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Rigid spherical t-designs and a theorem of Y. Hong

Dedicated to Professor Nagayoshi Iwahori on the occasion of his 60th birthday

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§ 0. Introduction.

Let $S^d=\{(x_1, x_2, \cdots, x_{d+1})\in R^{d+1} \mid x_1^2+x_2^2+\cdots+x_{d+1}^2=1\}$ be the unit sphere. A finite nonempty subset X of S^d is called a spherical t-design (after Delsarte-Goethals-Seidel [4]) if

$$\frac{1}{|S^d|}\int_{S^d} f(x)dw(x)=\frac{1}{|X|}\sum_{x\in X} f(x)$$

for all polynomials $f(x)=f(x_1, x_2, \cdots, x_{d+1})$ of degree $\leqq t$. The reader is referred to [4, 5, 6, 1, 2, 3] for the discussion of basic properties and examples of spherical t-designs.

In the circle S^1, it is easy to see that the $k+1$ vertices of a regular $(k+1)$-gon with $k\geqq t$ (embedded in S^1) form a t-design. In [7] Y. Hong proved the following results for spherical t-designs in S^1.

(i) If $|X|\leqq 2t+1$, then X must be a regular $(k+1)$-gon with $t\leqq k\leqq 2t$,

(ii) If $|X|=2t+2$, then X must be a union of two regular $(t+1)$-gons,

(iii) If $|X|\geqq 2t+3$, then there are infinitely many non-group type spherical t-designs, where group type means a union of regular (k_i+1)-gons with $k_i\geqq t$.

This result of Hong suggested the existence of spherical t-designs in abundance. The existence of spherical t-designs in S^d for any t and d was proved by Seymour-Zaslavsky [9] in a very general context.

Some time ago (cf. [2, 3]) the present author introduced the following concept of *rigid* spherical t-designs, and asked whether there are many such spherical t-designs.

DEFINITION 1.1. We call $X=\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n\}$ is a *non-rigid* (or deformable) spherical t-design in S^d, if for any given $\varepsilon>0$ there exists another spherical t-design $X'=\{\vec{x}'_1, \vec{x}'_2, \cdots, \vec{x}'_n\}$ such that $\|\vec{x}_i-\vec{x}'_i\|<\varepsilon$ (for $1\leqq i\leqq n$) and there exists no orthogonal transformation O in R^{d+1} with $O\vec{x}_i=\vec{x}'_i$ $(1\leqq i\leqq n)$.

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DEFINITION 1.2. We call X *rigid* (or non-deformable) if it is not non-rigid.

I was eventually led to conceive the following conjectures.

CONJECTURE 1. For each fixed pair of t and d, if $|X|$ is sufficiently large (i. e., greater than a certain number $f(t, d)$ depending only on t and d), then X is non-rigid.

CONJECTURE 2. For each fixed pair of t and d, there are only finitely many rigid spherical t-designs up to orthogonal transformations.

REMARKS. (i) Conjecture 2 implies Conjecture 1. (ii) Tight t-designs are examples of rigid spherical t-designs. There are some other known rigid spherical t-designs, but they are very rare (at least as far as I am aware of at the present time). (iii) It seems that a rigid spherical t-design may represent a stable state (from the viewpoint of moments) of finitely many particles in S^d. So, the classification problem (if at all possible) may be an interesting question from the viewpoint of physics. This question will be even more interesting if Conjecture 2 is proved to be true.

In the present paper, we restrict ourselves to the case S^1 (i. e., $d=1$). We prove Conjectures 1 and 2 for S^1 by completing classification of rigid t-designs in S^1. Namely, we prove the following:

THEOREM 1. *If X is a rigid spherical t-design in S^1, then X consists of $k+1$ vertices of a regular $(k+1)$-gon with $t \leqq k \leqq 2t$.*

The proof of Theorem 1, which is given in the subsequent sections, is not very difficult. The implicit function theorem plays a key role. We remark that our proof of Theorem 1 (for $d=1$) suggests that our method should also work for $d \geqq 2$. In fact, I have been able to prove Conjecture 1 for some special cases, including the case of $t=1$ and arbitrary d. Some of these results for $d \geqq 2$ will be discussed in a subsequent paper. Also we remark that our proof of Theorem 1 has some similarity to the proof in Hong [7]. The present paper may be regarded as giving a re-interpretation and a clarification of the meaning of Hong [7].

§1. An implicit function theorem.

For the convenience of reader, we state the implicit function theorem in a form ready to use in our proof. Proof of this theorem can be found in any advanced calculus book.

AN IMPLICIT FUNCTION THEOREM. *Let* $u(\vec{x})=(u_1(\vec{x}), \cdots, u_m(\vec{x}))$ *be a* C^1*-class function from* R^n *to* R^m *(with* $m<n$*) with* $u(\vec{0})=\vec{0}$ *and defined on a neighborhood* D *of* $\vec{0}=(x_i^0)=(x_1^0, \cdots, x_n^0)\in R^n$. *Let* V *be the inverse image of* $\vec{0}\in R^m$ *in* D. *Suppose that the Jacobian*

$$\frac{D(u_1, u_2, \cdots, u_m)}{D(x_1, x_2, \cdots, x_m)} \neq 0$$

at the point $(x_i^0)=(x_1^0, \cdots, x_n^0)$. *Then there exists a unique set of* C^1*-class functions* $\xi_\mu(x_{m+1}, \cdots, x_n)$ $(1\leqq\mu\leqq m)$ *defined on a neighborhood of* $(x_{m+1}^0, \cdots, x_n^0)$ *such that*

(i) $x_\mu^0=\xi_\mu(x_{m+1}^0, \cdots, x_n^0)$ $(1\leqq\mu\leqq m)$, *and*

(ii) $(\xi_1(x_{m+1}, \cdots, x_n), \cdots, \xi_m(x_{m+1}, \cdots, x_n), x_{m+1}, \cdots, x_n)\in V$.

§2. Proof of Theorem 1.

Let X be a spherical t-design in S^1. If $|X|\leqq 2t+1$, then by Hong [7] X must be a regular $(k+1)$-gon (with $t\leqq k\leqq 2t$), and so X must be a rigid t-design. Therefore, we assume that $|X|\geqq 2t+2$, and we show that X is not rigid. Theorem 1 is obtained as an immediate consequence of the following:

LEMMA 2. *For any* $Y=\{z_1, z_2, \cdots, z_{2t+1}\}$ *of distinct* $2t+1$ *points* $z_1, z_2, \cdots, z_{2t+1}$ *in* S^1 *(identified with the unit circle in Gauss' complex plane), there exist another set* $Y'=\{z_1', z_2', \cdots, z_{2t+1}'\}$ *with* $\|z_i-z_i'\|$ *arbitrary small and* $Y\neq Y'$ *such that*

$$\sum_{i=1}^{2t+1} f(z_i)=\sum_{i=1}^{2t+1} f(z_i')$$

for any homogeneous harmonic polynomial f *of degree* $1, 2, \cdots, t$.

Note that Y' may be an image of Y under an orthogonal transformation.

Lemma 2⇒Theorem 1. Suppose $|X|\geqq 2t+2$. Choose any $Y\subset X$ with $|Y|=2t+1$. Then move Y slightly to Y' according to Lemma 2. Then the set $X'=(X-Y)\cup Y'$ is close to X but not obtained by an orthogonal transformation of X. Thus X is not rigid.

PROOF OF LEMMA 2. Let us write $z=e^{2\pi\sqrt{-1}\theta}$ (or $z_i=e^{2\pi\sqrt{-1}\theta_i}$). A basis of the space of harmonic polynomials of degree k $(k\geqq 1)$ consists of two functions $\sin k\theta$ and $\cos k\theta$. Therefore, by the implicit function theorem mentioned in the previous section, we only have to prove the following statement. Let $Y=\{z_1, z_2, \cdots, z_{2t+1}\}$. Then Y has a subset Z with $|Z|=2t$ such that

$$\frac{D(u_1, u_2, \cdots, u_{2t})}{D(x_1, x_2, \cdots, x_{2t})} \neq 0 \tag{1}$$

at $\vec{0}=(x_1^0, x_2^0, \cdots, x_{2t+1}^0)$ where $x_1, x_2, \cdots, x_{2t}$ are those corresponding to the $2t$ indices of the $2t$-element subset Z of Y and

$$u_{2k-1}(x_1, \cdots, x_{2t+1})=\sum_{i=1}^{2t+1} \sin k(\theta_i+x_i)-\sum_{i=1}^{2t+1} \sin k\theta_i \quad (1 \leqq k \leqq t),$$

$$u_{2k}(x_1, \cdots, x_{2t+1})=\sum_{i=1}^{2t+1} \cos k(\theta_i+x_i)-\sum_{i=1}^{2t+1} \cos k\theta_i \quad (1 \leqq k \leqq t).$$

The above condition (1) is equivalent to the condition

$$\begin{vmatrix} \cos\theta_1 & \cos\theta_2 & \cdots & \cos\theta_{2t} \\ \sin\theta_1 & \sin\theta_2 & \cdots & \sin\theta_{2t} \\ \cos 2\theta_1 & \cos 2\theta_2 & \cdots & \cos 2\theta_{2t} \\ \sin 2\theta_1 & \sin 2\theta_2 & \cdots & \sin 2\theta_{2t} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cos t\theta_1 & \cos t\theta_2 & \cdots & \cos t\theta_{2t} \\ \sin t\theta_1 & \sin t\theta_2 & \cdots & \sin t\theta_{2t} \end{vmatrix} \neq 0.$$

This, in turn, is equivalent to

$$\begin{vmatrix} z_1 & z_2 & \cdots & z_{2t} \\ z_1^{-1} & z_2^{-1} & \cdots & z_{2t}^{-1} \\ z_1^2 & z_2^2 & \cdots & z_{2t}^2 \\ z_1^{-2} & z_2^{-2} & \cdots & z_{2t}^{-2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ z_1^t & z_2^t & \cdots & z_{2t}^t \\ z_1^{-t} & z_2^{-t} & \cdots & z_{2t}^{-t} \end{vmatrix} \neq 0,$$

and is equivalent to

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_{2t} \\ \cdots & \cdots & \cdots & \cdots \\ z_1^{t-1} & z_2^{t-1} & \cdots & z_{2t}^{t-1} \\ z_1^{t+1} & z_2^{t+1} & \cdots & z_{2t}^{t+1} \\ \cdots & \cdots & \cdots & \cdots \\ z_1^{2t} & z_2^{2t} & \cdots & z_{2t}^{2t} \end{vmatrix} = \Big(\sum_{1 \leqq i_1 \leqq \cdots \leqq i_t \leqq 2t} z_{i_1} z_{i_2} \cdots z_{i_t} \Big) \Delta(z_1, z_2, \cdots, z_{2t}) \neq 0 ,$$

where $\Delta(z_1, z_2, \cdots, z_{2t}) = \prod_{1 \leqq i < j \leqq 2t} (z_i - z_j)$. (Note that the above determinant is a Schur function, cf. [8].)

Thus the proof of Lemma 2 is complete from the following Lemma 3 which is straightforwardly proved.

LEMMA 3. *Let* $Y = \{z_1, z_2, \cdots, z_{2t+1}\}$ *be a set of distinct* $2t+1$ *complex numbers. Then there exists a subset* $Z = \{z'_1, z'_2, \cdots, z'_{2t}\}$ *of* Y *with* $|Z| = 2t$ *such that*

$$\sum_{1 \leqq i_1 < \cdots < i_t \leqq 2t} z'_{i_1} z'_{i_2} \cdots z'_{i_t} \neq 0 .$$

This completes the proof of Lemma 2, hence of Theorem 1.

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