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Some remarks on dressing transformations

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M. Semenov-Tian-Shansky [7] has shown that the hamiltonian structure of the Zakharov-Shabat dressing transformations of integrable systems theory is best understood in terms of the notions of Poisson Lie group and Lie bialgebra introduced by V. G. Drinfel'd [3]. The purpose of this note is to present some aspects of the theory of dressing transformations in fairly simple terms. Although it will not appear here explicitly, many of our ideas are motivated by the theory of symplectic groupoids [4] [6] [8] and, in particular, Karasev's construction in [4] of the symplectic groupoid associated with a Poisson Lie group.

1. Grouped Poisson structures

A Poisson structure on a Lie group $\boldsymbol{G}$ is said to be *grouped* if the multiplication map $m: \boldsymbol{G}\times\boldsymbol{G}\to\boldsymbol{G}$ is a Poisson mapping. It follows directly from this definition, as Drinfel'd has observed, that the Poisson structure given by the bivector field Π is grouped if and only if, for each g and h in $\boldsymbol{G}$,

$$\Pi(gh)=Tl_g\Pi(h)+Tr_h\Pi(g) \tag{1.1}$$

where l_g and r_h are left and right translation mappings from $\boldsymbol{G}$ to itself. From this condition, we may derive the following infinitesimal criterion for a Poisson structure to be grouped.

1.1 THEOREM. *The Poisson structure Π on $\boldsymbol{G}$ is grouped if and only if the following two conditions are satisfied*:

(i) $\mathscr{L}_X\mathscr{L}_Y\Pi=0$ *for every right-invariant vector field X and left invariant vector field Y on $\boldsymbol{G}$; i.e. $\mathscr{L}_Y\Pi$ is left-invariant whenever Y is left-invariant or $\mathscr{L}_X\pi$ is right-invariant whenever X is right-invariant.*

(ii) $\Pi(e)=0$, *where e is the identity element of $\boldsymbol{G}$.*

PROOF. Trivializing the tangent bundle of $\boldsymbol{G}$ by right translations, we may identify the tensor field Π with a mapping $\pi: \boldsymbol{G}\to g\wedge g$. In terms of π, Drinfel'd's condition (1.1) becomes the cocycle identity

$$\pi(gh)=\pi(g)+\mathrm{Ad}_g[\pi(h)] \tag{1.2}$$

for all g and h in $\boldsymbol{G}$. Setting $g=h=e$ in (1.2), we find that $\pi(e)=0$, which gives (ii). To prove (i), we choose any element y of $\mathfrak{g}$ and substitute $\exp(ty)$ for h in (1.2). Since the group of right translations by $\exp(ty)$ is generated by the *left*-invariant vector field Y associated with y, differentiating by t at $t=0$ gives

$$Y\pi(g)=\mathrm{Ad}_g[Y\pi(e)], \tag{1.3}$$

which, since we are using the *right* trivialization of TG, says that the derivative $Y\pi$ corresponds to a *left*-invariant tensor field on $\boldsymbol{G}$. But differentiating π by the generator Y of right translations corresponds to taking the Lie derivative $\mathscr{L}_Y\Pi$, so $\mathscr{L}_Y\Pi$ is left-invariant. Since each right-invariant vector field X generates a 1-parameter group of left translations, we must have $\mathscr{L}_X\mathscr{L}_Y\Pi=0$, which proves (ii).

Conversely, assuming (ii) and integrating it, using (i) as the initial condition, gives the cocycle identity (1.2). Q.E.D.

1.2 REMARKS. (i) There are two special cases where the theorem is obvious, which suggested the final statement. The first is the case of an abelian Lie group, in which the conditions (i) and (ii) are equivalent to linearity of the components of the Poisson structure in normal coordinates. The second is the case of classical r-matrix structures of the coboundary form $\Pi(g)=Tl_g r-Tr_g r$, where r is a 2-vector of $\mathfrak{g}$.

(ii) It might be interesting to consider Poisson structures which satisfy condition (i), without requiring (ii). Such structures include arbitrary sums of right and left invariant structures. In the case of a vector space, they include the targets for momentum mappings which are not coadjoint-equivariant.

More generally, one might look at this condition of "vanishing second derivatives" for tensors of other types on Lie groups, as a generalization of the conditions of right-, left-, or bi-invariance.

(iii) Theorem 1.1 has also been obtained by Pierre Dazord (private communication).

2. Infinitesimal dressing transformations

The space $\Omega^1(P)$ of 1-forms on a Poisson manifold P has the structure of a Lie algebra under the bracket [1] [4] [5]

$$\{\omega_1, \omega_2\}=d[\Pi(\omega_1, \omega_2)]-\Pi\omega_1\rfloor d\omega_2+\Pi\omega_2\rfloor d\omega_1. \tag{2.1}$$

(In the formula above, we consider Π both as a bilinear form on 1-forms and as a mapping from 1-forms to vector fields by the rule $\Pi\omega\rfloor\eta=\Pi(\eta, \omega)$. Thus, $\Pi(df)$ is the hamiltonian vector field of the function f.)

The bracket (2.1) together with the mapping $-\Pi$ make $\Omega^1(P)$ into a *Lie algebroid* [1] [2]; i.e. $-\Pi$ is a Lie algebra homomorphism into the vector fields with the commutator Lie algebra structure, and the following derivation law is satisfied:

$$\{f\omega_1, \omega_2\}=f\{\omega_1, \omega_2\}-(-\Pi\omega_2\cdot f)\omega_1.$$

The following theorem is used in [4], but the proof given there is much less direct than the one below.

2.1 THEOREM. *The right [left] invariant 1-forms on a Poisson Lie group form a subalgebra with respect to the bracket* (2.1). *The corresponding Lie algebra structure on* $\mathfrak{g}^*$ *is equal to the one given by linearizing the Poisson structure on* G *at the identity element.*

PROOF. Assuming that ω_1 and ω_2 are right-invariant, we shall show that the interior product of any right-invariant vector field X with $\{\omega_1, \omega_2\}$ is constant. (The proof for the left-invariant forms is essentially the same.)

The bracket (2.1) can be rewritten in the form

$$\{\omega_1, \omega_2\}=-d[\Pi(\omega_1, \omega_2)]-\mathcal{L}_{\Pi\omega_1}\omega_2+\mathcal{L}_{\Pi\omega_2}\omega_1. \tag{2.2}$$

Then we have

$$X\rfloor\{\omega_1, \omega_2\}=-X\rfloor d[\Pi(\omega_1, \omega_2)]-X\rfloor\mathcal{L}_{\Pi\omega_1}\omega_2+X\rfloor\mathcal{L}_{\Pi\omega_2}\omega_1. \tag{2.3}$$

Analyzing the right hand side of (2.3) term by term, we have first by the derivation property of the Lie derivative with respect to the contraction of tensors:

$$\begin{aligned} X\rfloor d[\Pi(\omega_1, \omega_2)]&=\mathcal{L}_X[\Pi(\omega_1, \omega_2)] \\ &=(\mathcal{L}_X\Pi)(\omega_1, \omega_2)+\Pi(\mathcal{L}_X\omega_1, \omega_2)+\Pi(\omega_1, \mathcal{L}_X\omega_2) \\ &=(\mathcal{L}_X\Pi)(\omega_1, \omega_2)+\Pi\omega_2\rfloor\mathcal{L}_X\omega_1-\Pi\omega_1\rfloor\mathcal{L}_X\omega_2. \end{aligned} \tag{2.4}$$

Next,

$$X \lrcorner \mathscr{L}_{\Pi\omega_1}\omega_2 = \mathscr{L}_{\Pi\omega_1}(X \lrcorner \omega_2) - [\Pi\omega_1, X] \lrcorner \omega_2 = -[\Pi\omega_1, X] \lrcorner \omega_2$$

since $X \lrcorner \omega_2$ is constant.

Now

$$-[\Pi\omega_1, X] \lrcorner \omega_2 = \mathscr{L}_X(\Pi\omega_1) \lrcorner \omega_2 = ((\mathscr{L}_X\Pi)\omega_1 + \Pi\mathscr{L}_X\omega_1) \lrcorner \omega_2.$$

So

$$X \lrcorner \mathscr{L}_{\Pi\omega_1}\omega_2 = (\mathscr{L}_X\Pi)(\omega_2, \omega_1) - \Pi\omega_2 \lrcorner \mathscr{L}_X\omega_1. \tag{2.5}$$

Similarly,

$$X \lrcorner \mathscr{L}_{\Pi\omega_2}\omega_1 = (\mathscr{L}_X\Pi)(\omega_1, \omega_2)\Pi\omega_1 \lrcorner \mathscr{L}_X\omega_2. \tag{2.6}$$

Substituting (2.4)-(2.6) into (2.3) and cancelling terms, we get

$$X \lrcorner \{\omega_1, \omega_2\} = (\mathscr{L}_X\Pi)(\omega_1, \omega_2). \tag{2.7}$$

Since $\mathscr{L}_X\Pi$, ω_1, and ω_2 are all right invariant, the right hand side of (2.7) is constant, and so the first part of our theorem is proven.

For the second part of the theorem, we note that the linearized Poisson structure on T_e^*G is given by the rule

$$\{df(e), dg(e)\} = d\{f, g\}(e),$$

which agrees with (2.1) since $\Pi(e)=0$. Q.E.D.

2.2 REMARKS. (i) Note that the brackets induced on right- and left- invariant 1-forms are equal rather than opposite to one another.

(ii) The simply connected group G^* whose Lie algebra is $\mathfrak{g}^*$ is called by Drinfel'd [3] the *dual group* of the Poisson Lie group (G, Π). Applying the Lie algebra homomorphism $-\Pi$ to the right invariant 1-forms, we get an action of the Lie algebra $\mathfrak{g}^*$ on G by vector fields vanishing at the identity. Integration thus gives a local (and global if these vector fields are complete) action of the group G^* on G which leaves the identity element fixed. These are the *dressing transformations* whose hamiltonian nature was explained in [7].

Theorem 4.5 of [4] suggests that the action of G^* on G is always globally defined. But in fact the proof of this fact, as well as that of Theorem 4.5, relies on the *assumption* (made 2 paragraphs before Theorem 3.1) that certain vector fields are complete. Thus, the question of the existence of a global phase space for a Poisson Lie group must be considered as open in the most general case.

(iii) To show that the dressing transformations give a *Poisson* action of G^* on G, it is possible to use an infinitesimal criterion like that in Theorem 1. In a sequel to this paper, we plan to return to this point, as well as to describe in some detail the dressing transformations for grouped Poisson structures on $SU(2)$.

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