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The theta functions of sublattices of the Leech lattice II

Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

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Introduction.

Let $\mathcal{L}$ be the Leech lattice. The automorphism group of $\mathcal{L}$ is denoted by $\cdot 0$, as usual. $\cdot 0$ possesses a maximal subgroup which is isomorphic to a split extension of the Mathieu group M_{24} by an elementary abelian group of order 2^{12}. This subgroup is sometimes called the monomial subgroup of $\cdot 0$ and is denoted by $2^{12}M_{24}$. For $\pi \in \cdot 0$, let

$$\mathcal{L}_{\pi}=\{v \in \mathcal{L} \mid v \circ \pi = v\}$$

$$\Theta_{\pi}(z)=\sum_{v \in \mathcal{L}_{\pi}} \exp(\pi i z \langle v, v\rangle) \qquad \text{(theta function of } \mathcal{L}_{\pi})$$

where $\langle\, ,\rangle$ denotes an inner product of the ambient Euclidean space of $\mathcal{L}$.

The purpose of this note is to express $\Theta_{\pi}(z)$ $(\pi \in 2^{12}M_{24})$ in terms of Jacobi theta functions $\theta_i(z)$ $(2 \leqq i \leqq 4)$ explicitly and to study a question raised by Conway-Norton for functions $\dfrac{\Theta_{\pi}(z)}{\eta_{\pi}(z)}$ (cf. Table 1~4 in § 3 and see § 2.3 for the definition of $\eta_{\pi}(z)$). If $\pi \in M_{24}$, this was done in [6]. But if π is outside M_{24}, the structure of $\mathcal{L}_{\pi}$ is rather simple compared with that of $\mathcal{L}_{\pi'}$ for $\pi' \in M_{24}$. So we will describe the structure of $\mathcal{L}_{\pi}$ $(\pi \in 2^{12}M_{24}-M_{24})$ explicitly in terms of lattices L, L_0, Λ and Λ_0 introduced in § 1. Then the expressions of $\Theta_{\pi}(z)$ can be obtained immediately from Lemma 1 in § 1. We note that, for $\pi \in \cdot 0-2^{12}M_{24}$, the calculations of $\Theta_{\pi}(z)$ and a question of Conway-Norton have been dealt with by M. L. Lang [7].

The present paper contains many interesting facts which were first pointed out by Koike [3] and [4]. Some of his results in [4], however, depend on those of the present paper (and also of Lang [7]). So we will prove some of these facts in [4] from somewhat different point of view from [4].

The authors would like to express their sincere thanks to Prof. Koike for his many invaluable suggestions.

Notations :

$$q=\exp(2\pi iz)$$

$$\eta(z)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad \text{the Dedekind } \eta\text{-function}$$

$$\theta_2(z)=\sum_{n=\ \infty}^{\infty}\exp\left\{\pi iz\left(n+\frac{1}{2}\right)^2\right\}=2q^{1/8}\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2$$

$$\theta_3(z)=\sum_{n=\ \infty}^{\infty}\exp(\pi izn^2)=\prod_{n=1}^{\infty}(1-q^n)(1+q^{n-(1/2)})^2$$

$$\theta_4(z)=\sum_{n=-\infty}^{\infty}(-1)^n\exp(\pi izn^2)=\prod_{n=1}^{\infty}(1-q^n)(1-q^{n-(1/2)})^2 .$$

§1. Theta functions of some integral lattices.

1.1. Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$ be independent vectors of an Euclidean space over the real number field. Suppose

$$\langle \boldsymbol{e}_i, \boldsymbol{e}_j\rangle = l_i\delta_{ij} \qquad (l_i\in \boldsymbol{Z})$$

where $\langle , \rangle$ is an inner product in the Euclidean space and δ_{ij} is the Kronecker symbol. Now we define some integral lattices:

$$L=L(l_1, l_2, \cdots, l_n)=\sum_{i=1}^{n}\boldsymbol{Z}\boldsymbol{e}_i$$

$$L_0=L_0(l_1, \cdots, l_n)=\left\{\sum_{i=1}^{n}x_i\boldsymbol{e}_i\in L \;\middle|\; \sum_i x_i\equiv 0 \bmod 2\right\}$$

$$\Lambda=\Lambda(l_1, \cdots, l_n)=L\cup\left(\frac{1}{2}\sum_i \boldsymbol{e}_i+L\right)$$

$$\Lambda_0=\Lambda_0(l_1, \cdots, l_n)=L_0\cup\left(\frac{1}{2}\sum_i \boldsymbol{e}_i+L_0\right).$$

For Λ_0, we assume that n is even. Then if $l_i\equiv l_j \bmod 2$ and $\sum_i l_i\equiv 0 \bmod 4$, Λ_0 is an integral lattice. If $\sum_i l_i\equiv 0 \bmod 2$ (resp. $l_i\equiv 0 \bmod 2$), $\sqrt{2}\Lambda$ (resp. Λ) is an integral lattice. We will use the notations such as $L(1^{r_1}2^{r_2}\cdots)$. For example, $L(1^5 3)$ means $L(1, 1, 1, 1, 1, 3)$.

LEMMA 1. *The theta functions of these lattices are expressed as follows in terms of Jacobi theta-functions* $\theta_2(z)$, $\theta_3(z)$, $\theta_4(z)$:

(i) $\Theta_L(z)=\prod_{i=1}^{n}\theta_3(l_iz)$

(ii) $\Theta_{L_0}(z)=\frac{1}{2}\left\{\prod_{i=1}^{n}\theta_3(l_iz)+\prod_{i=1}^{n}\theta_4(l_iz)\right\}$

(iii) $\Theta_\Lambda(z)=\prod_{i=1}^{n}\theta_2(l_iz)+\prod_{i=1}^{n}\theta_3(l_iz)$

(iv) $\Theta_{\Lambda_0}(z)=\frac{1}{2}\left\{\prod_{i=1}^{n}\theta_2(l_iz)+\prod_{i=1}^{n}\theta_3(l_iz)+\prod_{i=1}^{n}\theta_4(l_iz)\right\}$.

PROOF. (i) and (iii) are immediate. Let $\varepsilon(\boldsymbol{v})=\sum_k x_k$ for $v=\sum_k x_k\boldsymbol{e}_k$. Then we have

$$\Theta_{L_0}(z)=\sum_{v\in L_0}\exp(\pi iz\langle\boldsymbol{v},\boldsymbol{v}\rangle)=\frac{1}{2}\sum_{x_k\in Z}(1+(-1)^{\varepsilon(v)})\exp\left\{\pi iz\left(\sum_k l_kx_k^2\right)\right\},$$

which yields (ii). For (iv), setting $L_1=\frac{1}{2}\sum_k \boldsymbol{e}_k+L_0$,

$$\sum_{v\in L_1}\exp(\pi iz\langle\boldsymbol{v},\boldsymbol{v}\rangle)=\frac{1}{2}\sum_{x_k\in Z}(1+(-1)^{\varepsilon(v)})\exp\left\{\pi iz\left(\sum_k l_k\left(x_k+\frac{1}{2}\right)^2\right)\right\}$$
$$=\frac{1}{2}\sum_{x_k\in Z}\exp\left\{\pi iz\left(\sum_k l_k\left(x_k+\frac{1}{2}\right)^2\right)\right\},$$

because $\sum_{n=-\infty}^{\infty}(-1)^n\exp\left\{\pi iz\left(n+\frac{1}{2}\right)^2\right\}=0$. Thus we get (iv).

Example 1. Let $n=8$ and $l_1=l_2=\cdots=l_8=1$. Then $\Lambda_0(1^8)$ is isomorphic to the E_8-lattice (the lattice of root system of type E_8). In fact, if we let

$$\alpha_i=(-1)^{i-1}(\boldsymbol{e}_i-\boldsymbol{e}_{i+1})\ (1\leqq i\leqq 6),\quad \alpha_7=-(\boldsymbol{e}_6+\boldsymbol{e}_7),\quad \alpha_8=-\frac{1}{2}\sum_{i=1}^{8}\boldsymbol{e}_i,$$

then $\alpha_1,\cdots,\alpha_8$ are a basis of Λ_0 and form a fundamental root system of type E_8. Thus the theta function of the E_8-lattice is, by Lemma 1,

$$\theta(z,E_8)=\frac{1}{2}\{\theta_2(z)^8+\theta_3(z)^8+\theta_4(z)^8\}.$$

Furthermore $L_0(1^n)$ is isomorphic to the lattice of the root system of type D_n and the theta-function of $L_0(1^n)$ is

$$\theta(z,D_n)=\frac{1}{2}\{\theta_3(z)^n+\theta_4(z)^n\}.$$

LEMMA 2. (i) *$\Lambda_0(1^3 5)$ is isomorphic to the lattice of root system of type A_4 and we have*

$$\theta(z, A_4)=\frac{\eta(z)^5}{\eta(5z)}+25\cdot\frac{\eta(5z)^5}{\eta(z)}. \tag{1.1}$$

(ii) *$\Lambda_0(1.5^3)$ is isomorphic to the lattice corresponding to the matrix $5A_4^{-1}$ and we have*

$$\theta(z, 5A_4^{-1})=\frac{\eta(z)^5}{\eta(5z)}+5\cdot\frac{\eta(5z)^5}{\eta(z)}. \tag{1.2}$$

(iii) *$\Lambda_0(1^5.3)$ is isomorphic to the E_6-lattice and we have*

$$\theta(z, E_6)=\frac{\eta(z)^9}{\eta(3z)^3}+81\cdot\frac{\eta(3z)^9}{\eta(z)^3}. \tag{1.3}$$

(iv) *$\Lambda_0(1.3^5)$ is isomorphic to the lattice corresponding to the matrix $3E_6^{-1}$ and we have*

$$\theta(z, 3E_6^{-1})=\frac{\eta(z)^9}{\eta(3z)^3}+9\cdot\frac{\eta(3z)^9}{\eta(z)^3}. \tag{1.4}$$

PROOF. For the proofs of (1.1)~(1.4), firstly observe that both sides of each of (1.1)~(1.4) are modular forms whose level, weight and character are exactly the same (for (1.1)~(1.2), we have level 5, weight 2 and character $\left(\frac{*}{5}\right)$ and for (1.3)~(1.4), level 3, weight 3 and character $\left(\frac{*}{3}\right)$) and secondly observe that the first several coefficients of the Fourier expansions of both modular forms exactly coincide (cf. Hecke [2; p. 811]). We note that, for the computations of the Fourier coefficients of the left hand side of (1.1)~(1.4), the formulas obtained from Lemma 1 should be used: for example,

$$\theta(z, A_4)=\frac{1}{2}\{\theta_2(z)^3\theta_2(5z)+\theta_3(z)^3\theta_3(5z)+\theta_4(z)^3\theta_4(5z)\}.$$

Now we will show the first statements of (i)~(iv):

(i) Let e_1, e_2, e_3, and e_4 be independent vectors with $l_1=l_2=l_3=1$ and $l_4=5$. Let $\alpha_1=e_1-e_2$, $\alpha_2=-(e_2-e_3)$, $\alpha_3=-(e_1+e_2)$ and $\alpha_4=-\frac{1}{2}\sum_{i=1}^{4} e_i$. Then it is easily seen that $\alpha_1, \cdots, \alpha_4$ are a basis of $\Lambda_0(1^3.5)$ and form a fundamental root system of type A_4. We note that we have

$$A_4=\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad 5A_4^{-1}=\begin{pmatrix} 4 & -3 & 2 & -1 \\ -3 & 6 & -4 & 2 \\ 2 & -4 & 6 & -3 \\ -1 & 2 & -3 & 4 \end{pmatrix}.$$

(ii) Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ and $\boldsymbol{e}_4$ be independent vectors with $l_1=1$, $l_2=l_3=l_4=5$. Then vectors

$$-\frac{1}{2}\sum \boldsymbol{e}_i, \quad \boldsymbol{e}_1+\boldsymbol{e}_4, \quad \frac{1}{2}(-3\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_3-\boldsymbol{e}_4), \quad 2\boldsymbol{e}_1$$

are a basis of $\Lambda_0(1.5^3)$ and yield the matrix $5A_4^{-1}$. This proves (ii).

(iii) Let $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_6$ be independent vectors with $l_1=\cdots=l_5=1$ and $l_6=3$ and let $\alpha_i=(-1)^{i-1}(\boldsymbol{e}_i-\boldsymbol{e}_{i+1})$ $(1\leqq i\leqq 4)$, $\alpha_5=-(\boldsymbol{e}_4+\boldsymbol{e}_5)$, $\alpha_6=\frac{1}{2}\sum_{i=1}^{6}\boldsymbol{e}_i$. Then $\alpha_1, \cdots, \alpha_6$ are a basis of $\Lambda_0(1^5.3)$ and form a fundamental root system of type E_6. Furthermore $\alpha_1, \cdots, \alpha_6$ yield matrices

$$E_6=\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}, \quad 3E_6^{-1}=\begin{pmatrix} 4 & -5 & 6 & -3 & -4 & 2 \\ -5 & 10 & -12 & 6 & 8 & -4 \\ 6 & -12 & 18 & -9 & -12 & 6 \\ -3 & 6 & -9 & 6 & 6 & -3 \\ -4 & 8 & -12 & 6 & 10 & -5 \\ 2 & -4 & 6 & -3 & -5 & 4 \end{pmatrix}.$$

(iv) It is easy to see that the matrix $3E_6^{-1}$ is equivalent to

$$(\sharp) \qquad \begin{pmatrix} 4 & 1 & 1 & 1 & -2 & -2 \\ 1 & 4 & 1 & 1 & 1 & -2 \\ 1 & 1 & 4 & 1 & 1 & 1 \\ 1 & 1 & 1 & 4 & 1 & 1 \\ -2 & 1 & 1 & 1 & 4 & 1 \\ -2 & -2 & 1 & 1 & 1 & 4 \end{pmatrix}.$$

Let $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_6$ be independent vectors with $l_1=1$ and $l_2=\cdots=l_5=3$. Then vectors

$$\frac{1}{2}(\boldsymbol{e}_1-\boldsymbol{e}_2-\boldsymbol{e}_3+\boldsymbol{e}_4+\boldsymbol{e}_5+\boldsymbol{e}_6), \quad \frac{1}{2}\sum_{i=1}^{6}\boldsymbol{e}_i, \quad \frac{1}{2}(\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_3-\boldsymbol{e}_4+\boldsymbol{e}_5+\boldsymbol{e}_6), \quad 2\boldsymbol{e}_1$$

$$\frac{1}{2}(\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3-\boldsymbol{e}_4-\boldsymbol{e}_5+\boldsymbol{e}_6), \quad \frac{1}{2}(\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_3-\boldsymbol{e}_4-\boldsymbol{e}_5-\boldsymbol{e}_6)$$

are a basis of $\Lambda_0(1.3^5)$ and yield the matrix (♯) given above. This proves (iv).

1.2. Here we note that some isomorphisms between L, L_0, Λ, Λ_0 yield well known formulas which hold between $\theta_2(z)$, $\theta_3(z)$, $\theta_4(z)$.

Let $\Lambda_0=\Lambda_0(1.7)$. Then a basis of Λ_0 is $\frac{1}{2}(\boldsymbol{e}_1+\boldsymbol{e}_2)$, $2\boldsymbol{e}_1$ ($l_1=1$, $l_2=7$) and the corresponding matrix w. r. t. this basis is $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$. On the other hand, a basis of another lattice $\Lambda(2.14)$ is $\frac{1}{2}(\boldsymbol{e}_1+\boldsymbol{e}_2)$, $\boldsymbol{e}_1$ ($l_1=2$, $l_2=14$) and the corresponding matrix is $\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$. Thus we see that $\Lambda_0(1.7)$ is isomorphic to $\Lambda(2.14)$. This yields, by Lemma 1,

$$\frac{1}{2}\{\theta_2(z)\theta_2(7z)+\theta_3(z)\theta_3(7z)+\theta_4(z)\theta_4(7z)\}=\theta_2(2z)\theta_2(14z)+\theta_3(2z)\theta_3(14z)\,. \tag{1.5}$$

(cf. [6 ; (T15)])

Similarly the following well known formulas can be obtained by showing the isomorphisms of lattices given in the parenthesis:

(1.6) $\theta_2(z)^4+\theta_4(z)^4=\theta_3(z)^4$ $(\Lambda_0(1^4)\cong L(1^4))$

(1.7) $\theta_2(z)\theta_2(3z)+\theta_4(z)\theta_4(3z)=\theta_3(z)\theta_3(3z)$ $(\Lambda_0(1.3)\cong L(1.3))$

(1.8) $\theta_3(z)^2=\theta_2(2z)^2+\theta_3(2z)^2$ $(L(1^2)\cong \Lambda(2^2))$

(1.9) $\frac{1}{2}\{\theta_3(z)^2+\theta_4(z)^2\}=\theta_3(2z)^2$ $(L_0(1^2)\cong L(2^2))$.

The followings are obtained from the product formulas of $\theta_i(z)$ ($i=2, 3, 4$) and $\eta(z)$:

$$\theta_2(z)=2\frac{\eta(2z)^2}{\eta(z)} \tag{1.10}$$

$$\theta_3(2z)=\frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2} \tag{1.11}$$

$$\theta_4(2z)=\frac{\eta(z)^2}{\eta(2z)}\,. \tag{1.12}$$

From these we see

$$\theta_2(z)^2=2\theta_2(2z)\theta_3(2z) \tag{1.13}$$

$$\theta_4(2z)^2=\theta_3(z)\theta_4(z)\,. \tag{1.14}$$

Finally let

$$\theta_1'(z)=\theta_2(z)\theta_3(z)\theta_4(z)\,. \tag{1.15}$$

Then we have, by (1.10)$\sim$(1.12),

$$\theta_1'(z)=2\eta(z)^3\,. \tag{1.16}$$

§ 2. Leech lattice and Frame shapes.

2.1. The Leech lattice $\mathcal{L}$ can be described as follows:

$$\mathcal{L}=\bigcup_{X\in\mathcal{G}}\left\{\left(\frac{1}{2}\boldsymbol{e}_X+\mathcal{L}_0\right)\cup\left(\frac{1}{4}\boldsymbol{e}_\Omega+\frac{1}{2}\boldsymbol{e}_X+\mathcal{L}_1\right)\right\} \qquad \text{(cf. [8: p. 708])} \tag{2.1}$$

where we use the following notations (1)$\sim$(4):

(1) $\Omega=\{\infty, 0, 1, \cdots, 22\}$ is a 24-points set which may be identified with a projective line over the field F_{23} of 23 elements and $\mathcal{G}$ is the binary Golay code,

(2) $\boldsymbol{e}_\infty, \boldsymbol{e}_0, \cdots, \boldsymbol{e}_{22}$ is a basis of a 24 dimensional Euclidean space with

$$\langle \boldsymbol{e}_i, \boldsymbol{e}_j\rangle=2\delta_{ij} \qquad (i, j\in\Omega)$$

(3) $\mathcal{L}_\delta=\left\{\sum_{i\in\Omega}x_i\boldsymbol{e}_i \mid x_i\in\boldsymbol{Z},\ \sum_{i\in\Omega}x_i\equiv\delta \bmod 2\right\}$ for $\delta=0$ or 1,

(4) for $X\subseteqq\Omega$, $\boldsymbol{e}_X=\sum_{i\in X}\boldsymbol{e}_i$.

The Mathieu group M_{24} is, by definition, a subgroup of the symmetric group $S(\Omega)$ on Ω which leaves $\mathcal{G}$ invariant. For $\sigma\in M_{24}$, we set

$$\boldsymbol{e}_i\circ\sigma=\boldsymbol{e}_{i^\sigma} \qquad (i\in\Omega)\,. \tag{2.2}$$

Then we see from (2.1) that M_{24} leaves $\mathcal{L}$ invariant. Thus M_{24} is a subgroup of the automorphism group $\cdot 0$ of the Leech lattice $\mathcal{L}$.

Let $C\in\mathcal{G}$. We define a linear transformation ε_C as follows:

$$\boldsymbol{e}_i\circ\varepsilon_C=\begin{cases}-\boldsymbol{e}_i & i\in C\\ \boldsymbol{e}_i & i\notin C\,.\end{cases} \tag{2.3}$$

Then we have $\varepsilon_C\in\cdot 0$ by (2.1). Clearly a subgroup $U=\{\varepsilon_C \mid C\in\mathcal{G}\}$ is an elementary subgroup of order 2^{12} and we have

$$U\triangleleft U.M_{24}\subset\cdot 0\,.$$

The subgroup $U.M_{24}$ is sometimes called the monomial subgroup of $\cdot 0$ and we write $2^{12}.M_{24}$ for $U.M_{24}$.

For $\pi\in\cdot 0$, let

$$\mathcal{L}_\pi=\{\boldsymbol{v}\in\mathcal{L}\mid \boldsymbol{v}\circ\pi=\boldsymbol{v}\},$$

$$\Theta_\pi(z)=\sum_{v\in\mathcal{L}_\pi}\exp(\pi iz\langle \boldsymbol{v},\boldsymbol{v}\rangle).$$

One of the purposes of this note is to calculate $\Theta_\pi(z)$ for every $\pi\in 2^{12}.M_{24}$. $\Theta_\pi(z)$ for π outside $2^{12}.M_{24}$ have been calculated by M. L. Lang [7].

2.2. For $\pi\in\cdot 0$, a characteristic polynomial of π w. r. t. the action of π on $\mathcal{L}$ can be written in the form

$$\prod_{t=1}^{\infty}(X^t-1)^{r_t}\qquad (r_t\in\boldsymbol{Z}).$$

Then we associate with π a symbol, called a Frame shape of π,

$$\prod_{t=1}^{\infty}t^{r_t}.$$

A list of Frame shapes of all elements (conjugate classes) of $\cdot 0$ is given in Table I of [5]. For a symbol $\pi=\prod_t t^{r_t}$ (not necessarily a Frame shape), we set

$$\deg(\pi)=\sum_t tr_t$$

$$wt(\pi)=\frac{1}{2}\sum_t r_t$$

$$r(\pi)=r_1.$$

Furthermore, if $\pi'=\prod_t t^{r'_t}$, we define a product $\pi\cdot\pi'$ and π^{-1} naturally as follows:

$$\pi\cdot\pi'=\prod_t t^{r_t+r'_t},$$

$$\pi^{-1}=\prod_t t^{-r_t}.$$

Following Koike [3], it is convenient to say that

π is of type F if $wt(\pi)=0$
π is of type C if $r_t\geqq 0$ for all t
π is of type E if $wt(\pi)>0$ but $r_t<0$ for some t.

We note that $\Theta_\pi(z)=1$ for π of type F, because $2wt(\pi)=\sum_t r_t$ is equal to the rank of $\mathcal{L}_\pi$. If $\pi\in M_{24}$, then the Frame shape of π is of type C and

$\Theta_\pi(z)$ is calculated in [6]. Besides these π in M_{24}, we have four Frame shapes of type C

$$2^3 6^3,\quad 4^2 8^2,\quad 4.20,\quad 2.22 \tag{2.4}$$

which are all in the monomial subgroup $2^{12}M_{24}$. There are 24 Frame shapes of type E in $2^{12}M_{24}$ which will be listed up in (2.7)$\sim$(2.9) below, while there are 21 Frame shapes of type E outside $2^{12}M_{24}$.

2.3. For a Frame shape $\pi=\prod_t t^{r_t}$, we set

$$\eta_\pi(z)=\prod_t \eta(tz)^{r_t}$$

where $\eta(z)$ is the Dedekind η-function. In Table I of [3], Koike identified $\eta_\pi(z)$ for each π of type E with an Eisenstein series. Inspecting the table, we get a following important theorem:

THEOREM 1 (Koike [3; Table I]). *Suppose π is of type E. Then the following* (i) *and* (ii) *are equivalent*:

(i) $wt(\pi)=wt(\pi^p)$ *for some prime divisor p of the order of π.*
(ii) $\eta_\pi(z)$ *is not eigenfunction for a Hecke operator $T(p)$.*
Furthermore, if (i) *or* (ii) *holds, then we have*

$$\eta_\pi(z)=\eta_{\pi^p}(z)-(r(\pi)-r(\pi^p))\,\eta_{\pi^p}(pz)\,. \tag{2.5}$$

REMARKS. (1) There are twelve Frame shapes satisfying the conditions (i) or (ii):

$$\frac{1^8 4^8}{2^8},\quad \frac{1^5 3.6^4}{2^4},\quad \frac{1^4 8^4}{2^2 4^2},\quad \frac{1^3 5.10^2}{2^2},\quad \frac{2.3^3 12^3}{1.4.6^3},\quad \frac{1^2 3^2 4^2 12^2}{2^2 6^2},\quad \frac{1^3 12^3}{2.3.4.6},$$

$$\frac{1^2 16^2}{2.8},\quad \frac{1^2 9.18}{2.3},\quad \frac{1.2.18^2}{6.9},\quad \frac{1.4.7.28}{2.14},\quad \frac{2.3.5.30}{6.10}.$$

We note that, for these Frame shapes π, we have

$$\mathcal{L}_\pi=\mathcal{L}_{\pi^p} \text{ and so } \Theta_\pi(z)=\Theta_{\pi^p}(z) \qquad \text{(cf. M. L. Lang [7])}. \tag{2.6}$$

(2) Also G. Mason [10] has pointed out the equivalence of (i) and (ii) in Theorem 1.

2.4. For a Frame shape $\pi=\prod_t t^{r_t}$, the level N_π of π is defined as the least positive integer N satisfying the following conditions (i) and (ii):

(i) $t|N$ for any t with $r_t\neq 0$

(ii) $\sum_t \frac{N}{t} r_t \equiv 0 \mod 24.$

For a Hall divisor Q of N_π, we define a symbol $\pi * W_{Q,N_\pi}$ as

$$\pi * W_{Q,N_\pi} = \prod_t \left(\frac{Qt}{(Q,t)^2}\right)^{r_t}$$

and set

$$S(\pi)=\{\pi * W_{Q,N_\pi} \text{ for all Hall divisors of } N_\pi\}.$$

Also the following interesting fact was observed by Koike:

THEOREM 2 (Koike [4; Th. 4.2]). *Let* $\pi'\in S(\pi)$ *and* $\deg(\pi')=0$. *Then* $\pi\cdot\pi'^{-1}$ *is a Frame shape of* $\cdot 0$ *of type F. If we set* $\Gamma_\pi=\langle\Gamma_0(N_\pi), W_{Q,N}\,|\,\pi * W_{Q,N}=\pi\rangle$, $\boldsymbol{C}\left(\frac{\eta_{\pi'}(z)}{\eta_\pi(z)}\right)$ *is the field of modular functions w.r.t.* Γ_π. *In particular,* Γ_π *is of genus* 0 *and* $\frac{\eta_{\pi'}(z)}{\eta_\pi(z)}$ *coincides with a Thompson series of some element of Fischer-Griess's Monster up to constant.*

Now we classify Frame shapes of type E in more details: For π of type E, we say that

π is of type E_1 if $S(\pi)=\{\pi\}$
π is of type E_2 if $\deg(\pi')=0$ for any $\pi'\in S(\pi)-\{\pi\}$
π is of type E_3 otherwise.

REMARK. In [4], Frame shapes of type E_3 are called those of non self-conjugate type.

Frame shapes of each type in the monomial group $2^{12}M_{24}$ are as follows:

(2.7) Type E_1 (8 elements)

$$\frac{1^8 4^8}{2^8},\ \frac{2^4 8^4}{4^4},\ \frac{1^4 8^4}{2^2 4^2},\ \frac{1^2 3^2 4^2 12^2}{2^2 6^2},\ \frac{2^2 16^2}{4.8},\ \frac{1^2 16^2}{2.8},\ \frac{2.6.8.24}{4.12},\ \frac{1.4.7.28}{2.14},$$

(2.8) Type E_2 (8 elements)

$$\frac{2^{16}}{1^8},\ \frac{4^8}{2^4},\ \frac{2^6 4^4}{1^4},\ \frac{2^4 6^4}{1^2 3^2},\ \frac{8^4}{4^2},\ \frac{2^3 4.8^2}{1^2},\ \frac{4^2 12^2}{2.6},\ \frac{2^2 14^2}{1.7},$$

(2.9) Type E_3 (8 elements)

$$\frac{1^4 2.6^5}{3^4},\quad \frac{2^5 3^4 6}{1^4},\quad \frac{1^2 2.10^3}{5^2},\quad \frac{2^3 5^2 10}{1^2},\quad \frac{1.6.10.15}{3.5},\quad \frac{2.3.5.30}{1.15},$$

$$\frac{1^2 4.6^2 12}{3^2},\quad \frac{2^2 3^2 4.12}{1^2}.$$

2.5. If π is of type E_3, then $S(\pi)$ consists of three Frame shapes π_1, π_2, π_3 and just one elements π_0 of degree 0:

$$S(\pi)=\{\pi_0, \pi_1, \pi_2, \pi_3\}.$$

THEOREM 3. *Let i, j, k be any permutation of the letters* 1, 2, 3. *Notations being as above, then we have the following* (1)∼(5):

(1) *The symbols* $\pi_i\pi_0^{-1}$ *and* $\pi_i\pi_j\pi_k^{-1}\pi_0^{-1}$ *are Frame shapes of* $\cdot 0$.

(2) *Let* $c_i=r(\pi_0\pi_i^{-1})$ $(i=1, 2, 3)$. *Then we have*

$$\frac{\pi_0}{\pi_1}+c_1=\frac{\pi_0}{\pi_2}+c_2=\frac{\pi_0}{\pi_3}+c_3$$

where the η*-product* $\frac{\eta_{\pi_0}(z)}{\eta_{\pi_i}(z)}$ *associated with the symbols* $\pi_0\pi_i^{-1}$ *are expressed simply by* $\frac{\pi_0}{\pi_i}$.

Using similar expressions,

(3) $$\frac{\pi_i}{\pi_j}=1+(c_i-c_j)\frac{\pi_i}{\pi_0}$$

(4) $$\frac{\pi_0\pi_k}{\pi_i\pi_j}=\frac{\pi_0}{\pi_k}+(c_k-c_i)(c_k-c_j)\frac{\pi_k}{\pi_0}+2c_k-c_i-c_j$$

(*the formulas of the symmetrisation, cf. Table* 3 *of* [1])

(5) $$\frac{\pi_i\pi_j}{\pi_0\pi_k}=\frac{c_k-c_i}{c_j-c_i}\frac{\pi_i}{\pi_0}+\frac{c_k-c_j}{c_i-c_j}\frac{\pi_j}{\pi_0}.$$

COROLLARY. (6) $$\frac{\pi_i\pi_j}{\pi_0}=\frac{1}{c_i-c_j}(\pi_i-\pi_j).$$

(7) $$\frac{\pi_0\pi_i}{\pi_j}=\pi_0+(c_i-c_j)\pi_i.$$

(8) $$\frac{\pi_i\pi_j}{\pi_k}=\frac{c_k-c_i}{c_j-c_i}\pi_i+\frac{c_k-c_j}{c_i-c_j}\pi_j.$$

In particular, $\frac{\pi_l\pi_m}{\pi_n}$ *are modular forms, where* l, m, n *are distinct elements of* $\{0, 1, 2, 3\}$.

PROOF OF THEOREM 3. (1) This can be seen from Table I of [5].
(2) By Table 3 of [1], each of the $\frac{\eta_{\pi_0}(z)}{\eta_{\pi_i}(z)}$ $(i=1, 2, 3)$ generates the same function field of genus 0 and $\frac{\eta_{\pi_0}(z)}{\eta_{\pi_i}(z)}+c_i$ has the constant term 0. (2) follows from these facts.

(3) $\frac{\pi_i}{\pi_j}=\frac{\pi_i}{\pi_0}\cdot\frac{\pi_0}{\pi_j}=\frac{\pi_i}{\pi_0}\left(\frac{\pi_0}{\pi_i}+c_i-c_j\right)=1+(c_i-c_j)\frac{\pi_i}{\pi_0}.$

(4) $$\frac{\pi_0\pi_k}{\pi_i\pi_j}=\frac{\pi_0}{\pi_i}\left(1+(c_k-c_j)\frac{\pi_k}{\pi_0}\right)=\frac{\pi_0}{\pi_i}+(c_k-c_j)\frac{\pi_k}{\pi_i}$$
$$=\frac{\pi_0}{\pi_k}+c_k-c_i+(c_k-c_j)\left(1+(c_k-c_i)\frac{\pi_k}{\pi_0}\right),$$

which yields (4).

(5) Let $f=\frac{\pi_0}{\pi_k}$. Then we have

$$\frac{\pi_0\pi_k}{\pi_i\pi_j}=\frac{(f+c_k-c_i)(f+c_k-c_j)}{f}$$

and so

$$\frac{\pi_i\pi_j}{\pi_0\pi_k}=\frac{c_k-c_i}{c_j-c_i}\,\frac{1}{f+c_k-c_i}+\frac{c_k-c_j}{c_i-c_j}\,\frac{1}{f+c_k-c_j}.$$

Then (2) yields (5), q. e. d.

§ 3. $\Theta_\pi(z)$ and $j_\pi(z)$ for $\pi\in 2^{12}M_{24}$.

3.1. In this section, we will calculate $\Theta_\pi(z)$ for $\pi\in 2^{12}M_{24}$ and will study a question raised by Conway-Norton [1; p. 315] for a function $j_\pi(z)=\frac{\Theta_\pi(z)}{\eta_\pi(z)}$. They conjectured that $j_\pi(z)$ coincides with a Thompson series of some element of Fischer-Griess's Monster up to constant. Since this was done in [6] for $\pi\in M_{24}$, we will deal with the case $\pi\in 2^{12}M_{24}-M_{24}$. Such π are listed up in (2.4) and (2.7)$\sim$(2.9). Besides the notations introduced in § 2.1 for the Leech lattice, we will use the followings:

$$V_\pi=\{v\in \boldsymbol{R}^{24}\mid v\circ\pi=v\}, \qquad (\mathcal{L}_0)_\pi=V_\pi\cap\mathcal{L}_0.$$

Let $\pi\in 2^{12}M_{24}-M_{24}$. Then π can be written $\pi=\sigma\varepsilon_C$ where $\sigma\in M_{24}$ and C is a nontrivial code (an element $\neq\emptyset$ of the Golay code). A cycle U of a permutation σ is called a π-cycle if $|U\cap C|=$even. If the length of a cycle U of σ is denoted by $l(U)$, the Frame shape of σ may be expressed as $\prod_U l(U)$ where U run over all cycles of σ.

LEMMA 3. *The Frame shape of $\pi=\sigma\varepsilon_C$ may be written*

$$\Pi' l(U)\cdot\Pi''\left(\frac{2l(U)}{l(U)}\right)$$

where Π' denotes the product over all π-cycles and Π'' denotes the product over the remaining cycles of σ.

A code X is called π-admissible if X is a union of some π-cycles. If $\mathcal{L}_\pi\cap\left(\frac{1}{2}\boldsymbol{e}_X+\mathcal{L}_0\right)\neq\emptyset$, then X must be π-admissible, but the converse is not necessarily true. A vector $\boldsymbol{f}=\sum_{i\in X}\varepsilon_i\boldsymbol{e}_i$ $(\varepsilon_i=\pm1)$ is called π-admissible if X is a π-admissible code and then we say that $\boldsymbol{f}$ belongs to X.

3.2. Firstly we deal with four Frame shapes of type C in (2.4). The results are given in the following table:

Table 1

π	$\mathcal{L}_\pi$	$\Theta_\pi(z)$	
$2^3 6^3$	$\Lambda_0(2^3 6^3)$	$\theta_3(2z)^3\theta_3(6z)^3-6\eta_\pi(z)$	$12+$
$4^2 8^2$	$\Lambda(8^4)\cong L_0(4^4)$	$\theta(4z, D_4)$	$8\|4+$
4.20	$L(4.20)$	$\theta_3(4z)\theta_3(20z)$	$40\|2+$
2.22	$\Lambda_0(2.22)$	$\theta_3(2z)\theta_3(22z)-2\eta_\pi(z)$	$44+$

In this table, the second column shows the structure of $\mathcal{L}_\pi$ by using the notations in § 1. The expressions of $\Theta_\pi(z)$ are different from those obtained from Lemma 1 for some π. The last column shows discrete subgroups of $SL(2, \boldsymbol{R})$ by using the notations in [1] and implies that the function $\frac{\Theta_\pi(z)}{\eta_\pi(z)}$ is a generator of the field of modular functions w. r. t. the discrete subgroup. Thus a conjecture of Conway-Norton is true for π of type C.

Now we will begin the proof of the results in Table 1.

(1) The case $\pi=2^3 6^3$. Let σ be an element of M_{24} with a Frame

shape $1^6 3^6$ (=cycle decomposition). Then there exists a dodecade D of the Golay code $\mathcal{G}$ on which σ induces a permutation $1^3 3^3$ (cf. [6 ; Table 1]). Let $\pi=\sigma\varepsilon_D$, where ε_D is an element defined in (2.3). Then π is in $2^{12}M_{24}$ and, by Lemma 3, of Frame shape $2^3 6^3$. Let $D'=\Omega-D$. σ induces a permutation $1^3 3^3$ on D' and so, if $C_1, \cdots, C_6$ are cycles of the permutation which σ induces on D', we have $|C_i|=1$ for three cycles of the C_i $(1\leqq i\leqq 6)$ and $|C_i|=3$ for the remaining three cycles. Let $\boldsymbol{f}_i=\sum_{k\in C_i}\boldsymbol{e}_k$. Then it is easy to see $V_\pi=\left\{\sum_{i=1}^{6}x_i\boldsymbol{f}_i \mid x_i\in \boldsymbol{R}\right\}$ and $(\mathcal{L}_0)_\pi=\left\{\sum_{i=1}^{6}x_i\boldsymbol{f}_i \mid x_i\in \boldsymbol{Z},\ \sum_{i=1}^{6}x_i\equiv 0 \mod 2\right\}\cong L_0(2^3 6^3)$. Clearly D' is the only one π-admissible code and $\frac{1}{2}\boldsymbol{e}_{D'}\in\mathcal{L}$. Then it follows from (2.1) that

$$\mathcal{L}_\pi=(\mathcal{L}_0)_\pi\cup\left(\frac{1}{2}\boldsymbol{e}_{D'}+(\mathcal{L}_0)_\pi\right) \qquad \left(\boldsymbol{e}_{D'}=\sum_{i=1}^{6}\boldsymbol{f}_i\right)$$

and so $\mathcal{L}_\pi\cong\Lambda_0(2^3 6^3)$. Therefore we get, by Lemma 1,

$$\Theta_\pi(z)=\frac{1}{2}\sum_{i=2}^{4}\theta_i(2z)^3\theta_i(6z)^3 .$$

The right-hand side is equal to

$$\theta_3(2z)^3\theta_3(6z)^3-\frac{3}{2}\theta_1'(2z)\theta_1'(6z) \qquad \text{by (1.7) and (1.15)}$$

$$=\theta_3(2z)^3\theta_3(6z)^3-6\eta_\pi(z) \qquad \text{by (1.16).}$$

Now we have, by (1.11),

$$j_\pi(z)=\frac{\Theta_\pi(z)}{\eta_\pi(z)}=\eta_\delta(z)-6 \qquad \left(\delta=\frac{2^{12}6^{12}}{1^6 3^6 4^6 12^6}\right)$$

which implies that $j_\pi(z)$ is a generator for 12+ by Table 3 of [1].

(2) The case $\pi=4^2 8^2$. Let σ and C be an element of Frame shape 4^6 of M_{24} and an octad as follows:

$$\sigma=(\underline{\infty}, 21, \underline{0}, \underline{6})(\underline{1}, 7, 2, 19)(\underline{3}, 15, 5, \underline{10})(9, 17, 20, 14)(4, 12, 22, 11)(\underline{8}, 16, 18, \underline{13})$$

$$C=\{\infty, 0, 1, 3, 6, 8, 10, 13\} \qquad \text{(cf. [9 ; p. 215 and Table I]).}$$

$C_1=\{\infty, 0, 1, 2, 6, 7, 19, 21\}$ which is a union of the first two cycles of σ is also an octad. Let $\pi=\sigma\varepsilon_C$. Then π is, by Lemma 3, of Frame shape $4^2 8^2$. Let

$$\boldsymbol{f}_1=\boldsymbol{e}_3+\boldsymbol{e}_{15}+\boldsymbol{e}_5-\boldsymbol{e}_{10}, \quad \boldsymbol{f}_2=\boldsymbol{e}_9+\boldsymbol{e}_{17}+\boldsymbol{e}_{20}+\boldsymbol{e}_{14},$$

$$\boldsymbol{f}_3=\boldsymbol{e}_4+\boldsymbol{e}_{12}+\boldsymbol{e}_{22}+\boldsymbol{e}_{11}, \quad \boldsymbol{f}_4=\boldsymbol{e}_8+\boldsymbol{e}_{16}+\boldsymbol{e}_{18}-\boldsymbol{e}_{13}.$$

Then we easily see $V_\pi=\left\{\sum_{i=1}^{4} x_i\boldsymbol{f}_i \mid x_i\in\boldsymbol{R}\right\}$ and $(\mathcal{L}_0)_\pi=\left\{\sum_{i=1}^{4} x_i\boldsymbol{f}_i \mid x_i\in\boldsymbol{Z}\right\}\cong L(8^4)$. (Note that $\boldsymbol{f}_i\in\mathcal{L}_0$ $(1\leqq i\leqq 4)$). There are just two admissible octads to which $\boldsymbol{f}_1+\boldsymbol{f}_2$ and $\boldsymbol{f}_3+\boldsymbol{f}_4$ belong, but we have $\frac{1}{2}(\boldsymbol{f}_1+\boldsymbol{f}_2)$, $\frac{1}{2}(\boldsymbol{f}_3+\boldsymbol{f}_4)\notin\mathcal{L}_\pi$. Then we have $\mathcal{L}_\pi\subset\mathcal{L}_0\cup\left(\frac{1}{2}\boldsymbol{e}_{\Omega-C_1}+\mathcal{L}_0\right)$ by (2.1) and so we must have

$$\mathcal{L}_\pi=(\mathcal{L}_0)_\pi\cup\left(\frac{1}{2}\sum_{i=1}^{4}\boldsymbol{f}_i+(\mathcal{L}_0)_\pi\right)$$

and so $\mathcal{L}_\pi\cong\Lambda(8^4)\cong L_0(4^4)$. Therefore

$$\Theta_\pi(z)=\theta_2(8z)^4+\theta_3(8z)^4=\frac{1}{2}\{\theta_3(4z)^4+\theta_4(4z)^4\}=\theta(4z, D_4)\,.$$

Let $f(z)=\dfrac{\theta(z, D_4)^4}{\eta_\delta(z)}$ $(\delta=1^8 2^8)$ which is, by [6], a generator for $1^8 2^8$. Then we have $j_\pi(z)=f(4z)^{1/4}$, which implies that $j_\pi(z)$ is a generator for $8|4+$.

(3) The case $\pi=4.20$. Let σ and C be an element of M_{24} and an octad as follows:

$$\sigma=(\underline{\infty}, \underline{9})(\underline{2}, 5)(\underline{0}, 3, \underline{1}, \underline{15}, 19, 12, 18, 8, 4, \underline{10})(6, 13, 14, 7, 11, 16, 17, 21, 22, \underline{20})\,,$$

where elements of C are shown by underlined letters. Let $\pi=\sigma\varepsilon_C$. π is of Frame shape 4.20. Let

$$\boldsymbol{f}_1=\boldsymbol{e}_\infty-\boldsymbol{e}_9\,,$$

$$\boldsymbol{f}_2=\boldsymbol{e}_0+\boldsymbol{e}_3-\boldsymbol{e}_1+\boldsymbol{e}_{15}+\boldsymbol{e}_{19}+\boldsymbol{e}_{12}+\boldsymbol{e}_{18}+\boldsymbol{e}_8+\boldsymbol{e}_4-\boldsymbol{e}_{10}\,.$$

Then we have $V_\pi=\{x_1\boldsymbol{f}_1+x_2\boldsymbol{f}_2 \mid x_i\in\boldsymbol{R}\}$ and $(\mathcal{L}_0)_\pi=\{x_1\boldsymbol{f}_1+x_2\boldsymbol{f}_2 \mid x_i\in\boldsymbol{Z}\}\cong L(4.20)$. We note that a vector $\boldsymbol{f}_1+\boldsymbol{f}_2$ is π-admissible but $\frac{1}{2}(\boldsymbol{f}_1+\boldsymbol{f}_2)\notin\mathcal{L}$. So we must have $\mathcal{L}_\pi\subset\mathcal{L}_0$ and so $\mathcal{L}_\pi=(\mathcal{L}_0)_\pi\cong L(4.20)$. Thus

$$\Theta_\pi(z)=\theta_3(4z)\theta_3(20z)$$

and then, by (1.11),

$$j_\pi(z)=\eta_\delta(z) \qquad \left(\delta=\frac{4^4 20^4}{2^2 8^2 10^2 40^2}\right).$$

This implies that $j_\pi(z)$ is a generator for $40|2+$ by Table 3 of [1].

(4) The case $\pi=2.22$. Let σ be an element of Frame shape $1^2 11^2$ of M_{24} and D be a dodecade on which σ induces a permutation 1.11. Then $\pi=\sigma\varepsilon_D$ is of Frame shape 2.22. Similarly as in the case $\pi=2^3 6^3$, we see

$$(\mathcal{L}_0)_\pi \cong L_0(2.22)$$

$$\mathcal{L}_\pi = (\mathcal{L}_0)_\pi \cup \left(\frac{1}{2} e_{\Omega-D} + (\mathcal{L}_0)_\pi\right) \cong \Lambda_0(2.22)$$

and so

$$\Theta_\pi(z) = \frac{1}{2} \sum_{i=2}^{4} \theta_i(2z)\theta_i(22z) .$$

Now we apply the formula

$$4\eta(z)\eta(11z) = \theta_3(z)\theta_3(11z) - \theta_2(z)\theta_2(11z) - \theta_4(z)\theta_4(11z)$$

(cf. (T24) in Appendix of [6]).
Then

$$\Theta_\pi(z) = \theta_3(2z)\theta_3(22z) - 2\eta_\pi(z)$$

and so

$$j_\pi(z) = \eta_\delta(z) - 2 \qquad \left(\delta = \frac{2^4 22^4}{1^2 4^2 11^2 44^2}\right)$$

$j_\pi(z)$ is a generator for 44+ by Table 3 of [1].

3.3. It is convenient to study Frame shapes of type E_2 before dealing with those of type E_1. For $\Theta_\pi(z)$ for π of type E_2, we have the following theorem:

THEOREM 4. *Let π be a Frame shape of type E_2 in $2^{12}M_{24}$ and $S(\pi) = \{\pi, \pi'\}$ where $\deg(\pi') = 0$. Then we have*

(3.1) $$\Theta_\pi(z) = \eta_{\pi'}(z) + r(\pi')\eta_\pi(z)$$

and the structure of $\mathcal{L}_\pi$ are as follows:

Table 2

π	$\mathcal{L}_\pi$	the group for $j_\pi(z)$
$\frac{2^{16}}{1^8}$	$\Lambda_0(2^8) \cong \sqrt{2}E_8$	2−
$\frac{4^8}{2^4}$	$L_0(2^4) \cong \sqrt{2}D_4$	4−
$\frac{2^6 4^4}{1^4}$	$\Lambda(4^6)$	4−
$\frac{2^4 6^4}{1^2 3^2}$	$\Lambda_0(2^2 6^2)$ $\cong \Lambda(4.12) \oplus \Lambda(4.12)$	6+3

$\frac{8^4}{4^2}$	$L(4^2)$	$8-$
$\frac{2^3 4.8^2}{1^2}$	$L(4)\oplus\Lambda(8^3)$	$8-$
$\frac{4^2 12^2}{2.6}$	$\Lambda(8.24)$	$12+3$
$\frac{2^2 14^2}{1.7}$	$\Lambda(4.28)$	$14+7$

COROLLARY. *A conjecture of Conway-Norton is true for Frame shapes of type* E_2 *in* $2^{12}M_{24}$.

PROOF. This follows from (3.1) and Theorem 2 in § 2.

REMARK. It was first pointed out by Koike [4; Prop. 4.1] that the formulas (3.1) hold for all Frame shapes of type E_2 (not necessarily in $2^{12}M_{24}$). We note, however, that, in his proof, the explicit expressions of $\Theta_\pi(z)$ given in the present paper or M. L. Lang [7] are used.

Now we begin the proof of Theorem 4, i.e. the results of Table 2 and (3.1).

(1) The case $\pi=\frac{2^{16}}{1^8}$. Let C be an octad and $\pi=\varepsilon_{\Omega-C}$. Then π is of Frame shape $\frac{2^{16}}{1^8}$, $V_\pi=\left\{\sum_{i\in C} x_i\boldsymbol{e}_i \mid x_i\in\boldsymbol{R}\right\}$ and $(\mathcal{L}_0)_\pi=\left\{\sum_{i\in C} x_i\boldsymbol{e}_i \mid x_i\in\boldsymbol{Z},\ \sum_{i\in C} x_i\equiv 0 \bmod 2\right\}\cong L_0(2^8)$. Then we see from (2.1) that

$$\mathcal{L}_\pi=(\mathcal{L}_0)_\pi\cup\left(\frac{1}{2}\boldsymbol{e}_C+(\mathcal{L}_0)_\pi\right)\cong\Lambda_0(2^8)\,.$$

Thus we have

$$\begin{aligned}\Theta_\pi(z)&=\frac{1}{2}\sum_{i=2}^{4}\theta_i(2z)^8\\&=\frac{1}{2}\{(\theta_3(2z)^4-\theta_2(2z)^4)^2+2\theta_2(2z)^4\theta_3(2z)^4+\theta_4(2z)^8\}\\&=\theta_4(2z)^8+\frac{\theta_2(z)^8}{16}&&\text{by (1.6) and (1.14)}\\&=\eta_{\pi'}(z)+16\eta_\pi(z)\quad\left(\pi'=\frac{1^{16}}{2^8}\right)&&\text{by (1.10) and (1.12).}\end{aligned}$$

This proves Theorem 4 for $\pi=\frac{2^{16}}{1^8}$.

(2) The case $\pi=\frac{4^8}{2^4}$. Let

$$\sigma=(18, \underline{20})(\underline{10}, 12)(\underline{9}, 11)(\underline{8}, 15)(2, \underline{19}, 5, \underline{16})(1, 21, 17, 13)(0, 6, 14, 22)(\underline{\infty}, 7, \underline{3}, 4)$$

and C be the set of underlined letters.
Then $\sigma\in M_{24}$, and C is an octad. Let $\pi=\sigma\varepsilon_C$. Then π is of Frame shape $\frac{4^8}{2^4}$. Let

$$\boldsymbol{f}_1=\boldsymbol{e}_{19}+\boldsymbol{e}_5-\boldsymbol{e}_{16}-\boldsymbol{e}_2\,,\quad \boldsymbol{f}_2=\boldsymbol{e}_1+\boldsymbol{e}_{21}+\boldsymbol{e}_{17}+\boldsymbol{e}_{13}\,,$$

$$\boldsymbol{f}_3=\boldsymbol{e}_0+\boldsymbol{e}_6+\boldsymbol{e}_{14}+\boldsymbol{e}_{22}\,,\quad \boldsymbol{f}_4=\boldsymbol{e}_\infty+\boldsymbol{e}_7-\boldsymbol{e}_3-\boldsymbol{e}_4$$

which yield a basis of V_π over $\boldsymbol{R}$. Since any two of 4-cycles of σ yield an octad, $\boldsymbol{f}_i+\boldsymbol{f}_j$ $(1\leqq i,\ j\leqq 4)$ are π-admissible. Then we see from (2.1) that $\mathcal{L}_\pi\ni\frac{1}{2}(\boldsymbol{f}_i+\boldsymbol{f}_j)$ $(1\leqq i,\ j\leqq 4)$ and these generate $\mathcal{L}_\pi$. So, setting $\boldsymbol{f}'_i=\frac{1}{2}\boldsymbol{f}_i$, we get

$$\mathcal{L}_\pi=\left\{\sum_{i=1}^{4}x_i\boldsymbol{f}'_i \mid x_i\in\boldsymbol{Z},\ \sum_{i=1}^{4}x_i\equiv 0 \mod 2\right\}\cong L_0(2^4)\,.$$

Thus

$$\begin{aligned}\Theta_\pi(z)&=\frac{1}{2}\{\theta_3(2z)^4+\theta_4(2z)^4\}\\ &=\frac{1}{2}\theta_2(2z)^4+\theta_4(2z)^4 && \text{by (1.6)}\\ &=\eta_{\pi'}(z)+8\eta_\pi(z)\quad\left(\pi'=\frac{1^8}{2^4}\right) && \text{by (1.10) and (1.12).}\end{aligned}$$

This proves Theorem 4 for $\pi=\frac{4^8}{2^4}$.

(3) The case $\pi=\frac{2^6 4^4}{1^4}$. Let

$$\sigma=(\underline{3}, \underline{14})(\underline{5}, \underline{17})(4, 6, 22, 7)(8, 9, 18, 10)(11, 15, 12, 20)(13, 19, 16, 21)$$

where ∞, 0, 1 and 2 are fixed points of σ and

$$C=\{\infty, 0, 1, 2, 3, 5, 14, 17\}\,.$$

Then $\sigma\in M_{24}$ and C is an octad. If we put $\pi=\sigma\varepsilon_C$, π is of Frame shape $\frac{2^6 4^4}{1^4}$ and a basis of V_π is as follows:

$$\boldsymbol{f}_1=\boldsymbol{e}_3-\boldsymbol{e}_{14}\,,\quad \boldsymbol{f}_2=\boldsymbol{e}_5-\boldsymbol{e}_{17}\,,\quad \boldsymbol{f}_3=\boldsymbol{e}_4+\boldsymbol{e}_6+\boldsymbol{e}_{22}+\boldsymbol{e}_7\,,\quad \boldsymbol{f}_4=\boldsymbol{e}_8+\boldsymbol{e}_9+\boldsymbol{e}_{18}+\boldsymbol{e}_{10}\,,$$

$$\boldsymbol{f}_5=\boldsymbol{e}_{11}+\boldsymbol{e}_{15}+\boldsymbol{e}_{12}+\boldsymbol{e}_{20}\,,\quad \boldsymbol{f}_6=\boldsymbol{e}_{13}+\boldsymbol{e}_{19}+\boldsymbol{e}_{16}+\boldsymbol{e}_{21}\,.$$

There are just two π-admissible octads to which $\boldsymbol{f}_3 \pm \boldsymbol{f}_4$ and $\boldsymbol{f}_5 \pm \boldsymbol{f}_6$ belong and just four π-admissible dodecades to one of which $(\boldsymbol{f}_1+\boldsymbol{f}_2)+(\boldsymbol{f}_3+\boldsymbol{f}_5)$ belong. Then we see from (2.1) that $\mathcal{L}_\pi \ni \frac{1}{2}(\boldsymbol{f}_1+\boldsymbol{f}_2)+\frac{1}{2}(\boldsymbol{f}_3+\boldsymbol{f}_5)$, $\frac{1}{2}(\boldsymbol{f}_3 \pm \boldsymbol{f}_4)$, $\frac{1}{2}(\boldsymbol{f}_5 \pm \boldsymbol{f}_6)$ and these five vectors, together with $\boldsymbol{f}_1$, yield a basis of $\mathcal{L}_\pi$. Let

$$\boldsymbol{f}'_1=\boldsymbol{f}_1, \quad \boldsymbol{f}'_2=\boldsymbol{f}_2, \quad \boldsymbol{f}'_3=\frac{1}{2}(\boldsymbol{f}_3+\boldsymbol{f}_4), \quad \boldsymbol{f}'_4=\frac{1}{2}(\boldsymbol{f}_3-\boldsymbol{f}_4),$$

$$\boldsymbol{f}'_5=\frac{1}{2}(\boldsymbol{f}_5+\boldsymbol{f}_6), \quad \boldsymbol{f}'_6=\frac{1}{2}(\boldsymbol{f}_5-\boldsymbol{f}_6).$$

Then $\frac{1}{2}\sum_{i=1}^{6} \boldsymbol{f}'_i=\frac{1}{2}(\boldsymbol{f}_1+\boldsymbol{f}_2)+\frac{1}{2}(\boldsymbol{f}_3+\boldsymbol{f}_5)$ and so, setting $\mathcal{L}'=\sum_{i=1}^{6} \boldsymbol{Z}\boldsymbol{f}'_i$,

$$\mathcal{L}_\pi=\mathcal{L}' \cup \left\{\frac{1}{2}\sum_{i=1}^{6} \boldsymbol{f}'_i+\mathcal{L}'\right\} \cong \Lambda(2^6).$$

Thus

$$\begin{aligned}\Theta_\pi(z) &= \theta_2(4z)^6+\theta_3(4z)^6 \\ &= (\theta_3(4z)^2-\theta_2(4z)^2)(\theta_3(4z)^4-\theta_2(4z)^4) \\ &\quad +\theta_2(4z)^2\theta_3(4z)^4+\theta_2(4z)^4\theta_3(4z)^2 \\ &= \theta_4(2z)^2\theta_4(4z)^4+\theta_2(4z)^2\theta_3(4z)^2\theta_3(2z)^2 \\ &= \theta_4(2z)^2\theta_4(4z)^4+\frac{1}{16}\theta_2(z)^4\theta_2(2z)^2 \\ &= \eta_{\pi'}(z)+4\eta_\pi(z) \quad \left(\pi'=\frac{1^4 2^6}{4^4}\right).\end{aligned}$$

This proves Theorem 4 for $\pi=\frac{2^6 4^4}{1^4}$.

(4) The case $\pi=\frac{2^4 6^4}{1^2 3^2}$. Let σ be an element of Frame shape $1^6 3^6$ of M_{24} and C be an octad on which σ induces a permutation $1^2 3^2$. Then $\pi=\sigma\varepsilon_{\Omega-C}$ is of Frame shape $\frac{2^4 6^4}{1^2 3^2}$. Now it is easy to see $\mathcal{L}_\pi \cong \Lambda_0(2^2 6^2)$ and so, by (1.7) and (1.11),

$$\begin{aligned}\Theta_\pi(z) &= \frac{1}{2}\sum_{i=2}^{4} \theta_i(2z)^2\theta_i(6z)^2 \\ &= \theta_4(2z)^2\theta_4(6z)^2+\frac{1}{4}\theta_2(z)^2\theta_2(3z)^2 \\ &= \eta_{\pi'}(z)+4\eta_\pi(z) \quad \left(\pi'=\frac{1^4 3^4}{2^2 6^2}\right).\end{aligned}$$

This proves Theorem 4 for $\pi=\frac{2^4 6^4}{1^2 3^2}$.

(5) The case $\pi=\frac{8^4}{4^2}$. Let

$$\sigma=(18,\underline{20})(10,12)(9,\underline{11})(8,15)(2,\underline{19},5,16)(\underline{1},21,17,13)(\underline{0},6,14,22)(\underline{\infty},7,\underline{3},\underline{4}) \in M_{24},$$

$$C=\{\infty,0,1,3,4,11,19,20\}.$$

Then $\pi=\sigma\varepsilon_C$ is of Frame shape $\frac{8^4}{4^2}$ and $\mathcal{L}_\pi=\{x(\boldsymbol{e}_{10}+\boldsymbol{e}_{12})+y(\boldsymbol{e}_8+\boldsymbol{e}_{15}) \mid x, y\in \boldsymbol{Z}\}$. Thus

$$\Theta_\pi(z)=\theta_3(4z)^2=\theta_4(2z)^2+\theta_2(4z)^2=\eta_{\pi'}(z)+4\eta_\pi(z) \quad \left(\pi'=\frac{1^4}{2^2}\right).$$

(6) The case $\pi=\frac{2^3 4.8^2}{1^2}$. Let

$$\sigma=(\infty)(0)(1,2)(3,17,14,5)(4,9,6,18,22,10,7,8)(13,20,19,11,16,15,21,12)$$

$$C=\{\infty,0,1,2,3,5,14,17\}.$$

Then $\pi=\sigma\varepsilon_C$ is of Frame shape $\frac{2^3 4.8^2}{1^2}$. Let

$$\boldsymbol{f}_1=\boldsymbol{e}_1-\boldsymbol{e}_2,\quad \boldsymbol{f}_2=\boldsymbol{e}_3-\boldsymbol{e}_{17}+\boldsymbol{e}_{14}-\boldsymbol{e}_5,\quad \boldsymbol{f}_3=\sum_{i\in U}\boldsymbol{e}_i,\quad \boldsymbol{f}_4=\sum_{i\in V}\boldsymbol{e}_i$$

where U and V are the 8-cycles of σ. The set of π-admissible codes consist of two dodecades to which $\boldsymbol{f}_2+\boldsymbol{f}_3$ and $\boldsymbol{f}_2+\boldsymbol{f}_4$ belong and a co-octad (a code of 16-elements) to which $\boldsymbol{f}_3\pm\boldsymbol{f}_4$ belong. Let $\mathcal{L}'=\left\langle \boldsymbol{f}_2, \frac{1}{2}(\boldsymbol{f}_3+\boldsymbol{f}_4), \frac{1}{2}(\boldsymbol{f}_3-\boldsymbol{f}_4)\right\rangle \cong L(8^3)$ and $\mathcal{L}''=\left\langle \frac{1}{2}(\boldsymbol{f}_2+\boldsymbol{f}_3), \boldsymbol{f}_2, \frac{1}{2}(\boldsymbol{f}_3+\boldsymbol{f}_4)\right\rangle$. Then we see from (2.1) that

$$\mathcal{L}''=\mathcal{L}'\cup\left(\frac{1}{2}(\boldsymbol{f}_2+\boldsymbol{f}_3)+\mathcal{L}'\right)\cong \Lambda(8^3)$$

$$\mathcal{L}_\pi=\langle \boldsymbol{f}_1\rangle\oplus\mathcal{L}''\cong L(4)\oplus\Lambda(8^3).$$

Thus

$$\begin{aligned}\Theta_\pi(z)&=\theta_3(4z)(\theta_2(8z)^3+\theta_3(8z)^3)\\&=\theta_3(4z)\theta_3(2z)(\theta_2(8z)^2+\theta_3(8z)^2-\theta_2(8z)\theta_3(8z))\\&=\theta_3(4z)\theta_3(2z)\{(\theta_2(8z)-\theta_3(8z))^2+\theta_2(8z)\theta_3(8z)\}\\&=\theta_3(4z)\theta_3(2z)\theta_4(2z)^2+\frac{1}{2}\theta_3(4z)\theta_3(2z)\theta_2(4z)^2\end{aligned}$$

$$=\eta_{\pi'}(z)+2\eta_{\pi}(z) \quad \left(\pi'=\frac{1^2 2.4^3}{8^2}\right).$$

(7) The case $\pi=\dfrac{4^2 12^2}{2.6}$. Let

$\sigma=(\underline{10}, 12)(\underline{\infty}, 4, \underline{3}, 7)(8, 9, \underline{18}, 15, 11, 20)(0, 13, \underline{5}, \underline{6}, 1, \underline{16}, \underline{14}, 21, 2, 22, 17, 19)$

and C be the set of underlined letters which is an octad. Then $\pi=\sigma\varepsilon_C$ is of Frame shape $\dfrac{4^2 12^2}{2.6}$ and $\mathcal{L}_\pi \cong \Lambda(8.24)$. Thus

$$\begin{aligned}\Theta_\pi(z)&=\sum_{i=2}^{3}\theta_i(8z)\theta_i(24z)=\frac{1}{2}\sum_{i=3}^{4}\theta_i(2z)\theta_i(6z)\\&=\theta_4(2z)\theta_4(6z)+\frac{1}{2}\theta_2(2z)\theta_2(6z)\\&=\eta_{\pi'}(z)+2\eta_\pi(z) \quad \left(\pi'=\frac{1^2 3^2}{2.6}\right).\end{aligned}$$

(8) The case $\pi=\dfrac{2^2 14^2}{1.7}$. Let σ be an element of Frame shape 1.2.7.14 of M_{24} and C be an octad on which σ induces a permutation 1.7. Then $\pi=\sigma\varepsilon_C$ is of Frame shape $\dfrac{2^2 14^2}{1.7}$ and $\mathcal{L}_\pi \cong \Lambda(4.28)$. Thus

$$\begin{aligned}\Theta_\pi(z)&=\frac{1}{2}\sum_{i=2}^{4}\theta_i(2z)\theta_i(14z)=\theta_4(2z)\theta_4(14z)+\frac{1}{2}\theta_2(z)\theta_2(7z)\\&=\eta_{\pi'}(z)+2\eta_\pi(z) \quad \left(\pi'=\frac{1^2 7^2}{2.14}\right)\end{aligned}$$

where we used a variation of (1.5):

$$\theta_2(z)\theta_2(7z)=\theta_2(2z)\theta_2(14z)+\theta_3(2z)\theta_3(14z)-\theta_4(2z)\theta_4(14z).$$

3.4. For Frame shapes of type E_1 listed in (2.7), the results are given in the following table:

Table 3

π	$\delta=\pi^2$	$\mathcal{L}_\pi$	the group for $j_\pi(z)$
$\dfrac{1^8 4^8}{2^8}$	$\dfrac{2^{16}}{1^8}$	$\Lambda_0(2^8)$	4+
$\dfrac{1^4 8^4}{2^2 4^2}$	$\dfrac{4^8}{2^4}$	$L_0(2^4)$	8+

$\frac{1^2 3^2 4^2 12^2}{2^2 6^2}$	$\frac{2^4 6^4}{1^2 3^2}$	$\Lambda_0(2^2 6^2)$	12+
$\frac{1^2 16^2}{2.8}$	$\frac{8^4}{4^2}$	$L(4^2)$	16+
$\frac{1.4.7.28}{2.14}$	$\frac{2^2 14^2}{1.7}$	$\Lambda(4.28)$	28+
$\frac{2^4 8^4}{4^4}$	—	$L(4^4)$	8|2+
$\frac{2^2 16^2}{4.8}$	—	$L(4.8)$	16|2+
$\frac{2.6.8.24}{4.12}$	—	$L(4.12)$	24|2+

For the first five Frame shapes in Table 3, we have the following theorem:

THEOREM 5. *Let π be one of the first five Frame shapes in Table 3. Then we have $wt(\pi)=wt(\pi^2)$ and π^2 is of type E_2. Let $\delta=\pi^2$ and $S(\delta)=\{\delta, \delta'\}$ where $\deg(\delta')=0$. Then we have*

$$\eta_\delta(2z)\eta_{\delta'}(z)=\eta_\pi(z)^2 \tag{3.2}$$

$$j_\pi(z)=\frac{\eta_{\delta'}(z)}{\eta_\pi(z)}+c_\pi\frac{\eta_\pi(z)}{\eta_{\delta'}(z)}+r(\delta'), \quad c_\pi=r(\delta')(r(\pi)-r(\delta)) \tag{3.3}$$

and $j_\pi(z)$ is a generator of the function field for $N_\pi+$.

PROOF. We will prove (3.3) and the last statement, because the other statements are immediate. We have, by (2.6) and (3.1) in Th. 4,

$$j_\pi(z)=\frac{\Theta_\pi(z)}{\eta_\pi(z)}=\frac{\eta_{\delta'}(z)}{\eta_\pi(z)}+r(\delta')\frac{\eta_\delta(z)}{\eta_\pi(z)}$$

and, by (2.5) and (3.2),

$$\frac{\eta_\delta(z)}{\eta_\pi(z)}=1+(r(\pi)-r(\delta))\frac{\eta_\delta(2z)}{\eta_\pi(z)}=1+(r(\pi)-r(\delta))\frac{\eta_\pi(z)}{\eta_{\delta'}(z)}$$

from which (3.3) follows immediately. Then the last statement follows from the formulas of symmetrisation in Table 3 of [1].

For each of the last three Frame shapes in Table 3, we will give an element σ of M_{24} and an octad C which are needed to construct the Frame shape. (Elements of C are shown by underlined letters in the expressions of σ):

$\frac{2^4 8^4}{4^4}$ $\sigma=(8,15)(9,11)(10,12)(\underline{18},\underline{20})(\underline{\infty},7,\underline{3},\underline{4})(0,6,14,\underline{22})(1,\underline{21},17,13)(2,19,5,\underline{16})$

$\frac{2^2 16^2}{4.8}$ $\sigma=(\infty)(\underline{0})(1,2)(\underline{3},\underline{17},14,\underline{5})(13,20,19,\underline{11},16,15,21,12)(\underline{4},\underline{9},6,18,22,10,\underline{7},8)$

$\frac{2.6.8.24}{4.12}$ $\sigma=(10,12)(\underline{\infty},\underline{4},\underline{3},7)(8,9,18,15,\underline{11},\underline{20})(\underline{0},13,5,6,\underline{1},16,14,21,2,22,17,\underline{19})$.

Now $\mathcal{L}_\pi$ and $\Theta_\pi(z)$ can be calculated quite similarly as in the previous cases of type C or E_2. We leave the details of the proof to the readers.

3.5. In this paragraph, we deal with Frame shapes of type E_3, but it is much better to mention not only Frame shapes in $2^{12}M_{24}$ but also those outside $2^{12}M_{24}$. If π is of type E_3, then $S(\pi)$ consists of three Frame shapes of type E_3 and just one element of degree 0.

THEOREM 6. (1) *Let π be a Frame shape of type E_3. Then the structure of $\mathcal{L}_\pi$ is given in Table 4 below. In this table, * shows that π is a Frame shape outside $2^{12}M_{24}$.*
(2) *Let $S(\pi)=\{\pi_0,\pi_1,\pi_2,\pi_3\}$ $(\deg(\pi_0)=0)$ and let i, j, k be a permutation of the letters 1, 2, 3. Then the followings hold:*

(3.4) $$\Theta_{\pi_i}(z)=\eta_{\pi_0}(z)+\eta_{\pi_i}(z)+(c_j-c_k)(\eta_{\pi_j}(z)-\eta_{\pi_k}(z))$$

(3.5) $$j_{\pi_i}(z)=\frac{\eta_{\pi_0}(z)}{\eta_{\pi_i}(z)}+c(\pi_j)\cdot\frac{\eta_{\pi_j}(z)}{\eta_{\pi_0}(z)}+c(\pi_k)\cdot\frac{\eta_{\pi_k}(z)}{\eta_{\pi_0}(z)}+1$$

where the c_i $(i=1,2,3)$ are the constants defined in (2) of Theorem 3 in § 2.5 and $c(\pi_i)=(c_i-c_j)(c_i-c_k)$. For each π of type E_3, $c(\pi)$ is given in Table 4 below.

Table 4

π		$\mathcal{L}_\pi$	$c(\pi)$	π		$\mathcal{L}_\pi$	$c(\pi)$
$\frac{2^5 3^4 6}{1^4}$		$\Lambda_0(2.6^5)$	72	$\frac{2^3 5^2 10}{1^2}$		$\Lambda_0(2.10^3)$	20
$\frac{1^4 2.6^5}{3^4}$		$\Lambda_0(2^5 6)$	-8	$\frac{1^2 2.10^3}{5^2}$		$\Lambda_0(2^3 10)$	-4
$\frac{1^5 3.6^4}{2^4}$	*	$\Lambda_0(1.3^5)$	9	$\frac{1^3 5.10^2}{2^2}$	*	$\Lambda_0(1.5^3)$	5

$\frac{2^2 3^2 4.12}{1^2}$	$\Lambda(4.12^3)$	12		$\frac{2^2 9.18}{1.6}$	*	$\Lambda_0(12.36)$	6
$\frac{1.2^2 3.12^2}{4^2}$ *	$L_0(1.3^3)$	-3		$\frac{1.2.18^2}{6.9}$	*	$\Lambda_0(4.12)$	-2
$\frac{1^2 4.6^2 12}{3^2}$	$\Lambda(4^3 12)$	4		$\frac{1^2 9.18}{2.3}$	*	$\Lambda_0(6.18)$	3
$\frac{1.6.10.15}{3.5}$	$\Lambda_0(2.30)$	2					
$\frac{2.3.5.30}{1.15}$	$\Lambda_0(6.10)$	2					
$\frac{2.3.5.30}{6.10}$ *	$\Lambda_0(1.15)$	-1					

REMARKS. (1) It is easy to see from the expressions of $j_\pi(z)$ in (3.5) that $j_\pi(z)$ coincide with no Thompson series of Monster's elements. Thus, for Frame shapes of type E_3, a conjecture of Conway-Norton is not true. More precisely, M. L. Lang has shown that the fixing group of $j_\pi(z)$ is $\Gamma_0(N_\pi)$ (N_π=the level of π). (2) We see from (4) of Theorem 3 that $c(\pi)$ is a constant which is associated with a Frame shape $\pi\cdot\pi_0^{-1}$ in the formula of the symmetrisation of Table 3 of [1]. (3) A generator of the function field for $N_\pi+$ is expressed as follows:

$$\frac{\eta_{\pi_0}(z)}{\eta_\pi(z)}+\sum_{i=1}^{3} c(\pi_i)\cdot\frac{\eta_{\pi_i}(z)}{\eta_{\pi_0}(z)}.$$

The proof of Theorem 6. For $\pi\in 2^{12}M_{24}$, the proof that the structure of $\mathcal{L}_\pi$ is as in Table 4 is left to the readers, while, for $\pi\notin 2^{12}M_{24}$, the structure of $\mathcal{L}_\pi$ can be seen from Lang [7] (also see § 4). Now we will prove (3.4) for $\pi=\frac{2^5 3^4 6}{1^4}$. We have

$$S(\pi)=\left\{\pi_0=\frac{1.2^4 3^5}{6^4},\ \pi_1=\pi,\ \pi_2=\frac{1^4 2.6^5}{3^4},\ \pi_3=\frac{1^5 3.6^4}{2^4}\right\}.$$

$$c_1=5\,,\quad c_2=-3\,,\quad c_3=-4\,.$$

Since $\mathcal{L}_\pi\cong\Lambda_0(2.6^5)$, we have, by Lemma 2 in § 1,

$$\Theta_\pi(z)=\frac{\eta(2z)^9}{\eta(6z)^3}+9\frac{\eta(6z)^9}{\eta(2z)^3}.$$

But we have $\frac{2^9}{6^3}=\pi_0\pi_2\pi_3^{-1}$, $\frac{6^9}{2^3}=\pi_1\pi_3\pi_0^{-1}$ and $c_1-c_3=9$. Thus, by using the expressions of η-products similar to those in Theorem 3,

$$(3.6)\qquad \begin{aligned}\Theta_{\pi_1}(z)&=\frac{\pi_0\pi_2}{\pi_3}+(c_1-c_3)\frac{\pi_1\pi_3}{\pi_0}\\&=\pi_0+(c_2-c_3)\pi_2+\pi_1-\pi_3 &&\text{by corollary of Theorem 3}\\&=\pi_0+\pi_1+(c_2-c_3)(\pi_2-\pi_3) &&\text{(Note } c_2-c_3=1).\end{aligned}$$

This yields (3.4) for π_1. Similarly it follows from the structure of $\mathcal{L}_\pi$ and Lemma 2 that

$$(3.7)\qquad \begin{aligned}\Theta_{\pi_2}(z)&=\frac{\pi_0\pi_2}{\pi_3}+(c_1-c_3)^2\cdot\frac{\pi_1\pi_3}{\pi_0}\\&=\pi_0+(c_2-c_3)\pi_2+(c_1-c_3)(\pi_1-\pi_3) &&\text{(Corollary of Theorem 3)}\\&=\pi_0+\pi_2+(c_1-c_3)(\pi_1-\pi_3)\,, &&(c_2-c_3=1)\end{aligned}$$

$$(3.8)\qquad \begin{aligned}\Theta_{\pi_3}(z)&=\frac{\pi_0\pi_2}{\pi_1}+(c_1-c_3)\cdot\frac{\pi_1\pi_3}{\pi_2}\\&=\pi_0+(c_2-c_1)\pi_2-(c_2-c_1)\pi_1+(c_2-c_3)\pi_3\\& &&\text{(Corollary of Theorem 3)}\\&=\pi_0+\pi_3+(c_2-c_1)(\pi_2-\pi_1) &&(c_2-c_3=1).\end{aligned}$$

Also for Frame shapes in $S(\pi)$ $\left(\pi=\frac{2^3 5^2 10}{1^2}, \frac{2^2 3^2 4.12}{1^2}, \frac{2^2 9.18}{1.6} \text{ or } \frac{1.6.10.15}{3.5}\right)$, it follows from Lemma 2 in §1 or Lemma 4∼6 below that exactly the same formulas as (3.6)∼(3.8) hold if the notations π_1, π_2, π_3 for elements of $S(\pi)$ are chosen as is seen from the arrangement of Frame shapes of $S(\pi)$ in Table 4. Note that, when π_1, π_2, π_3 are chosen in such a way, we always have $c_2-c_3=1$. Thus (3.4) hold for all Frame shapes of type E_3. Then (3.5) follows from (3.4) and Corollary of Theorem 3.

LEMMA 4. *Let* $\mu=\frac{2^2 12^4}{1.3}$, $\mu'=\frac{2^2 3^4}{4.12}$, $\nu=\frac{6^2 4^4}{1.3}$ *and* $\nu'=\frac{1^4 6^2}{4.12}$. *Then we have*

$$\theta(z, \Lambda(4.12^3))=\eta_{\mu'}(z)+4\eta_\mu(z)$$

$$\theta(z, \Lambda(4^3 12))=\eta_{\nu'}(z)+4\eta_\nu(z)$$

$$\theta(z, L_0(1.3^3))=\eta_{\mu'}(z)+16\eta_\mu(z)\,.$$

LEMMA 5. *Let* $\mu=\frac{1^2 15^2}{3.5}$ *and* $\mu'=\frac{3^2 5^2}{1.15}$. *Then we have*

$$\theta(z, \Lambda_0(1.15))=\theta\left(z,\begin{bmatrix}4 & 1\\ 1 & 4\end{bmatrix}\right)=\eta_{\mu'}(z)-\eta_{\mu}(z),$$

$$\theta(z, \Lambda_0(3.5))=\theta\left(z,\begin{bmatrix}2 & 1\\ 1 & 8\end{bmatrix}\right)=\eta_{\mu'}(z)+\eta_{\mu}(z).$$

LEMMA 6. *Let* $\mu=\frac{9^3}{3}$ *and* $\mu'=\frac{1^3}{3}$. *Then we have*

$$\theta(z, \Lambda_0(2.6))=\theta\left(z,\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)=\eta_{\mu'}(z)+9\eta_{\mu}(z),$$

$$\theta(z, \Lambda_0(6.18))=\theta\left(z,\begin{bmatrix}6 & 3\\ 3 & 6\end{bmatrix}\right)=\eta_{\mu'}(z)+3\eta_{\mu}(z).$$

As in Lemma 2 of § 1, the proof of Lemma 4～6 can be done by comparing the Fourier coefficients of both sides of each equality. But, for that of Lemma 4, it can be also done by expressing the both sides of the equalities in terms of Jacobi theta functions.

§ 4. Concluding remarks.

4.1. In the previous section, we have shown that, for $\pi\in 2^{12}M_{24}-M_{24}$, the structure of $\mathcal{L}_\pi$ can be described in terms of rather elementary lattices L, L_0, Λ and Λ_0 introduced in § 1. In view of M. L. Lang [7], this is also true for any $\pi\in\cdot 0-2^{12}M_{24}$. Here we will make some remarks about Lang's results.

Let π be one of the following Frame shapes which are all in $\cdot 0-2^{12}M_{24}$:

$$\pi=\frac{3^9}{1^3},\ \frac{5^5}{1},\ \frac{3^3 6^3}{1.2},\ \frac{1^3 9^3}{3^2},\ \frac{1.2^2 3.12^2}{4^2}.$$

Then Lang introduced five matrices A, B, C, D and E to describe $\Theta_\pi(z)$. We note that

(i) A, B or C is equivalent to the matrices $3E_6^{-1}$, $5A_4^{-1}$ or $3D_4$ respectively and so $\mathcal{L}_\pi\cong\Lambda_0(1^5 3)$, $\Lambda_0(1^3 5)$ or $L_0(3^4)$ according as $\pi=\frac{3^9}{1^3}$, $\frac{5^5}{1}$ or $\frac{3^3 6^3}{1.2}$.

(ii) Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ and $\boldsymbol{e}_4$ be independent vectors with $l_1=1$, $l_2=l_3=3$ and $l_4=9$ in the notations in § 1.1. Then vectors $\frac{1}{2}(-\boldsymbol{e}_1+\boldsymbol{e}_2-\boldsymbol{e}_3+\boldsymbol{e}_4)$, $\boldsymbol{e}_1+\boldsymbol{e}_2$, $\boldsymbol{e}_1-\boldsymbol{e}_3$, $\frac{1}{2}(\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3+\boldsymbol{e}_4)$ form a basis of $\Lambda_0(1.3^2 9)$ and yield a matrix

$$\begin{pmatrix} 4 & 1 & 1 & 2 \\ 1 & 4 & 1 & 2 \\ 1 & 1 & 4 & -1 \\ 2 & 2 & -1 & 4 \end{pmatrix}$$

which is exactly the same as the matrix D introduced for $\pi=\frac{1^3 9^3}{3^2}$ by Lang. Thus we get

$$\mathcal{L}_\pi \cong \Lambda_0(1.3^2 9) \quad \text{and} \quad \Theta_\pi(z)=\frac{1}{2}\sum_{i=2}^{4}\theta_i(z)\theta_i(3z)^2\theta_i(9z) \quad \left(\pi=\frac{1^3 9^3}{3^2}\right).$$

(iii) Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ and $\boldsymbol{e}_4$ be vectors with $l_1=1$ and $l_2=l_3=l_4=3$. Then vectors $2\boldsymbol{e}_1$, $\boldsymbol{e}_1-\boldsymbol{e}_2$, $\boldsymbol{e}_1-\boldsymbol{e}_3$ and $\boldsymbol{e}_1-\boldsymbol{e}_4$ yield a basis of $L_0(1.3^3)$ and a matrix

$$\begin{pmatrix} 4 & 2 & 2 & 2 \\ 2 & 4 & 1 & 1 \\ 2 & 1 & 4 & 1 \\ 2 & 1 & 1 & 4 \end{pmatrix}$$

which is equal to the matrix E introduced for $\pi=\frac{1.2^2 3.12^2}{4^2}$ by Lang. Thus

$$\mathcal{L}_\pi \cong L_0(1.3^3) \quad \text{and} \quad \Theta_\pi(z)=\frac{1}{2}\sum_{i=3}^{4}\theta_i(z)\theta_i(3z)^3 \quad \left(\pi=\frac{1.2^2 3.12^2}{4^2}\right).$$

4.2. Let π be one of the following Frame shapes which are of type E_3 and outside $2^{12}M_{24}$:

$$\frac{1^5 3.6^4}{2^4}, \quad \frac{1^3 5.10^2}{2^2}, \quad \frac{2.3.5.30}{6.10}.$$

Then we have $wt(\pi)=wt(\pi^2)$ and π^2 is of type E_2, and so

$$\mathcal{L}_\pi=\mathcal{L}_{\pi^2}\cong \Lambda_0(1^5 3), \quad \Lambda_0(1^3 5) \quad \text{or} \quad \Lambda_0(1.15)$$

by (i) mentioned above for the first two π and a result $\left(\text{i.e. } \Theta_{\pi^2}(z)=\theta\left(z,\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\right)\right)$ of Lang for the third π. We note that Lemma 2 and Lemma 5 were suggested by these facts and (3.1) of Theorem 3.

References

[1] Conway, J.H. and S.P. Norton, Monstrous moonshines, Bull. London Math. Soc. **11** (1979), 308-339.

[2] Hecke, E., Mathematische Werke, Vandenhoeck & Ruprecht, Göttingen, 1959.

[3] Koike, M., Moonshines of PSL(2, q) and the automorphism group of Leech lattice, Japan. J. Math. **12** (1986), 283-323.
[4] Koike, M., Modular forms and the automorphism group of Leech lattice, to appear in Nagoya Math. J.
[5] Kondo, T., The automorphism group of Leech lattice and elliptic modular functions, J. Math. Soc. Japan **37** (1985), 337-362.
[6] Kondo, T. and T. Tasaka, The theta functions of sublattices of the Leech lattice, Nagoya Math. J. **101** (1986), 151-179.
[7] Lang, M. L., On a question Raised by Conway-Norton, to appear in J. Math. Soc. Japan.
[8] Tasaka, T., On even lattices of 2-square type and self-dual codes, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1982), 701-714.
[9] Todd, N. J., A representation of the Mathieu group M_{24} as a collineation group, Ann. Mat. Pura Appl. **71** (1966), 199-238.
[10] Mason, G., Finite Groups and Hecke Operators, Preprint.

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