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# *Fisher's inequality for block designs with finite group action*

Dedicated to Professor Nagayoshi Iwahori on the occasion of his 60th birthday

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## 1. Introduction.

A *block design* with parameters $(v, b, r, k, \lambda)$ is a pair $(X, B)$ with a $v$-set of *points* and with a family $B$ of $k$-subsets of $X$ (called *blocks*) such that any two points are contained in exactly $\lambda$ blocks, any points are contained in exactly $r$ blocks, and there are $b$ blocks. The *order* of the block design is defined to be the integer $n := r - \lambda$. These parameters satisfy the following relations:

$$(1) \qquad vr = bk, \qquad (v-1)\lambda = r(k-1).$$

The *incidence matrix* of the block design $(X, B)$ is the matrix $A$ of size $X \times B$ of which $(x, \beta)$-entry is 1 if $x \in \beta$ and 0 otherwise. Then the condition for block designs can be expressed by the incidence equations:

$$(2)\qquad \begin{aligned} &AJ=rJ, \qquad JA=kJ, \\ &AA^t=nI+\lambda J, \end{aligned}$$

where $I$ is the identity matrix and $J$ is the matrix all of whose entries are 1. From this we derive that if $\lambda<r$, then

$$(3)\qquad \det(AA^t)=rkn^{v-1}\neq 0,$$

which yields famous Fisher's inequality:

$$(4)\qquad v\leqq b.$$

A block design such that $v=b$ is called a *symmetric design*.

Next assume that a finite group $G$ acts on a block design $(X, B)$, that is, $G$ acts on $X$ and this action preserves $B$. In this case, we can again apply the above argument and obtain inequalities like as Fisher's one. Let $R$ be a commutative ring in which $rkn$ is invertible. Then the composition

$$RX \xrightarrow{\alpha'} RB \xrightarrow{\alpha} RX$$

of the linear maps $\alpha$, $\alpha'$ corresponding to the matrices $A$, $A^t$ is an $RG$-isomorphism. Thus

$$(5)\qquad RX\,|\,RB, \quad \text{and} \quad RX\cong RB \quad \text{for a symmetric design,}$$

where $RX\,|\,RB$ means that $RX$ is isomorphic to a direct summand of $RB$. This relation does not occur as frequently as it looks. It is a difficult problem to decide the condition for finite $G$-sets $X$ and $B$ to satisfy $RX\,|\,RB$ or $RX\cong RB$. Comparing the ranks of $G$-fixed-point submodules in (5), we have the following well-known orbit theorem ([De 68], 2.3.1 ; [Ts 82], Theorem 1.5.8):

$$(6)\qquad |X/G|\leqq |B/G|, \quad \text{and the equality holds for a symmetric design.}$$

(Remember that $(RX)^G\cong R[X/G]$.) In particular, if $G$ is transitive on $B$, then it is transitive on $X$.

We can furthermore derive various inequalities from (5). To show it, let $M:\mathbf{Hec}(G, R)\rightarrow\mathbf{Mod}_R$ be an $R$-additive functor of the category of permutation $RG$-modules (*Hecke category*) to the category of $RG$-modules. Then (5) yields that $M(RX)$ is isomorphic to a direct summand of $M(RB)$. Such a functor $M$ is called a *Hecke functor*, which corresponds with a cohomological $G$-functor ([Yo 83b]).

In Section 2, we apply this idea to the Ext-functor and prove Theorem

A. The following is a part of Theorem A.

THEOREM a. *Let $(P, B)$ be a block design with parameters $(v, b, r, k, \lambda)$ and with order $n := r-\lambda$ on which a finite group $G$ acts. Let $R$ be a commutative ring in which $rn$ is invertible and let $M$ be an $RG$-module. Then for any nonnegative integer $m$,*

$$\prod_{x \in X/G} H^m(G_x, M) \mid \prod_{\beta \in B/G} H^m(G_\beta, M).$$

*When the design is symmetric, the above cohomology groups are isomorphic.*

For example, when $m=1$ and $M=R$, the theorem yields several relations about the commutator groups which can be considered as weak forms of transfer theorems for finite groups. See Section 2d. Furthermore, (6) follows immediately from the case where $m=0$ and $M$ is a trivial $RG$-module. Almost all statements in Theorem A follows directly from Theorem B, but in order to prove Theorem A, the knowledge of category theory is unnecessary.

In Section 3, we construct a category which can substitute for the Hecke category. In fact, it is the $\boldsymbol{Z}_{(p)}$-additive category $\mathbf{Mc}=\mathbf{Mc}(G, e^p_{G,1}\Omega_{(p)})$ which is accompanied by an isomorphism-reflecting functor $\Phi$ into the Hecke category, where $\Omega$ is the Burnside ring functor and $e^p_{G,1}$ is a primitive idempotent of the Burnside ring of $G$ localized at $p$. This category is a kind of the *Mackey categories*. An object of $\mathbf{Mc}$ is a finite $G$-set and the hom-set of $Y$ to $X$ is generated by the pairs of $G$-maps $[X \xleftarrow{\lambda} A \xrightarrow{\mu} Y]$.

Now, let $(X, B)$ be a block design with action of a finite group $G$, and put $F := \{(x, \beta) \in X \times B \mid x \in \beta\}$ (the set of flags), so that there are canonical $G$-maps of $F$ to $X$ and $B$. Thus we obtain two morphisms in $\mathbf{Mc}$, that is, $\alpha := [X \leftarrow F \rightarrow B]$ and its transpose $\alpha' := [B \leftarrow F \rightarrow X]$ which are mapped to the incidence matrix and its transpose by the functor $\Phi$. Since $\Phi$ reflects isomorphisms, we have that if the prime $p$ does not divide $nr$, then $\alpha \circ \alpha'$ is an isomorphism. Thus we can apply the argument in the case of Hecke functors to additive functors from $\mathbf{Mc}$. An additive functor of $\mathbf{Mc}$ to the module category is called a *poly-Hecke functor*. Hence we have the following theorem which is a part of Theorem B.

THEOREM b. *Assume that a finite group $G$ acts on a block design $(X, B)$ with parameters $(v, b, r, k, \lambda)$ and order $n$. Let $p$ be a prime which does not divide $nr$ and let $\boldsymbol{M}$ be a poly-Hecke functor. Then $\boldsymbol{M}(X)$ is isomorphic to a direct summand of $\boldsymbol{M}(B)$. When the design is symmetric, $\boldsymbol{M}(X)$ and $\boldsymbol{M}(B)$ are isomorphic.*

Applying this theorem to the poly-Hecke functor given by the Burnside rings and character rings and counting the ranks, we have the following:

COROLLARY b.1. *We use the same notation as in the theorem. Let $P$ be a normal $p$-subgroup or a cyclic and central $p$-subgroup of $G$. Assume that the prime $p$ does not divide $n$. Then the following hold:*

(a) *If $p \nmid r$, then $|X^P/G| \leqq |B^P/G|$.*

(b) *If the design is symmetric, then $|X^P/G| = |B^P/G|$.*

COROLLARY b.2. *Assume that $p$ does not divide $nr$. Then*

$$\sum_{x \in X/G} c_p(G_x) \leqq \sum_{\beta \in B/G} c_p(G_\beta),$$

*where $c_p(H)$ is the number of $H$-conjugate classes of $p$-elements of the finite group $H$. For a symmetric design, the equality holds (even if $p|r$).*

These results are, in the more generalized forms, stated in Section 4.

*Notation and Terminology.*

For notation and terminology, we will refer the following books and papers: [De 68], [La 83], [Ts 82] for block designs with group action; [CE 56], [HS 71], [We 69] for homological algebra and cohomology theory of finite groups; [Ma 71] for category theory; [Dr 71], [Di 79], [Yo 80] for Mackey functors and $G$-functors; [Go 68] for finite group theory.

In particular, $(X, B)$ is a block design with parameters $(v, b, r, k, \lambda)$, $G$ is a finite group, $R$ is a commutative ring with identity element, and $p$ is a prime. A set having $n$ elements is called an $n$-set. The notation $H \leqq G$ means that $H$ is a subgroup of $G$. We put $A^g := g^{-1}Ag$ for $A \subseteqq G$ and $g \in G$. $O^p(G)$ denotes the normal subgroup generated by all $p'$-elements, and $G'$ denotes the commutator subgroup of $G$. For a finite (right) $G$-set $X$, the set of $G$-orbits is denoted by $X/G$ and the stabilizer of $x \in X$ is denoted by $G_x$. The ring $\boldsymbol{Z}_{(p)}$ is defined to be the subring of $\boldsymbol{Q}$ consisting of fraction $m/n$ with $n$ prime to $p$. For a set $X$, the free $R$-module over $X$ is denoted by $RX$ or $R[X]$. The hom-set of $X$ to $Y$ in a category $\boldsymbol{C}$ is denoted by $\boldsymbol{C}(X, Y)$. If $f: A \to B$ and $g: B \to C$, we write $g \circ f$ (not $f \circ g$) for the composed morphism $A \to C$, and we use $1_X$ (or simply 1) for the identity morphism of an object $X$.

## 2. Fisher's inequality for cohomology of groups.

In this section we shall prove some results about cohomology groups of finite groups acting on block designs which are prototypes of Theorem B.

We first summarize some well-known facts and notation about cohomology theory of finite groups in order to give a typical example of Hecke functors, and then we state Theorem A and its corollaries which are regarded as some sort of Fisher's inequality.

Throughout this section, we let $G$ be a finite group and $R$ a commutative ring with identity element (possibly $R=0$). Furthermore we introduce the following notation. For an $R$-module $N$, an integer $a$ and a rational prime $p$, we put

$$ {}_aN := \{x \in N \mid ax = 0\}, \qquad N/(a) := N/aN, $$
$$ N_{(p)} := \boldsymbol{Z}_{(p)} \otimes_Z N. $$

Note that if $N$ is a torsion group as abelian group, then $N_{(p)}$ is the $p$-torsion part of $N$, and that if $N$ is a finite abelian group, then $N_{(p)}$ is a Sylow $p$-subgroup of $N$. When $N$ is an $RG$-module, we write

$$ N^G := \{x \in N \mid xg = x \text{ for all } g \in G\}. $$

For two modules $M$ and $N$, the notation $M \mid N$ means that $M$ is isomorphic to a direct summand of $N$.

## a. Cohomology theory of finite groups.

Let $H^m(G, M)$ denote the $m$-th cohomology group of $G$ with coefficient $RG$-module $M$. Then there are familiar maps of three kinds for $H \leqq G$ and $g \in G$:

$$ \mathrm{res}_H : H^m(G, M) \longrightarrow H^m(H, M), $$
$$ \mathrm{cor}^G : H^m(H, M) \longrightarrow H^m(G, M), $$
$$ \mathrm{con}^g : H^m(H, M) \longrightarrow H^m(H^g, M) \qquad (H^g := g^{-1}Hg). $$

Of course there are the corresponding maps for the Tate cohomology groups $\hat{H}^m(H, M)$, $m \in \boldsymbol{Z}$. These maps satisfy the axioms of $G$-functors (see Section 3. b).

For any finite $G$-set $X$ and any $RG$-module $M$, put

$$ H^m(X; M) := \left\{ (\xi_x) \in \prod_{x \in X} H^m(G_x, M) \mid \xi_{xg} = \mathrm{con}^g(\xi_x) \right\}. $$

Since

$$ \mathrm{Ext}^0_{RG}(RX, M) = \mathrm{Hom}_{RG}(RX, M) \cong H^0(X; M), $$

we have that for any $m \geqq 0$,

$$ \mathrm{Ext}^m_{RG}(RX, M) \cong H^m(X; M). $$

Let $X$ and $Y$ be finite $G$-sets and let $\alpha : RY \to RX$ be an $RG$-homomorphism corresponding to a matrix $(a_{xy})$, that is,

$$\alpha : RY \longrightarrow RX ;\ y \longmapsto \sum_{x\in X} a_{xy} x .$$

Then the map $\alpha^* : \mathrm{Ext}^m_{RG}(RX, M) \to \mathrm{Ext}^m_{RG}(RY, M)$ induced by $\alpha$ yields, through the above isomorphism, the following map:

$$\alpha^* : H^m(X; M) \longrightarrow H^m(Y; M) ;\ (\xi_x) \longmapsto (\eta_y) ,$$

$$\eta_y := \sum_{x\in X/G_y} a_{xy} \operatorname{cor}^{G_y} \circ \operatorname{res}_{G_{xy}}(\xi_x) .$$

The functor $X \mapsto \mathrm{Ext}^m_{RG}(RX, M)$ becomes a Hecke functor corresponding to the $G$-functor $H \mapsto H^m(H, M)$. See Section 3. a and c.

## b. Incidence maps of block designs with group action.

We consider a block design $(X, B)$ with parameters $(v, b, r, k, \lambda)$ and with order $n$ on which a finite group $G$ acts. Two $RG$-homomorphisms corresponding to the incidence matrix $A$ and its transpose $A^t$ are given by

$$\alpha : RB \longrightarrow RX ;\ \beta \longmapsto \hat{\beta} := \sum_{x\in\beta} x ,$$

$$\alpha' : RX \longrightarrow RB ;\ x \longmapsto \hat{x} := \sum_{\beta\ni x} \beta .$$

Furthermore we define "(co-)augmentation maps" by

$$\varepsilon_X : RX \longrightarrow R ;\ x \longmapsto 1 ,$$

$$\varepsilon_B : RB \longrightarrow R ;\ \beta \longmapsto 1 ,$$

$$\varepsilon'_X : R \longrightarrow RX ;\ 1 \longmapsto \sum_{x\in X} x ,$$

$$\varepsilon'_B : R \longrightarrow RB ;\ 1 \longmapsto \sum_{\beta\in B} \beta .$$

Here $x \in X$ and $\beta \in B$. Then the following lemma follows immediately from the definition of block designs.

LEMMA 2.1. (i) $\alpha \circ \alpha' = n \cdot \mathrm{id} + \lambda \varepsilon'_X \circ \varepsilon_X$.

(ii) $\varepsilon_X \circ \alpha = k \varepsilon_B$, $\varepsilon_B \circ \alpha' = r \varepsilon_X$.

(iii) $\alpha \circ \varepsilon'_B = r \varepsilon'_X$, $\alpha' \circ \varepsilon'_X = k \varepsilon'_B$.

(iv) $\varepsilon_X \circ \varepsilon'_X = v \cdot \mathrm{id}_R$, $\varepsilon_B \circ \varepsilon'_B = b \cdot \mathrm{id}_R$.

(v) *The following diagram is commutative and its vertical lines are exact.*

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
n\cdot \mathrm{id} : \mathrm{Ker}\, \varepsilon_X & \longrightarrow & \mathrm{Ker}\, \varepsilon_B & \longrightarrow & \mathrm{Ker}\, \varepsilon_X \\
\downarrow & & \downarrow & & \downarrow \\
RX & \xrightarrow{\alpha'} & RB & \xrightarrow{\alpha} & RX \\
\varepsilon_X \downarrow & & \downarrow \varepsilon_B & & \downarrow \varepsilon_X \\
R & \xrightarrow{r} & R & \xrightarrow{k} & R \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0\,.
\end{array}
$$

(vi) *The following diagram is commutative and its vertical lines are exact.*

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
R & \xrightarrow{k} & R & \xrightarrow{r} & R \\
\varepsilon'_X \downarrow & & \downarrow \varepsilon'_B & & \downarrow \varepsilon'_X \\
RX & \xrightarrow{\alpha'} & RB & \xrightarrow{\alpha} & RX \\
\downarrow & & \downarrow & & \downarrow \\
n\cdot \mathrm{id} : \mathrm{Cok}\, \varepsilon'_X & \longrightarrow & \mathrm{Cok}\, \varepsilon'_B & \longrightarrow & \mathrm{Cok}\, \varepsilon'_X \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0\,.
\end{array}
$$

LEMMA 2.2. *Assume that the design $(X, B)$ is symmetric, that is, $v=b$. If $nr$ is invertible in $R$, then $\alpha$ and $\alpha'$ induce an isomorphism $RX \cong RB$. If $n$ is invertible, then $\alpha$ and $\alpha'$ induce $RG$-isomorphisms $\mathrm{Ker}\, \varepsilon_X \cong \mathrm{Ker}\, \varepsilon_B$ and $\mathrm{Cok}\, \varepsilon'_X \cong \mathrm{Cok}\, \varepsilon'_B$.*

PROOF. Assume that $nr$ is invertible in $R$. In order to prove that $\alpha' : RX \to RB$ is an $RG$-isomorphism, we may assume that $R=\boldsymbol{Z}[1/nr]$. Then $\alpha \circ \alpha'$ is an $RG$-automorphism of $RX$ by the reason as in Introduction, and so $\alpha'$ is a split monomorphism. Since $R$-modules $RX$ and $RB$ have the same ranks, we conclude that $\alpha'$ is an isomorphism. Assume next that $n$ is invertible. The vertical lines of Lemma 2.1 (v) and (vi) are $R$-split. Thus we may assume that $R=\boldsymbol{Z}[1/n]$. Since $n\cdot\mathrm{id}$ is an automorphism and since $\mathrm{Ker}\, \varepsilon_X$, $\mathrm{Ker}\, \varepsilon_B$, $\mathrm{Cok}\, \varepsilon'_X$ and $\mathrm{Cok}\, \varepsilon'_B$ are equal in rank, the statement of the lemma holds.

## c. Theorem A and its proof.

PROPOSITION 2.3. *Assume that $n$ is invertible in $R$. Let $N$ be an $RG$-module and let $m$ be a nonnegative integer. Then the following hold:*

(i) *The following sequence of $R$-modules is exact:*

$$0 \longrightarrow {}_{rk}\mathrm{Ext}^m_{RG}(R, N) \xrightarrow{\bar{\varepsilon}_X^*} \mathrm{Ext}^m_{RG}(RX, N) \xrightarrow{(\alpha\alpha')^*}$$
$$\longrightarrow \mathrm{Ext}^m_{RG}(RX, N) \xrightarrow{\bar{\varepsilon}_X'^*} \mathrm{Ext}^m_{RG}(R, N)/(rk) \longrightarrow 0,$$

*where $\bar{\varepsilon}_X^*$ and $\bar{\varepsilon}_X'^*$ are maps induced by $\varepsilon_X$ and $\varepsilon_X'$. In particular, if $r\cdot\mathrm{id}$ is an isomorphism on $\mathrm{Ext}^m_{RG}(R, N)$, then $\alpha^*$ induces*

$$\mathrm{Ext}^m_{RG}(RX, N) \mid \mathrm{Ext}^m_{RG}(RB, N).$$

(ii) *$\alpha^*$ induces $\mathrm{Ker}\,\varepsilon_X'^* \mid \mathrm{Ker}\,\varepsilon_B'^*$ and $\mathrm{Cok}\,\varepsilon_X^* \mid \mathrm{Cok}\,\varepsilon_B^*$.*

(iii) *When the design $(X, B)$ is symmetric, that is, $v=b$, the above relations of direct summand give isomorphisms.*

(iv) $r\cdot\mathrm{Ker}(\alpha^* : \mathrm{Ext}^m_{RG}(RX, N) \longrightarrow \mathrm{Ext}^m_{RG}(RB, N))=0,$

$r\cdot\mathrm{Cok}\,(\alpha'^* : \mathrm{Ext}^m_{RG}(RB, N) \longrightarrow \mathrm{Ext}^m_{RG}(RX, N))=0.$

(v) *If $b\cdot\mathrm{id}$ is invertible on $\mathrm{Ext}^m_{RG}(R, N)$, then*

$$\mathrm{Ker}\,\alpha^* \cong {}_k\mathrm{Ext}^m_{RG}(R, N) = {}_r\mathrm{Ext}^m_{RG}(R, N).$$

PROOF. For any $RG$-module $M$, we put

$$E^m(M) := \mathrm{Ext}^m_{RG}(M, N).$$

(i) Applying long exact sequence of Ext to the exact sequences of the both sides of the exact sequence of the diagram of Lemma 2.1 (v) and (vi), we have the diagrams of $R$-modules as in Figures 1 and 2 with exact vertical and horizontal lines. By an easy diagram chase, we conclude that

$$\varepsilon_0 : {}_{rk}E^m(R) \longrightarrow \mathrm{Ker}\,(\alpha\alpha')^*$$

and

$$\varepsilon_0' : \mathrm{Cok}\,(\alpha\alpha')^* \longrightarrow E^m(R)/(rk)$$

are isomorphisms. This prove the first part of (i). Assume next that $r\cdot\mathrm{id}$ is isomorphism on $E^m(R)$. Since $nr=k(r^2-b\lambda)$, we then have that ${}_{rk}E^m(R) =E^m(R)/(rk)=0$, and so $\alpha'^*\circ\alpha^* : E^m(RX)\to E^m(RB)\to E^m(RX)$ is an isomorphism, as required.

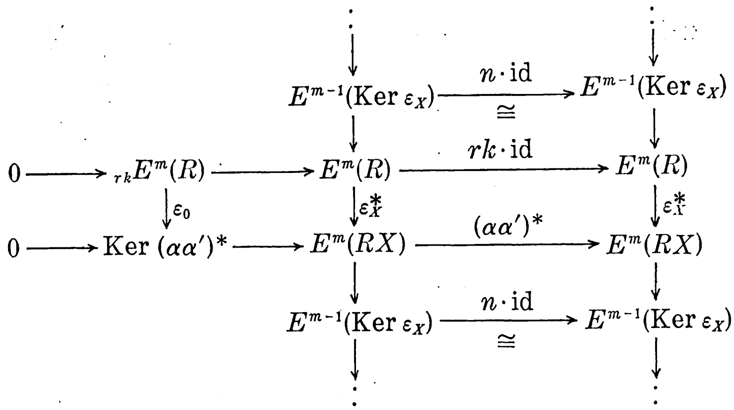


Figure 1.

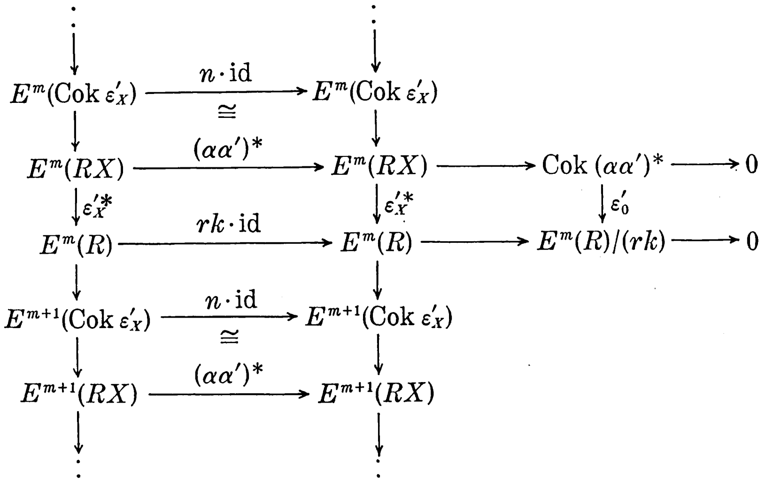


Figure 2.

(ii) In fact, $(\alpha\cdot\alpha')^*$ induces automorphisms $n\cdot\mathrm{id}$ on $\mathrm{Ker}\,\varepsilon'^*_X$ and on $\mathrm{Cok}\,\varepsilon^*_X$. See Figures 3 and 4.

(iii) This follows immediately from Lemma 2.2 and long exact sequences given by Lemma 2.1 (v), (vi).

(iv) Consider the commutative diagrams in Figures 3 and 4 induced by the diagram of Lemma 2.1 (v), (vi). Note that $n\cdot\mathrm{id}$ is an isomorphism.

(v) Since $k|nr$ and $bk=vr$, we have that ${}_rE^m(R)={}_kE^m(R)$. By Lemma 2.1 (ii),

$$\alpha^*\circ\varepsilon^*_X=(\varepsilon_X\circ\alpha)^*=k\varepsilon^*_B : E^m(R) \longrightarrow E^m(RB)\,,$$

$$\begin{array}{ccccc}
n\cdot\mathrm{id}: E^m(\mathrm{Cok}\,\varepsilon'_X) & \longrightarrow & E^m(\mathrm{Cok}\,\varepsilon'_B) & \longrightarrow & E^m(\mathrm{Cok}\,\varepsilon'_X) \\
\downarrow & & \downarrow & & \downarrow \\
E^m(RX) & \xrightarrow{\alpha^*} & E^m(RB) & \xrightarrow{\alpha'^*} & E^m(RX) \\
\downarrow{\scriptstyle \varepsilon'^*_X} & & \downarrow{\scriptstyle \varepsilon'^*_B} & & \downarrow{\scriptstyle \varepsilon'^*_X} \\
E^m(R) & \xrightarrow{r\cdot\mathrm{id}} & E^m(R) & \xrightarrow{k\cdot\mathrm{id}} & E^m(R)
\end{array}$$

Figure 3.

$$\begin{array}{ccccc}
E^m(R) & \xrightarrow{k\cdot\mathrm{id}} & E^m(R) & \xrightarrow{r\cdot\mathrm{id}} & E^m(R) \\
\downarrow{\scriptstyle \varepsilon^*_X} & & \downarrow{\scriptstyle \varepsilon^*_B} & & \downarrow{\scriptstyle \varepsilon^*_X} \\
E^m(RX) & \xrightarrow{\alpha^*} & E^m(RX) & \xrightarrow{\alpha'^*} & E^m(RX) \\
\downarrow & & \downarrow & & \downarrow \\
n\cdot\mathrm{id}: E^m(\mathrm{Ker}\,\varepsilon_X) & \longrightarrow & E^m(\mathrm{Ker}\,\varepsilon_B) & \longrightarrow & E^m(\mathrm{Ker}\,\varepsilon_X)
\end{array}$$

Figure 4.

and so

$$\varepsilon^*_X({}_kE^m(R)) \subseteqq \mathrm{Ker}\,\alpha^* \subseteqq \mathrm{Ker}\,(\alpha\circ\alpha')^*\,.$$

Now the conclusion follows from (ii) and the injectivity of $\varepsilon^*_X$ on ${}_{rk}E^m(R)$.

Now, the $R$-module $H^m(X; N)$ ($\cong \mathrm{Ext}^m(RX, N)$) defined in Section 2. a is isomorphic to $\prod_{x\in X/G} H^m(G_x, N)$, where $x$ runs over a complete set of representatives of $G$-orbits. In this view point, the maps defined in Section 2. b induce the following.

$$\begin{aligned}
\mathrm{cor}^G_X : \prod_{x\in X/G} H^m(G_x, N)_{(p)} &\longrightarrow H^m(G, N)_{(p)}\,; \\
(\xi_x)_{x\in X/G} &\longmapsto \sum_{x\in X/G} \mathrm{cor}^G_{G_x}(\xi_x)\,, \\
\mathrm{res}^G_X : H^m(G, N)_{(p)} &\longrightarrow \prod_{x\in X/G} H^m(G_x, N)_{(p)}\,; \\
\eta &\longmapsto (\mathrm{res}^G_{G_x}(\eta))_{x\in X/G}\,, \\
\alpha^* : \prod_{x\in X/G} H^m(G_x, N)_{(p)} &\longrightarrow \prod_{\beta\in B/G} H^m(G_\beta, N)_{(p)}\,; \\
(\xi_x) &\longmapsto (\zeta_\beta)\,, \\
\zeta_\beta := \sum_{x\in X/G} \sum_{\substack{g\in G_x\backslash G/G_\beta \\ : xg\in\beta}} &\mathrm{cor}^{G_\beta}\circ\mathrm{res}_{G_{xg},\beta}\circ\mathrm{con}^g(\xi_x)\,.
\end{aligned}$$

Similarly, we can define $\mathrm{cor}^G_B$ and $\mathrm{res}^G_B$. The maps $\mathrm{cor}^G_X$ and $\mathrm{res}^G_X$ correspond to $\varepsilon'^*_X$ and $\varepsilon^*_X$, respectively.

THEOREM A. *Let $(X, B)$ be a block design with parameters $(v, b, r, k, \lambda)$ and with order $n$ on which a finite group $G$ acts. Let $m$ be an integer, $p$ a prime not dividing $n$ and $N$ an $RG$-module.*

(i) $\mathrm{Ker\,cor}_X^G \mid \mathrm{Ker\,cor}_B^G$,

$\mathrm{Cok\,res}_X^G \mid \mathrm{Cok\,res}_B^G$.

(ii) *Assume further that $p$ does not divide $r$. Then*

$$\prod_{x\in X/G} H^m(G_x, N)_{(p)} \,\Big|\, \prod_{\beta\in B/G} H^m(G_\beta, N)_{(p)}\,.$$

(iii) *Assume that $(X, B)$ is symmetric, that is, $v=b$, then the above direct summands give isomorphisms.*

*The maps induced by $\alpha^*$ gives the injections for direct summands and the isomorphism. Furthermore, these relations hold also for the Tate cohomology groups.*

PROOF. We put $R' := \boldsymbol{Z}_{(p)}\otimes_{\boldsymbol{Z}} R$ and $N' := R'\otimes_R N$. Then $n$ is invertible in the ring $R'$, and for any subgroup $H$, there are natural isomorphisms

$$H^m(H, N') \cong R'\otimes_R H^m(H, N) \cong H^m(H, N)_{(p)}\,.$$

Thus we may assume from the beginning that $R=R'$. Then the theorem follows immediately from Proposition 2.2. Next we consider about the Tate cohomology groups. Let $I$ be the augmentation ideal of $RG$ and $H$ a subgroup of $G$. Then by the dimension shifters, Shapiro's lemma and Frobenius reciprocity, we have that $\hat{H}^m(H, N) \cong \hat{H}^{m+1}(H, I\otimes_R N)$. (See [We 69], 4.1.6 and 3.7.14). This isomorphism is commutative with corestrictions, restrictions and conjugations. Since the Tate cohomology groups of positive degree coincide the ordinary ones, the results in this case follow from the case of ordinary cohomology groups. The theorem is proved.

## d. Corollaries for Theorem A.

COROLLARY A.1. *Under the same notation as in the theorem, assume that $G$ acts block-transitively on $(X, B)$.*

(i) *If $p$ does not divide $r$, then $H^m(G_x, N)$ is isomorphic to the direct summand of $H^m(G_\beta, N)$ together with the injection*

$$\alpha^* : \xi \longmapsto \sum_{\substack{g\in G_x\backslash G/G_\beta \\ : xg\in\beta}} \mathrm{cor}^G\circ\mathrm{res}_{G_{xg},\beta}\circ\mathrm{con}^g(\xi)\,.$$

(ii) *The above $\alpha^*$ gives*

$$\mathrm{Ker}\,(\mathrm{cor}_{G_x}^G) \mid \mathrm{Ker}\,(\mathrm{cor}_{G_\beta}^G)\,,$$

$$\mathrm{Cok}(\mathrm{res}^G_{G_x}) \mid \mathrm{Cok}(\mathrm{res}^G_{G_\beta}).$$

(iii) *If the design is symmetric, then* (i) *and* (ii) *give isomorphisms of* $H^m(G_x, N)$ *and* $H^m(G_\beta, N)$, *etc.*

First and second (co-)homology groups of finite groups are familiar objects in group theory. For example, we have that for any finite group $H$,

$$Ab(H) = \hat{H}^{-2}(H, \boldsymbol{Z}) \quad (= H/H', \text{ the abelianized group}),$$

$$M(H) = H^2(H, \boldsymbol{C}^*) \quad (\text{the Schur multiplier}).$$

(Here we denote by $H'$ the commutator group of $H$.) So Theorem A gives the following results for these cases.

COROLLARY A.2. *Let $p$ be a prime which does not divide $n$.*

(i) *Assume that $p$ does not divide $r$ or that the order of $Ab(G)$ is prime to $p$. Then*

$$\prod_{x \in X/G} Ab(G_x)_{(p)} \mid \prod_{\beta \in B/G} Ab(G_\beta)_{(p)}.$$

(ii) *Assume that $G$ acts block-transitively on $(X, B)$. Let $x \in X$ and $\beta \in B$. Let $P_x$ (resp. $P_\beta$) be a Sylow $p$-subgroup of $G_x$ (resp. $G_\beta$). Then*

$$\frac{P_x \cap G'}{P_x \cap G'_x} \,\Big|\, \frac{P_\beta \cap G'}{P_\beta \cap G'_\beta}.$$

(iii) *Assume that $(X, B)$ is symmetric. Under the same assumption as in* (ii), *there is an isomorphism*

$$\frac{P_x \cap G'}{P_x \cap G'_x} \cong \frac{P_\beta \cap G'}{P_\beta \cap G'_\beta}.$$

Of course, we can write the generalizations of (ii) and (iii) to the non-block-transitive case. Furthermore, there is the similar result about Schur multipliers as this corollary.

## 3. Brief outline of transfer theory of finite groups.

Throughout this section, $G$ is a finite group, $\mathbf{Set}^G_f$ is the category of finite $G$-sets and $G$-maps, and $R$ is a commutative ring.

### a. Mackey functors and $G$-functors.

In this subsection, we will define Mackey functors and $G$-functors, and state some properties of them. The details are found in [Gr 71], [Dr 72] and [Yo 80].

DEFINITION. Let $\boldsymbol{E}$ be a category with finite limits and finite coproducts. Let $\boldsymbol{B}$ be a category. A *Mackey functor* $\boldsymbol{M}: \boldsymbol{E} \to \boldsymbol{B}$ is a functor which to each morphism $f: X \to Y$ assigns two morphisms $f^*: M(Y) \to M(X)$ and $f_*: M(X) \to M(Y)$ and satisfies the following properties:

(M.1) The contravariant part of this functor sends finite coproducts to products. (In particular, $M(\mathbf{0}) = \mathbf{1}$ and $M(X+Y) \cong M(X) \times M(Y)$.)

(M.2) For any pull-back diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & D, \end{array}$$

the diagram

$$\begin{array}{ccc} M(A) & \xrightarrow{h_*} & M(B) \\ {\scriptstyle k^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ M(C) & \xrightarrow[g_*]{} & M(D) \end{array}$$

is commutative.

The property (M.2) is called the pull-back property. A morphism between Mackey functors is naturally defined. So we have the category $\mathbf{Mc}[\boldsymbol{E}, \boldsymbol{B}]$ of Mackey functors.

DEFINITION. Let $\boldsymbol{L}, \boldsymbol{M}, \boldsymbol{N}$ be Mackey functors of the category $\boldsymbol{E}$ to the module category $\mathbf{Mod}_R$ for a commutative ring $R$. A *pairing* $\rho: \boldsymbol{L} \times \boldsymbol{M} \to \boldsymbol{N}$ is a family of $R$-bilinear maps

$$\rho_X: L(X) \times M(X) \longrightarrow N(X)\,;\ (\alpha, \beta) \longmapsto \alpha \cdot \beta \qquad (X \in \boldsymbol{E})$$

which satisfies for each morphism $f: X \to Y$ in $\boldsymbol{E}$ the following properties:

(P.1) $\quad f^*(\alpha') \cdot f^*(\beta') = f^*(\alpha' \cdot \beta'), \quad \alpha' \in L(Y), \beta' \in M(Y).$

(P.2) $\quad f_*(\alpha \cdot f^*(\beta')) = f_*(\alpha) \cdot \beta', \quad \alpha \in L(X), \beta' \in M(Y).$

(P.3) $\quad f_*(f^*(\alpha') \cdot \beta) = \alpha' \cdot f_*(\beta), \quad \alpha' \in L(Y), \beta \in M(X).$

The properties (P.2) and (P.3) are Frobenius reciprocity.

DEFINITION. A Mackey functor $\boldsymbol{A}: \boldsymbol{E} \rightarrow \mathbf{Mod}_R$ is called a *Mackey ring* if there is a pairing $\boldsymbol{A} \times \boldsymbol{A} \rightarrow \boldsymbol{A}$ which makes $\boldsymbol{A}(X)$ into an $R$-algebra with an algebra homomorphism $f^*: A(Y) \rightarrow A(X)$ preserving the identity element for each $f: X \rightarrow Y$. Similarly we can define a (right) *Mackey module* over a Mackey ring. Morphisms of these objects are naturally defined.

LEMMA 3.1. *Let $\boldsymbol{A}$ be a Mackey ring and $\boldsymbol{M}$ a Mackey $\boldsymbol{A}$-module from $\boldsymbol{E}$ to $\mathbf{Mod}_R$. Then $\boldsymbol{M}(X)$ is an $\boldsymbol{A}(\mathbf{1})$-module for each object $X$ of $\boldsymbol{E}$ by $\mu \cdot \alpha := X^*(\mu) \cdot \alpha$, where the unique morphism $X \rightarrow \mathbf{1}$ is denoted by $X$ itself. Furthermore, the $R$-linear maps $f^*: \boldsymbol{M}(Y) \rightarrow \boldsymbol{M}(X)$ and $f_*: \boldsymbol{M}(X) \rightarrow \boldsymbol{M}(Y)$ induced by any $f: X \rightarrow Y$ are $\boldsymbol{A}(\mathbf{1})$-homomorphism.*

PROOF. Easy.

DEFINITION. Let $G$ be a finite group. A *$G$-functor* $\boldsymbol{a}$ consists of a family of $R$-modules $\boldsymbol{a}(H)$, $H \leqq G$, and families of $R$-linear maps of three kinds as follows:

$$\mathrm{cor}^K : \boldsymbol{a}(H) \longrightarrow \boldsymbol{a}(K) \,;\ \alpha \longmapsto \alpha^K \quad (H \leqq K \leqq G)\,,$$

$$\mathrm{res}_H : \boldsymbol{a}(K) \longrightarrow \boldsymbol{a}(H) \,;\ \beta \longmapsto \beta_H \quad (H \leqq K \leqq G)\,,$$

$$\mathrm{con}^g : \boldsymbol{a}(H) \longrightarrow \boldsymbol{a}(H^g) \,;\ \alpha \longmapsto \alpha^g \quad (H \leqq G,\, g \in G)\,.$$

Furthermore they must satisfy the following axioms for any $D, H, K, L \leqq G$, $g, g' \in G$, $\alpha \in \boldsymbol{a}(H)$ and $\beta \in \boldsymbol{a}(K)$:

(G.1) $\qquad \alpha^H = \alpha\,, \quad (\alpha^K)^L = \alpha^L \qquad$ if $H \leqq K \leqq L$;

(G.2) $\qquad \beta_K = \beta\,, \quad (\beta_H)_D = \beta_D \qquad$ if $D \leqq H \leqq K$;

(G.3) $\qquad \alpha^h = \alpha\,, \quad (\alpha^g)^{g'} = \alpha^{gg'} \qquad$ if $h \in H$;

(G.4) $\qquad (\alpha^K)^g = (\alpha^g)^{K^g}\,, \quad (\beta_H)^g = (\beta^g)_{H^g} \qquad$ if $H \leqq K$;

(G.5) $\qquad (\alpha^L)_K = \sum_{HgK} {\alpha^g_{H^g \cap K}}^K \qquad$ if $H, K \leqq L$,

where $g$ runs over a complete set of representatives of $H \backslash L / K$.

Morphisms between $G$-functors, pairings, "rings", "modules" and so on are defined by the similar way as Mackey functors.

DEFINITION. If the $G$-functor $\boldsymbol{a}$ furthermore satisfies the following property, it is called to be *cohomological*:

(C) $\qquad {\beta_H}^K = |K : H| \beta \qquad$ if $H \leqq K \leqq G$, $\beta \in \boldsymbol{a}(K)$.

LEMMA 3.2 (Dress). *Let $G$ be a finite group. The category* $\mathbf{Mc}[\mathbf{Set}_f^G, \mathbf{Mod}_R]$ *of Mackey functors of finite $G$-sets to $R$-modules is equivalent to the category of $G$-functors.*

PROOF (Outline). Let $\boldsymbol{M}$ be a Mackey functor. We set $\boldsymbol{a}(H) := \boldsymbol{M}(H\backslash G)$. For any $H \leqq K \leqq G$ and $g \in G$, there are canonical $G$-maps $H\backslash G \to K\backslash G$ and $H\backslash G \to H^g\backslash G$. Thus applying the contravariant part and the covariant part to them, we have maps cor, res, con. Conversely, let $\boldsymbol{a}$ be a $G$-functor. For each finite $G$-set $X$, we set

$$\boldsymbol{M}(X) := \{(\xi_x) \in \prod_{x \in X} \boldsymbol{a}(G_x) \mid \xi_{xg} := \mathrm{con}^g(\xi_x) \text{ for } g \in G\}.$$

Furthermore for a $G$-map $f: X \to Y$, we define $R$-linear maps

$$f^* : \boldsymbol{M}(Y) \longrightarrow \boldsymbol{M}(X)\,;\ (\eta_y) \longmapsto (\xi_x)\,, \qquad \xi_x := \mathrm{res}_{G_x}(\eta_{f(x)})\,,$$

$$f_* : \boldsymbol{M}(X) \longrightarrow \boldsymbol{M}(Y)\,;\ (\xi_x) \longmapsto (\eta_y)\,, \qquad \eta_y := \sum_{x \in f^{-1}(y)/G_y} \mathrm{cor}^{G_y}(\xi_x)\,,$$

where $x$ runs over a complete set of representatives of $G_y$-orbits in $f^{-1}(y)$. Then we obtain a Mackey functor $X \mapsto \boldsymbol{M}(X)$.

*Example.* Let $M$ be an $RG$-module. Each $G$-map $f: X \to Y$ between finite $G$-sets induces an $RG$-homomorphism $RX \to RY$ and its transposition $RY \to RX$. Thus for each nonnegative integer $m$, we have maps

$$\mathrm{Ext}_{RG}^m(RX, M) \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathrm{Ext}_{RG}^m(RY, M)\,,$$

which makes the assignment $X \mapsto \mathrm{Ext}_{RG}^m(RX, M)$ into a Mackey functor. The corresponding $G$-functor is given by $H(\leqq G) \mapsto H^m(H, M)$ together with corestrictions, restrictions and conjugations and this $G$-functor is cohomological. If $L \times M \to N$ is a $G$-pairing of $RG$-modules, then we have a pairing

$$\mathrm{Ext}_{RG}^m(RX, L) \times \mathrm{Ext}_{RG}^n(RX, M) \longrightarrow \mathrm{Ext}_{RG}^{m+n}(RX, N)$$

by combining the cup products for Ext with the diagonal map $RX \times RX \to RX$.

## b. Burnside rings.

For a category $\boldsymbol{E}$ and its object $X$, the *comma category* $\boldsymbol{E}/X$ is the category of all morphisms into $X$. Refer to [Ma 71], p. 46. We identify $\boldsymbol{E}/\mathbf{1}$ and $\boldsymbol{E}$, and so we sometimes write $X \to \mathbf{1}$ as simply $X$.

We consider the case where $\boldsymbol{E} = \mathbf{Set}_f^G$. Then the comma category $\boldsymbol{E}/X$

has finite coproducts and finite products:

$$(A \longrightarrow X)+(B \longrightarrow X)=(A+B \longrightarrow X),$$

$$(A \longrightarrow X)\times(B \longrightarrow X)=(A\times_X B \longrightarrow X),$$

where $A\times_X B$ is the fiber product. Isomorphism classes of $\mathrm{Set}_f^G/X$ make a semi-ring by coproducts and products.

DEFINITION. Let $\Omega(X)$ denote the Grothendieck ring of $\mathbf{Set}_f^G/X$. Then the assignment $X\to\Omega(X)$ becomes a Mackey ring called the *Burnside ring functor* as follows. Each $G$-map $f: X\to Y$ defines a pair of adjoint functors:

$$(\mathrm{Set}_f^G)/X \underset{f^*}{\overset{\Sigma_f}{\rightleftarrows}} (\mathrm{Set}_f^G)/Y,$$

where $\Sigma_f$ is defined by the composition with $f$ and the pullback functor $f^*$ is defined by

$$f^*: (A \longrightarrow Y) \longmapsto (A\times_Y X \longrightarrow X).$$

(See [Jo 77].) Since $\Sigma_f$ preserves coproducts and $f^*$ preserves both of coproducts and products, they make the assignment $\Omega: X\mapsto\Omega(X)$ into a Mackey ring. Furthermore for any commutative ring $R$, we have a Mackey ring $R\otimes\Omega$ which sends $X$ to $R\otimes_Z\Omega(X)$. In particular, we put $\Omega_{(p)}:=\boldsymbol{Z}_{(p)}\otimes_Z\Omega$.

DEFINITION. The *Burnside ring* $\omega(G)$ of a finite group $G$ is the Grothendieck ring of $\mathbf{Set}_f^G$ with respect to disjoint unions and cartesian products. See [Di 79]. For $H\leqq K\leqq G$ and $g\in G$, we have the following mappings:

$$\mathrm{res}_H: \omega(K) \longrightarrow \omega(H)\,;\ [X] \longmapsto [X_H].$$

$$\mathrm{con}^g: \omega(H) \longrightarrow \omega(H^g)\,;\ [X] \longmapsto [X^g].$$

$$\mathrm{ind}^K: \omega(H) \longrightarrow \omega(K)\,;\ [H/D] \longmapsto [K/D].$$

The assignment $\omega: H(\leqq G)\mapsto\omega(H)$ together with ind, res, con makes a $G$-functor corresponding to the Burnside ring functor $\Omega$. Furthermore, we have a $G$-functor $\omega_{(p)}: H\mapsto\boldsymbol{Z}_{(p)}\otimes_Z\omega(H)$ which corresponds to $\Omega_{(p)}$.

LEMMA 3.3 ([Di 79], 6.2.3). *Every Mackey functor $\boldsymbol{M}$ can be regarded as an $\Omega$-module by*

$$\boldsymbol{M}(X)\times\Omega(X) \longrightarrow \boldsymbol{M}(X)\,;\ (m,[A \xrightarrow{\alpha} X]) \longmapsto \alpha_*\alpha^*(m).$$

*Furthermore, if* ($\boldsymbol{a}$, cor, res, con) *is a G-functor corresponding to* $\boldsymbol{M}$, *then the action of* $\omega(G)$ *on* $\boldsymbol{a}(G)$ *is given by* $\alpha\cdot[H\backslash G]:=\mathrm{cor}^G\cdot\mathrm{res}_H(\alpha)$.

LEMMA 3.4 (Di79], 1.2.2). *Let* $C(G)$ *be the set of conjugate classes of subgroups of* $G$. *For each subgroup* $H$ *of* $G$, *we define a linear map* $\varphi_H$ *of* $\omega(G)$ *to* $\boldsymbol{Z}$ *by* $\varphi_H([X]):=|X^H|$. *Then the map*

$$\varphi:=(\varphi_H):\omega(G)\longrightarrow\prod_{(H)\in C(G)}\boldsymbol{Z}$$

*is an injective ring homomorphism with finite cokernel.*

A finite group $G$ is called *p-perfect* provided $G$ has no normal subgroups of index $p$. We denote by $O^p(G)$ the subgroup of $G$ generated by all $p'$-elements of $G$.

LEMMA 3.5 ([Yo 83a]). *Let* $p$ *be a prime. There are primitive idempotents* $e^p_{G,Q}$ *of* $\boldsymbol{Z}_{(p)}\otimes_{\boldsymbol{Z}}\omega(G)$ *corresponding to the classes* $(Q)$ *of p-perfect subgroups* $Q$ *of* $G$ *such that*

$$\varphi_H(e^p_{G,Q})=\begin{cases}1 & \text{if } O^p(H) \text{ is } G\text{-conjugate to } Q\\ 0 & \text{otherwise,}\end{cases}$$

*where we extended linearly the definition of* $\varphi_H$ *to* $\omega_{(p)}(G)$. *Furthermore,* $e^p_{G,Q}$ *is a linear combination of elements of the form* $[H\backslash G]$ *where* $O^p(H)=Q$.

DEFINITION. Let $\boldsymbol{M}$ be a Mackey functor of $\mathbf{Set}^G_f$ to $\mathbf{Mod}_R$, where $R=\boldsymbol{Z}_{(p)}$ (or more generally $R$ is a commutative ring such that $J(R)$ contains $pR$). By Lemmas 3.1 and 3.3, we have a Mackey subfunctor

$$e^p_{G,Q}\boldsymbol{M}:X\longmapsto X^*(e^p_{G,Q})\cdot\boldsymbol{M}(X)\,.$$

A Mackey functor $\boldsymbol{M}$ such that $e^p_{G,1}\boldsymbol{M}=\boldsymbol{M}$ (that is, $\boldsymbol{M}$ is a Mackey module over $e^p_{G,1}\Omega_{(p)}$) is called a *poly-Hecke* functor. A $G$-functor corresponding to a poly-Hecke functor is called to be *poly-cohomological* with respect to $p$. For example, a cohomological $G$-functor over $R$ is poly-cohomological.

LEMMA 3.6. *Let* $\boldsymbol{a}$ *be a G-functor over a ring* $R$ *such that* $pR\subseteq J(R)$. *The following statements are equivalent*:

(a) $\boldsymbol{a}$ *is poly-cohomological.*

(b) $\mathrm{res}:\boldsymbol{a}(H)\to\boldsymbol{a}(H_p)$ *is injective for any subgroup* $H$ *with a Sylow p-subgroup* $H_p$.

(c) $\mathrm{cor}:\boldsymbol{a}(H_p)\to\boldsymbol{a}(H)$ *is surjective for any subgroup* $H$ *with a Sylow p-subgroup* $H_p$.

PROOF. Put $e:=e^p_{G,1}$. Let $H$ be any subgroup of $G$ with a Sylow $p$-subgroup $P$. Then by Lemma 3.5, we have that $e_P\ (=\mathrm{res}_P(e))=1$ and $e_H=f^H\ (=\mathrm{ind}^H(f))$ for some $f\in\omega_{(p)}(P)$. Thus the lemma follows easily from these equalities and Frobenius reciprocity.

DEFINITION. Let $\boldsymbol{m}$ be a $G$-functor over $R$. For any subgroup $H$ of $G$, let $\boldsymbol{m}'(H)$ be the $R$-submodule of $\boldsymbol{m}(H)$ generated by all elements of the form $\mu_D{}^H-|E:D|\mu^H$, where $D\leqq E\leqq H$ and $\mu\in\boldsymbol{m}(H)$. Then $\boldsymbol{m}'$ is a $G$-subfunctor of $\boldsymbol{m}$ and $\boldsymbol{m}^{\mathrm{coh}}:=\boldsymbol{m}/\boldsymbol{m}'$ is a cohomological $G$-functor. When $\boldsymbol{M}$ is a Mackey functor corresponding to $\boldsymbol{m}$, we can define $\boldsymbol{M}^{\mathrm{coh}}$ to be the Mackey functor corresponding to $\boldsymbol{m}^{\mathrm{coh}}$. We call $\boldsymbol{M}^{\mathrm{coh}}$ and $\boldsymbol{m}^{\mathrm{coh}}$ the *cohomologicalizations* of $\boldsymbol{M}$ and $\boldsymbol{m}$, respectively. For example, $(R\otimes\Omega)^{\mathrm{coh}}$ is isomorphic to the Mackey functor $X\mapsto\mathrm{Ext}^0_{RG}(RX, R)$.

LEMMA 3.7. *Let $R$ be a commutative ring such that $pR\subseteqq J(R)$. Let $\boldsymbol{A}$ be a Mackey ring over $R$ and let $\boldsymbol{M}$ be a poly-Hecke and simple Mackey module over $\boldsymbol{A}$. Assume that each component $\boldsymbol{A}(X)$ is finitely generated as an $R$-module. Then $p\boldsymbol{M}=0$ and the $G$-functor corrresponding to $\boldsymbol{M}$ is cohomological.* (Remark: A Mackey functor corresponding to a cohomological $G$-functor is called a *Hecke functor*. See 3.c.)

PROOF. Let $\boldsymbol{a}$ and $\boldsymbol{m}$ be the $G$-functors corresponding to $\boldsymbol{A}$ and $\boldsymbol{M}$, so that $\boldsymbol{m}$ is a poly-cohomological and simple "module" over the "ring" $\boldsymbol{a}$. First note that each component $\boldsymbol{m}(H)$ is a finitely generated $R$-module. This fact follows, for example, from the representability of Mackey $\boldsymbol{A}$-modules (Lemma 3.8). Thus Nakayama's lemma yields that $p\boldsymbol{m}=J(R)\boldsymbol{m}=0$. Next assume that $\boldsymbol{m}$ is not cohomological. Then the subfunctor $\boldsymbol{m}'$ that was defined as above is an "$\boldsymbol{a}$-submodule", and so $\boldsymbol{m}^{\mathrm{coh}}=\boldsymbol{m}/\boldsymbol{m}'=0$ by the simplicity of $\boldsymbol{m}$. Thus each $\boldsymbol{m}(H)$ is generated by elements of the form $\mu_D{}^H-|E:D|\mu$, where $D<E\leqq H\leqq G$ and $\mu\in\boldsymbol{m}(H)$. On the other hand, choose a minimal subgroup $H$ such that $\boldsymbol{m}(H)\neq 0$. Then by Lemma 3.6, $H$ is a $p$-subgroup, and so $\boldsymbol{m}(H)$ is generated by elements $p\mu$, $\mu\in\boldsymbol{m}(H)$. This is a contradiction.

## c. Mackey categories and Hecke categories.

DEFINITION. Let $\boldsymbol{A}$ be a Mackey ring of $\mathbf{Set}^G_f$ to $\mathbf{Mod}_R$. Then the *Mackey category* $\mathbf{Mc}(G,\boldsymbol{A})$ is the category whose objects are finite $G$-sets and whose hom-set of $Y$ to $X$ is the abelian group $\boldsymbol{A}(X\times Y)$. The composition is defined by the following bilinear map:

$$\boldsymbol{A}(X\times Y)\times\boldsymbol{A}(Y\times Z)\xrightarrow{\pi_{12}^{*}\times\pi_{23}^{*}}\boldsymbol{A}(X\times Y\times Z)\times\boldsymbol{A}(X\times Y\times Z)$$
$$\xrightarrow{\text{multi}}\boldsymbol{A}(X\times Y\times Z)\xrightarrow{\pi_{13}^{*}}\boldsymbol{A}(X\times Z),$$

where $\pi_{ij}$ denotes the projection from $X\times Y\times Z$ to its $(i, j)$-factor. The Mackey category $\mathbf{Mc}(G, \boldsymbol{A})$ is an $R$-additive category with finite biproducts. The isomorpohism $\boldsymbol{A}(X\times Y)\cong\boldsymbol{A}(Y\times X)$ induced by the transpose $X\times Y\cong Y\times X$ gives the self-duality of the Mackey category: $\mathbf{Mc}(G, \boldsymbol{A})\cong\mathbf{Mc}(G, \boldsymbol{A})^{\mathrm{op}}$.

There exist covariant and contravariant functors $(-)_!$ and $(-)^!$ of $\mathbf{Set}_f^G$ to $\mathbf{Mc}(G, \boldsymbol{A})$ defined by $X_!=X^!=X$ and

$$f_!:=\langle f, 1_X\rangle_*(1_{\boldsymbol{A}(X)})\in\boldsymbol{A}(Y\times X),$$
$$f^!:=\langle 1_X, f\rangle_*(1_{\boldsymbol{A}(X)})\in\boldsymbol{A}(X\times Y).$$

The pair of functors $(-)^!$ and $(-)_!$ defines a Mackey functor $\mathbf{Set}_f^G\to\mathbf{Mc}(G, \boldsymbol{A})$.

Any homomorphism $\theta:\boldsymbol{A}\to\boldsymbol{B}$ of Mackey rings induces a functor $\hat{\theta}:\mathbf{Mc}(G, \boldsymbol{A})\to\mathbf{Mc}(G, \boldsymbol{B})$. For example, the canonical homomorphism $\theta:\Omega\to\boldsymbol{A}$ (the $X$-component $\Omega(X)\to\boldsymbol{A}(X)$ maps $[f:A\to X]$ to $f_*f^*(1_{\boldsymbol{A}(X)})$) induces $\tilde{\theta}$ such that

$$\tilde{\theta}([A\xrightarrow{\langle\lambda, m\rangle}X\times Y])=\langle\lambda, m\rangle_*(1)=\lambda_!\circ\mu^!\in\boldsymbol{A}(X\times Y).$$

LEMMA 3.8. *The category of Mackey $\boldsymbol{A}$-modules is equivalent to the $R$-additive functor category* $\mathbf{Add}_R[\mathbf{Mc}(G, \boldsymbol{A})^{\mathrm{op}}, \mathbf{Mod}_R]$.

PROOF (Outline). When $\boldsymbol{M}$ is a Mackey $\boldsymbol{A}$-module, each $\alpha\in\boldsymbol{A}(X\times Y)$ induces an $R$-linear map $\boldsymbol{M}(X)\to\boldsymbol{M}(Y)$ by $m\mapsto\pi_{2*}(\pi_1^*(m)\cdot\alpha)$, where $\pi_1$ and $\pi_2$ are the projections from $X\times Y$. By the pullback formula (M.2) and the Frobenius reciprocity, we have a contravariant functor of the Mackey category. Conversely, let $\boldsymbol{M}$ be an $R$-additive contravariant functor of $\mathbf{Mc}(G, \boldsymbol{A})$ to $\mathbf{Mod}_R$. Put $\boldsymbol{M}^*:=\boldsymbol{M}\circ(-)_!$ and $\boldsymbol{M}_*:=\boldsymbol{M}\circ(-)^!$, so that for each $G$-map $f:X\to Y$,we have $R$-linear maps

$$f_*:=\boldsymbol{M}_*(f)=\boldsymbol{M}((X\xrightarrow{\langle 1, f\rangle}X\times Y)(1_{\boldsymbol{A}(X)}))\,;\,\boldsymbol{M}(X)\longrightarrow\boldsymbol{M}(Y),$$
$$f^*:=\boldsymbol{M}^*(f)=\boldsymbol{M}((X\xrightarrow{\langle f, 1\rangle}Y\times X)(1_{\boldsymbol{A}(X)}))\,;\,\boldsymbol{M}(Y)\longrightarrow\boldsymbol{M}(X).$$

Then we obtain a Mackey functor $\boldsymbol{M}:f\to f_*, f^*$. Furthermore, an $\boldsymbol{A}$-module structure of $\boldsymbol{M}$ is given by

$$\boldsymbol{A}(X)\xrightarrow{\delta_*}\boldsymbol{A}(X\times X)=\mathrm{End}(X)\xrightarrow{\text{can}}\mathrm{End}_R(\boldsymbol{M}(X)),$$

where End$(X)$ is the endomorphism ring in $\mathbf{Mc}(G, \boldsymbol{A})$ and $\delta: X\to X\times X$ is the diagonal $G$-map.

*Example.* We consider the Mackey category $\mathbf{Mc}(G, \Omega)$. A morphism of $Y$ to $X$ in this category is a difference of morphisms of the form $[\langle\lambda, \mu\rangle: A\to X\times Y]$, where $\lambda: A\to X$ and $\mu: A\to Y$. The composition of $[\langle\lambda, \mu\rangle: A\to X\times Y]$ and $[\langle\nu, \kappa\rangle: B\to Y\times Z]$ is $[\langle\lambda\circ\nu', \kappa\circ\mu'\rangle: C\to X\times Z]$, where $\nu'$ and $\mu'$ are defined by the following pullback square:

$$\begin{array}{ccccc} C & \xrightarrow{\mu'} & B & \xrightarrow{\kappa} & Z \\ {\scriptstyle\nu'}\big| & & \big|{\scriptstyle\nu} & & \\ A & \xrightarrow[\mu]{} & Y & & \\ {\scriptstyle\lambda}\big| & & & & \\ X & & & & \end{array}$$

Any Mackey functor $\boldsymbol{M}$ is a Mackey $\Omega$-module. The corresponding contravariant functor maps $[\langle\lambda, \mu\rangle: A\to X\times Y]\in\Omega(X\times Y)$ to

$$\boldsymbol{M}(\langle\lambda, \mu\rangle):=\mu_*\circ\lambda^*: \boldsymbol{M}(X) \longrightarrow \boldsymbol{M}(A) \longrightarrow \boldsymbol{M}(Y).$$

DEFINITION. The *Hecke category* $\mathbf{Hec}(G, R)$ is the category in which objects are finite $G$-sets and a morphism of $Y$ to $X$ is an $R$-matrix $(a_{xy})_{x\in X, y\in Y}$ of size $X\times Y$ with $a_{xg,yg}=a_{x,y}$ for any $x\in X$, $y\in Y$, $g\in G$. Compositions are defined by the product of matrices. Clearly, $\mathbf{Hec}(G, R)$ is self-dual (by the transpositions of matrices) and equivalent to the category of permutation $RG$-modules by the functor

$$X \longmapsto RX,$$

$$((a_{xy}): Y \longrightarrow X) \longmapsto (RY \longrightarrow RX: y \longmapsto \sum_{x\in X} a_{xy}y).$$

A contravariant $R$-additive functor of $\mathbf{Hec}(G, R)$ to $\mathbf{Mod}_R$ is called a *Hecke functor*. There is an $R$-additive functor

$$\Phi: \mathbf{Mc}(G, R\otimes\Omega) \xrightarrow{\text{can}} \mathbf{Mc}(G, R\otimes\Omega^{\text{coh}}) \xrightarrow{\cong} \mathbf{Hec}(G, R)$$

such that $\Phi(X)=X$ and

$$\Phi: [A \xrightarrow{h} X\times Y] \longmapsto (|h^{-1}(x, y)|)_{xy}.$$

Thus any Hecke functor can be regarded as a Mackey functor. Remember that all Mackey functor is an $R\otimes\Omega$-module (Lemma 3.3).

LEMMA 3.9 (See [Yo 83]). *The above functor $\Phi$ induces an equivalent between the category of Hecke functors and the category of cohomological G-functors.*

## d. Isomorphisms and split morphisms in a Mackey category.

We can now prove a proposition which is essential to prove Theorem B in the next section. Let $\Phi$ denote the functor of $\mathbf{Mc}(G, \Omega_{(p)})$ to $\mathbf{Hec}(G, \boldsymbol{Z}_{(p)})$ defined as above. We first note that there is a canonical ring homomorphism of the Burnside ring $\omega(G)$ to the center of $\mathbf{Mc}(G, R\otimes\Omega)$ (the center of a category is defined to be the set of endo-natural transformations of the identity functor of this category). In fact, the action of $[E]\in\omega(G)$ over $R\otimes\Omega(X\times Y)$ is defined by $[E]\cdot[h: A\to X\times Y]:=(X\times Y)^*([E])\cdot h=[E\times A\to A \to X\times Y]$. Furthermore, the functor $\Phi$ maps $[E]\cdot h$ to $|E|\Phi(h)$. In particular,

$$\Phi(e_{G,1}^{\pi}h)=\varphi_1(e_{G,1}^{\pi})\Phi(h)=\Phi(h) \qquad \text{for any morphism } h.$$

LEMMA 3.10. *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be skeletally small R-additive categories. Let $\Phi: \boldsymbol{C}\to\boldsymbol{D}$ be an R-additive functor. Assume that for every simple object S of $\mathbf{Add}_R[\boldsymbol{C}^{\mathrm{op}}, \mathbf{Mod}_R]$, there exists an object T of $\mathbf{Add}_R[\boldsymbol{D}^{\mathrm{op}}, \mathbf{Mod}_R]$ such that $S=T\circ\Phi$. Then $\Phi$ reflects split epimorphisms.*

PROOF. Let $f: X\to Y$ be a morphism in $\boldsymbol{C}$ such that $\Phi(f)$ is a split epimorphism. Suppose $f$ is not a split epimorphism. We denote the contravariant hom-functor for each object $Z$ by $H_Z: A\mapsto \boldsymbol{C}(A, Z)$. Then $H_f: H_X \to H_Y$ is not an epimorphism in the functor category $\mathbf{Add}_R[\boldsymbol{C}^{\mathrm{op}}, \mathbf{Mod}_R]$. The image $\mathrm{Im}(H_f)$ of $H_f$ is a subfunctor of $H_Y$ such that $\mathrm{Im}(H_f)(Z)=f\circ\boldsymbol{C}(Z, X) \subsetneqq \boldsymbol{C}(Z, Y)$. There exists a maximal subfunctor $M$ of $H_Y$ containing $\mathrm{Im}(H_f)$. Put $S:=H_Y/M$, so that $S$ is a simple functor. By the assumption, there exists $T\in\mathbf{Add}_R[D^{\mathrm{op}}, \mathbf{Mod}_R]$ such that $S=T\circ\Phi$. Now, since $\Phi(f)$ is a split epimorphism, there is a morphism $d: \Phi(Y)\to\Phi(X)$ such that $\Phi(f)\circ d=1_{\Phi(Y)}$. Thus $T(d)\circ S(f)=T(d)\circ T(\Phi(f))=1_{S(Y)}$, and so $S(f): S(Y)\to S(X)$ is injective. Since $S(f)$ maps $1_Y \bmod M(Y)$ to $f \bmod M(X)=0$, we have that $1_Y\in M(Y)$. For any $g: Z\to Y$ in $\boldsymbol{C}$, we have that $M(g): M(Y)\to M(Z)\,;\, 1_Y\mapsto g$, and so $M(Z)= \boldsymbol{C}(Z, Y)=H_Y(Z)$ for any $Z$. This contradicts the fact that $M$ is a maximal subfunctor.

REMARK. If we apply the theory of radicals of additive categories ([Ke 64], [Mi 70], [Ba 75]), this lemma is trivial. In fact, the radical of $\boldsymbol{C}$ is the inverse image of the radical of $\boldsymbol{D}$ by $\Phi$. This fact is applied to determine the central idempotents of the Mackey categories ([Yo 85]).

DEFINITION. For any $R$-additive category $\boldsymbol{C}$, we define a category $\boldsymbol{C}^+$ as follows. An object of $\boldsymbol{C}^+$ is a pair $(X, e)$ of an object $X$ of $\boldsymbol{C}$ and $e=e^2 \in \mathrm{End}(X)$, and a morphism of $(X, e)$ to $(Y, f)$ is a morphism $\lambda \in \boldsymbol{C}(X, Y)$ such that $f\lambda e=\lambda$. An embedding of $\boldsymbol{C}$ into $\boldsymbol{C}^+$ is given by $X \mapsto (X, 1)$, $f \mapsto f$, and so we identify $\boldsymbol{C}$ as a subcategory of $\boldsymbol{C}^+$. For any idempotent $e=e^2 \in \mathrm{End}_C(X)$, we have a biproduct diagram

$$(X, e) \underset{e}{\overset{e}{\rightleftarrows}} (X, 1) \underset{f}{\overset{f}{\leftrightarrows}} (X, f),$$

where $f:=1-e$. Any $R$-additive contravariant functor of $\boldsymbol{C}$ to an idempotent splitting category can be uniquely extended to $\boldsymbol{C}^+$. In particular. we have an equivalence

$$\mathbf{Add}_R[\boldsymbol{C}^{\mathrm{op}}, \mathbf{Mod}_R] \cong \mathbf{Add}_R[\boldsymbol{C}^{+\mathrm{op}}, \mathbf{Mod}_R].$$

In fact, a unique extension of such a functor $F$ is given by $F^+ : (X, e) \mapsto F(e)F(X)$. Furthermore any $R$-additive functor $\Phi : \boldsymbol{C} \to \boldsymbol{D}$ induces $\Phi^+ : \boldsymbol{C}^+ \to \boldsymbol{D}^+$. For example, $\mathbf{Hec}(G, R)^+$ is equivalent to the category of trivial source modules (see [La83]) and any Hecke functor is extended to a functor from this category.

PROPOSITION 3.11. *The canonical functors* $\Phi_A : \mathbf{Mc}(G, \boldsymbol{A}) \mapsto \mathbf{Mc}(G, \boldsymbol{A}^{\mathrm{coh}})$ *and* $\Phi_A^+ : \mathbf{Mc}(G, \boldsymbol{A})^+ \to \mathbf{Mc}(G, \boldsymbol{A}^{\mathrm{coh}})^+$ *reflect isomorphisms, split epimorphisms and split monomorphisms.*

PROOF. Since Mackey categories are self-dual, this is clear from Lemmas 3.7, 3.8, and 3.10.

COROLLARY 3.12. *Let $R$ be a commutative ring with* $pR \subseteqq J(R)$. *Put* $\Omega' := e_{G,1}^{\mathfrak{p}}(R \otimes \Omega)$ *and we consider morphisms in the Mackey category* $\mathbf{Mc}(G, \Omega')$.

(i) *An endomorphism* $\nu$ $(\in \Omega'(X \times X))$ *of $X$ is an isomorphism provided* $\det(\Phi(\nu)) \not\equiv 0 \pmod p$, *where $\Phi$ is the functor to* $\mathbf{Hec}(G, R)$ *defined in Section* 3.c.

(ii) *The morphism* $X^! = e_{G,1}^{\mathfrak{p}}[\langle 1, X \rangle : X \to X \times \mathbf{1}]$ *of* $\mathbf{1}$ *to $X$ is a split monomorphism provided* $|X|$ *is prime to $p$.*

(iii) *The morphism* $X_! = e_{G,1}^{\mathfrak{p}}[\langle X, 1 \rangle : X \to \mathbf{1} \times X]$ *of $X$ to* $\mathbf{1}$ *is a split epimorphism provided* $|X|$ *is prime to $p$.*

PROOF. The functor $\Phi$ is the composition of the canonical functor $\Phi_{\Omega'} : \mathbf{Mc}(G, \Omega') \to \mathbf{Mc}(G, \Omega'^{\mathrm{coh}})$ and the equivalent $\mathbf{Mc}(G, \Omega'^{\mathrm{coh}}) \cong \mathbf{Hec}(G, R)$. Thus (i) is clear. The image of the morphism of (ii) under $\Phi$ is the matrix of size $X\times\mathbf{1}$ all of which components are 1. When $|X|$ is prime to $p$, this matrix has clearly a $G$-invariant left inverse. Thus (ii) holds. Taking its transpose, (iii) follows.

DEFINITION. A Mackey ring $\boldsymbol{A}$ is called *X-projective* if the map $X_*:\boldsymbol{A}(X)\to\boldsymbol{A}(\mathbf{1})$ is surjective.

Let $\boldsymbol{A}$ be a Mackey ring and $X$ a finite $G$-set. We denote by $\pi_i^n: X^n \to X^{n-1}$ the projection which omits the $i$-th factor, $0\leqq i\leqq n$. Then we have chain complexes in $\mathbf{Mc}(G, \boldsymbol{A})$

$$Am(X, \boldsymbol{A}): \mathbf{0} \longrightarrow \mathbf{1} \xrightarrow{d^0} X \xrightarrow{d^1} X^2 \xrightarrow{d^2} \cdots$$

$$Am(X, \boldsymbol{A})^{\mathrm{op}}: \mathbf{0} \longleftarrow \mathbf{1} \xleftarrow{d_1} X \xleftarrow{d_2} X^2 \xleftarrow{d_2} \cdots$$

defined by $d^{n-1}:=\sum_i(-1)^i\pi_i^{n!}$ and $d_n:=\sum_i(-1)^i\pi_{i!}^n$.

LEMMA 3.13. *Let $\boldsymbol{A}$ be an X-projective Mackey ring. Then the above complexes $Am(X, \boldsymbol{A})$ and $Am(X, \boldsymbol{A})^{\mathrm{op}}$ have contracting homotopy.*

PROOF. For any $\boldsymbol{M}\in[\mathbf{Mc}(G, \boldsymbol{A})^{\mathrm{op}}, \mathbf{Mod}_R]$, the complex $\boldsymbol{M}(Am(X, \boldsymbol{A}))$ is exact. See [Dr 73], Sectiom 3, Corollary 1 or [Di 79], Proposition 6.1.6. Applying this fact to representable functors, the existence of a contracting homotopy follows. Explicitly, the homotopy $h^n: X^n\to X^{n+1}$ is inductively constructed as follows. Take $a\in\boldsymbol{A}(X)$ such that $X_*(a)=1$. Define $h^n$ to be the image of $a$ by

$$\boldsymbol{A}(X) \xrightarrow{\pi_1^*} \boldsymbol{A}(X\times X^n) \xrightarrow{(1\times\Delta)_*} \boldsymbol{A}(X\times X^n\times X^n),$$

where $\Delta: X^n\to X^n\times X^n$ is the diagonal map. Then we have that

$$h^{n+1}\circ d^n+d^{n-1}\circ h^n=1\,.$$

The statement for $Am(X, \boldsymbol{A})^{\mathrm{op}}$ follows from taking the dual.

## 4. Fisher's inequality for poly-Hecke functors.

Throughout this section, $(X, B)$ denotes a block design with parameters $(v, b, r, k, \lambda)$ and with order $n:=r-\lambda$ on which a finite group $G$ acts. Let $F$ be the set of flags:

$$F:=\{(x, \beta)\in X\times B \mid x\in\beta\}\subseteqq X\times B\,.$$

Then $F$ is a $G$-subset of $X\times B$ with projections $\lambda: F\to X$ and $\mu: F\to B$. Thus we can regard the pair of morphisms $\alpha:=\langle\lambda, \mu\rangle: F\to X\times B$ as a morphism in the Mackey category $\mathrm{Mc}(G, \Omega)$, of which image by the functor $\Phi$ is the incidence matrix of $(X, B)$. Using this morphism $\alpha$ instead of the incidence matrix, we obtain some generalizations of Fisher's inequality.

## a. Theorem B and its proof.

Before we state the theorem, we write some trivial relations between parameters of the block design.

LEMMA 4.1. (i) *$vr=bk$, $v\lambda=rk-n$.*

(ii) *$rn=(r^2-b\lambda)k$.*

(iii) *Let $p$ be a prime which does not divide $n$. Then $p|r$ implies $p\nmid v$. Furthermore, $p\nmid r$ implies $p\nmid k$.*

THEOREM B. *Let $p$ be a prime which does not divide $n$ and let $R$ be a commutative ring with $pR\subseteq J(R)$. Let $\boldsymbol{A}:\mathrm{Set}_f^G\to\mathrm{Mod}_R$ be a Mackey ring and let $\boldsymbol{M}$ be a Mackey module over $\boldsymbol{A}$ such that $\boldsymbol{M}$ is poly-Hecke functor as a Mackey functor.*

(1) *The following relations of direct summands induced by the $\boldsymbol{A}(\mathbf{1})$-homomorphism*

$$\mu_*\circ\lambda^*:\boldsymbol{M}(X)\longrightarrow\boldsymbol{M}(F)\longrightarrow\boldsymbol{M}(B)$$

*hold. The notation "$M|N$" means that $M$ is isomorphic to a direct summand of $N$ as $\boldsymbol{A}(\mathbf{1})$-modules.*

(a) *$\boldsymbol{M}(X)\,|\,\boldsymbol{M}(B)$ if $p$ does not divide $r$.*

(b) *If $\boldsymbol{A}$ is $X$-projective (e.g., $p$ does not divide $v$), then*

$$\mathrm{Ker}\,(X_*:\boldsymbol{M}(X)\longrightarrow\boldsymbol{M}(\mathbf{1}))\,|\,\boldsymbol{M}(B)\,,$$

$$\mathrm{Cok}\,(X^*:\boldsymbol{M}(\mathbf{1})\longrightarrow\boldsymbol{M}(X))\,|\,\boldsymbol{M}(B)\,.$$

(c) *If $\boldsymbol{A}$ is $B$-projective (e.g., $p$ does not divide $b$), then $\mathrm{Ker}\,X_*\,|\,\mathrm{Ker}\,B_*$ and $\mathrm{Cok}\,X^*\,|\,\mathrm{Cok}\,B^*$.*

(2) *Assume that the design $(X, B)$ is symmetric, that is, $v=b$. Then $\mu_*\circ\lambda^*:\boldsymbol{M}(X)\to\boldsymbol{M}(B)$ induces the following isomorphisms. The notation "$\cong$" stands for an isomorphism of $\boldsymbol{A}(\mathbf{1})$-modules.*

(a) *$\boldsymbol{M}(X)\cong\boldsymbol{M}(B)$ if $p$ does not divide $r$.*

(b) *$\mathrm{Ker}\,X_*\cong\mathrm{Ker}\,B_*$ and $\mathrm{Cok}\,X^*\cong\mathrm{Cok}\,B^*$ if $\boldsymbol{A}$ is $X$-projective.*

PROOF. We may assume that $\boldsymbol{A}$ is a poly-Hecke functor because $\boldsymbol{A}$ is the direct sum of $e_{G,Q}^p\boldsymbol{A}$'s, where $Q$ is a $p$-perfect subgroup of $G$, and $e_{G,Q}^p\boldsymbol{A}$ annihilate $\boldsymbol{M}$ if $Q\neq 1$. We consider the commutative diagram of $R$-additive functors as follows:

$$\text{(1)} \qquad \begin{array}{ccccc} \Phi : \mathbf{Mc}(\Omega')^+ & \xrightarrow{\Phi_{\Omega'}} & \mathbf{Mc}(\Omega'^{\mathrm{coh}})^+ & \xrightarrow{\cong} & \mathbf{Hec}(G, R)^+ \\ J\downarrow & & \downarrow K & & \downarrow \mathrm{inc} \\ \mathbf{Mc}(\boldsymbol{A})^+ & \xrightarrow[\Phi_A]{} & \mathbf{Mc}(\boldsymbol{A}^{\mathrm{coh}})^+ & & \mathbf{Mod}_{RG} \end{array}$$

Here, we put $\Omega' := e^{p}_{G,1}(R \otimes \Omega)$, $\mathbf{Mc}(\boldsymbol{A}) := \mathbf{Mc}(G, \boldsymbol{A})$, etc., and $J$ and $K$ are the functors induced by the canonical homomorphisms $\Omega' \to \boldsymbol{A}$ and $\Omega'^{\mathrm{coh}} \to \boldsymbol{A}^{\mathrm{coh}}$. By Proposition 3.12 and Corollary 3.13, $\Phi$ and $\Phi_A$ reflect isomorphisms. We regard the Mackey $\boldsymbol{A}$-module $\boldsymbol{M}$ as a contravariant functor from $\mathbf{Mc}(\boldsymbol{A})^+$. Furthermore, we use same symbols for a morphism of $\mathbf{Mc}(\Omega')$ and its image by the functor $J$. Define the "incidence morphisms" by

$$\text{(2)} \qquad \alpha := e^{p}_{G,1}[F \xrightarrow{\langle \lambda, \mu \rangle} X \times B] \in \Omega'(X \times B),$$

$$\text{(2)}' \qquad \alpha' := e^{p}_{G,1}[F \xrightarrow{\langle \mu, \lambda \rangle} B \times X] \in \Omega'(B \times X).$$

Since $\boldsymbol{M}$ is a poly-Hecke functor, $\boldsymbol{M}(\alpha) = \mu_* \circ \lambda^* : \boldsymbol{M}(X) \to \boldsymbol{M}(B)$.

Assume first that $p$ does not divide $r$. Let $A$ be the incidence matrix of $(X, B)$. Then the functor $\Phi$ maps $\alpha$ to $A$ and $\alpha'$ to $A^t$. By the assumption, $\det(AA^t) = rkn^{v-1}$ is a unit of $R$. Thus it follows directly from Corollary 3.12 that $\alpha \circ \alpha'$ is an automorphism of $X$ in the Mackey category $\mathbf{Mc}(\Omega')$, whence $\boldsymbol{M}(\alpha') \circ \boldsymbol{M}(\alpha) : \boldsymbol{M}(X) \to \boldsymbol{M}(B) \to \boldsymbol{M}(X)$ is an isomorphism. Since $\boldsymbol{M}(\alpha)$ and $\boldsymbol{M}(\alpha')$ are $\boldsymbol{A}(\mathbf{1})$-homomorphisms by Lemma 3.1, we have that $\boldsymbol{M}(X)$ is isomorphic to the direct summand of $\boldsymbol{M}(B)$, proving (1.a). Assume further that $(X, B)$ is symmetric. Then the incidence matrix $A$ is a nonsingular square matrix, and so $\alpha' : X \to B$ is an isomorphism in $\mathbf{Mc}(\boldsymbol{A})$, whence $\boldsymbol{M}(\alpha) : \boldsymbol{M}(X) \to \boldsymbol{M}(B)$ is also an isomorphism. This proves (2.a).

We next prove (1.b). So we assume that $\boldsymbol{A}$ is $X$-projective. We identify $\mathbf{Mc}(\boldsymbol{A})$, etc. with a subcategory of $\mathbf{Mc}(\boldsymbol{A})^+$, etc. by the canonical embedding $Y \mapsto (Y, 1_Y)$. Let $Am(X, \Omega')^{\mathrm{op}} = \{X^n, d_n\}$ be the chain complex defined in Section 3.d. Then by the definition, $d_1 = X_!$ and $d_2 = \pi_{1!} - \pi_{2!}$, where $\pi_i : X^2 \to X$, $i = 1, 2$, are the projections in the category of finite $G$-sets. Let $\{h_n : X^n \to X^{n+1}\}$ be a contracting homotopy (Lemma 3.13), so that

$$\text{(3)} \qquad d_1 \circ h_0 = 1_1, \qquad h_0 \circ d_1 + d_2 \circ h_1 = 1_X.$$

Put $e := d_2 \circ h_1 : X \to X$. Then $e$ is an idempotent and there is a biproduct diagram in $\mathbf{Mc}(\boldsymbol{A})^+$:

$$\text{(4)} \qquad (X, e) \underset{q}{\overset{i}{\rightleftarrows}} X \underset{d_1}{\overset{h_0}{\leftrightarrows}} \mathbf{1},$$

where $i:=e:(X, e)\to X=(X, 1)$ and $q:=e:X=(X, 1)\to(X, e)$. Thus $\boldsymbol{M}(e)\boldsymbol{M}(X)=\boldsymbol{M}(X, e)\cong\mathrm{Cok}\,\boldsymbol{M}(d_1)=\mathrm{Cok}\,X^*$. Furthermore, $d_2:X^2\to X$ has a (split) epi-mono factorization

$$d_2:X^2\xrightarrow{q\circ d_2}(X, e)\xrightarrow{i}X, \tag{5}$$

where $q\circ d_2$ is a split epimorphism with the right inverse $h_1\circ e:(X, e)\to X^2$. We must show that the endomorphism of $(X, e)$ in $\mathbf{Mc}(\boldsymbol{A})^+$

$$(X, e)\xrightarrow{i}X\xrightarrow{\alpha'}B\xrightarrow{\alpha}X\xrightarrow{q}(X, e) \tag{6}$$

is an isomorphism, where we identify $\alpha$ (resp. $\alpha'$) with $J(\alpha)$ (resp. $J(\alpha')$). The statement of (1.b) about cokernels follows directly from this fact.

We put $\gamma:=\alpha\circ\alpha'$ and $C:=\Phi(\gamma)$. Then in the category $\mathbf{Hec}(G, R)$, we have that

$$n\Phi(d_2)=\Phi(\gamma)\circ\Phi(d_2)\,. \tag{7}$$

Note that if we embed $\mathbf{Hec}(G, R)$ into $\mathbf{Mod}_{RG}$, then $\Phi(d_2)$ corresponds to the linear map $RX^2\to RX\,;(x, y)\mapsto x-y$, and so (7) follows from the definition of block designs. See also Lemma 2.1. Next, applying the functor $\Psi:=\Phi_A:\mathbf{Mc}(\boldsymbol{A})^+\to\mathbf{Mc}(\boldsymbol{A}^{\mathrm{coh}})^+$ to the morphisms in (5) and (6) we have a diagram

$$\begin{array}{ccc} X^2 & & X^2 \\ {\scriptstyle qd_2}\downarrow & & \downarrow{\scriptstyle qd_2} \\ (X, e) & \xrightarrow{q\gamma i} & (X, e) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{\gamma} & X. \end{array} \qquad (d_2: X^2\to X) \tag{8}$$

The square is commutative. Apply the functor $\Psi$, so that we obtain a diagram in $\mathbf{Mc}(\boldsymbol{A}^{\mathrm{coh}})^+$

$$\begin{array}{ccc} X^2 & \xrightarrow{n\cdot\mathrm{id}} & X^2 \\ {\scriptstyle \Psi qd_2}\downarrow & & \downarrow{\scriptstyle \Psi qd_2} \\ (X, e) & \xrightarrow{\Psi q\gamma i} & (X, e) \\ {\scriptstyle \Psi i}\downarrow & & \downarrow{\scriptstyle \Psi i} \\ X & \xrightarrow{\Psi\gamma=KC} & X. \end{array} \qquad (\Psi d_2: X^2\to X) \tag{9}$$

By (7), the outside rectangle is commutative. Using the fact that $q\circ d_2$ is a split epimorphism, $i$ is a split monomorphism, and the down square is commutative, we have that $\Psi q\gamma i=n\cdot\mathrm{id}$, and so $\Psi q\gamma i$ is an automorphism of $(X, e)$ in $\mathbf{Mc}(\boldsymbol{A}^{\mathrm{coh}})^+$. Since $\Psi$ reflects isomorphisms, we conclude that $q\gamma i$ in

(6) is an isomorphism in $\mathbf{Mc}(\boldsymbol{A})^+$, as required. Hence $\operatorname{Cok} X^*$ is isomorphic to a direct summand of $\boldsymbol{M}(B)$ with injection

$$\operatorname{Cok} X^* \cong \boldsymbol{M}(e)\boldsymbol{M}(X) \xrightarrow{\boldsymbol{M}(q)} \boldsymbol{M}(X) \xrightarrow{\boldsymbol{M}(\alpha)} \boldsymbol{M}(B).$$

The self-duality of the Mackey functors yields the statement about the kernel. The proof of (1.b) is complete.

We will prove (1.c). So assume that $\boldsymbol{A}$ is $B$-projective. Then by Lemma 3.13, the complex $Am(B, \Omega')^{\mathrm{op}} = \{d'_n : B^n \to B^{n-1}\}$ has a contracting homotopy $\{k_n : B^n \to B^{n+1}\}$. We claim first that $\boldsymbol{A}$ is $X$-projective. By Lemma 2.1 (ii), we have that $kB_! = X_! \circ \Psi(\alpha)$ in $\mathbf{Mc}(\boldsymbol{A}^{\mathrm{coh}})$, and so $k \cdot \mathrm{id}_1 = X_! \circ \Psi(\alpha) \circ \Psi(h_0)$. Suppose $k \cdot \mathrm{id}_1$ is not an isomorphism in $\mathbf{Mc}(\boldsymbol{A}^{\mathrm{coh}})$. Then $p|k$, and so $p$ does not divide $v = |X|$ by Lemma 4.1, whence Corollary 3.12 implies that $\boldsymbol{A}$ is $X$-projective. Suppose next that $k \cdot \mathrm{id}_1$ is an isomorphism in $\mathbf{Mc}(\boldsymbol{A}^{\mathrm{coh}})$. In this case, $X_!$ has a right inverse in $\mathbf{Mc}(\boldsymbol{A}^{\mathrm{coh}})$, and so $X_!$ has also a right inverse in $\mathbf{Mc}(\boldsymbol{A})$ by Proposition 3.11. Thus it follows from the definition of the composition in $\mathbf{Mc}(\boldsymbol{A})$ that $X_* : \boldsymbol{A}(X) \to \boldsymbol{A}(\mathbf{1})$ is surjective. In either case $\boldsymbol{A}$ is $X$-projective, as required. Now, as before, let $\{h_n : X^n \to X^{n+1}\}$ be a contracting homotopy for the complex $Am(X, \boldsymbol{A}) = \{X^n, d_n\}$ and put $e := d_2 \circ h_1 : X \to X$. Furthermore, we put $f := d'_2 \circ k_1 : B \to B$. Then $e$ and $f$ are idempotents in $\mathbf{Mc}(\boldsymbol{A})$. We consider the composition of morphisms in $\mathbf{Mc}(\boldsymbol{A})^+$ as follows:

$$(X, e) \xrightarrow{i} X \xrightarrow{\alpha'} B \xrightarrow{q'} (B, f) \xrightarrow{i'} B \xrightarrow{\alpha} X \xrightarrow{q} (X, e), \tag{10}$$

where $i, i'$ are canonical injections and $q, q'$ are canonical projections. We will prove that the morphism $q\alpha i' q' \alpha' i$ is an isomorphism in $\mathbf{Mc}(\boldsymbol{A})^+$. In fact, we can show that $\Psi(q\alpha i' q' \alpha' i) = n \cdot \mathrm{id}$ in $\mathbf{Mc}(\boldsymbol{A}^{\mathrm{coh}})$ by considering the commutative diagram in $\mathbf{Mc}(\boldsymbol{A}^{\mathrm{coh}})$ in Figure 5. Thus by Proposition 3.11, we have that $q\alpha i' q' \alpha' i$ is an isomorphism, as required. By the similar way as in the proof of (1.b), we have that

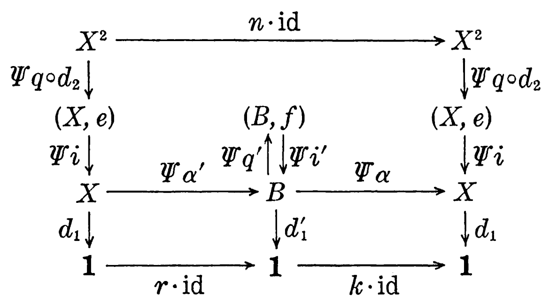


Figure 5.

$$\text{(11)}\quad \mathrm{Cok}\,X^* \cong \boldsymbol{M}(e)\boldsymbol{M}(X) \xrightarrow{\text{inc}} \boldsymbol{M}(X) \xrightarrow{\boldsymbol{M}(\alpha)} \boldsymbol{M}(B) \xrightarrow{\text{pr}} \boldsymbol{M}(f)\boldsymbol{M}(B) \cong \mathrm{Cok}\,B^*$$

is a split monomorphism, proving (1.c).

Finally we will prove (2.b). In this case, the dual design $(B, X)$ is also a symmetric design. Thus by the similar way as the proof of (1.c), we have that the $X$-projectivity of $\boldsymbol{A}$ implies the $B$-projectivity of $\boldsymbol{A}$. We use the notation in the proof of (1.c). Then the morphism $(q\alpha i')(q'\alpha' i)$ defined in (10) is an automorphism of $(X, e)$. On the other hand, applying the same fact to the dual design $(B, X)$, we have that $(q'\alpha' i)(q\alpha i')$ is an automorphism of $(B, f)$. Thus $q\alpha i'$ is an isomorphism in $\mathbf{Mc}(\boldsymbol{A})^+$. Hence in the symmetric case, the $R$-linear map (11) is an isomorphism, proving (2.b). The theorem is proved.

REMARK. (1) Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be Mackey functors from $\mathbf{Set}_f^G$ to $\mathbf{Mod}_R$. Then $\boldsymbol{M}$ is called to be *X-projective* if the canonical morphism $\boldsymbol{M}_X \to \boldsymbol{M}$ is a split epimorphism, where $\boldsymbol{M}_X$ is a Mackey functor defined by $\boldsymbol{M}_X : Y \mapsto \boldsymbol{M}(X \times Y)$. For example, any poly-Hecke functor is $G_p\backslash G$-projective, where $G_p$ is a Sylow $p$-subgroup of $G$, and $\boldsymbol{M}_X$ is $X$-projective. See [Dr 73]. If $\boldsymbol{M}$ is an $X$-projective, then we can construct an $X$-projective Mackey ring $\mathbf{End}(\boldsymbol{M})$ which acts on $\boldsymbol{M}$ as follows. The composition with the functor

$$\Sigma_X : \mathrm{Set}_f^G/X \longrightarrow \mathrm{Set}_f^G\,;\,(A \longrightarrow X) \longmapsto A$$

gives a Mackey functor $\boldsymbol{M}_{|X}$ from $\mathrm{Set}_f^G/X$. A $G$-map $f : X \to Y$ induces morphisms $\boldsymbol{M}_{|X} \to \boldsymbol{M}_{|Y}$ and $\boldsymbol{M}_{|Y} \to \boldsymbol{M}_{|X}$. We define a Mackey functor $\mathbf{Hom}(\boldsymbol{M}, \boldsymbol{N})$ by

$$\mathbf{Hom}(\boldsymbol{M}, \boldsymbol{N}) : X \longmapsto \mathrm{Hom}(\boldsymbol{M}_{|X}, \boldsymbol{N}_{|X}) \cong \mathrm{Hom}(\boldsymbol{M}_X, \boldsymbol{N}).$$

We put $\mathbf{End}(\boldsymbol{M}) := \mathbf{Hom}(\boldsymbol{M}, \boldsymbol{M})$. Then the composition of morphisms of Mackey functors makes the Mackey functor $\mathbf{End}(\boldsymbol{M})$ into a Mackey ring. If $\boldsymbol{M}$ is $X$-projective, then $\mathbf{End}(\boldsymbol{M})$ is $X$-projective as a Mackey ring. So if we accept this fact, the assumption that $\boldsymbol{A}$ is $X$-projective in the theorem can be replaced by the $X$-projectivity of $\boldsymbol{M}$.

(2) Theorem B does not imply Theorem A. Because the Mackey categories do not possess commutative diagrams as in Lemma 2.1, we are compelled to assume $X$- or $B$-projectivities for $\boldsymbol{A}$.

(3) Theorem B holds also for block designs with repeated blocks. Furthermore, it is also correct when $v=1$ and $p|n>0$, because $\det(AA^t) = rkn^{v-1} \neq 0$. In this case, Corollary A.1 yields the well-known fact that if $H$ is a subgroup of $G$ of index prime to $p$, then $(H/H')_{(p)}$ is isomorphic to a direct summand of $(G/G')_{(p)}$. This is a foundation of transfer theorems

of finite groups. See [Yo 80].

## b. Corollaries.

There are many Mackey functors and $G$-functors. We are interested to apply Theorem B to particular Mackey functors. As before, $(X, B)$ denotes a block design on which a finite group $G$ acts.

COROLLARY B.1. *Let $p$ be a prime which does not divide $n$. Let $P$ be a $p$-subgroup of $G$ with the normalizer $N := N_G(P)$. Then the following hold:*

(1) (a) *Assume that $p \nmid r$ or $p \nmid b$. Then $|X^P/N| \leqq |B^P/N|$, where $X^P/N$ is the set of $N$-orbits of $X^P$. In particular, if $G$ is a $p$-group, then $|X^G| \leqq |B^G|$.*

(b) *If $p|r$, then $|X^P/N| - 1 \leqq |B^P/N|$.*

(c) *If the design $(X, B)$ is symmetric, then $|X^P/N| = |B^P/N|$. In particular, if $G$ is a $p$-group, then $|X^G| = |B^G|$.*

(2) (a) $|(X \times B)^P/N| \leqq |(B \times B)^P/N|$.

(b) $|(X \times X)^P/N| - 1 \leqq |(X \times B)^P/N|$.

(c) *If $p \nmid r$ or $p \nmid b$, then $|(X \times X)^P/N| \leqq |(X \times B)^P/N|$.*

(d) *If the design is symmetric, then*

$$|(X \times X)^P/N| = |(X \times B)^P/N| = |(B \times B)^P/N| .$$

PROOF. For any $p$-subgroup $Q$ of $G$ and any finite $G$-set $Y$, let $\Omega_Q(Y)$ be the subgroup of $\Omega(Y)$ generated by the elements of the form $[D \backslash G \to Y]$, where $D$ is a subgroup of $Q$. Then $\Omega_Q$ is a Mackey subfunctor of $\Omega$. We now define a Mackey functor

$$\boldsymbol{M} := \boldsymbol{Z}_{(p)} \otimes (\Omega_P / \sum_{Q<P} \Omega_Q) .$$

Then $\boldsymbol{M}$ is a poly-Hecke functor by Lemma 3.6 and the $\boldsymbol{Z}_{(p)}$-rank of $\boldsymbol{M}(Y)$ equals to $|Y^P/N_G(P)|$. We take $\boldsymbol{Z}_{(p)} \otimes \Omega$ as $\boldsymbol{A}$ in Theorem B. Thus if $p \nmid r$, then (1.a) follows directly from Theorem B (1.b). If $p \nmid b$, then $\boldsymbol{A}$ is $B$-projective by Corollary 3.12, and so (1.a) holds by Theorem B (1.c). When $p|r$, we have that $p \nmid v$, and so $\boldsymbol{A}$ is $X$-projective, whence (1.b) holds by Theorem B (1.b). Assume that $(X, B)$ is symmetric. Then $p \nmid r$ or $p \nmid v = b$, and so (1.c) follows from (1.a).

Next, we will prove (2). The Mackey functor $\boldsymbol{M}_B : Y \mapsto \boldsymbol{M}(Y \times B)$ is $B$-projective. Thus (2.a) follows from Theorem B (1.c). (We choose $\mathbf{End}(\boldsymbol{M}_B)$ as $\boldsymbol{A}$ in Theorem B. See the remark after Theorem B.) Similarly (2.b) follows from the $X$-projectivity of $\boldsymbol{M}_X$. Furthermore, in the case where

$p \nmid r$, (2.c) follows from Theorem B (1.a). In the case where $p \nmid b$, (2.c) follows from Theorem B (1.c). Assume that $(X, B)$ is symmetric. Applying Theorem B (2.a) to $\boldsymbol{M}_X$ and $\boldsymbol{M}_B$, we have that (2.d) holds if $p \nmid r$. So assume further that $p|r$. Then $v=b$ is prime to $p$, and so we can use Theorem B (2.b), and then (2.d) follows. The proof is complete.

There are similar results for the character rings.

COROLLARY B.2. *When $P$ is a cyclic $p$-subgroup, all of the statements of Corollary* B.1 *remains true even if we replace $N$ by $C_G(P)$.*

PROOF (Outline). Let $t$ be a generator of $P$ and let $C:=C_G(P)$. For any subgroup $H$ of $G$, let $R_t(H)$ be the subgroup of the character ring $R(H)$ of $H$ consisting of all virtual characters $\theta$ such that $\theta(h)=0$ if $h$ is not $G$-conjugate to $t$. Then $R_t$ is a $G$-subfunctor of the character ring functor $R$. Let $\boldsymbol{M}$ be the Mackey functor corresponding to the $G$-functor $\boldsymbol{Z}_{(p)} \otimes R_t$. Then the rank of $\boldsymbol{M}(Y)$ for a finite $G$-set $Y$ equals to $|Y^P/C|$. Thus this corollary is proved by the similar way as Corollary B.1.

COROLLARY B.3. *Let $p$ be a prime which does not divide $n$. Assume that $G$ acts transitively on $B$, and so on $X$. Let $P$ be a $p$-subgroup* (*resp. $p$-element*) *of $G$. We take $x \in X$ and $\beta \in B$. Then the following hold*:

(a) *Assume that $p \nmid r$ or $p \nmid b$ and that any two subgroups* (*resp. elements*) *$P_1$ and $P_2$ of $G_\beta$ which are $G$-conjugate to $P$ are $G_\beta$-conjugate to each other. Then any two subgroups* (*resp. elements*) *$P_1$ and $P_2$ of $G_x$ which are $G$-conjugate to $P$, if they exist, are $G_x$-conjugate to each other.*

(b) *Assume that $P$ acts fixed point freely on $B$. Then any two subgroups* (*resp. elements*) *$P_1$ and $P_2$ of $G_x$ which are $G$-conjugate to $P$, if they exist, are $G_x$-conjugate.*

PROOF. This follows from Corollaries B.1 and B.2 and the fact that $|B^P/N|=1$, where $N=N_G(P)$ (resp. $C_G(P)$), if and only if any two subgroups (resp. elements) $P_1$ and $P_2$ of $G_\beta$ which are $G$-conjugate to $P$ are $G_\beta$-conjugate.

REMARK. In general, if $Y$ is a finite $G$-set and $P$ is a subgroup (resp. an element) of $G$ with $N:=N_G(P)$ (resp. $N:=C_G(P)$), then $|Y^P/N|$ equals to

$$|Y^P/N| = \sum_{y \in Y/G} c(G_y, P),$$

where for any subgroup $H$ of $G$, $c(H, P)$ is the number of $G_y$-conjugate classes of $G$-conjugates of $P$ contained in $H$. (Of course, $Y^P$ is the set of elements of $Y$ fixed by $P$.)

COROLLARY B.4. *Let $p$ be a prime which does not divide $n$. Let $\boldsymbol{a}$ be a poly-cohomological G-functor. Assume that $G$ acts transitively on $B$. Take $x \in X$ and $\beta \in B$. Then the following hold:*

(a) *Assume that $p \nmid r$. Then $\boldsymbol{a}(G_x) \mid \boldsymbol{a}(G_\beta)$.*

(b) *If $p \mid v$, then*

$$\mathrm{Ker}\,(\mathrm{cor} : \boldsymbol{a}(G_x) \longrightarrow \boldsymbol{a}(G)) \mid \boldsymbol{a}(G_\beta)\,.$$

$$\mathrm{Cok}\,(\mathrm{res} : \boldsymbol{a}(G) \longrightarrow \boldsymbol{a}(G_x)) \mid \boldsymbol{a}(G_\beta)\,.$$

(c) *If $p \nmid b$, then*

$$\mathrm{Ker}\,(\mathrm{cor} : \boldsymbol{a}(G_x) \longrightarrow \boldsymbol{a}(G)) \mid \mathrm{Ker}\,(\mathrm{cor} : \boldsymbol{a}(G_\beta) \longrightarrow \boldsymbol{a}(G))\,,$$

$$\mathrm{Cok}\,(\mathrm{res} : \boldsymbol{a}(G) \longrightarrow \boldsymbol{a}(G_x)) \mid \mathrm{Cok}\,(\mathrm{res} : \boldsymbol{a}(G) \longrightarrow \boldsymbol{a}(G_\beta))\,.$$

PROOF. This follows from Theorem B and Lemma 3.2.

REMARK. In the block transitive case, the map $\alpha^*$ of $\boldsymbol{a}(G_x)$ to $\boldsymbol{a}(G_\beta)$ induced by the incidence map $\alpha$ is explicitly given by the following. See Section 2d.

$$\alpha^*(\xi) := \sum_{\substack{g \in G_x \backslash G/G_\beta \\ : xg \in \beta}} \mathrm{cor}^{G_\beta} \circ \mathrm{res}_{G_{xg},\beta} \circ \mathrm{con}^g(\xi)\,.$$

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