculation, we have

$$\int_{d'}^{\xi}\left(2\int_{\sigma}^{\xi}g_1(\rho)d\rho\right)^{-1/2}d\sigma=\int_{d'}^{\xi}\left(2\int_{\sigma}^{\xi}g'(b_3)(\rho-b_3)d\rho\right)^{-1/2}d\sigma \tag{4.9}$$
$$=\frac{1}{(-g'(b_3))^{1/2}}\log\frac{b_3-d'+((d'-b_3)^2-(\xi-b_3)^2)^{1/2}}{b_3-\xi},$$

$$\left|\frac{\int_{\sigma}^{\xi}g_2(\rho)d\rho}{\int_{\sigma}^{\xi}g_1(\rho)d\rho}\right|=\left|\frac{\int_{\sigma}^{\xi}r_1(\rho)(\rho-b_3)^2d\rho}{\frac{1}{2}g'(b_3)((\sigma-b_3)^2-(\xi-b_3)^2)}\right| \tag{4.10}$$
$$\leqq\frac{2r_*}{-3g'(b_3)}(2b_3-\sigma-\xi)\leqq\frac{4r_*}{-3g'(b_3)}(b_3-d')$$

where $r_*=\max_{b_2\leqq\rho\leqq b_3}|r_1(\rho)|$, $d'<(d'+b_3)/2\leqq\xi<b_3$. By the power series expansion, we have

$$(1+Y)^{-1/2}=\sum_{j=0}^{\infty}c_jY^j \quad \text{for} \quad |Y|<1 \ (=\text{radius of convergence})$$

where $c_j=(-1/2)(-1/2-1)(-1/2-2)\cdots(-1/2-(j-1))/j!$. Using the above expansion and the estimate (4.10), we have the following expansion:

$$\left(2\int_{\sigma}^{\xi}g(\rho)d\rho\right)^{-1/2}=\left(2\int_{\sigma}^{\xi}g_1(\rho)d\rho\right)^{-1/2} \tag{4.11}$$
$$\times\sum_{j=0}^{\infty}c_j\left(\frac{\int_{\sigma}^{\xi}g_2(\rho)d\rho}{\int_{\sigma}^{\xi}g_1(\rho)d\rho}\right)^j \quad \text{for} \quad d'<(d'+b_3)/2\leqq\xi<b_3.$$

For any $\varepsilon>0$, take d' close to b_3 and fix it, so that the following estimate is derived from (4.10):

$$\left|1-\sum_{j=0}^{\infty}c_j\left(\frac{\int_{\sigma}^{\xi}g_2(\rho)d\rho}{\int_{\sigma}^{\xi}g_1(\rho)d\rho}\right)^j\right|\leqq\varepsilon \tag{4.12}$$

for $\xi\in[(d'+b_3)/2, b_3)$ and $\sigma\in[d', \xi]$. Integrating (4.7) with σ from d' to ξ, we have

(4.13)
$$1-\varepsilon \leqq \frac{\int_{d'}^{\xi}\left(2\int_{\sigma}^{\xi} g(\rho)d\rho\right)^{-1/2} d\sigma}{\int_{d'}^{\xi}\left(2\int_{\sigma}^{\xi} g_1(\rho)d\rho\right)^{-1/2} d\sigma} \leqq 1+\varepsilon$$

for $\xi \in [(d'+b_3)/2, b_3)$. Using (4.8), (4.9) and (4.13), we have the following:

(4.14)
$$1-\varepsilon \leqq \varliminf_{\xi \uparrow b_3} \frac{s(\xi)}{(-g'(b_3))^{-1/2} \log \dfrac{1}{b_3-\xi}} \leqq \varlimsup_{\xi \uparrow b_3} \frac{s(\xi)}{(-g'(b_3))^{-1/2} \log \dfrac{1}{b_3-\xi}} \leqq 1+\varepsilon$$

for any $\varepsilon > 0$. Consequently we have

$$\lim_{\xi \uparrow b_3} \frac{s(\xi)}{(-g'(b_3))^{-1/2} \log \dfrac{1}{b_3-\xi}} = 1.$$

Hereafter we take d' near b_3 and fix it so that $B(\xi, \sigma) \geqq 1/2$ for $\xi \in [(d'+b_3)/2, b_3)$ and $\sigma \in [d', \xi]$. We put

$$B(\xi, \sigma) = \left(1 + \frac{\int_{\sigma}^{\xi} g_2(\rho)d\rho}{\int_{\sigma}^{\xi} g_1(\rho)d\rho}\right)^{-1/2},$$

$$F(\xi, \sigma) = \left(2\int_{\sigma}^{\xi} g_1(\rho)d\rho\right)^{1/2}.$$

Then the second term of the right-hand side of (4.8) can be written as

$$\int_{d'}^{\xi} \frac{1}{F(\xi, \sigma)} B(\xi, \sigma)d\sigma = \int_{0}^{\xi-d'} \frac{1}{F(\xi, \xi-\eta)} B(\xi, \xi-\eta)d\eta \equiv I(\xi).$$

In what follows we shall estimate $dI/d\xi$:

(4.15)
$$\begin{aligned}\frac{d}{d\xi} I(\xi) &= \frac{1}{F(\xi, d')} B(\xi, d') + \int_{0}^{\xi-d'} \frac{\partial}{\partial\xi} \frac{1}{F(\xi, \xi-\eta)} \cdot B(\xi, \xi-\eta)d\eta \\ &\quad + \int_{0}^{\xi-d'} \frac{1}{F(\xi, \xi-\eta)} \frac{\partial}{\partial\xi} B(\xi, \xi-\eta)d\eta \\ &\geqq \frac{1}{2} \frac{1}{F(\xi, d')} + \frac{1}{2}\int_{0}^{\xi-d'} \frac{\partial}{\partial\xi} \frac{1}{F(\xi, \xi-\eta)} d\eta\end{aligned}$$

$$+\int_0^{\xi-d'}\frac{1}{F(\xi,\,\xi-\eta)}\frac{\partial}{\partial\xi}B(\xi,\,\xi-\eta)d\eta.$$

Here we have used the fact that $(\partial/\partial\xi)(1/F(\xi,\,\xi-\eta))\geqq 0$. On the other hand, one can easily check that

$$\frac{d}{d\xi}\left(\frac{\int_{\xi-\eta}^{\xi}g_2(\rho)d\rho}{\int_{\xi-\eta}^{\xi}g_1(\rho)d\rho}\right)$$

is bounded in $(d'+b_3)/2\leqq\xi<b_3$ and $0<\eta\leqq\xi-d'$, hence so is $(\partial/\partial\xi)B(\xi,\,\xi-\eta)$. Therefore there exists a constant M such that $|(\partial/\partial\xi)B(\xi,\,\xi-\eta)|\leqq M$ for $\xi\in[(d'+b_3)/2,\,b_3)$ and $0<\eta\leqq\xi-d'$. Consequently we have

$$\begin{aligned}(4.16)\quad \frac{\partial}{\partial\xi}I(\xi)&\geqq\frac{1}{2}\frac{\partial}{\partial\xi}\int_0^{\xi-d'}\frac{1}{F(\xi,\,\xi-\eta)}d\eta-M\int_0^{\xi-d'}\frac{1}{F(\xi,\,\xi-\eta)}d\eta\\ &=\frac{1}{2}\frac{\partial}{\partial\xi}\int_{d'}^{\xi}\frac{1}{F(\xi,\,d')}d\sigma-M\int_{d'}^{\xi}\frac{1}{F(\xi,\,d')}d\sigma\\ &=\frac{1}{2}(-g'(b_3))^{-1/2}\left(\frac{1}{b_3-\xi}+\frac{(b_3-\xi)\cdot((d'-b_3)^2-(b_3-\xi)^2)^{-1/2}}{b_3-d'+((d'-b_3)^2-(b_3-\xi)^2)^{1/2}}\right)\\ &\quad-\frac{M}{2}(-g'(b_3))^{-1/2}\log\left(\frac{b_3-d'+((d'-b_3)^2-(b_3-\xi)^2)^{1/2}}{b_3-\xi}\right).\end{aligned}$$

Combining this and (4.8), we obtain

$$\lim_{\xi\uparrow b_3}\frac{d}{d\xi}s(\xi)=\infty.$$

We can deal with the case $\xi\to d$ in quite the same manner as above except that we use the following decomposition instead of (4.8):

$$\begin{aligned}s(\xi)=&\int_{b_1}^{e_1}\left(2\int_{\sigma}^{b_1}g(\rho)d\rho+2\int_{d}^{\xi}g(\rho)d\rho\right)^{-1/2}d\sigma\\ &+\int_{e_1}^{d}\left(2\int_{\sigma}^{b_1}g(\rho)d\rho+2\int_{d}^{\xi}g(\rho)d\rho\right)^{-1/2}d\sigma+\int_{d}^{\xi}\left(2\int_{\sigma}^{\xi}g(\rho)d\rho\right)^{-1/2}d\sigma.\end{aligned}$$

The details are omitted. Thus we have completed the proof of Lemma 4.1.

By Lemma 4.1, the equation (4.7), which is rewritten as

$$s(\xi)=\vartheta^{1/2},$$

has exactly two solutions $\xi_1<\xi_2$ in the interval $(d,\,b_3)$, if the parameter

$\vartheta>0$ is adequately large. Moreover we have

$$s'(\xi_1)<0, \quad s'(\xi_2)>0. \tag{4.17}$$

Consequently, for any sufficiently large $\vartheta>0$, problem (4.2) for $f(\xi)=\vartheta g(\xi)$ has exactly two nontrivial solutions $V^{(1)}$ and $V^{(2)}$, which correspond to ξ_1 and ξ_2 respectively, one can easily check that

$$b_1<V^{(1)}(z)<V^{(2)}(z)<b_3 \quad \text{in} \quad -1<z<1.$$

By the aid of the almost same method as in K. Maginu [14], we can use (4.17) to investigate the signature of the linearized first eigenvalues $\lambda_{V^{(0)}}$, $\lambda_{V^{(1)}}$ and $\lambda_{V^{(2)}}$ (see (3.5) for the definition) and we conclude that $\lambda_{V^{(0)}}>0$, $\lambda_{V^{(1)}}<0$ and $\lambda_{V^{(2)}}>0$, where $V^{(0)}(z)\equiv b_1$. Thus we complete the proof of Proposition 3.

In what follows we shall first construct a family of solutions of (3.1), $\{v_\zeta^{(2)}\}_{0<\zeta<\zeta_*}$ such that $v_\zeta^{(2)}$ behaves like $V^{(2)}$ in $Q(\zeta)$ and takes values near b_1 in $D_1\cup D_2$ and that $\mu_1(v_\zeta^{(2)})>0$ holds for small $\zeta>0$. Here we denoted by $\mu_1(v_\zeta^{(2)})$ the first eigenvalue of the eigenvalue problem (3.5) for the family $\{v_\zeta^{(2)}\}_{0<\zeta<\zeta_*}$. We set the function $\Psi_*(x_1)=\Phi_{V^{(2)}}(x_1)+\rho_*$, where $\rho_*>0$ is a small constant such that $\lambda_{V^{(2)}}\Phi_{V^{(2)}}(x_1)-\rho_* f'(V^{(2)}(x_1))>0$ for any $x_1\in[-1, 1]$. (Recall that $V^{(2)}(-1)=V^{(2)}(1)=b_1$ and $f'(b_1)<0$.) Now we define a function $W_\zeta(x)$ on $\Omega(\zeta)$ as follows:

$$W_\zeta(x_1, x')\equiv\begin{cases} b_1+\dfrac{\zeta}{2}\Big(\dfrac{dV^{(2)}}{dx_1}(1-2\zeta)-\delta_*(\zeta)\dfrac{d\Psi_*}{dx_1}(1-2\zeta)\Big) \\ \qquad \text{for} \quad x\in D_1\cup D_2\cup(R_1(\zeta)\cap\{x_1\geqq 1-\zeta\})\cup(R_2(\zeta)\cap\{x_1\leqq -1+\zeta\}) \\ b_1-\dfrac{1}{2\zeta}\Big(\dfrac{dV^{(2)}}{dx_1}(1-2\zeta)-\delta_*(\zeta)\dfrac{d\Psi_*}{dx_1}(1-2\zeta)\Big)\cdot(x_1-1+2\zeta)\cdot(x_1-1) \\ \qquad \text{for} \quad x\in R_1(\zeta)\cap\{1-2\zeta\leqq x_1<1-\zeta\} \\ V^{(2)}(x_1)-\delta_*(\zeta)\Psi_*(x_1) \quad \text{for} \quad x\in\Gamma(\zeta) \\ W_\zeta(-x_1, x') \quad \text{for} \quad x\in R_2(\zeta)\cap\{-1+\zeta<x_1\leqq -1+2\zeta\}, \end{cases}$$

where we have put $\delta_*(\zeta)=(V^{(2)}(1-2\zeta)-b_1)/\Psi_*(1-2\zeta)$. It is easily seen that $\delta_*(\zeta)>0$ and that $\lim_{\zeta\to 0}\delta_*(\zeta)=0$.

LEMMA 4.2. *For each small $\zeta>0$, W_ζ belongs to $C^1(\overline{\Omega(\zeta)})$ and satisfies*

$$\Delta W_\zeta+f(W_\zeta)>0$$

in $\Omega(\zeta)-\{x_1=1-2\zeta$ or $1-\zeta$ or $-1+2\zeta$ or $-1+\zeta\}$,

$$\frac{\partial W_\zeta}{\partial \nu}=0 \quad on \quad \partial\Omega(\zeta).$$

PROOF. One can check $W_\zeta \in C^1(\overline{\Omega(\zeta)})$ by a simple calculation. In $D_1 \cup D_2 \cup (R_1(\zeta) \cap \{x_1>1-\zeta\}) \cup (R_2(\zeta) \cap \{x_1<-1+\zeta\})$, we have $\Delta W_\zeta=0$ and $W_\zeta(x) \leqq b_1$ for small $\zeta>0$ by virture of $(dV^{(2)}/dx_1)(1)<0$ and $\lim_{\zeta\to 0}\delta_*(\zeta)=0$. Then by (IV-1), we obtain the desired inequality. In $R_1(\zeta) \cup \{1-2\zeta< x_1<1-\zeta\}$, $\Delta W_\zeta=(-1/2\zeta)((dV^{(2)}/dx_1)(1-2\zeta)-\delta^*(\zeta)(d\Psi_*/dx_1)(1-2\zeta))>0$ and $W_\zeta(x) \leqq b_1$ for small $\zeta>0$. Therefore we can obtain the desired inequality in the same way as above. The same is true for the region $R_2(\zeta) \cup \{-1+\zeta<x_1<-1+2\zeta\}$. In $\Gamma(\zeta)$, we calculate as follows:

$$\Delta W_\zeta + f(W_\zeta) = \frac{\partial^2}{\partial x_1^2}(V^{(2)}(x_1)-\delta_*(\zeta)\Psi_*(x_1))$$
$$+f(V^{(2)})-\delta_*(\zeta)\Psi_* f'(V^{(2)})+\delta_*(\zeta^2)\Psi_*{}^2\Xi_\zeta$$
$$\left(\text{where } \Xi_\zeta(x_1)=\int_0^1(1-\tau)f''(V^{(2)}(x_1)-\tau\delta_*(\zeta)\Psi_*(x_1))d\tau\right)$$
$$=\frac{d^2V^{(2)}}{dx_1^2}+f(V^{(2)})-\delta_*(\zeta)\left(\frac{d^2\Psi_*}{dx_1^2}+f'(V^{(2)})\Psi_*\right)+\delta_*(\zeta)^2\Psi_*{}^2\Xi_\zeta$$
$$=\delta_*(\zeta)\{(\lambda_{V^{(2)}}\Phi_{V^{(2)}}(x_1)-\rho_* f'(V^{(2)}(x_1)))+\delta_*(\zeta)\Psi_*{}^2\Xi_\zeta(x_1)\}>0$$

holds in $\Gamma(\zeta)$ for sufficiently small $\zeta>0$ since $\lim_{\zeta\to 0}\delta_*(\zeta)=0$. Thus we have completed the proof of Lemma 4.2.

By Lemma 4.2, W_ζ is a "weak lower solution" in the sense of D. H. Sattinger [20] for small $\zeta>0$. It follows that if $u_\zeta(t, x)$ is a solution of (2.1) with initial data $u_0=W_\zeta$, then $u_\zeta(t, x)$ is monotone increasing in t. Moreover, by the comparison theorem, $u_\zeta(t, x)<b_3$ for all $t \geqq 0$. Arguing as in [20] and also in the proof of Theorem 1, we have

$$\text{(4.18)} \qquad \begin{cases} \lim_{t\to\infty} u_\zeta(t, x)=v_\zeta^{(2)}(x) & \text{in } \Omega(\zeta), \\ \lim_{\zeta\to 0} \|v_\zeta^{(2)}-b_1\|_{L^2(D_1\cup D_2)}=0, \end{cases}$$

where $v_\zeta^{(2)}$ is some solution of (3.1),

$$b_1<v_\zeta^{(2)}(x) \leqq b_3 \quad \text{in} \quad \Omega(\zeta).$$

Here we put $v_\zeta^{(0)} \equiv b_1$. Then $v_\zeta^{(0)}$, $v_\zeta^{(2)}$ are both solutions of (3.1) and satisfies $v_\zeta^{(0)}<v_\zeta^{(2)}$ in $\Omega(\zeta)$. Moreover, by virtue of (4.18), $v_\zeta^{(2)}$ is stable

from below, while $v_\zeta^{(0)}$ is clearly a stable solution. Applying Theorem 4.4 of Matano [15], we see that there exists another solution $v_\zeta^{(1)}$ of (3.1) that is unstable and satisfies

$$b_1 \equiv v_\zeta^{(0)} < v_\zeta^{(1)} < v_\zeta^{(2)} \leqq b_3. \tag{4.19}$$

Considering the stability properties of these solutions, we find that

$$\mu_1(v_\zeta^{(0)}) \geqq 0, \quad \mu_1(v_\zeta^{(1)}) \leqq 0, \quad \mu_1(v_\zeta^{(2)}) \geqq 0 \tag{4.20}$$

for all small $\zeta > 0$. Let $\{\zeta_m\}_{m=1}^\infty$ be an arbitrary sequence of positive numbers converging to 0. By Theorem 3, we can choose a subsequence $\{\kappa_m\}_{m=1}^\infty \subset \{\zeta_m\}_{m=1}^\infty$ and solutions $\tilde{V}^{(0)}$, $\tilde{V}^{(1)}$ and $\tilde{V}^{(2)}$ af (4.2) such that

$$\lim_{m\to\infty} \sup_{x\in Q(\kappa_m)} |v_\zeta^{(i)}(x) - \tilde{V}^{(i)}(x_1)| = 0 \quad (i=0, 1, 2).$$

It is clear that $\tilde{V}^{(0)} \leqq \tilde{V}^{(1)} \leqq \tilde{V}^{(2)}$ and that $\{\tilde{V}^{(0)}, \tilde{V}^{(1)}, \tilde{V}^{(2)}\} \subset \{V^{(0)}, V^{(1)}, V^{(2)}\}$. Since we have

$$\overline{\lim_{m\to\infty}}\, \mu_1(v_{\kappa_m}^{(1)}) \leqq 0$$

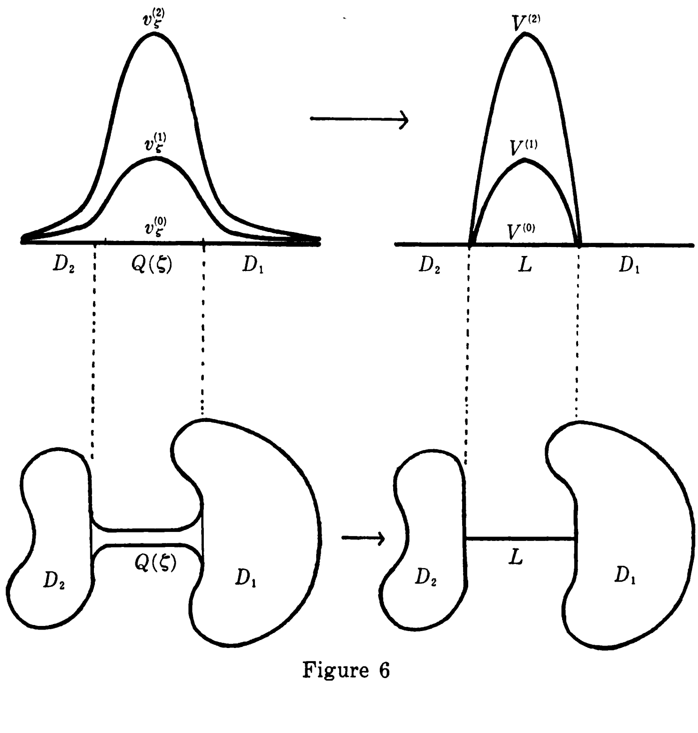

Figure 6

by virtue of (4.20), the latter half of Theorem 3 implies that $\tilde{V}^{(1)}=V^{(1)}$. (Recall that $\lambda_{V^{(0)}}>0$, $\lambda_{V^{(1)}}<0$ and $\lambda_{V^{(2)}}>0$.) A similar argument shows that

$$\tilde{V}^{(0)}=V^{(0)}, \quad \tilde{V}^{(2)}=V^{(2)}.$$

Since $\{\zeta_m\}_{m=1}^{\infty}$ was chosen arbitrary, we see that $v_\zeta^{(i)}|_{Q(\zeta)}$ converges to $V^{(i)}$ as $\zeta\to 0$ for $i=0, 1, 2$. Thus we have obtained the following theorem:

THEOREM 4. *There exists a constant* $\vartheta_0>0$ *such that for any* $\vartheta\geqq\vartheta_0$ *and for any sufficiently small* $\zeta>0$ *the problem* (3.1) *has precisely three solutions* $v_\zeta^{(0)}<v_\zeta^{(1)}<v_\zeta^{(2)}$. *Moreover three solutions satisfy*

$$\lim_{\zeta\to 0}\sup_{x\in D_1\cup D_2}|v_\zeta^{(i)}(x)-b_1|=0 \quad (i=0, 1, 2)$$

$$\lim_{\zeta\to 0}\sup_{x\in Q(\zeta)}|v_\zeta^{(i)}(x_1, x')-V^{(i)}(x_1)|=0 \quad (i=0, 1, 2)$$

$$\varliminf_{\zeta\to 0}\mu_1(v_\zeta^{(0)})>0, \quad \varlimsup_{\zeta\to 0}\mu_1(v_\zeta^{(1)})<0, \quad \varliminf_{\zeta\to 0}\mu_1(v_\zeta^{(2)})>0$$

where $\mu_1(v_\zeta^{(i)})$ *denotes the first eigenvalue of the eigenvalue problem* (3.5) *for the family* $\{v_\zeta^{(i)}\}_{0<\zeta<\zeta_*}$ $(i=0, 1, 2)$.

References

[1] Agmon, S., Douglis, A. and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I., Comm. Pure Appl. Math. **XII** (1959), 623-727.

[2] Amann, H., On the existence of positive solutions of nonlinear elliptic boundary value problems, Indiana Univ. Math. J. **21** (1971), 125-146.

[3] Casten, R. G. and C. J. Holland, Instability results for reaction diffusion equation with Neumann boundary conditions, J. Differential Equations **27** (1978), 266-273.

[4] Chafee, N., Asymptotic behavior for solutions of a one-dimensional parabolic equation with homogeneous Neumann boundary conditions, J. Differential Equations **18** (1975), 111-134.

[5] Chafee, N. and E. F, Infante, A bifurcation problem for a nonlinear partial differential equation of parabolic type, Applicable Anal. **4** (1974), 17-37.

[6] Chavel, I. and E. A. Feldman, Spectra of domains in compact manifolds, J. Funct. Anal. **30** (1978), 198-222.

[7] Coddington, E. A. and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

[8] Courant, R. and D. Hilbert, Method of Mathematical Physics, Vol. I, Wiley-Interscience, New York, 1953.

[9] Gilbarg, D. and N. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, New York, 1977.

[10] Hale, J. K. and J. Vegas, A nonlinear parabolic equation with varying domain, Arch.

Rational Mech. Anal. **86** (1984), 99-123.
[11] Jimbo, S., On a semilinear diffusion equation on a Riemannian manifold and its stable equilibrium solutions, Proc. Japan Acad. **60** (1984), 349-352.
[12] Keyfitz, B. L. and H. J. Kuiper, Bifurcation resulting from changes in domain in a reaction-diffusion equation, J. Differential Equations **47** (1983), 378-405.
[13] Kishimoto, K. and H. F. Weinberger, The spatial homogeneity of stable equilibria of some reaction-diffusion systems on convex domains, J. Differential Equations **58** (1985), 15-21.
[14] Maginu, K., Stability of stationary solutions of a semilinear parabolic partial differential equation, J. Math. Anal. Appl. **63** (1978), 224-243.
[15] Matano, H., Asymptotic behavior and stability of solutions of semilinear diffusion equations, Publ. Res. Inst. Math. Sci. **15** (1979), 401-454.
[16] Matano, H., Existence of nontrivial unstable sets for equilibriums of strongly order-preserving systems, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **30** (1984), 645-673.
[17] Matano, H. and M. Mimura, Pattern formation in competition-diffusion systems in nonconvex domains, Publ. Res. Inst. Math. Sci. **19** (1983), 1049-1079.
[18] Ozawa, S., Singular variation of domains and eigenvalues of the Laplacian, Duke Math. J. **48** (1981), 767-778.
[19] Protter, M. H. and H. F. Weinberger, Maximum Principles in Differential Equationns, Prentice-Hall, Englewood Cliffs, 1967.
[20] Sattinger, D. H., Monotone methods in nonlinear elliptic and parabolic boundary value problems, Indiana Univ. Math. J. **21** (1972), 979-1000.
[21] Taylor, M. E., Pseudodifferential Operators, Princeton Univ. Press, Princeton, 1981.
[22] Vegas, J. M., Bifurcation caused by perturbing the domain in an elliptic equation, J. Differential Equations **48** (1983), 189-226.

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