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Another look at the Frame shapes of finite groups

Dedicated to Professor N. Iwahori on his 60th birthday

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§ 0. Introduction.

Let $\eta(z)$ be the Dedekind eta function:

$$\eta(z)=e^{\pi i z/12}\prod_{n=1}^{\infty}(1-e^{2\pi i n z}).$$

$\eta(z)$ is defined on the upper half plane $\mathcal{H}=\{z\in \boldsymbol{C}|\mathrm{Im}(z)>0\}$ and is holomorphic on $\mathcal{H}$. It is known also that $\eta(z)$ does not vanish on $\mathcal{H}$.

The Dedekind eta function $\eta(z)$ admits the following transformation formula under the action of $SL_2(\boldsymbol{Z})$. Let

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\boldsymbol{Z}).$$

Then

$$\eta(Az)\overset{\text{def}}{=}\eta\left(\frac{az+b}{cz+d}\right)=v(A)(cz+d)^{1/2}\eta(z)$$

where $v(A)$ is a 24-th root of unity and a branch of $(cz+d)^{1/2}$ is suitably chosen. The precise value of $v(A)$ was first found by Petersson [5].

For each natural number n, define

$$\eta_n(z)=\prod_{d|n}\eta(dz)^{\mu(n/d)}$$

where μ is the Möbius function, which is defined as follows

(1) $\mu(1)=1$,
(2) $\mu(p_1p_2\cdots p_r)=(-1)^r$, $p_1, p_2, \cdots, p_r$ distinct primes,
(3) $\mu(n)=0$ if n is divisible by a square >1.

We call $\eta_n(z)$ the n-th *cyclotomic Dedekind η-function.*

The purpose of this paper is to establish some transformation formulas for a function $f(z)$ of the form

$$f(z)=\prod_{1\neq n|N}\eta_n(z)^{r_n}$$

where N is a natural number and $r_n\in\boldsymbol{Z}$ under the action of $\Gamma_0(N)$. The action of its Atkin-Lehner involutions on $\eta_n(z)$ with $n|N$ is also investigated.

Before we discuss the product of cyclotomic Dedekind η-functions any further, let us make a brief review on the so called *Frame shapes* of a finite group.

Let Λ be the Leech lattice and G be the group of all isometries (the Conway group .0). Let $\varphi_g(x)$ be the characteristic polynomial of g acting on Λ. Then $\varphi_g(x)$ can be written

$$\varphi_g(x)=\prod_{t\leq 24}(x^t-1)^{s_t}, \qquad s_t\in\boldsymbol{Z}.$$

The symbol $\pi_g=\prod_t t^{s_t}$ is called the *Frame shape* of g on Λ (see [2], [3], [4]). Using the Frame shape π_g of g, define

$$\eta_g(z)=\prod_t \eta(tz)^{s_t}.$$

It can be shown that there exists an integer N such that $\eta_g(z)$ is a modular form of *level* N, *weight* $(1/2)\sum s_t$, and with some character. Furthermore, if g fixes no element of Λ, then $\sum s_t=0$, and $\eta_g(z)$ is a modular function of $\Gamma_0(N)$. Similarly, we can define the Frame shapes and the corresponding products of the Dedekind η-functions for a finite group G with respect to a representation over the field $\boldsymbol{Q}$ of the rational numbers.

The notion of Frame shapes is a special case of the following. A symbol $\pi=\prod_t t^{s_t}$ is called a *generalized permutation* if $t\in\boldsymbol{N}$, $s_t\in\boldsymbol{Z}$, and $s_t=0$ except for finitely many t. Define

$$\eta_\pi(z)=\prod_t \eta(tz)^{s_t}.$$

It is easy to show (Lemma 2.2) that if π is a generalized permutation, then $\eta_\pi(z)$ can be expressed uniquely as a product of some cyclotomic Dedekind η-functions.

For a function

$$f(z)=\prod_t \eta(tz)^{s_t}$$

we define

(1) *degree* of $f(z)=\deg(f(z))=\sum_t ts_t$,

(2) *weight* of $f(z)=wt(f(z))=(1/2)\sum_t s_t$.

If $f(z)=\eta_{\pi_g}(z)$, $g\in G=$Conway group .0 acting on Λ, then $\sum ts_t=24$ for all g and $\sum s_t$ is the number of eigenvalue 1. Since $\sum_{d|n}\mu(n/d)=0$ if $n\geqq 2$, the cyclotomic Dedekind functions $\eta_n(z)$ have weight 0. By the transformation formula, we see that $\eta(z)=\eta_1(z)$ is of weight 1/2. The cyclotomic Dedekind η-functions may be viewed as building blocks of more general product of the Dedekind η-functions. Modular functions or forms have been studied in connection with the Frame shapes of finite group (the Frame shapes of the pair Conway group—Leech lattice in particular). The author, however, believes that it is worthwhile to investigate the subject through a new product defined above. Some motivation of such an investigation will be given later in the Introduction.

Among other things, the following theorem will be proved. In the theorem, $\varphi(n)$ is the Euler function and $\rho(n)=\prod_{p|n}(1-p)$, where p ranges over the primes dividing n.

(2.9) THEOREM. *Let*

$$f(z)=\prod_{\substack{n|N\\ n>1}}\eta_n(z)^{r_n},$$

where N is a natural number and $r_n\in\boldsymbol{Z}$. Then $f(z)$ is invariant under $\Gamma_0(N)$ if and only if the following conditions hold:

(1) $\sum_{n|N}\varphi(n)r_n\equiv 0 \pmod{24}$,

(2) $\sum_{n|N}\frac{N}{n}\rho(n)r_n\equiv 0 \pmod{24}$,

(3) $\sum_{n|p^e}r_n\equiv 0 \pmod 2$ *for all prime divisors p of N, and $p^e|N$, $(p, N/p^e)=1$.*

In this paper, only modular *functions* of the shape

$$\prod\eta_n(t)^{r_n} \qquad n\geqq 2$$

are discussed. If $\eta_1(x)=\eta(z)$ is allowed in the product, then we obtain modular *forms* (of positive weights). J. Mason has made a very interesting observation: Let σ be an element of .0 and let

$$f_\sigma(z)=\prod\eta_n(t)^{r_n}$$

be the corresponding product associated with σ. Moreover, assume $r_1\neq 0$. Then $f_\sigma(z)$ fails to be an eigenfunction of the Hecke operator $T(p)$ for a prime p if and only if the order of σ is divisible by p but $r_p=0$, namely η_p does not appear in the expression of $f_\sigma(z)$. There are only 12 conjugacy classes which have this property. Relations between multiplicative forms and the products in terms of cyclotomic Dedekind functions appear to be worth being investigated.

§ 1. Dedekind eta function and its transformation formula —known results.

For $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z})$, we have

$$\eta(Az)=v(A)\sqrt{cz+d}\,\eta(z) .$$

Here $\sqrt{cz+d}$ is chosen so that $-\frac{\pi}{2}<\arg\sqrt{cz+d}\leqq\frac{\pi}{2}$ and

$$v(A)=\begin{cases} \left(\dfrac{d}{c}\right)^{*} \exp\left\{\dfrac{\pi i}{12}[(a+d)c-db(c^2-1)-3c]\right\}, & \text{if } c \text{ is odd} \\ \left(\dfrac{c}{d}\right)_{*} \exp\left\{\dfrac{\pi i}{12}[(a+d)c-bd(c^2-1)+3d-3-3cd]\right\}, & \text{if } c \text{ is even.} \end{cases}$$
$(i=\sqrt{-1})$

The symbols $(d/c)^*$ and $(c/d)_*$ are extensions of Jacobi symbol. Define

$$\left(\frac{d}{c}\right)^{*}=\left(\frac{d}{|c|}\right), \quad \left(\frac{0}{\pm 1}\right)^{*}=1 ,$$

$$\left(\frac{c}{d}\right)_{*}=\varepsilon\left(\frac{c}{|d|}\right) \qquad \text{where } \varepsilon=(-1)^{\frac{\operatorname{sgn}(c)-1}{2}\cdot\frac{\operatorname{sgn}(d)-1}{2}} ,$$

$$\left(\frac{0}{\pm 1}\right)_{*}=1 .$$

In particular, $(d/c)^*=1/(d/c)^*$ and $(c/d)_*=1/(c/d)_*$ for all possible c, d. This fact will be used freely. We also put $(a/1)=1$ for all $a\in\boldsymbol{Z}$.

The following lemma, which has probably been known, is useful when $v(A)$ is calculated for many $A\in SL_2(\boldsymbol{Z})$ simultaneously.

(1.1) LEMMA. *Suppose that c and d are both odd. Then the first expression of $v(M)$ and the second are equal. Hence,*

$$v(A)=\begin{cases} \left(\dfrac{d}{c}\right)^{*} \exp\left\{\dfrac{\pi i}{12}[(a+d)c-bd(c^2-1)-3c]\right\} , & \textit{if } c \textit{ is odd} \\ \left(\dfrac{c}{d}\right)_{*} \exp\left\{\dfrac{\pi i}{12}[(a+d)c-bd(c^2-1)+3d-3-3cd]\right\}, & \textit{if } d \textit{ is odd.} \end{cases}$$

PROOF. Suppose that c, d are both odd. We need only to show

$$\left(\frac{d}{c}\right)^* \exp\left\{\frac{\pi i}{12}(-3c)\right\} = \left(\frac{c}{d}\right)_* \exp\left\{\frac{\pi i}{12}(3d-3-3cd)\right\}. \tag{1}$$

By definition

$$\left(\frac{c}{d}\right)_* \Big/ \left(\frac{d}{c}\right)^* = \left(\frac{c}{|d|}\right)(-1)^{\frac{\operatorname{sgn}(c)-1}{2}\cdot\frac{\operatorname{sgn}(d)-1}{2}}\left(\frac{d}{|c|}\right)$$
$$= (-1)^{((c-1)/2)\cdot((d-1)/2)}.$$

The last equality was obtained by the Quadratic Reciprocity Law for Jacobi symbols:

$$\left(\frac{n}{|m|}\right)\left(\frac{m}{|n|}\right) = (-1)^{\frac{m-1}{2}\cdot\frac{n-1}{2}+\frac{\operatorname{sgn}(m)-1}{2}\cdot\frac{\operatorname{sgn}(n)-1}{2}}.$$

Therefore, the ratio of the right hand side and the left of (1) is equal to

$$(-1)^{\frac{c-1}{2}\cdot\frac{d-1}{2}}\cdot\exp\left\{\frac{\pi i}{4}(c-1)(d-1)\right\} = (-1)^{\frac{c-1}{2}\cdot\frac{d-1}{2}}(-1)^{\frac{c-1}{2}\cdot\frac{d-1}{2}} = 1$$

as desired.

§ 2. Cyclotomic Dedekind η functions.

Let $\Phi_n(x)$ be the n-th cyclotomic polynomial. Then

$$\Phi_n(x) = \prod_{d|n}(x^d-1)^{\mu(n/d)}.$$

Define the n-th *cyclotomic Dedekind η-function* as follows

$$\eta_n(z) = \prod_{d|n}\eta(dz)^{\mu(n/d)}.$$

(2.1) LEMMA. *We have*

(1) $\deg(\eta_n(z)) = \varphi(n)$, *where φ is the Euler function, and*;

(2) $wt(\eta_1(z)) = 1/2$ *and* $wt(\eta_n(z)) = 0$ *if* $n \geqq 2$.

PROOF. By definition,

$$\deg(\eta_n(z)) = \sum_{d|n} d\mu\left(\frac{n}{d}\right) = \varphi(n).$$

Since $\eta_1(z) = \eta(z)$, $wt(\eta_1(z)) = 1/2$. Suppose $n \geqq 2$. Then

$$wt(\eta_n(z)) = \sum_{d|n}\mu\left(\frac{n}{d}\right) = 0.$$

(2.2) LEMMA. *Let $\pi=\prod t^{s_t}$ be a generalized permutation and $\eta_\pi(z)=\prod \eta(t)^{s_t}$. Then $\eta_\pi(z)$ can be expressed uniquely as a product of cyclotomic Dedekind η-functions.*

PROOF. Let t_0 be the largest t appearing in π (with $s_{t_0}\neq 0$). Consider

$$\eta_\pi(z)\eta_{t_0}(z)^{-s_{t_0}}=\prod_{t'} \eta(t')^{s_{t'}}\,.$$

All t' appearing on the right hand side are smaller than t_0. Hence by induction, $\eta_\pi(z)$ is expressible as a product of $\eta_n(z)$'s. We note that $\eta_1(z)=\eta(z)$.

We next show that such an expression obtained above is unique for a given π. It suffices to show that if

$$\prod_n \eta_n(z)^{s_n}=1\,,$$

then all $s_n=0$. The following argument is due to Koike [2]. Suppose false and let n_0 be the largest number with $s_{n_0}\neq 0$. Leave only $\eta(nz)^{s_n}$ on the left side and move all others to the right side, then one gets

$$\eta(nz)^{s_n}=\prod_t \eta(tz)^{s'_t}\,. \tag{2}$$

Expand both sides as power series in $q=\exp(\pi i/12)z$. First we must have

$$ns_t=\sum_t ts'_t\,.$$

Factoring out the power of q, we get

$$1-s_nq^n+\cdots=1-s'_{t_0}q^{t_0}+\cdots$$

where t_0 is the smallest t appearing on the right side of (2) with $s_t\neq 0$. This is impossible. This proves the uniqueness of the expression. This completes the proof of the lemma.

Lemmas 2.1 and 2.2 show that $\eta_\pi(z)$, for a generalized permutation π, can be expressed as a product of $\eta_1(z)^{2wt(\pi)}$ and some functions $\eta_n(z)$ of weight 0. We will investigate transformation formulas of $\eta_n(z)$ under some congruence subgroups and its Atkin-Lehner involutions.

(2.3) DEFINITION. $\rho(n)=\prod_{p|n}(1-p)$ where n is a natural number and p runs over all prime divisors of n. Define $\rho(1)=1$.

(2.4) LEMMA. *We have*

(1) $\sum_{d|n} d\mu(d)=\rho(n)$,

(2) $\prod_{d|n} d^{|\mu(d)|}$ *is a square if n is not a power of a prime.*

PROOF. Let $n=p_1^{e_1}\cdots p_r^{e_r}$, p_i distinct primes. The function $f(d)=d\mu(d)$ from $\boldsymbol{N}$ to $\boldsymbol{Z}$ is obviously multiplicative and so

$$\sum_{d|n} d\mu(d)=\prod_{i=1}^{r}\left(\sum_{j=0}^{ei} p_i^j\,\mu(p_i^j)\right)=\prod_{i=1}^{r}(1-p_i)=\rho(n)\,.$$

As for (2), we may assume $r\geqq 2$ and $e_1=\cdots=e_r=1$, and so

$$\prod_{d|n} d^{|\mu(d)|}=\prod_{d|n} d=(p_1\cdots p_r)^e$$

for some e by symmetry.

$$\prod_{d|n} d=\prod p_i \prod (p_ip_j)\cdots\prod(p_1\cdots p_n)$$

and so

$$e=1+\binom{r-1}{1}+\binom{r-1}{2}+\cdots+\binom{r-1}{r-1}$$
$$=(1+1)^{r-1}=2^{r-1}\equiv 0 \pmod 2\,.$$

(2.5) LEMMA. *Suppose $n\geqq 2$. Then the following holds.*

(1) *Suppose cn is odd and $d\neq 0$. Then*

$$\prod_{m|n}\left(\frac{d}{cm}\right)^{*\mu(m)}=\begin{cases}\left(\dfrac{d}{p}\right), & \text{if } n \text{ is a power of a prime } p,\\ 1, & \text{otherwise.}\end{cases}$$

(2) *Suppose d is odd and $c\neq 0$. Then*

$$\prod_{m|n}\left(\frac{cm}{d}\right)_*^{\mu(m)}=\begin{cases}\left(\dfrac{p}{|d|}\right), & \text{if } n \text{ is a power of a prime } p\\ 1, & \text{otherwise.}\end{cases}$$

PROOF. Suppose $n=p^e$, cn odd, and $d\neq 0$. Then

$$\prod_{m|p^e}\left(\frac{d}{cm}\right)^{*\mu(m)}=\left(\frac{d}{c}\right)^*\left(\frac{d}{cp}\right)^*=\left(\frac{d}{|c|}\right)\left(\frac{d}{|c|p}\right)=\left(\frac{d}{p}\right).$$

Suppose $n=p^e$, d odd, and $c\neq 0$. Then

$$\prod_{m\mid p^e}\left(\frac{cm}{d}\right)_*^{\mu(m)}=\left(\frac{c}{d}\right)_*\left(\frac{cp}{d}\right)_*=\left(\frac{c}{|d|}\right)\left(\frac{cp}{|d|}\right)(-1)^{2\cdot\frac{\operatorname{sgn}(c)-1}{2}\cdot\frac{\operatorname{sgn}(d)-1}{2}}$$
$$=\left(\frac{p}{|d|}\right).$$

Next we assume that n is not a power of a prime. Suppose cn odd and $d\neq 0$. Then

$$\prod_{m\mid n}\left(\frac{d}{cm}\right)^{*\mu(m)}=\prod_{m\mid n}\left(\frac{d}{|c|m}\right)^{\mu(m)}=\left(\frac{d}{|c|}\right)^{\Sigma\mu(m)}\cdot\prod_{m\mid n}\left(\frac{d}{m}\right)^{\mu(m)}$$
$$=\prod_{m\mid n}\left(\frac{d}{m}\right)^{|\mu(m)|}=1$$

by Lemma 2.4. If d is odd and $c\neq 0$, then

$$\prod_{m\mid n}\left(\frac{cm}{d}\right)_*^{\mu(m)}=\prod_{m\mid n}\left[\left(\frac{cm}{|d|}\right)(-1)^{\frac{\operatorname{sgn}(c)-1}{2}\cdot\frac{\operatorname{sgn}(d)-1}{2}}\right]^{\mu(m)}$$
$$=\prod_{m\mid n}\left(\frac{cm}{|d|}\right)^{\mu(m)}=\left(\frac{c}{|d|}\right)^{\Sigma\mu(m)}\cdot\prod_{m\mid n}\left(\frac{m}{|d|}\right)^{|\mu(m)|}=1$$

again by Lemma 2.4. This completes the proof.

(2.6) THEOREM. *Suppose* $n\geqq 2$, $n|N$ *and* $A=\begin{pmatrix}a & b\\ cN & d\end{pmatrix}\in\Gamma_0(N)$. *Then*

$$\eta_n(Az)=\varepsilon\exp\frac{\pi i}{12}B\eta_n(z)$$

where

$$\varepsilon=\begin{cases}\left(\frac{d}{p}\right), & \text{if } cN \text{ odd, and } n \text{ is a power of a prime } p,\\[2mm] \left(\frac{p}{|d|}\right), & \text{if } d \text{ odd, and } n \text{ is a power of a prime } p,\\[2mm] 1, & \text{otherwise}\end{cases}$$

and

$$B=\begin{cases}c(a+d-bcdN-3)\frac{N}{n}\rho(n)+bd\varphi(n), & \text{if } cN \text{ odd}\\[2mm] c(a+d-bcdN-3d)\frac{N}{n}\rho(n)+bd\varphi(n), & \text{if } d \text{ odd}\end{cases}$$

where $\rho(n)=\prod_{p\mid n}(1-p)$, *and* $\varphi(n)$ *is the Euler function.*

PROOF. We first note that $d\neq 0$, since $ad-bcN=1$ and $N\geqq 2$. By definition

$$\eta_n(Az)=\prod_{m\mid n}\eta\left(m\frac{az+b}{cNz+d}\right)^{\mu(n/m)}$$

$$=\prod_{m\mid n}\eta\left(\frac{a(mz)+mb}{\frac{cN}{m}(mz)+d}\right)^{\mu(n/m)}$$

$$=\prod_{m\mid n}\left\{v\begin{pmatrix} a & mb \\ \frac{cN}{m} & d\end{pmatrix}(cNz+d)^{1/2}\eta(mz)\right\}^{\mu(n/m)}$$

$$=\left[\prod_{m\mid n}\left\{v\begin{pmatrix} a & mb \\ \frac{cN}{m} & d\end{pmatrix}\right\}^{\mu(n/m)}\right]\eta_n(z)\,.$$

Put $k=\prod_{m\mid n}v\begin{pmatrix} a & bm \\ \frac{cN}{m} & d\end{pmatrix}^{\mu(n/m)}$.

Case 1. cN odd.

In this case, we have

$$k=\left[\prod_{m\mid n}\left(\frac{d}{\frac{cN}{m}}\right)^{*\mu(n/m)}\right]\exp\frac{\pi i}{12}B$$

where

$$B=\sum_{m\mid n}\left[(a+d)\frac{cN}{m}-bmd\left(\frac{c^2N^2}{m^2}-1\right)-\frac{3cN}{m}\right]\mu\left(\frac{n}{m}\right).$$

We compute, by Lemma 2.5,

$$\prod_{m\mid n}\left(\frac{d}{\frac{cN}{m}}\right)^{*\mu(n/m)}=\prod_{m\mid n}\left(\frac{d}{\frac{cNm}{n}}\right)^{*\mu(m)}=\begin{cases}\left(\frac{d}{p}\right), & \text{if } n \text{ is a power of a prime } p,\\ 1\,, & \text{otherwise.}\end{cases} \tag{2}$$

We next compute, by Lemma 2.3,

$$B=\sum_{m\mid n}\left[c(a+d-bcdN-3)\frac{N}{m}\,\mu\left(\frac{n}{m}\right)+bdm\;\mu\left(\frac{n}{m}\right)\right]$$

$$=c(a+d-bcdN-3)\frac{N}{n}\rho(n)+bd\;\varphi(n)\,. \tag{3}$$

Combine (2) and (3) to obtain the assertion of the theorem if cN is odd.

Case 2. d odd and $c\neq 0$.

In this case, we have

$$k=\left[\prod_{m\mid n}\left(\frac{\frac{cN}{m}}{d}\right)_*^{\mu(n/m)}\right]\exp\frac{\pi i}{12}B$$

where

$$B=\sum_{m\mid n}\left[(a+d)\frac{cN}{m}-bmd\left(\frac{c^2N^2}{m^2}-1\right)+3d-3-3\frac{cNd}{m}\right]\mu\left(\frac{n}{m}\right).$$

We compute, by Lemma 2.5,

$$\prod_{m\mid n}\left(\frac{\frac{cN}{m}}{d}\right)_*^{\mu(n/m)}=\prod_{m\mid n}\left(\frac{\frac{cNm}{n}}{d}\right)_*^{\mu(m)}=\begin{cases}\left(\dfrac{p}{|d|}\right), & \text{if } n \text{ is a power of a prime } p,\\ 1\,, & \text{otherwise.}\end{cases}$$

We next compute, by Lemma 2.4,

$$B=\sum_{m\mid n}\left[c(a+d-bcdN-3d)\frac{N}{m}\mu\left(\frac{n}{m}\right)+bdm\,\mu\left(\frac{n}{m}\right)+(3d-3)\mu\left(\frac{n}{m}\right)\right]$$

$$=c(a+d-bcdN-3d)\frac{N}{n}\rho(n)+bd\,\varphi(n)\,.$$

Finally we treat:

Case 3. $c=0$.

In this case $a=d=1$ or $a=d=-1$. Therefore,

$$\eta_n(Az)=\eta_n(z+bd)=\prod_{m\mid n}\eta(mz+bdm)^{\mu(n/m)}$$

$$=\exp\frac{\pi i}{12}B\cdot\eta_n(z)$$

where

$$B=bd\sum_{m\mid n}m\mu\left(\frac{n}{m}\right)=bd\,\varphi(n)\,.$$

Therefore, this case is subsumed in the previous cases (Note $(p/1)=1$).

(2.7) DEFINITION. For each $n\geqq 2$, and N with $n\mid N$,

$$\Gamma_0(N,n)=\{A\in\Gamma_0(N)\mid \eta_n(Az)=\eta_n(z)\}\,.$$

(2.8) COROLLARY. *Let* $l=24/(24,\rho(N))$. *Then*

$$\Gamma_0(N)/\Gamma_0(N, N) \cong \begin{cases} \boldsymbol{Z}/2\boldsymbol{Z}, & \textit{if } N=p^e \textit{ for some prime } p \textit{ and } e \textit{ and } l=1, \\ \boldsymbol{Z}/6\boldsymbol{Z}, & \textit{if } N=p^e \textit{ for some prime } p \textit{ for } e \textit{ and } l=3, \\ \boldsymbol{Z}/l\boldsymbol{Z}, & \textit{otherwise.} \end{cases}$$

PROOF. By Theorem 2.6, if $A\in\Gamma_0(N)$, then we have

$$\eta_N(Az)=k\eta_N(z)$$

where k is a 24-th root of unity. Since $\varphi(N)\equiv 0 \pmod{\prod_{p|N}(p-1)}$, k is a l-th root of unity unless $N=p^e$ and l is odd. Suppose $N=p^e$ and l is odd. Then $p\equiv 1 \pmod 8$. Choose d such that $(d/p)=-1$. Then there exist integers a, b such that $ad-bN=1$. Putting $A=\begin{pmatrix} a & b \\ N & d\end{pmatrix}$, we get $k^{2l}=1\neq k^l$ in this case. Therefore, the order of $\Gamma_0(N)/\Gamma_0(N, N)$ is at most as stated in the corollary.

Put $A=\begin{pmatrix} 1 & 0 \\ N & 1\end{pmatrix}$. Then the corresponding k is equal to

$$\exp\frac{\pi i}{12}(\pm\rho(N)) .$$

Hence, $\Gamma_0(N)/\Gamma_0(N, N)$ contains an element of order l. This proves the result.

(2.9) THEOREM. *Let*

$$f(z)=\prod_{\substack{n|N \\ n>1}}\eta_n(z)^{r_n} ,$$

where N is a natural number and $r_n\in\boldsymbol{Z}$. Then $f(z)$ is invariant under $\Gamma_0(N)$ if and only if the following conditions hold:

(1) $\sum_{n|N}\varphi(n)r_n\equiv 0 \pmod{24}$,

(2) $\sum_{n|N}\frac{N}{n}\rho(n)r_n\equiv 0 \pmod{24}$,

(3) $\sum_{n|p^e} r_n\equiv 0 \pmod 2$ *for all prime divisors p of N, and $p^e|N$, $(p, N/p^e)=1$.*

PROOF. Let ε, B be the constants appearing in Theorem (2.6). Put $\varepsilon_n=\varepsilon$ and $B_n=B$. Then for an element $\begin{pmatrix} a & b \\ cN & d\end{pmatrix}\in\Gamma_0(N)$, we have

$$f(Az)=Kf(z)$$

where

$$K=\left[\prod_{n\mid N}\varepsilon_n^{r_n}\right]\exp\left[\frac{\pi i}{12}\sum_{n\mid N}B_n r_n\right].$$

We compute

$$\sum_{n\mid N}B_n r_n=\sum_{n\mid N}\left[c(a+d-bcdN-3\delta)\frac{N}{n}\rho(n)r_n+bd\,\varphi(n)r_n\right],$$

where $\delta=1$ if cN is odd, and $\delta=d$, if d is odd, and

$$\prod_{n\mid N}\varepsilon_n^{r_n}=\prod_{i=1}^{r}\left(\frac{d}{p_i}\right)^{\Sigma r_n},\quad \text{or}\quad \prod_{i=1}^{r}\left(\frac{p_i}{|d|}\right)^{\Sigma r_n}$$

where the summation ranges over all divisors of $p_i^{e_i}$ if $N=p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}$. It is obvious that the conditions (1), (2), and (3) are sufficient.

To prove necessity, consider the action of

$$A_1=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\in\Gamma_0(N).$$

Then $\sum B_n r_n=\sum\varphi(n)r_n$ and $\prod\varepsilon_n^{r_n}=1$.

Hence $\sum\varphi(n)r_n\equiv 0 \pmod{24}$. Next consider

$$A_2=\begin{pmatrix}1 & 0\\ N & 1\end{pmatrix}.$$

We have $\prod\varepsilon_n^{r_n}=1$ and $\sum B_n r_n=-\sum(N/n)\rho(n)r_n$. Hence (2) must hold. To shows that (3) is necessary, choose $d(>0)$ such that d is not a quadratic residue modulo a fixed prime p_i, $1\leqq i\leqq r$, but d is a quadratic residue mod p_j, $j\neq i$, $1\leqq j\leqq r$. The Chinese Remainder Theorem guarantees the existence of such d. Since $(d, N)=1$, there exist a, b with $ad-bN=1$. Applying

$$A_3=\begin{pmatrix}a & b\\ N & d\end{pmatrix},$$

we obtain

$$\prod\varepsilon_n^{r_n}=\prod_{p_i}\varepsilon_n^{r_n}=(-1)^{\Sigma r_n}\qquad \text{if } N \text{ is odd.}$$

Hence (3) is necessary if N is odd. Suppose N is even. Then d is odd and so

$$\prod\varepsilon_n^{r_n}=\prod_{i=1}^{r}\left(\frac{p_i}{|d|}\right)^{\Sigma r_n}$$

where $\sum$ ranges over all divisors of $p_i^{e_i}$, $1\leqq i\leqq r$. We have

$$\left(\frac{2}{|d|}\right)=\begin{cases} 1, & \text{if } d\equiv 1,7 \mod 8 \\ -1, & \text{if } d\equiv 3,5 \mod 8 \end{cases}$$

$$\left(\frac{p_i}{|d|}\right)=\left(\frac{|d|}{p_i}\right), \quad \text{if } p_i\equiv 1 \mod 4$$

$$\left(\frac{p_i}{|d|}\right)=-\left(\frac{|d|}{p_i}\right), \quad \text{if } p_i\equiv 3 \mod 4 .$$

Therefore the Chinese Remainder Theorem guarantees the existence of d such that

$$\left(\frac{p_i}{|d|}\right)=-1$$

for only one i. Hence (3) is necessary for this case also. This completes the proof of the theorem.

§ 3. Action of Atkin-Lehner involutions on cyclotomic Dedekind η-functions.

If a pair of natural numbers e and N satisfies $e|N$ and $(e, N/e)=1$, then e is said to be a strict divisor of N. We write $e\|N$.

(3.1) DEFINITION. Let N be a natural number and e be a strict divisor of N. Define

$$W_e=\begin{pmatrix} ae & b \\ cN & de \end{pmatrix}, \quad a, b, c, d\in \boldsymbol{Z}, \quad ade-bc\frac{N}{e}=1 .$$

W_e is an *Atkin-Lehner* involution of $\Gamma_0(N)$. W_e normalizes $\Gamma_0(N)$ modulo the scalar matrices.

Let t be a positive integer. Then one computes directly that

$$\eta(t W_e(z))=v\begin{pmatrix} a\delta & bt/\delta \\ cN\delta/et & de/\delta \end{pmatrix}\left(\frac{cNz+de}{\delta}\right)^{1/2}\eta\left(\frac{et}{\delta^2}z\right)$$

where $\delta=(e, t)$. We will apply W_e to the cyclotomic Dedekind η-function $\eta_n(z)$ with $n\geqq 2$ and $n|N$. Since W_e for a general e is a product of some Atkin-Lehner involutions associated with powers of primes, we assume $e=p^r$ and $N=p^r\cdot f$, $(f, p)=1$. For a natural number m such that $m|N$, let m_p be a power of p such that $m_p\|m$ and $m_f=m/m_p$.

(3.2) DEFINITION. We write $f(z)\sim g(z)$ if a function $f(z)$ is a constant multiple of another function $g(z)$.

(3.3) THEOREM. *Suppose* $e=p^r$, $e\|N$, $N=e\cdot f$, $n\geqq 2$, *and* $n|N$. *Then*

$$\eta_n(W_e(z))\sim\begin{cases}\prod_{j=0}^{r}\eta_{p^jn}(z)\,, & \text{if } p\nmid n\,,\\ [\eta_{epn_f/n_p}(z)]^{-1}\,, & \text{otherwise.}\end{cases}$$

We first prove the following lemma.

(3.4) LEMMA. *Under the same assumption as above,*

$$\eta_n(W_e(z))\sim\begin{cases}\eta_n(ez)\,, & \text{if } p\nmid n\\ \left[\eta_{pn_f}\left(\dfrac{e}{n_p}z\right)\right]^{1}, & \text{if } p|n\,.\end{cases}$$

PROOF.

$$\eta_n(W_e(z))\sim\prod_{m|n}\eta\left(\frac{em}{m_p^2}z\right)^{\mu(n/m)}$$

$$=\prod_{m|n}\eta\left(\frac{em_f}{m_p}z\right)^{\mu(n_p/m_p)\mu(n_f/m_f)}.$$

Suppose $p\nmid n$. Then $n_p=1=m_p$. Hence

$$\eta_n(W_e(z))\sim\prod_{m|n}\eta(emz)^{\mu(n/m)}=\eta_n(ez)\,.$$

Suppose $n_p=p^{r'}$, $r'\geqq 1$. It suffices to consider only m with $m_p=p^{r'}$ or $m_p=p^{r'-1}$. Hence

$$\frac{em_f}{m_p}=p^{r-r'}m_f \quad\text{or}\quad p^{r-r'+1}m_f\,.$$

Therefore,

$$\eta_n(W_e(z))\sim\prod_{m|n_f}\eta(p^{r-r'}mz)^{\mu(n_f/m)}\prod_{m|n_f}\eta(p^{r-r'+1}mz)^{-\mu(n_f/m)}$$

$$=\left[\prod_{m|pn_f}\eta(p^{r-r'}mz)^{\mu(pn_f/m)}\right]^{-1}$$

$$=\eta_{pn_f}(p^{r-r'}z)^{-1}$$

$$=\eta_{pn_f}\left(\frac{e}{n_p}z\right)^{-1}.$$

This completes the proof of the lemma.

PROOF OF THEOREM (3.3). First suppose $p\nmid n$.

$$\eta_n(ez)=\prod_{m\mid n}\eta(mez)^{\mu(n/m)}=\prod_{m\mid n}\eta(mez)^{\mu(ne/me)}$$

$$=\prod_{m\mid ne}\eta(mz)^{\mu(ne/m)}/\prod_{\substack{m\mid ne\\ e\nmid m}}\eta(mz)^{\mu(ne/m)}$$

$$=\eta_{ne}(z)/(*)$$

where

$$(*)=\prod_{\substack{m\mid np^r\\ m_p\leq p^{r-1}}}\eta(mz)^{\mu(ne/m)}=\prod_{\substack{m\mid np^r\\ m_p=p^{r-1}}}\eta(mz)^{\mu(ne/m)}$$

$$=\prod_{m\mid n}\eta(p^{r-1}mz)^{-\mu(n/m)}=[\eta_n(p^{r-1}z)]^{-1}\,.$$

Hence

$$\eta_n(ez)=\eta_n(p^rz)=\eta_{np^r}(z)\eta_n(p^{r-1}z)\,.$$

Repeating this process, we get

$$\eta_n(W_e(z))\sim\eta_n(ez)=\prod_{j=0}^{r}\eta_{np^j}(z)\,.$$

Suppose next $n=p^{r'}\cdot n_f$, $r'\geqq 1$. We compute

$$\eta_{pn_f}(p^{r-r'}z)=\prod_{m\mid pn_f}\eta(p^{r-r'}mz)^{\mu(pn_f/m)}$$

$$=\prod_{m\mid pn_f}\eta(p^{r-r'}mz)^{\mu(p^{r-r'+1}\cdot n_f/p^{r-r'}\cdot m)}$$

$$=\eta_{p^{r-r'+1}n_f}(z)/(*)$$

where

$$(*)=\prod_{\substack{m\mid p^{r-r'+1}n_f\\ p^{r-r'}\nmid m}}\eta(mz)^{\mu(p^{r-r'+1}n_f/m)}=1\,.$$

Hence

$$\eta_{pn_r}(p^{r-r'}z)=\eta_{p^{r-r'+1}n_f}(z)\,.$$

Therefore,

$$\eta_n(W_e(z))\sim\eta_{p^{r-r'+1}n_f}(z)^{-1}\,,\qquad \text{if } p\mid n\,.$$

This completes the proof of Theorem 3.3.

(3.5) COROLLARY. *We have*

(1) $\eta_n(W_p(z))\sim\eta_{pn}(z)\eta_n(z)$, *if* $e=p$, $n\mid N$, *and* $p\nmid n$.

(2) $\eta_p(W_e(z))\sim\eta_e(z)^{-1}$, *if* $n=p$, *and* $e=p^r$.

(3) $\eta_e(W_e(z))\sim\eta_p(z)^{-1}$, *if* $e=p^r$.

To make $f(z)\sim g(z)$ into $f(z)=Cg(z)$ with an explicit constant C, we need to calculate the Petersson constants. Let us assume as before $n\geqq 2$, $n|N$, $e\|N$, $e=p^{r}$, $N=ef$. Then $\eta_n(W_e(z))=C\cdot f(z)$ where $f(z)$ is the function appearing in Theorem 3.3, and

$$C=\prod_{m|n} v\begin{pmatrix} a(e,m) & bm/(e,m) \\ cN(e,m)/em & de/(e,m)\end{pmatrix}^{\mu(n/m)}$$

$$=\prod_{m|n} v\begin{pmatrix} am_p & bm_f \\ cf/m_f & de/m_p\end{pmatrix}^{\mu(n/m)}.$$

Suppose first:

Case 1. cf odd.

If $d=0$, then $\det(W_e)=-bcN=e$ and so $N=e$, and $bc=-1$. Hence $n=p^{r'}$, $r'\geqq 1$. In this case, we have

$$C=v\begin{pmatrix} an & b \\ c & 0\end{pmatrix} v\begin{pmatrix} an/p & b \\ c & 0\end{pmatrix}^{-1}$$

$$=\exp\frac{\pi i}{12}\left[(anc-3c)-\left(\frac{an}{p}c-3c\right)\right]$$

$$=\exp\left[\frac{\pi i}{12}ac\,\varphi(n)\right].$$

We next assume $d\neq 0$. Then

$$C=\left[\prod_{m|n}\left(\frac{\dfrac{de}{m_p}}{\dfrac{cf}{m_f}}\right)^{*\,\mu(n/m)}\right]\exp\frac{\pi i}{12}B=C_1\exp\frac{\pi i}{12}B$$

where

$$B=\sum_{m|n}\left[\left(am_p+\frac{de}{m_p}\right)\frac{cf}{m_f}-bm_f\frac{de}{m_p}\left(\frac{c^2f^2}{m_f^2}-1\right)-3\frac{cf}{m_f}\right]\mu\left(\frac{n}{m}\right).$$

We first compute C_1.

$$C_1=\prod_{m|n}\left(\frac{\dfrac{de}{m_p}}{\dfrac{|c|f}{m_f}}\right)^{\mu(n/m)}=\prod_{m|n}\left[\left(\frac{d}{|c|}\right)\left(\frac{d}{\dfrac{f}{m_f}}\right)\left(\frac{\dfrac{e}{m_p}}{|c|}\right)\left(\frac{\dfrac{e}{m_p}}{\dfrac{f}{m_f}}\right)\right]^{\mu(n/m)}$$

$$=\prod_{m|n}\left[\left(\frac{d}{|c|}\right)\left(\frac{d}{f}\right)\left(\frac{d}{m_f}\right)\left(\frac{e}{|c|}\right)\left(\frac{m_p}{|c|}\right)\left(\frac{e}{f}\right)\left(\frac{m_p}{f}\right)\left(\frac{e}{m_f}\right)\left(\frac{m_p}{m_f}\right)\right]^{\mu(n/m)}.$$

We have $\sum_{m|n} \mu(n/m)=0$, since $n \geqq 2$ and by Lemma 2.4 $\prod_{m|n} m^{|\mu(m)|}$ is a square if n is not a power of a prime. We divide it into four subcases

Subcase 1. $n=p^{r'}$. So $n_f=1$

$$C_1=\prod_{m|n}\left[\left(\frac{m_p}{|c|}\right)\left(\frac{m_p}{f}\right)\right]^{\mu(n/m)}$$

$$=\left(\frac{p^{r'}}{|c|}\right)\left(\frac{p^{r'}}{f}\right)\left(\frac{p^{r'-1}}{|c|}\right)\left(\frac{p^{r'-1}}{f}\right)=\left(\frac{p}{|c|}\right)\left(\frac{p}{f}\right)=\left(\frac{p}{|c|f}\right).$$

Subcase 2. $n=q^s$, q a prime, $q \neq p$.

$$C_1=\prod_{m|n}\left[\left(\frac{d}{m_f}\right)\left(\frac{e}{m_f}\right)\right]^{\mu(n/m)}=\left(\frac{de}{q^s}\right)\left(\frac{de}{q^{s-1}}\right)=\left(\frac{de}{q}\right).$$

Subcase 3. $n=p^{r'}q^s$.

$$C_1=\prod_{m|n}\left[\left(\frac{d}{m_f}\right)\left(\frac{m_p}{|c|}\right)\left(\frac{m_p}{f}\right)\left(\frac{e}{m_f}\right)\left(\frac{m_p}{m_f}\right)\right]^{\mu(n/m)}.$$

It suffices to consider only m such that $p^{r'-1}q^{s'-1}|m$. Hence

$$C_1=\left(\frac{p^{r'}}{q^{s'}}\right)\left(\frac{p^{r'-1}}{q^{s'}}\right)\left(\frac{p^{r'}}{q^{s'-1}}\right)\left(\frac{p^{r'\ 1}}{q^{s'-1}}\right)=\left(\frac{p}{q^{s'}}\right)\left(\frac{p}{q^{s'-1}}\right)=\left(\frac{p}{q}\right).$$

Subcase 4. *n is divisible by three distinct primes.*

$$C_1=\prod_{m|n}\left(\frac{m_p}{m_f}\right)^{\mu(n/m)}=\prod_{m|n_f}\left[\left(\frac{p^{r'}}{m}\right)\left(\frac{p^{r'-1}}{m}\right)\right]^{\mu(n_f/m)}$$

$$=\prod_{m|n_f}\left(\frac{p}{m}\right)^{\mu(n_f/m)}$$

$$=1.$$

Summarizing, we obtain (still under the assumption that cf is odd),

$$C_1=\begin{cases}\left(\frac{p}{|c|f}\right), & \text{if } n \text{ is a power of } p, \\ \left(\frac{de}{q}\right), & \text{if } n \text{ is a power of a prime } q\neq p, \\ \left(\frac{p}{q}\right), & \text{if } n=p^r q^s,\ p\neq q, \\ 1, & \text{otherwise.}\end{cases}$$

We next calculate B. We need the following lemma.

(3.6) LEMMA. *Let N be a natural number and e be a strict divisor of N. Put $f=N/e$. If n is a divisor of N, write $n_e=(n, e)$ and $n_f=(n, f)$. Under this notation, the following holds.*

(1) $$\sum_{m\mid n}\frac{m_e}{m_f}\mu\left(\frac{n}{m}\right)=\frac{1}{n_f}\varphi(n_e)\rho(n_f).$$

(2) $$\sum_{m\mid n}\frac{1}{m_f}\mu\left(\frac{n}{m}\right)=\begin{cases}0, & \text{if } n_e>1\\ \dfrac{1}{n_f}\rho(n_f), & \text{if } n_e=1.\end{cases}$$

PROOF.

(1) $$\sum_{m\mid n}\frac{m_e}{m_f}\mu\left(\frac{n}{m}\right)=\sum_{m\mid n}m_e\mu\left(\frac{n_e}{m_e}\right)\cdot\frac{1}{m_f}\mu\left(\frac{n_f}{m_f}\right)$$
$$=\left(\sum_{m\mid n_e}m_e\mu\left(\frac{n_e}{m}\right)\right)\left(\sum_{m\mid n_f}\frac{1}{m}\mu\left(\frac{n_f}{m}\right)\right)$$
$$=\varphi(n_e)\cdot\frac{1}{n_f}\sum_{m\mid n_f}m\mu(m)$$
$$=\frac{1}{n_f}\varphi(n_e)\rho(n_f)$$

by Lemma 2.4.

(2) $$\sum_{m\mid n}\frac{1}{m_f}\mu\left(\frac{n}{m}\right)=\sum_{m\mid n}\frac{1}{m_f}\mu\left(\frac{n_f}{m_f}\right)\mu\left(\frac{n_e}{m_e}\right)$$
$$=\left[\frac{1}{n_f}\sum_{m\mid n_f}m\,\mu(m)\right]\left[\sum_{m\mid n_e}\mu(m)\right]$$
$$=\frac{1}{n_f}\rho(n_f)=\begin{cases}0, & \text{if } n_e>1\\ 1, & \text{if } n_e=1.\end{cases}$$

This completes the proof of Lemma 3.6.

We can now compute B.

$$B=\sum_{m\mid n}\left[acm_p\frac{f}{m_f}+dc\frac{N}{m}-bc^2df\frac{N}{m}+bd\frac{e}{m_p}m_f-3c\frac{f}{m_f}\right]\mu\left(\frac{n}{m}\right).$$

Apply Lemma 3.6 with $(e, f)=(e, f)$, or $(1, N)$, or (f, e) to conclude

$$B=ac\varphi(n_p)\frac{f}{n_f}\rho(n_f)+(dc-bdc^2f)\frac{N}{n}\rho(n)+bd\frac{e}{n_p}\rho(n_p)\varphi(n_f)-3c\delta\,,$$

where $\delta=0$ if $n_p>1$, $\delta=(f/n)\rho(n)$ if $n_p=1$.

Since $\rho(n)|\varphi(n)$ for an arbitrary n and so

$$B\equiv 0 \mod \rho(n)\,.$$

Recall $l=24/(24,\rho(n))$. We conclude that

$$\begin{cases} C^2=1\,, & \text{if } l=1\,,\\ C^6=1\,, & \text{if } l=3\,,\\ C^l=1\,, & \text{otherwise}. \end{cases}$$

Next we treat

Case 2. *de* odd.

First we note that $c\neq 0$, since $e>1$. Let

$$C_1=\left[\prod_{m|n}\left(\frac{\frac{cf}{m_f}}{\frac{de}{m_p}}\right)_*^{\mu(n/m)}\right].$$

Then

$$C_1=\left[\prod_{m|n}\left(\frac{\frac{cf}{m_f}}{\frac{|d|e}{m_p}}\right)^{\mu(n/m)}\right]$$

$$=\prod_{m|n}\left[\left(\frac{c}{|d|}\right)\left(\frac{c}{e}\right)\left(\frac{c}{m_p}\right)\left(\frac{f}{|d|}\right)\left(\frac{f}{e}\right)\left(\frac{f}{m_p}\right)\left(\frac{m_f}{|d|}\right)\left(\frac{m_f}{e}\right)\left(\frac{m_f}{m_p}\right)\right]^{\mu(n/m)}$$

$$=\prod_{m|n}\left[\left(\frac{c}{m_p}\right)\left(\frac{f}{m_p}\right)\left(\frac{m_f}{|d|}\right)\left(\frac{m_f}{e}\right)\left(\frac{m_f}{m_p}\right)\right]^{\mu(n/m)}.$$

Subcase 1. $n=p^{r'}$.

$$C_1=\prod_{m|n}\left[\left(\frac{c}{m_p}\right)\left(\frac{f}{m_p}\right)\right]^{\mu(n/m)}=\left(\frac{cf}{p}\right).$$

Subcase 2. $n=q^s$, q a prime $q\neq p$.

$$C_1=\prod_{m|n}\left[\left(\frac{m_f}{|d|}\right)\left(\frac{m_f}{e}\right)\right]^{\mu(n/m)}=\left(\frac{q}{|d|e}\right).$$

Subcase 3. $n=p^{r'}q^{s}$.

$$C_1=\left(\frac{q}{p}\right)$$

just as in the case in which cf is odd.

Subcase 4. n is divisible by three distinct primes

$$C_1=1$$

again by an analogous argument as in the case in which cf odd.

Summarizing, we have

$$C_1=\begin{cases}\left(\dfrac{cf}{p}\right), & \text{if } n \text{ is a power of } p,\\[2mm] \left(\dfrac{q}{|d|e}\right), & \text{if } n \text{ is a prime of } q,\ q\neq p,\\[2mm] \left(\dfrac{q}{p}\right), & \text{if } n=p^{r'}q^{s},\ p\neq q,\\[2mm] 1, & \text{otherwise.}\end{cases}$$

Let us next calculate B when de is odd.

$$B=\sum_{m|n}\left[\left(am_p+\frac{de}{m_p}\right)\frac{cf}{m_f}-bm_f\frac{de}{m_p}\left(\frac{c^2f^2}{m_f^2}-1\right)+3\frac{de}{m_p}-3-3\frac{cf}{m_f}\frac{de}{m_p}\right]\mu\left(\frac{n}{m}\right)$$

$$=ac\varphi(n_p)\frac{f}{n_f}\rho(n_f)+(dc-bdc^2f)\frac{N}{n}\rho(n)$$

$$+bd\frac{e}{n_p}\rho(n_p)\varphi(n_f)+3de\delta_1-3cd\frac{N}{n}\rho(n)$$

where

$$\delta_1=\begin{cases}0, & \text{if } n_f>1,\\[2mm] \dfrac{1}{n_p}\rho(n_p), & \text{if } n_f=1.\end{cases}$$

Therefore, we conclude again

$$B\equiv 0 \pmod{\rho(n)}.$$

Summarizing all the results above, we obtain

(3.6) THEOREM. *Suppose* $e=p^r$, $e\| N$, $f=N/e$, $n\geqq 2$ *and* $n|N$. *Then*

$\eta_n(W_e(z))=C\cdot f(z)$ *where*

$$f(z)=\begin{cases}\prod_{j=0}^{r}\eta_{p^j n}(z)\,, & \text{if } p\nmid n\,,\\ [\eta_{epn_f/n_p}(z)]^{-1}\,, & \text{if } p|n\,,\end{cases}$$

and $C=C_1\exp(\pi i/12)B$ *where*

$$C_1=\begin{cases}\text{cf odd} & & \text{de odd} & \\ \left(\frac{p}{|c|f}\right) & \text{or} & \left(\frac{cf}{p}\right), & \text{if } n|e\,,\\ \left(\frac{de}{q}\right) & \text{or} & \left(\frac{q}{|d|e}\right), & \text{if } n=q^b\ (q\neq p)\,,\\ \left(\frac{p}{q}\right) & \text{or} & \left(\frac{q}{p}\right), & \text{if } n=p^a q^b\ (p\neq q)\,,\\ & 1 & , & \text{otherwise}\end{cases}$$

and

$$B=ac\,\varphi(n_p)\frac{f}{n_f}\rho(n_f)+(dc-bdc^2f)\frac{N}{n}\rho(n)+bd\frac{e}{n_p}\rho(n_p)\varphi(n_f)+B_1$$

where

$$B_1=\begin{cases}0\,, & \text{if cf odd, and } n|f\,,\\ -3c\frac{f}{n}\rho(n)\,, & \text{if cf odd, and } n\nmid f\,,\\ -3cd\frac{N}{n}\rho(n)\,, & \text{if de odd, and } n|e\,,\\ -3cd\frac{N}{n}\rho(n)+3\frac{de}{n}\rho(n)\,, & \text{if de odd, and } n\nmid e.\end{cases}$$

(3.7) COROLLARY. *Under the same assumption as in Theorem* (3.6), *we have*

$$\begin{cases}C^2=1\,, & \text{if } l=1\\ C^6=1\,, & \text{if } l=3\\ C^l=1\,, & \text{otherwise}\end{cases}$$

where $l=24/(24,\rho(n))$.

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