J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
35 (1988), 149-162.

L^p-boundedness of pseudo-differential operators satisfying Besov estimates II

By Mitsuru SUGIMOTO

(Communicated by H. Komatsu)

0. Introduction

The author [17] has shown that the theory of Besov spaces plays an important role for studying the L^2-boundedness of pseudo-differential operators with non-regular symbols. In the present paper, we shall consider the general L^p-boundedness ($p\neq 2$).

There have already been many results on the $L^p(\boldsymbol{R}^n)$-boundedness of pseudo-differential operators with non-regular symbols; for example, Nagase [12], [13], [14], [15], Mossaheb-Okada [10], Coifman-Meyer [3], Muramatu-Nagase [11], Wang-Li [18], Yamazaki [19], Miyachi-Yabuta [9], and Miyachi [8]. They treat the classes of non-regular symbols which generalize Hörmander's class $S_{\rho,\delta}^{m_{\rho,p}}$ ($0\leqq\delta\leqq\rho\leqq 1$). Here $m_{\rho,p}=-n(1-\rho)\times|1/p-1/2|$, and $m_{\rho,p}$ is the critical order for the $L^p(\boldsymbol{R}^n)$-boundedness; see Hörmander [5]. In the case $\rho\neq 0$ or $\delta\neq 0$, many authors discuss the $L^p(\boldsymbol{R}^n)$-boundedness under very weak regularity conditions for symbols, while there are few in the case $\rho=\delta=0$.

In this paper, we shall discuss the case $\rho=\delta=0$. In this direction, the following theorem, due to Coifman-Meyer [3], is fundamental.

THEOREM A. *Let $1\leqq p<\infty$ and let λ, λ' be integers $\geqq 2n$. Then there exists a constant C such that the estimate*

$$(0.1)\qquad \|\sigma(X, D)f\|_{L^p(\boldsymbol{R}^n)}\leqq C\|\sigma(x, \xi)\|_{C_{(0,|n/p-n/2|)}^{(\lambda,\lambda')}(\boldsymbol{R}_{(n,n)}^{(n,n)})}\|f\|_{H^p(\boldsymbol{R}^n)}$$

holds for all $\sigma\in C_{(0,|n/p-n/2|)}^{(\lambda,\lambda')}$ and all $f\in\mathscr{S}(\boldsymbol{R}^n)\cap H^p(\boldsymbol{R}^n)$.

Here the space $C_{(0,|n/p-n/2|)}^{(\lambda,\lambda')}$ is one of generalized classes of $S_{0,0}^{-n|1/p-1/2|}$, and the superscript order λ (resp. λ') denotes differentiability conditions for symbols with respect to the variable x (resp. ξ); see Sugimoto [17], Definition 1.1.5. On the other hand, $H^p(\boldsymbol{R}^n)$ denotes the Hardy space introduced by Fefferman-Stein [4], and the space $H^p(\boldsymbol{R}^n)$ coincides with the space $L^p(\boldsymbol{R}^n)$ as a Banach space if $1<p<\infty$, while not if $p=1$. Furthermore, we cannot replace $H^1(\boldsymbol{R}^n)$ by $L^1(\boldsymbol{R}^n)$ in Theorem A in the case $p=1$; see Miyachi [8], Section 5.

Recently Miyachi [8] has generalized the symbol class $C^{(\lambda,\lambda')}_{(0,|n/p-n/2|)}$ in terms of Lipschitz spaces and proved the following theorem in ralation to Theorem A.

THEOREM B. *Let* $0<p\leqq 2$ *(resp.* $2<p<\infty$*) and let* λ, λ' *be real numbers such that* $\lambda>n/2$, $\lambda'>n/p$ *(resp.* $\lambda>n/p$, $\lambda'>n/2$*). Then there exists a constant C such that the estimate*

$$(0.2)\qquad \|\sigma(X, D)f\|_{L^p(\boldsymbol{R}^n)}\leqq C\|\sigma(x, \xi)\|_{B^{(\lambda,\lambda')}_{\infty,\infty,(0,|n/p-n/2|)}(\boldsymbol{R}^{(n,n)}_{(n,n)})}\|f\|_{H^p(\boldsymbol{R}^n)}$$

holds for all $\sigma\in\mathscr{S}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ *and all* $f\in\mathscr{S}(\boldsymbol{R}^n)\cap H^p(\boldsymbol{R}^n)$.

Here we have used the notations given in Sugimoto [17] instead of Miyachi [8]. The space $B^{(\lambda,\lambda')}_{\infty,\infty,(0,|n/p-n/2|)}$ is a Lipschitz space version of the space $C^{(\lambda,\lambda')}_{(0,|n/p-n/2|)}$, and estimate (0.2) is sharper than estimate (0.1); see Corollary 1.3.1 in Sugimoto [17]. But Theorem B does not necessarily include Theorem A, since the assumption $\sigma\in\mathscr{S}$ is rather restrictive.

The purpose of the present paper is not only to remove this restriction, but also to replace the inequalities $\lambda>n/2$, $\lambda'>n/p$ in Theorem B by the equalities $\lambda=n/2$, $\lambda'=n/p$, in the case $0<p\leqq 2$. That is, our main result is the following (cf. Theorem 1.1).

THEOREM C. *Let* $0<p\leqq 2$. *Then there exists a constant C such that the estimate*

$$\|\sigma(X, D)f\|_{L^p(\boldsymbol{R}^n)}\leqq C\|\sigma(x, \xi)\|_{B^{(n/2,n/p)}_{\infty,\min(p,1),(0,|n/p-n/2|)}(\boldsymbol{R}^{(n,n)}_{(n,n)})}\|f\|_{H^p(\boldsymbol{R}^n)}$$

holds for all $\sigma\in B^{(n/2,n/p)}_{\infty,\min(p,1),(0,|n/p-n/2|)}$ *and all* $f\in\mathscr{S}(\boldsymbol{R}^n)\cap H^p(\boldsymbol{R}^n)$.

Theorem C is an improvement of Theorem B in the case $0<p\leqq 2$, since

$$\bigcup_{\lambda>n/2,\lambda'>n/p} B^{(\lambda,\lambda')}_{\infty,\infty,(0,|n/p-n/2|)}\subsetneqq B^{(n/2,n/p)}_{\infty,\min(p,1),(0,|n/p-n/2|)}.$$

See Remarks 1.3.

The more precise statement of the main result is given in Section 1. Section 2 through Section 5 are devoted to the proof of it. In Section 6, we give some other estimates of various types. Our proof is essentially a repetition of that of Miyachi [8], but we need some other new ideas as used in Sugimoto [17].

The author is grateful to Professor Tosinobu Muramatu for valuable advice and discussions.

1. Main theorem

In this section, we shall state the main theorem of the present paper. For this purpose, we use the weighted Besov spaces on product spaces of dimension "$2n$", which are introduced by Sugimoto [17]. Throughout this section, we take $n=n_1+\cdots+n_N=\tilde{n}_1+\cdots+\tilde{n}_{\tilde{N}}$ ($n_r, \tilde{n}_s \in \boldsymbol{N}$; $r=1, \cdots, N$, $s=1, \cdots, \tilde{N}$) and use the following notations:

Let $0<q\leqq\infty$, $\boldsymbol{\lambda}=(\lambda_1, \cdots, \lambda_N)\in \boldsymbol{R}^N$, $\lambda'\in\boldsymbol{R}$, $\boldsymbol{\rho}=(\rho_1, \cdots, \rho_{\tilde{N}})\in\boldsymbol{R}^{\tilde{N}}$, $\boldsymbol{n}=(n_1, \cdots, n_N)$, and $\tilde{\boldsymbol{n}}=(\tilde{n}_1, \cdots, \tilde{n}_{\tilde{N}})$. Then we set

$$B^{(\lambda,\lambda')}_{\infty,q,(0,\rho)}=B^{(\lambda,\lambda')}_{\infty,q,(0,\rho)}(\boldsymbol{R}^{(n,n)}_{(n,\tilde{n})})$$
$$=\{\sigma\in\mathscr{S}'(\boldsymbol{R}^{2n});\ \|\sigma\|^{(\lambda,\lambda')}_{q,\rho}<+\infty\},$$

where

$$\|\sigma\|^{(\lambda,\lambda')}_{q,\rho}=\|\sigma\|_{B^{(\lambda,\lambda')}_{\infty,q,(0,\rho)}}$$
$$=\|2^{j\cdot\lambda+k\lambda'}\|\mathscr{F}^{-1}\Phi_{j,k}\mathscr{F}\sigma\|_{L^\infty_{(0,\rho)}(R_{(n,\tilde{n})})}\|_{l^q}$$
$$=\{\sum_{\substack{j\geqq 0\\ k\geqq 0}}(2^{j\cdot\lambda+k\lambda'}\|(\mathscr{F}^{-1}\Phi_{j,k}\mathscr{F}\sigma)(x,\xi)\cdot\omega_\rho(\xi)\|_{L^\infty(R^n_x\times R^n_\xi)})^q\}^{1/q}$$

(with a slight modification in the case of $q=\infty$). Here $\boldsymbol{j}=(j_1, \cdots, j_N)\in(\boldsymbol{N}\cup\{0\})^N$, $k\in\boldsymbol{N}\cup\{0\}$, $\omega_\rho(\xi)=\omega^{\tilde{n}}_\rho(\xi)$ and $\Phi_{j,k}(y,\eta)=\Phi_j(y)\Phi_k(\eta)=\Phi^n_j(y)\Phi^n_k(\eta)$; see also Definitions 1.1.2 and 1.1.3 in Sugimoto [17].

REMARK 1.1. If $N=\tilde{N}=1$, the space $B^{(\lambda,\lambda')}_{\infty,\infty,(0,\rho)}$ coincides with the symbol class $S^{-\rho}_{0,0}(\lambda,\lambda')$ which was introduced by Miyachi [8].

In this paper, we shall consider pseudo-differential operators of the following type.

DEFINITION 1.1. Let $\sigma(x,\xi)$ be a function on $\boldsymbol{R}^n_x\times\boldsymbol{R}^n_\xi$ and let f be a tempered distribution on $\boldsymbol{R}^n$ whose Fourier transform is a function. If $\sigma(x,\xi)\hat{f}(\xi)\in L^1(\boldsymbol{R}^n_\xi)$, then we set

$$\sigma(X,D)f(x)=(2\pi)^{-n}\int e^{ix\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)d\xi. \tag{1.1}$$

We shall say that the operator $\sigma(X,D)$ is well-defined for f if and only if $\sigma(x, \xi)\hat{f}(\xi) \in L^1(\boldsymbol{R}^n_\xi)$ for almost all x.

Now, we shall state the main theorem of this paper. Theorem C in Section 0 is a special case of it.

THEOREM 1.1. *Let $0<p\leqq 2$. Then there exists a constant C such that the estimate*

$$\|\sigma(X, D)f\|_{L^p(\boldsymbol{R}^n)} \leqq C\|\sigma(x, \xi)\|_{\min(p,1),\tilde{n}/p-\tilde{n}/2}^{(n/2,n/p)}\|f\|_{H^p(\boldsymbol{R}^n)} \tag{1.2}$$

holds for all $\sigma \in B^{(n/2,n/p)}_{\infty,\min(p,1),(0,\tilde{n}/p-\tilde{n}/2)}$ and all $f \in H^p(\boldsymbol{R}^n)\cap FL^1(\boldsymbol{R}^n)$.

Here H^p denotes the Hardy space introduced by Fefferman-Stein [4], and $FL^1=\{f\in\mathscr{S}'; \hat{f}\in L^1(\boldsymbol{R}^n)\}$. We remark that $B^{(n/2,n/p)}_{\infty,\min(p,1),(0,\tilde{n}/p-\tilde{n}/2)}\subset L^\infty_{x,\xi}$ (see Theorems 1.3.2 and 1.3.5 in Sugimoto [17]) and the operator $\sigma(X, D)$ is well-defined for $f\in FL^1$. Since $H^p\cap FL^1$ is dense in H^p (see Calderón-Torchinsky [2], Theorem 1.8 (v)), the operator $\sigma(X, D)$: $H^p\cap FL^1\to L^p$ can be extended to a bounded operator $\sigma(X, D)_e$: $H^p\to L^p$ such that $\sigma(X, D)_{e|H^p\cap FL^1}=\sigma(X, D)$.

REMARK 1.2. In Theorem 1.1, we can have the sharpest estimate if we take $N=n$, and the other cases are corollaries of this special case; see Sugimoto [17] Theorem 1.3.9.

REMARK 1.3. Theorem 1.1 is an improvement of Theorem B in the case $0<p\leqq 2$. In fact, it holds that

$$\bigcup_{\lambda>n/2,\lambda'>n/p} B^{(\lambda,\lambda')}_{\infty,\infty,(0,|n/p-n/2|)}\subset B^{(n/2,n/p)}_{\infty,\min(p,1)/2,(0,|n/p-n/2|)} \subsetneqq B^{(n/2,n/p)}_{\infty,\min(p,1),(0,|n/p-n/2|)}.$$

See Theorem 1.3.2 in Sugimoto [17].

REMARK 1.4. The sharpness of the order $(n/2, n/p)$ is essentially discussed in Miyachi [8], Section 5. The sharpness of the order $\tilde{n}/p-\tilde{n}/2$ is also discussed in Miyachi [7], Section 5.

By virtue of Corollary 1.3.1 in Sugimoto [17], we can easily have the following Corollary, which states that symbols belonging to $S^{-n|1/p-1/2|}_{0,0}$ generate $H^p(\boldsymbol{R}^n)-L^p(\boldsymbol{R}^n)$-bounded pseudo-differential operators $(0<p\leqq 2)$.

COROLLARY 1.1. *Let $0<p\leqq 2$ and let $\sigma(x, \xi)$ be a function such that $\langle\xi\rangle^{n/p-n/2}\partial^\alpha_x\partial^\beta_\xi\sigma(x, \xi)$ is bounded-continuous for*

$$|\alpha|\leqq[n/2]+1,\quad |\beta|\leqq[n/p]+1, \tag{i}$$

or

(ii) $$\alpha \in \{0, 1\}^n, \quad |\beta| \leqq [n/p]+1.$$

Then the pseudo-differential operator $\sigma(X, D)$ *can be extended to a bounded operator from* $H^p(\boldsymbol{R}^n)$ *into* $L^p(\boldsymbol{R}^n)$.

2. Rapidly decreasing symbols

We shall prove Theorem 1.1 in the following sections. In the rest of this paper, the letter C (sometimes with some subscripts) denotes a constant which may be different in each occasion, and C_0^∞ denotes the set of all smooth functions with compact support. In this section, we shall reduce Theorem 1.1 to another form which can be treated more easily.

DEFINITION 2.1. We say that a function $\sigma(x, \xi) \in L^\infty_{x,\xi}$ is rapidly decreasing with respect to ξ if and only if $\sigma(x, \xi)\langle\xi\rangle^M \in L^\infty_{x,\xi}$ for all real number M. Here $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$.

Rapidly decreasing symbols have the following fundamental property.

LEMMA 2.1. *Let* $0<p\leqq 2$ *and let* $\sigma(x, \xi)$ *be a function on* $\boldsymbol{R}^n_x \times \boldsymbol{R}^n_\xi$ *which is rapidly decreasing with respect to* ξ. *Then the operator* $\sigma(X, D)$ *is well-defined for* $f \in H^p(\boldsymbol{R}^n)$ *and has the following estimate for* $M>n/p$:

$$\|\sigma(X, D)f\|_{L^\infty} \leqq C\|\sigma(x, \xi)\langle\xi\rangle^M\|_{L^\infty_{x,\xi}}\|f\|_{H^p} \tag{2.1}$$

where C *is a constant depending only on* n, p, *and* M.

PROOF OF LEMMA 2.1. In the case $0<p\leqq 1$, we have

$$\begin{aligned}\|\sigma(X, D)f\|_{L^\infty} &\leqq C\int \|\sigma(x, \xi)\hat{f}(\xi)\|_{L^\infty_x} d\xi \\ &\leqq C\int \langle\xi\rangle^{-M}\cdot|\xi|^{n(1/p-1)}d\xi\cdot\|\sigma(x, \xi)\langle\xi\rangle^M\|_{L^\infty_{x,\xi}}\|f\|_{H^p} \\ &= C\|\sigma(x, \xi)\langle\xi\rangle^M\|_{L^\infty_{x,\xi}}\|f\|_{H^p}.\end{aligned}$$

Here we have used the fact that the Fourier transform $\hat{f}$ of $f \in H^p$ is a continuous function and the estimate $|\hat{f}(\xi)| \leqq C\|f\|_{H^p(R^n)}|\xi|^{n(1/p-1)}$ holds for $0<p\leqq 1$; see Fefferman-Stein [4], p. 174 or Calderón-Torchinsky [1], Theorem 4.4. In the case $1<p\leqq 2$, we have by Hölder's inequality and

Hausdorff-Young's inequality,

$$\begin{aligned}\|\sigma(X, D)f\|_{L^\infty} &\leqq C\| \, \|\sigma(x, \xi)\|_{L^p_\xi}\|_{L^\infty_x}\|\hat{f}\|_{L^q} \quad (1=1/p+1/q)\\ &\leqq C\|\langle\xi\rangle^{-M}\|_{L^p}\|\sigma(x, \xi)\langle\xi\rangle^M\|_{L^\infty_{x,\xi}}\|f\|_{L^p}\\ &\leqq C\|\sigma(x, \xi)\langle\xi\rangle^M\|_{L^\infty_{x,\xi}}\|f\|_{L^p}.\end{aligned}$$

Now, we shall prove the following:

CLAIM. In the proof of Theorem 1.1, we may assume, without loss of generality, that $\sigma(x, \xi)$ is rapidly decreasing with respect to ξ.

In order to prove this claim, we need the following lemma.

LEMMA 2.2. *Let $0<\boldsymbol{\lambda}\in \boldsymbol{R}^N$, $0<\lambda'\in \boldsymbol{R}$, $0<q\leqq\infty$, $\boldsymbol{\rho}_0, \boldsymbol{\rho}_1\in \boldsymbol{R}^{\tilde{N}}$ and $\kappa=[\lambda']+1$. Then there exists a constant C such that the estimate*

$$\|\sigma(x, \xi)\chi(\xi)\|^{(\lambda,\lambda')}_{q,\rho_0+\rho_1}\leqq C\|\chi(\xi)\|_{C^\kappa_{\rho_0}}\|\sigma(x, \xi)\|^{(\lambda,\lambda')}_{q,\rho_1}$$

holds for all $\chi\in C^\kappa_{\rho_0}=C^\kappa_{\rho_0}(\boldsymbol{R}^n_{\tilde{n}})$ and all $\sigma\in B^{(\lambda,\lambda')}_{\infty,q,(0,\rho_1)}=B^{(\lambda,\lambda')}_{\infty,q,(0,\rho_1)}(\boldsymbol{R}^{(n,n)}_{(n,\tilde{n})})$.

This lemma can be easily obtained from Corollary 1.3.2 in Sugimoto [17].

RROOF OF CLAIM. We assume that estimate (1.2) is proved for all $\sigma\in B^{(n/2,n/p)}_{\infty,\min(p,1),(0,\tilde{n}/p-\tilde{n}/2)}$ which is rapidly decreasing with respect to ξ, and all $f\in H^p\cap FL^1$. Let $k(x)$ be a function such that $\hat{k}\in C^\infty_0(\boldsymbol{R}^n)$ and $\hat{k}(\xi)=1$ for ξ near 0. We set $k_s(x)=s^{-n}k(x/s)$ for $0<s<1$ and $f_s=f*k_s$ for $f\in H^p\cap FL^1$. If we take $\chi\in C^\infty_0(\boldsymbol{R}^n)$ such that $\chi(\xi)=1$ for $\xi\in \operatorname{supp}\hat{k}$, then we have $\hat{f}_s(\xi)=\chi_s(\xi)\cdot\hat{f}_s(\xi)$, where $\chi_s(\xi)=\chi(s\xi)$. Since $\sigma(x, \xi)\chi_s(\xi)$ is rapidly decreasing with respect to ξ, we have, by Lemma 2.2,

$$\begin{aligned}(2.2)\qquad \|\sigma(X, D)f_s\|_{L^p} &= \|\sigma(X, D)\chi_s(D)f_s\|_{L^p}\\ &\leqq C\|\sigma(x, \xi)\chi_s(\xi)\|^{(n/2,n/p)}_{\min(p,1),\tilde{n}/p-\tilde{n}/2}\|f_s\|_{H^p}\\ &\leqq C_\chi\|\sigma\|^{(n/2,n/p)}_{\min(p,1),\tilde{n}/p-\tilde{n}/2}\|f_s\|_{H^p}\end{aligned}$$

for all $\sigma\in B^{(n/2,n/p)}_{\infty,\min(p,1),(0,\tilde{n}/p-\tilde{n}/2)}$ and all $f\in H^p\cap FL^1$. Here C_χ is a constant independent of σ, f and $0<s<1$. On the other hand f_s converges to f in $H^p(\boldsymbol{R}^n)$ $(0<p<\infty)$ as $s\downarrow 0$; see the proof of Theorem 1.8 (v) in Calderón-Torchinsky [2]. By estimate (2.2), we have that $\sigma(X, D)f_s$ converges to some function g in L^p as $s\downarrow 0$. Furthermore, $\sigma(X, D)f_s$ converges to $\sigma(X, D)f$ in L^∞. In fact,

$$\|\sigma(X, D)(f-f_s)\|_{L^\infty} \leqq C\|\sigma\|_{L^\infty_{x,\xi}}\|(1-\hat{k}(s\xi))\hat{f}(\xi)\|_{L^1_\xi} \longrightarrow 0$$

as $s \downarrow 0$. Hence $g=\sigma(X, D)f$ and we obtain Theorem 1.1 from estimate (2.2).

3. The case $p=2$

Theorem 2.1.2 in Sugimoto [17] and Lemma 2.1 give the following:

LEMMA 3.1. *There exists a constant C such that the estimate*

$$\|\sigma(X, D)f\|_{L^2(\mathbf{R}^n)} \leqq C\|\sigma(x, \xi)\|_{1,0}^{(n/2, n/2)}\|f\|_{L^2(\mathbf{R}^n)} \tag{3.1}$$

holds for all $\sigma \in B^{(n/2, n/2)}_{\infty,1,(0,0)}$ which is rapidly decreasing with respect to ξ, and all $f \in L^2(\mathbf{R}^n)$.

PROOF OF LEMMA 3.1. Theorem 2.1.2 in [17] gives the estimate (3.1) for $f \in \mathcal{S}$. Since $\mathcal{S}$ is dense in L^2, $f \in L^2$ can be approximated by a sequence $\{f_\nu\}_{\nu=1}^\infty \subset \mathcal{S}$. Then we have that $\sigma X, (D)f_\nu$ converges to some function g in L^2. On the other hand, by estimate (2.1) in Lemma 2.1, we have that $\sigma(X, D)f_\nu$ converges to $\sigma(X, D)f$ in L^∞. Hence $g=\sigma(X, D)f$ and we have lemma.

By virtue of Claim in Section 2, Lemma 3.1 implies Theorem 1.1 with $p=2$. Lemma 3.1 will be used also in the next section.

4. The case $0<p<1$

In this section, we shall prove Theorem 1.1 in the case $0<p<1$. Following Miyachi [7], [8], we shall prove this by the method of p-atom decomposition. (See Latter [6].)

For $0<p\leqq 1$, we call a function g on $\mathbf{R}^n$ a p-atom if there is a ball $B=B_g \subset \mathbf{R}^n$ such that

$$\operatorname{supp} g \subset B, \quad \|g\|_{L^\infty} \leqq |B|^{-1/p}, \quad \text{and}$$

$$\int g(x)x^\alpha dx=0 \quad \text{for} \quad |\alpha| \leqq [n/p-n]$$

($|B|$=Lebesgue measure of B). Every $f \in H^p(\mathbf{R}^n)$ can be represented as a linear combination of p-atoms;

$$f=\sum_{l=0}^\infty \lambda_l g_l, \; \lambda_l \in \mathbf{C}, \; g_l; \; p\text{-atom}, \tag{4.1}$$

$$C\Big(\sum_{l=0}^\infty |\lambda_l|^p\Big)^{1/p} \leqq \|f\|_{H^p} \leqq C'\Big(\sum_{l=0}^\infty |\lambda_l|^p\Big)^{1/p}.$$

Here the sum $\sum_{l=0}^{\infty}\lambda_l g_l$ is convergent in H^p, and C, C' are constants depending only on n and p.

REMARK 4.1. Condition (4.1) implies

$$\frac{C}{2}\Big(\sum_{l=0}^{M}|\lambda_l|^p\Big)^{1/p}\leqq\Big\|\sum_{l=0}^{M}\lambda_l g_l\Big\|_{H^p}$$

for all $M\geqq M_0$, if we take large enough M_0 depending on f.

Now, we shall prove Theorem 1.1 in the case $0<p<1$. We assume that $\sigma(x,\xi)$ is rapidly decreasing with respect to ξ (see Claim in Section 2) and decompose it into the sum of $\sigma_{j,k}(x,\xi)=\mathcal{F}^{-1}\Phi_{j,k}\mathcal{F}\sigma(x,\xi)$. We remark that $\sigma_{j,k}(x,\xi)$ is also rapidly decreasing with respect to ξ. Since $\sigma(X,D)f(x)=\sum_{j\geqq 0,k\geqq 0}\sigma_{j,k}(X,D)f(x)$ $(\sigma\in B^{(0,0)}_{\infty,1,(0,0)}, f\in FL^1)$, all we have to show is the following estimate:

$$\|\sigma_{j,k}(X,D)f\|_{L^p}\leqq C2^{j\cdot n/2+kn/p}\|\sigma_{j,k}(x,\xi)\omega_{\tilde{n}/p-\tilde{n}/2}(\xi)\|_{L^\infty_{x,\xi}}\|f\|_{H^p}.$$

Furthermore, by virtue of atomic decomposition theorem in H^p (see also Remark 4.1) and Lemma 2.1, it suffices to show the estimate

$$\|\sigma_{j,k}(X,D)f\|_{L^p}\leqq C2^{j\cdot n/2+kn/p}\|\sigma_{j,k}(x,\xi)\omega_{\tilde{n}/p-\tilde{n}/2}(\xi)\|_{L^\infty_{x,\xi}} \tag{4.2}$$

for all $f\in H^p(\boldsymbol{R}^n)$ which satisfies the following condition for some $r>0$:

$$\operatorname{supp} f\subset\{x;\ |x|\leqq r\},\quad \|f\|_{L^\infty}\leqq r^{-n/p},$$
$$\int f(x)x^\alpha dx=0\quad\text{for}\quad |\alpha|\leqq[n/p-n]. \tag{4.3}$$

Condition (4.3) implies the estimate $\|f\|_{L^2}\leqq Cr^{-n/p+n/2}$. Hence, by Hölder's inequality, Lemma 3.1, and Lemma 1.2.3 in Sugimoto [17], we have

$$\begin{aligned}\|\sigma_{j,k}(X,D)f\|_{L^p(|x|\leqq 2r)}&\leqq Cr^{n/p-n/2}\|\sigma_{j,k}(X,D)f\|_{L^2}\\&\leqq Cr^{n/p-n/2}\|\sigma_{j,k}(x,\xi)\|_{1,0}^{(n/2,n/2)}\|f\|_{L^2}\\&\leqq C2^{j\cdot n/2+kn/2}\|\sigma_{j,k}(x,\xi)\|_{L^\infty_{x,\xi}}\\&\leqq C2^{j\cdot n/2+kn/p}\|\sigma_{j,k}(x,\xi)\omega_{\tilde{n}/p-\tilde{n}/2}(\xi)\|_{L^\infty_{x,\xi}}.\end{aligned}$$

Since $\sigma_{j,k}(X,D)f(x)=C\int_{|\eta|\leqq r}K_{j,k}(x,\eta-x)f(\eta)d\eta$ and $\operatorname{supp}_\eta K_{j,k}(x,\eta)\subset\{\eta;\ |\eta|\leqq C2^k\}$ for all x, we have $\operatorname{supp}\sigma_{j,k}(X,D)f(x)\cap\{x;\ |x|\geqq 2r\}\subset\{x;\ |x|\leqq C2^{k+1}\}$. Here $K_{j,k}(x,\eta)=\mathcal{F}_\xi[\sigma_{j,k}(x,\xi)](\eta)$. Then Hölder's inequality gives

$$\|\sigma_{j,k}(X,D)f\|_{L^p(|x|\geqq 2r)}\leqq C2^{kn(1/p-1/2)}\|\sigma_{j,k}(X,D)f\|_{L^2}.$$

Hence, estimate (4.2) is reduced to the following estimate:

(4.4) $$\|\sigma_{j,k}(X, D)f\|_{L^2}\leqq C2^{j\cdot n/2+kn/2}\|\sigma_{j,k}(x, \xi)\omega_{\tilde{n}/p-\tilde{n}/2}(\xi)\|_{L^\infty_{x,\xi}}.$$

In order to show estimate (4.4), we shall take a function $\phi\in C_0^\infty(\boldsymbol{R}^n)$ such that $\phi(\xi)=1$ for ξ near 0, and decompose $\sigma_{j,k}(x, \xi)$ into the sum of $\sigma^{(1)}_{j,k}(x, \xi)=\sigma_{j,k}(x, \xi)\cdot\phi(r\xi)$ and $\sigma^{(2)}_{j,k}(x, \xi)=\sigma_{j,k}(x, \xi)\cdot(1-\phi(r\xi))$. Setting $K^{(l)}_{j,k}(x, \eta)=\mathscr{F}_\xi[\sigma^{(l)}_{j,k}(x, \xi)](\eta)$ $(l=1, 2)$, we obtain from condition (4.3)

$$\begin{aligned}\sigma^{(1)}_{j,k}(X, D)f(x)&=C\int_{|\eta|\leqq r}\{K^{(1)}_{j,k}(x, \eta-x)-\sum_{|\alpha|\leqq L-1}\partial_2^\alpha K^{(1)}_{j,k}(x, -x)\eta^\alpha/\alpha!\}f(\eta)d\eta\\&=C\int_0^1\int_{|\eta|\leqq r}L(1-t)^{L-1}\sum_{|\alpha|=L}\partial_2^\alpha K^{(1)}_{j,k}(x, -x+t\eta)(\eta^\alpha/\alpha!)f(\eta)dtd\eta,\\ \sigma^{(2)}_{j,k}(X, D)f(x)&=C\int_{|\eta|\leqq r}K^{(2)}_{j,k}(x, -x+\eta)f(\eta)d\eta,\end{aligned}$$

where $L=[n/p-n]+1$ and $\partial_2^\alpha K^{(1)}_{j,k}(x, z)=\partial_\eta^\alpha K^{(1)}_{j,k}(x, \eta)|_{\eta=z}$. From these equalities, we have

(4.5) $$\|\sigma^{(1)}_{j,k}(X, D)f\|_{L^2}\leqq Cr^{L+n-n/p}\sup_{|\alpha|=L,|\eta|\leqq r,0\leqq t\leqq 1}\|\partial_2^\alpha K^{(1)}_{j,k}(x, -x+t\eta)\|_{L^2_x},$$

(4.6) $$\|\sigma^{(2)}_{j,k}(X, D)f\|_{L^2}\leqq Cr^{n-n/p}\sup_{|\eta|\leqq r}\|K^{(2)}_{j,k}(x, -x+\eta)\|_{L^2_x}.$$

Furthermore, it holds that

$$\begin{aligned}\partial_2^\alpha K^{(1)}_{j,k}(x, -x+t\eta)&=\sigma_{j,k}(X, D)\omega_{\tilde{n}/p-\tilde{n}/2}(D)h^{(1)}_{t,\eta,\alpha,r}(x),\\ K^{(2)}_{j,k}(x, -x+\eta)&=\sigma_{j,k}(X, D)\omega_{\tilde{n}/p-\tilde{n}/2}(D)h^{(2)}_{\eta,r}(x),\end{aligned}$$

where $h^{(1)}_{t,\eta,\alpha,r}$, $h^{(2)}_{\eta,r}$ are the inverse Fourier transform of the functions

$$\begin{aligned}\mathscr{F}h^{(1)}_{t,\eta,\alpha,r}(\xi)&=\phi(r\xi)(-i\xi)^\alpha e^{-it\xi\cdot\eta}\omega_{\tilde{n}/2-\tilde{n}/p}(\xi),\\ \mathscr{F}h^{(2)}_{\eta,r}(\xi)&=(1-\phi(r\xi))e^{-i\xi\cdot\eta}\omega_{\tilde{n}/2-\tilde{n}/p}(\xi).\end{aligned}$$

By virtue of Lemma 3.1, we obtain

(4.7) $$\|\partial_2^\alpha K^{(1)}_{j,k}(x, -x+t\eta)\|_{L^2_x}\leqq C\|\sigma_{j,k}(x, \xi)\omega_{\tilde{n}/p-\tilde{n}/2}(\xi)\|^{(n/2,n/2)}_{1,0}\|h^{(1)}_{t,\eta,\alpha,r}\|_{L^2},$$

(4.8) $$\|K^{(2)}_{j,k}(x, -x+\eta)\|_{L^2_x}\leqq C\|\sigma_{j,k}(x, \xi)\omega_{\tilde{n}/p-\tilde{n}/2}(\xi)\|^{(n/2,n/2)}_{1,0}\|h^{(2)}_{\eta,r}\|_{L^2}.$$

By Lemma 2.2 and by Lemma 1.2.3 in Sugimoto [17], we have

(4.9) $$\begin{aligned}&\|\sigma_{j,k}(x, \xi)\omega_{\tilde{n}/p-\tilde{n}/2}(\xi)\|^{(n/2,n/2)}_{1,0}\\&\leqq C\|\omega_{\tilde{n}/p-\tilde{n}/2}(\xi)\|_{\sigma^\kappa_{\tilde{n}/2-\tilde{n}/p}}\|\sigma_{j,k}(x, \xi)\|^{(n/2,n/2)}_{1,\tilde{n}/p-\tilde{n}/2}\\&\leqq C2^{j\cdot n/2+kn/2}\|\sigma_{j,k}(x, \xi)\omega_{\tilde{n}/p-\tilde{n}/2}(\xi)\|_{L^\infty_{x,\xi}}.\end{aligned}$$

On the other hand, Plancherel's theorem gives

$$(4.10)\qquad \begin{aligned}&\sup_{|\alpha|=L,|\eta|\leqq r,0\leqq t\leqq 1}\|h^{(1)}_{t,\eta,\alpha,r}\|_{L^2}\\&=\sup_{|\alpha|=L,|\eta|\leqq r,0\leqq t\leqq 1} C\|\mathscr{F}h^{(1)}_{t,\eta,\alpha,r}\|_{L^2}\\&\leqq C\left\{\int|\phi(r\xi)|^2|\xi|^{2L+2(n/2-n/p)}d\xi\right\}^{1/2}\\&=Cr^{-L-n+n/p}\left\{\int|\phi(\zeta)|^2|\zeta|^{2L+2(n/2-n/p)}d\zeta\right\}^{1/2}\\&\leqq Cr^{-L-n+n/p},\end{aligned}$$

$$(4.11)\qquad \begin{aligned}\sup_{|\eta|\leqq r}\|h^{(2)}_{\eta,r}\|_{L^2}&\leqq Cr^{-n+n/p}\left\{\int|1-\phi(\zeta)|^2|\zeta|^{2(n/2-n/p)}d\zeta\right\}^{1/2}\\&\leqq Cr^{-n+n/p}.\end{aligned}$$

Combining estimates (4.5)-(4.11), we have estimate (4.4), and this completes the proof.

5. The case $1\leqq p<2$

Theorem 1.1 with the remainder case $1\leqq p<2$ is obtained from the results of Section 3 and Section 4 by the method of interpolation. We shall use "analytic interpolation theorem", due to Stein-Weiss [16], Calderón-Torchinsky [2].

Let $S=\{z\in \boldsymbol{C};\ 0\leqq \mathrm{Re}\ z\leqq 1\}$, Meas $(\boldsymbol{R}^n)=\{$Lebesgue measurable functions on $\boldsymbol{R}^n\}$, $C^\infty_{00}(\boldsymbol{R}^n)=\{f;\ \hat{f}\in C^\infty_0(\boldsymbol{R}^n),\ \hat{f}(\xi)=0$ for ξ near $0\}$, and $L^\infty_0(\boldsymbol{R}^n)=\{g\in L^\infty(\boldsymbol{R}^n);\ \mathrm{supp}\ g$ is compact$\}$. Suppose $\{T_z\}_{z\in S}$ is a family of linear operators mapping C^∞_{00} into Meas. We say that $\{T_z\}_{z\in S}$ is analytic in case the mapping $z\to\int T_zf(x)\cdot g(x)dx$ is analytic in the interior of S and bounded continuous on S for each $f\in C^\infty_{00}$ and $g\in L^\infty_0$.

LEMMA 5.1. *Let $0<p_0\leqq p_1<\infty$ and $1/p=(1-\theta)/p_0+\theta/p_1$, where $0<\theta<1$, and let $\{T_z\}_{z\in S}$ be an analytic family of operators such that the estimates*

$$\sup_{t\in\boldsymbol{R}}\|T_{it}f\|_{L^{p_0}}\leqq M_0\|f\|_{H^{p_0}},\quad \sup_{t\in\boldsymbol{R}}\|T_{1+it}f\|_{L^{p_1}}\leqq M_1\|f\|_{H^{p_1}},$$

and $\|T_\theta f\|_{L^\infty}\leqq M\|f\|_{H^p}$ hold for all $f\in C^\infty_{00}$. Then there exists a constant C such that the estimate

$$\|T_\theta f\|_{L^p}\leqq CM_0^{1-\theta}M_1^\theta\|f\|_{H^p}$$

holds for all $f \in C_{00}^\infty$. *Here* C *is a constant depending only on* n *and* p.

PROOF OF LEMMA 5.1. This lemma is an immediate consequence of Theorem 3.3 and its proof in Calderón-Torchinsky [2]. We leave the details to the reader. (See also Stein-Weiss [16].)

Now we shall prove Theorem 1.1 in the case $1 \leqq p < 2$. Let $0 < p_0 < 1$ and let $\sigma(x, \xi)$ be rapidly decreasing with respect to ξ. Decomposing $\sigma(x, \xi)$ into the sum of $\sigma_{j,k}(x, \xi) = \mathscr{F}^{-1}\Phi_{j,k}\mathscr{F}\sigma(x, \xi)$, we set

$$\sigma_{j,k}^z(x, \xi) = \omega_{\tilde{n}/p_0 - \tilde{n}/2}(\xi)^{z-\theta} e^{(z-\theta)^2} \sigma_{j,k}(x, \xi),$$

where $z \in S$ and $1/p = (1-\theta)/p_0 + \theta/2$. We remark that $\sigma_{j,k}(x, \xi) = \sigma_{j,k}^\theta(x, \xi)$ is rapidly decreasing with respect to ξ. By Lemma 2.2 and by Lemma 1.2.3 in Sugimoto [17], we have

$$\begin{aligned}
&\sup_{t \in R} \|\sigma_{j,k}^{it}(x, \xi)\|_{p_0, \tilde{n}/p_0 - \tilde{n}/2}^{(n/2, n/p_0)} \\
&\leqq C \sup_{t \in R} e^{-t^2} \|\omega_{\tilde{n}/p_0 - \tilde{n}/2}^{it-\theta}\|_{C^\kappa_{\tilde{n}/p_0 - \tilde{n}/p}} \|\sigma_{j,k}\|_{p_0, \tilde{n}/p - \tilde{n}/2}^{(n/2, n/p_0)} \\
&\leqq C \sup_{t \in R} e^{-t^2} a(|t|) \|\omega_{-(\tilde{n}/p_0 - \tilde{n}/2)\theta + \tilde{n}/p_0 - \tilde{n}/p}\|_{L^\infty} \\
&\qquad\qquad \cdot 2^{j \cdot n/2 + kn/p_0} \|\sigma_{j,k}(x, \xi)\omega_{\tilde{n}/p - \tilde{n}/2}(\xi)\|_{L^\infty_{x,\xi}} \\
&\leqq C 2^{j \cdot n/2 + kn/p_0} \|\sigma_{j,k}(x, \xi)\omega_{\tilde{n}/p - \tilde{n}/2}(\xi)\|_{L^\infty_{x,\xi}},
\end{aligned}$$

where $a|(t|)$ is a polynomial with respect to $|t|$. Similarly, we have

$$\sup_{t \in R} \|\sigma_{j,k}^{1+it}(x, \xi)\|_{1,0}^{(n/2, n/2)} \leqq C 2^{j \cdot n/2 + kn/2} \|\sigma_{j,k}(x, \xi)\omega_{\tilde{n}/p - \tilde{n}/2}(\xi)\|_{L^\infty_{x,\xi}}.$$

By virtue of Lemma 2.1, the family of operators $\{\sigma_{j,k}^z(X, D)\}_{z \in S}$ satisfies the assumption of analytic interpolation theorem (Lemma 5.1), and we have, for $f \in C_{00}^\infty$,

$$\|\sigma_{j,k}(X, D)f\|_{L^p} \leqq C 2^{j \cdot n/2 + kn/p} \|\sigma_{j,k}(x, \xi)\omega_{\tilde{n}/p - \tilde{n}/2}(\xi)\|_{L^\infty_{x,\xi}} \|f\|_{H^p}.$$

This implies estimate (1.2) for $\sigma \in B_{\infty, \min(p,1), (0, \tilde{n}/p - \tilde{n}/2)}^{(n/2, n/p)}$ which is rapidly decreasing with respect to ξ, and for $f \in C_{00}^\infty$. Lemma 2.1 and Claim in Section 2 give the general case since C_{00}^∞ is dense in H^p; see Calderón-Torchinsky [2], Theorem 1.8 (v).

6. Some additional results

To begin with, we shall give an L^p-estimates for pseudo-differential operators in the case $2 \leqq p \leqq \infty$.

THEOREM 6.1. *Let $2 \leqq p \leqq \infty$ and let $\boldsymbol{\rho} > \tilde{\boldsymbol{n}}/2$. Then there exists a constant C such that the estimate*

$$\|\sigma(X, D)f\|_{L^p(\boldsymbol{R}^n)} \leqq C\|\sigma(x, \xi)\|_{1,\rho}^{(0,n/2)}\|f\|_{L^p(\boldsymbol{R}^n)}$$

holds for all $\sigma \in \mathscr{S}(\boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n)$ and all $f \in L^p(\boldsymbol{R}^n) \cap FL^1(\boldsymbol{R}^n)$.

PROOF OF THEOREM 6.1. Let $\sigma_{j,k}(x, \xi) = \mathscr{F}^{-1}\Phi_{j,k}\mathscr{F}\sigma(x, \xi)$. By virtue of Lemma 2.2.3 in Sugimoto [17], we have

$$\begin{aligned}\|\sigma_{j,k}(X, D)f\|_{L^p(\boldsymbol{R}^n)} &\leqq C2^{kn/2}\sup_x \|\sigma_{j,k}(x, \xi)\|_{L_\xi^2}\|f\|_{L^p(\boldsymbol{R}^n)} \\ &\leqq C2^{kn/2}\|\sigma_{j,k}(x, \xi)\omega_\rho(\xi)\|_{L_{x,\xi}^\infty}\|f\|_{L^p(\boldsymbol{R}^n)}.\end{aligned}$$

Here we have used the fact $\|\omega_{-\rho}(\xi)\|_{L^2} < +\infty$. This implies the desired result.

Theorem 6.1 is an extension of Theorem C in Miyachi [8]. With the aid of this theorem (in the case of $p=2$), we can obtain some $H^p - L^p$ estimates for dual pseudo-differential operators.

DEFINITION 6.1. Let $\sigma(x, \xi) \in \mathscr{S}(\boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n)$. Then we shall define the operator $\sigma(X, D)^*: \mathscr{S}(\boldsymbol{R}^n) \to \mathscr{S}(\boldsymbol{R}^n)$ by

$$\sigma(X, D)^*f(y) = \int K(x, y-x)f(x)dx, \quad f \in \mathscr{S}(\boldsymbol{R}^n),$$

where $K(x, \eta) = \mathscr{F}_\xi[\sigma(x, \xi)](\eta)$.

We have the following estimates for the operator $\sigma(X, D)^*$.

THEOREM 6.2. (i) *Let $0 < q < 1$. Then there exists a constant C such that the estimate*

$$\|\sigma(X, D)^*f\|_{L^q(\boldsymbol{R}^n)} \leqq C\|\sigma(x, \xi)\|_{q,\tilde{n}/q-\tilde{n}/2}^{(n/q-n, n/q-n/2)}\|f\|_{H^q(\boldsymbol{R}^n)}$$

holds for all $\sigma \in \mathscr{S}(\boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n)$ and all $f \in \mathscr{S}(\boldsymbol{R}^n) \cap H^q(\boldsymbol{R}^n)$.

(ii) *Let $1 \leqq q < 2$, and let $\boldsymbol{\lambda} > \boldsymbol{n} - \boldsymbol{n}/q$, $\lambda' > n/2$. Then there exists a constant C such that the estimate*

$$\|\sigma(X, D)^*f\|_{L^q(\boldsymbol{R}^n)} \leqq C\|\sigma(x, \xi)\|_{1,\tilde{n}/q-\tilde{n}/2}^{(\lambda,\lambda')}\|f\|_{H^q(\boldsymbol{R}^n)}$$

holds for all $\sigma \in \mathscr{S}(\boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n)$ and all $f \in \mathscr{S}(\boldsymbol{R}^n) \cap H^q(\boldsymbol{R}^n)$.

(iii) *Let* $2 \leqq q < \infty$. *Then there exists a constant* C *such that the estimate*

$$\|\sigma(X, D)^* f\|_{L^q(R^n)} \leqq C\|\sigma(x, \xi)\|_{1,\tilde{n}/2-\tilde{n}/q}^{(n/2, n-n/q)} \|f\|_{L^q(R^n)}$$

holds for all $\sigma \in \mathscr{S}(\boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n)$ *and all* $f \in \mathscr{S}(R^n)$.

(iv) *Let BMO be the space of functions of bounded mean oscillation on* $\boldsymbol{R}^n$. *Then there exists a constant* C *such that the estimate*

$$\|\sigma(X, D)^* f\|_{BMO} \leqq C\|\sigma(x, \xi)\|_{1,\tilde{n}/2}^{(n/2, n)} \|f\|_{L^\infty(R^n)}$$

holds for all $\sigma \in \mathscr{S}(\boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n)$ *and all* $f \in \mathscr{S}(\boldsymbol{R}^n)$.

Theorem 6.2 is an extension of Theorem 2 in Miyachi [8].

Proof of Theorem 6.2. Parts (iii) and (iv) are easily obtained from Theorem 1.1 if we notice the dualities between L^p and L^q $(1/p+1/q=1)$ and between H^1 and BMO (see Fefferman-Stein [4]). The proof of part (i) is similar to that of Theorem 1.1, and we shall omit it. (Use Theorem 6.1, $p=2$, instead of Lemma 3.1.) Part (ii) is obtained from parts (i) and (iii) by the same method as Section 5.

The following theorem is an immediate consequence of Theorem 6.2 (ii) via a duality argument.

Theorem 6.3. (i) *Let* $2 < p < \infty$, *and let* $\boldsymbol{\lambda} > \boldsymbol{n}/p$, $\lambda' > n/2$. *Then there exists a constant* C *such that the estimate*

$$\|\sigma(X, D) f\|_{L^p(R^n)} \leqq C\|\sigma(x, \xi)\|_{1,\tilde{n}/q-\tilde{n}/2}^{(\lambda, \lambda')} \|f\|_{H^p(R^n)}$$

holds for all $\sigma \in \mathscr{S}(\boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n)$ *and all* $f \in \mathscr{S}(\boldsymbol{R}^n) \cap H^p(\boldsymbol{R}^n)$.

(ii) *Let* $\boldsymbol{\lambda} > \boldsymbol{0}$, $\lambda' > n/2$. *Then there exists a constant* C *such that the estimate*

$$\|\sigma(X, D) f\|_{BMO} \leqq C\|\sigma(x, \xi)\|_{1,\tilde{n}/2}^{(\lambda, \lambda')} \|f\|_{L^\infty(R^n)}$$

holds for all $\sigma \in \mathscr{S}(\boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n)$ *and all* $f \in \mathscr{S}(\boldsymbol{R}^n)$.

This theorem is essentially the same as Theorem 1, (iii) (iv), in Miyachi [8]. The present author does not know whether one can replace the inequalities $\boldsymbol{\lambda} > \boldsymbol{n}/p$, $\lambda' > n/2$ in Theorem 6.3 (i) (resp. $\boldsymbol{\lambda} > \boldsymbol{0}$, $\lambda' > n/2$ in Theorem 6.3 (ii)) by the equality $\boldsymbol{\lambda} = \boldsymbol{n}/p$, $\lambda' = n/2$ (resp. $\boldsymbol{\lambda} = \boldsymbol{0}$, $\lambda' = n/2$). But Theorem 6.1 with $p = \infty$ tells us that this is true in the case (ii) replacing the weight order $\tilde{n}/2$ by $\boldsymbol{\rho}$ such that $\boldsymbol{\rho} > \tilde{n}/2$.

Finally, we remark that the case $N=n$ is the sharpest result in Theorems 6.2 and 6.3. (Cf. Remark 1.2.)

References

[1] Calderón, A. P. and A. Torchinsky, Parabolic maximal functions associated with a distribution, Adv. in Math. **16** (1975), 1-64.

[2] Calderón, A. P. and A. Torchinsky, Parabolic maximal functions associated with a distribution II, Adv. in Math. **24** (1977), 101-171.

[3] Coifman, R. R. and Y. Meyer, Au-delà des opérateurs pseudo-différentiels, Astérisque vol. 57, Soc. Math. France, Paris, 1978.

[4] Fefferman, C. and E. M. Stein, H^p spaces of several variables, Acta Math. **129** (1972), 137-193.

[5] Hörmander, L., Pseudo-differential operators and hypoelliptic equations, Proc. Symp. on Singular Integrals, vol. 10, Amer. Math. Soc., Previdence, R.I., 1967, 138-183.

[6] Latter, R. H., A characterization of $H^p(\boldsymbol{R}^n)$ in terms of atoms, Studia Math. **62** (1978), 93-101.

[7] Miyachi, A., On some Fourier multipliers for $H^p(\boldsymbol{R}^n)$, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), 157-179.

[8] Miyachi, A., Estimates for pseudo-differential operators of class $S_{0,0}$, Math. Nachr. **133** (1987), 135-154.

[9] Miyachi, A. and K. Yabuta, L^p-boundedness of pseudo-differential operators with non-regular symbols, Bull. Fac. Sci. Ibaraki Univ. Ser. A **17** (1985), 1-20.

[10] Mossaheb, S. and M. Okada, Une classe d'opérateurs pseudo-différentiels bornés sur $L^r(\boldsymbol{R}^n)$, $1<r<\infty$, C. R. Acad. Sci. Paris Sér. A **285** (1977), 613-616.

[11] Muramatu, T. and M. Nagase, On sufficient conditions for the boundedness of pseudo-differential operators, Proc. Japan Acad. Ser. A Math. Sci. **55** (1979), 293-296.

[12] Nagase, M., The L^p-boundedness of pseudo-differential operators with non-regular symbols, Comm. Partial Differential Equations **2** (1977), 1045-1061.

[13] Nagase, M., On the boundedness of pseudo-differential operators in L^p-spaces, Sci. Rep. College Gen. Ed. Osaka Univ. **32** (1983), 9-19.

[14] Nagase, M., On a class of L^p-bounded pseudo-differential operators, Sci. Rep. College Gen. Ed. Osaka Univ. **33** (1984), 1-7.

[15] Nagase, M., On some classes of L^p-bounded pseudo-differential operators, Osaka Math. J. **23** (1986), 425-440.

[16] Stein, E. M. and G. Weiss, On the interpolation of analytic families of operators acting on H^p-spaces, Tôhoku Math. J. (2) **9** (1957), 318-339.

[17] Sugimoto, M., L^p-boundedness of pseudo-differential operators satisfying Besov estimates I, J. Math. Soc. Japan **40** (1988), 105-122.

[18] Wang, Roughuai and Li, Chengzhang, On the L^p-boundedness of several classes of pseudo-differential operators, Chinese Ann. Math. Ser. B **5** (1984), 193-213.

[19] Yamazaki, M., The L^p-boundedness of pseudo-differential operators with estimates of parabolic type and product type, J. Math. Soc. Japan **38** (1986), 199-225.

(Received May 27, 1987)

Institute of Mathematics
University of Tsukuba
Ibaraki,
305 Japan