J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
34 (1987), 573-597.

*Orbits and stabilizers of nilpotent elements of a graded semisimple Lie algebra**)

Dedicated to Professor Nagayoshi Iwahori on his sixtieth birthday

By Noriaki KAWANAKA

§1. Introduction.

Let G be a connected semisimple linear algebraic group over an algebraically closed field K of characteristic p, which may be zero or non-zero. Let D be either

(T) a torus over K

or

(C) a finite cyclic group whose order is not a multiple of p.

We assume that D acts morphically on G by algebraic group automorphisms. Let $X(D)=\mathrm{Hom}(D, K^{\times})$ be the character module of D, and g the Lie algebra of G.

For $\lambda\in X(D)$, we put

$$g(\lambda)=\{A\in g\,;\ d\cdot A=\lambda(d)A,\ d\in D\}. \tag{1.1}$$

This gives an $X(D)$-gradation (see 2.1) of g. Let $G(0)$ be the identity component of the group of D-fixed elements of G. Then $G(0)$ is reductive, and its adjoint action on g preserves $g(\lambda)$ for any $\lambda\in X(D)$. An element N of $g(\lambda)$ is called nilpotent if it is nilpotent as an element of g, i.e., if it is contained in the Lie algebra of a unipotent subgroup of G. The purpose of this paper is to study orbits $G(0)\cdot N$ and stabilizers $Z_{G(0)}(N)$ of nilpotent elements N of $g(\lambda)$. The main results are stated in Section 3.1. These can be summarized as follows.

Assume that the characteristic p is zero or a good prime [30; I, 4.3] for G.

(i) For a given $\lambda\in X(D)$, the nilpotent $G(0)$-orbits in $g(\lambda)$ are naturally parametrized by a finite set of weighted Dynkin diagrams (see 2.1) 'inde-

*) This work is partly supported by Grant-in-Aid for Scientific Research, The Ministry of Education, Science and Culture.

pendently' of p. In particular, the number of nilpotent orbits in $g(\lambda)$ is finite and 'independent' of p.

(ii) Let o be a nilpotent $G(0)$-orbit in $g(\lambda)$. Then, from the weighted Dynkin diagram associated with o (and known data on nilpotent G-orbits in g), one can extract the following information: the structure of a reductive part of $Z_{G(0)}(N)$, the dimension of o, an explicit representative of o, the weighted Dynkin diagram associated with the nilpotent orbit $G\cdot N$ in g. In particular, we have the following general fact. If we put

$$Z(G(0)\mid g(\lambda))=\{x\in G(0)\,;\ x\cdot A=A,\ A\in g(\lambda)\}$$

and

$$G(0)_{\lambda}=G(0)/Z(G(0)\mid g(\lambda)),$$

then the component group $Z_{G(0)_{\lambda}}(N)/Z_{G(0)_{\lambda}}(N)^0$ of $Z_{G(0)_{\lambda}}(N)$ is isomorphic to a direct product $\prod_i S_{n_i}$ of symmetric groups S_{n_i} of degrees $1\leqq n_i\leqq 5$.

The finiteness part in the second assertion of (i) is a special case of a result [23] due to Richardson. In view of a recent result of Kostant (unpublished) and Sekiguchi [27], one can say that (i), (ii) (for the case when $K=\boldsymbol{C}$ and D is generated by a Cartan involution of a real form of G) provides a parametrization and a set of representatives for the nilpotent orbits in a real semisimple Lie algebra, and also describes how a complex nilpotent orbit splits into real ones. If D is a torus and $\lambda\neq 0$, then the pair $(G(0), g(\lambda))$ is a prehomogeneous vector space [25] with a finite number of orbits. Thus, in that case, (i), (ii) may be considered as complements to the results of Sato and Kimura [25] and Kimura, Kasai and Yasukura [15]. See also [24], [20]. We should also mention works [33]-[35] of Vinberg which are closely related to ours.

Now we explain briefly how (i) and (ii) may be obtained. If the D-action on G is trivial, i.e., if $G=G(0)$, then (i) and (ii) are known facts. For example, it is well-known (Dynkin [6]) that, in characteristic zero, the nilpotent orbits in $g=g(0)$ are parametrized by a set of weighted Dynkin diagrams, and, although less-well-known, a completely analogous result holds in good positive characteristics also (see Section 2.1 for the detail). We shall get (i), (ii) by reducing the problems to this special and already solved case.

If we restricted our attention to the case of characteristic zero, then our task would become much easier. But, even if one is interested only in that case, it is often helpful to know that the orbital structure of the set of nilpotent elements of $g(\lambda)$ does not depend on the characteristic. See the example given in 3.4. Moreover, one of the applications we have in mind is related to our study [12]-[14] on generalized Gelfand-Graev representations of a reductive group over a finite field. This will be discussed elsewhere.

The paper is organized as follows. In Section 2, we collect more or less known materials which are needed in formulating or proving the main results: in 2.1, we review results on nilpotent orbits in g, in 2.2, we examine the $X(D)$-gradations (1.1) coming from various D-actions on G. In Section 3, after formulating the main results in 3.1, we first prove them, in 3.2, assuming that $p=0$ or $p\gg 0$, and explain, in 3.3, why the same results remain true in good positive characteristics. In 3.4, we give an example of actual calculations.

The author would like to express his heartfelt thanks to D. Vogan, who pointed out an overlooking in the author's talk on the content of this paper given at the 1987 Luminy conference "Unipotent orbits, …".

§ 2. Preliminaries.

2.1. Weighted Dynkin diagrams and nilpotent orbits.

Let Σ be an abstract reduced root system [3], and Π a fixed simple root system (or base) of Σ. We consider the set $H(\Sigma)=H(\Sigma,\Pi)$ of Z-valued functions on Π extended, by linearity, to Z-valued functions on Σ. An element h of $H(\Sigma)$ is called a weighted Dynkin diagram (associated with (Σ,Π)), because it can be considered as the Dynkin diagram of (Σ,Π) with weights $h(\alpha)$, $\alpha\in\Pi$. The automorphism group $\mathrm{Aut}(\Sigma)$ of Σ acts on $H(\Sigma)$ in an obvious way:

(2.1.1) $$h \longrightarrow h\circ\gamma, \qquad h\in H(\Sigma),\ \gamma\in\mathrm{Aut}(\Sigma).$$

We denote by $H(\Sigma)_+=H(\Sigma,\Pi)_+$ the subset of $H(\Sigma)$ which consists of elements h satisfying $h(\alpha)\geqq 0$ for any $\alpha\in\Pi$. The following lemma is easy.

(2.1.2) LEMMA. *For any $h\in H(\Sigma)$, there exists a unique element $h_+\in H(\Sigma)_+$ such that $h_+=h\circ w$ for some element w of the Weyl group $W(\Sigma)$ of Σ.*

Let G be a connected reductive linear algebraic group over an algebraically closed field K, and $g=\mathrm{Lie}\,G$ the Lie algebra of G. Let now Σ be the root system of G with respect to a maximal torus T. For an element h of $H(\Sigma)$, we put, for $i\in Z$,

(2.1.3) $$g(i)_h=\begin{cases} \bigoplus\limits_{\substack{\alpha\in\\ h(\alpha)=i}} g_\alpha & \text{if } i\neq 0\,;\\ \mathrm{Lie}\ T\oplus\Big(\bigoplus\limits_{\substack{\alpha\in\\ h(\alpha)=0}} g_\alpha\Big) & \text{if } i=0\,,\end{cases}$$

where g_α is the root subspace of g corresponding to $\alpha \in \Sigma$; and we understand that $g(i)_h = \{0\}$ if $i \neq 0$ and the set $\{\alpha \in \Sigma ; h(\alpha) = i\}$ is empty. Then (2.1.3) gives a Z-gradation of g. (In general, a set $\{g(m) ; m \in M\}$ of subspaces of a Lie algebra g indexed by an abelian group M is called an M-gradation of g, if g can be decomposed as a direct sum

$$g = \bigoplus_{m \in M} g(m)$$

and $\{g(m)\}$ satisfies

$$[g(l), g(m)] \subset g(l+m), \qquad l, m \in M.)$$

If G is semisimple and the characteristic p of K is either zero or large, then the converse is also true: any Z-gradation of g can be obtained as in (2.1.3) by choosing a suitable maximal torus T. Thus, using (2.1.2), we get the following:

(2.1.4) LEMMA. *Assume that G is semisimple and $p=\mathrm{char}(K)$ is zero or large. Let Π be a base of the root system Σ of G with respect to a fixed maximal torus T. Then the correspondence $h \to \{g(i)_h ; i \in Z\}$ defined by* (2.1.3) *is a bijection between $H(\Sigma, \Pi)_+$ and the set of G-conjugacy classes of Z-gradations of g.*

Let G and K be as in (2.1.4). Let N be a nilpotent element of g. Then, by the Jacobson-Morozov theorem, there exists a Lie algebra homomorphism $f_N : sl_2(K) \to g$ such that $f_N\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = N$. The element $H = H_N = f_N\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is called a characteristic of N.

(2.1.5) LEMMA (Dynkin [6]). (i) *Let N be a nilpotent element of g. Then two characteristics of N are conjugate under $Z_G(N)$.*

(ii) *Two nilpotent elements of g are G-conjugate if and only if their characteristics are.*

Let H be a characteristic of a nilpotent element N of g. We put

(2.1.6) $$g(i)_H = \{X \in g ; [H, X] = iX\}, \qquad i \in Z.$$

Then, by representation theory of $sl_2(K)$, this gives a Z-gradation of g. Hence, by (2.1.4), we get a well-defined map from the set of nilpotent G-orbits in g to the set $H(\Sigma)_+$. We denote the image of this map by $H(\Sigma)_n = H(\Sigma, \Pi)_n$, which is known to be independent of $p=\mathrm{char}(K)$ (provided that $p=0$ or $p \gg 0$). Then (2.1.5) implies that the set $H(\Sigma)_n$ parametrizes the nilpotent orbits of g.

Let now G be a connected reductive linear algebraic group over an algebraically closed field K. Let Σ be the root system of G with respect to a fixed maximal torus T, and Π a fixed base of Σ. For $h \in H(\Sigma, \Pi)_n$, we define a Z-gradation $\{g(i)_h\,;\ i \in Z\}$ of $g = \mathrm{Lie}\, G$ by (2.1.3). Let P_h, U_h and $G(0)_h$ be the connected closed subgroups of G with the Lie algebras $\bigoplus_{i \geqq 0} g(i)_h$, $\bigoplus_{i \geqq 1} g(i)_h$ and $g(0)_h$ respectively. Then P_h is a parabolic subgroup of G, and $G(0)_h$ and U_h are a Levi subgroup and the unipotent radical of P_h respectively.

(2.1.7) THEOREM. *Let the notations be as above. We assume that* $p = \mathrm{char}(K)$ *is zero or a good prime* [30 ; I, 4.3] *for* G. *Let* h *be an element of* $H(\Sigma, \Pi)_n$.

(i) *There exists an open, dense* $G(0)_h$*-orbit* o_h *in* $g(2)_h$. *When* $p=0$ *or* $p \gg 0$, *an element* N *of* $g(2)_h$ *(which is automatically nilpotent) is contained in* o_h *if and only if there exists a characteristic* H *of* N *such that the two gradations* (2.1.3) *and* (2.1.6) *of* g *coincide. In the following, we denote by* N_h *a representative of* o_h.

(ii) *The correspondence* $h \to O_h = G \cdot N_h$ *is a bijection between the set* $H(\Sigma, \Pi)_n$ *and the set of nilpotent* G*-orbits in* g.

(iii) $o_h = g(2)_h \cap O_h$.

(iv) $P_h \cdot N_h = \left(\bigoplus_{i \geqq 2} g(i)_h\right) \cap O_h = o_h + \left(\bigoplus_{i \geqq 3} g(i)_h\right)$. *In particular,* $P_h \cdot N_h$ *is open and dense in* $\bigoplus_{i \geqq 2} g(i)_h$.

(v) $Z_G(N_h) = Z_{G(0)_h}(N_h) Z_{U_h}(N_h)$ *(semi-direct product) ;* $Z_{G(0)_h}(N_h)$ *is reductive and* $Z_{U_h}(N_h)$ *is the unipotent radical of* $Z_G(N_h)$. *In particular,* $Z_G(N_h) \subset P_h$.

(vi) $\dim Z_{U_h}(N_h) = \dim(g(1)_h \oplus g(2)_h)$.

If p is zero, this is due to Dynkin [6] and Kostant [16]. The case $p \gg 0$ and the case when G is classical and p is a good prime were treated by Springer and Steinberg [30]. In the case when G is of type E_n ($n=6,7,8$) and p is good, this is implicitly contained in works [18], [19] of Mizuno. In its full generality, Theorem (2.1.7) was first formulated in [13]. But, unfortunately, we still have no intrinsic proof for this.

In the rest of this subsection, we review some of known results related to Theorem (2.1.7).

(2.1.8) The set $H(\Sigma)_n$ is explicitly determined in [30 ; IV] (resp. [7]) in the case when Σ is irreducible and of classical (resp. exceptional) type. One can also consult [4 ; 13.1].

(2.1.9) The (global) structure of the reductive group $Z_{G(0)_h}(N_h)$ $(h\in H(\Sigma)_n)$ is known; see [36], [30; IV] (resp. [1], [5], [28], [18], [19]) for the case when G is a classical (resp. exceptional) simple group. From these we see that the structure of $Z_{G(0)_h}(N_h)$ is independent of $p=\mathrm{char}(K)$, except for small deviation which can occur only when G contains a normal subgroup of type A_n, $n\geqq 1$.

(2.1.10) From (2.1.9), one can observe that, when G is adjoint, the component group $Z_{G(0)_h}(N_h)/Z_{G(0)_h}(N_h)^0$ of $Z_{G(0)_h}(N_h)$ is always isomorphic to a direct product $\prod_i S_{n_i}$ of symmetric groups of degrees $1\leqq n_i\leqq 5$.

(2.1.11) For $h\in H(\Sigma)_n$, an explicit representative of o_h (hence of O_h) can be found in [30; IV] (resp. in [18], [19]) when G is classical (resp. of type E_n, $n=6,7,8$). When G is of type F_4 or G_2, it is also easy to find an explicit representative in each o_h. See also Hirai [8].

In some references cited above, unipotent classes of G, rather than nilpotent classes of g, are studied. But, in good characteristics, information on unipotent classes can be translated into that on nilpotent classes (and vice versa) by [30; III, 3.12]. For nilpotent and unipotent classes in bad characteristics, we refer the reader to [4; 5.11].

2.2. D-actions and $X(D)$-gradations.

Let G be a connected linear algebraic group over an algebraically closed field K, and let D be a diagonalizable group over K acting morphically on G by algebraic group automorphisms. Let $G(0)$ be the connected component of the group of D-fixed elements of G.

(2.2.1) LEMMA. (i) $\mathrm{Lie}\,G(0)=\{A\in g\,;\ d\cdot A=A,\ d\in D\}$.

In (ii)–(v), *we assume that G is semisimple.*

(ii) *$G(0)$ is reductive.*

(iii) *There exists a D-stable maximal torus of G.*

(iv) *If D is a torus or a finite cyclic group, then there exists a D-stable Borel subgroup B and a D-stable maximal torus T such that $T\subset B$.*

(v) *If D is a torus, any D-stable torus is contained in $G(0)$.*

PROOF. (i) is proved in [2; p. 234]. For (ii)–(iv), see the proof of [30; II, 5.16]. (v) follows from the connectivity of D.

Now let D be as in Introduction. We are going to examine the $X(D)$-gradation (1.1) of g in each case (T) and (C) separately. We first consider the case (T), i. e. the case when D is a torus.

Let T be as in (2.2.1) (iv), and let Σ be the root system of G with respect to T. Since, for any $\alpha\in\Sigma$, the root subspace g_α of g is D-stable, it must be contained in some $g(\lambda)$, $\lambda\in X(D)$. The correspondence $\alpha\rightarrow\lambda$ defines a Z-linear map from (the Z-span of) Σ into $X(D)$. Conversely, given any linear map $f:\Sigma\rightarrow X(D)$, there exists a D-action on G such that $T\subset G(0)$ and that

$$g_\alpha\subset g(f(\alpha)), \qquad \alpha\in\Sigma.$$

Next we consider the case (C), i. e., the case when D is a finite cyclic group Z_m whose order m is not a multiple of $p=\mathrm{char}(K)$. Let B and T be as in (2.2.1) (iv). Let Σ be the root system with respect to T, and Π the simple root system of Σ corresponding to B. We fix a generator δ of D. Then δ induces an automorphism of Σ preserving Π. More precisely, for $\alpha\in\Sigma$, let X_α be the root vector contained in a Chevalley basis of g, and let

$$K\ni\xi\longrightarrow x_\alpha(\xi)=\exp\xi X_\alpha\in G$$

be the corresponding root subgroup of G. Then we can define the δ-action on Σ by

$$\delta(x_\alpha(\xi))=x_{\delta\alpha}(c(\alpha)\xi), \qquad \alpha\in\Sigma,\ \xi\in K \tag{2.2.2}$$

or, equivalently, by

$$\delta X_\alpha=c(\alpha)X_{\delta\alpha}, \qquad \alpha\in\Sigma \tag{2.2.3}$$

with some $c(\alpha)\in K^\times$. By (2.2.2), it is easy to see that the action of δ can be written as

$$\delta(g)=t\gamma(g)t^{-1}, \qquad g\in G, \tag{2.2.4}$$

with some element $t\in T$ and a graph automorphism γ of G characterized by

$$\gamma(x_\alpha(\xi))=x_{\delta\alpha}(\xi), \qquad \alpha\in\pm\Pi,\ \xi\in K.$$

(See [31 ; § 10] for the detail.)

(2.2.5) LEMMA. *Suppose we are in the case* (C). *Let* $T,\Sigma,\delta,\cdots$ *be as above. We have the following.*

(i) $T\cap G(0)$ *is a maximal torus of* $G(0)$; *and its Lie algebra is equal to Lie* $T\cap g(0)$.

(ii) *For* $\alpha\in\Sigma$, *let* $l(\alpha)$ *be the length of the D-orbit* $(\alpha)=\{\alpha, \delta\alpha, \delta^2\alpha, \cdots\}$, *and let*

$$C(\alpha)=\prod_{i=0}^{l(\alpha)-1} c(\delta^i\alpha)\,.$$

Then $C(\alpha)$ *is a root of unity in* K. *If* $n(\alpha)$ *is the order of* $C(\alpha)$, *then* $l(\alpha)n(\alpha)$ *divides the order* m *of* D.

(iii) *For* $\alpha\in\Sigma$, *let* (α) *and* $l(\alpha)$ *be as in* (ii). *Then one of the following two cases occurs*:

(iii a) $l(\alpha)=1$, *or* $l(\alpha)\geqq 2$ *and any two roots in* (α) *are orthogonal to each other*;

(iii b) $l(\alpha)=2$, *and* (α) *generates a root system of type* A_2.

(iv) *Let* λ_1 *be a fixed generator of* $X(D)\cong \mathrm{Hom}(Z_m, K^\times)$. *Using the notations in* (ii), *we put, for* $\alpha\in\Sigma$,

$$\mathfrak{g}_{(\alpha)}=\bigoplus_{i=0}^{l(\alpha)\ 1} \mathfrak{g}_{\delta^i\alpha}\,,$$

and take an integer $t=t(\alpha)$ *such that*

$$C(\alpha)=\Big(\frac{tm}{n(\alpha)}\lambda_1\Big)(\delta)\,. \tag{2.2.6}$$

Then

$$\mathfrak{g}_{(\alpha)}=\bigoplus_{i=0}^{l(\alpha)-1}\Big\{\mathfrak{g}_{(\alpha)}\cap\mathfrak{g}\Big(\Big(\frac{tm}{l(\alpha)n(\alpha)}+\frac{im}{l(\alpha)}\Big)\lambda_1\Big)\Big\}\,.$$

Moreover, each summand on the right hand side is 1-*dimensional and spanned by an element of the form*

$$X_{(\alpha)}=\sum_{i=0}^{l(\alpha)-1} a_i X_{\delta^i\alpha} \tag{2.2.7}$$

with some $a_i\in K^\times$, $0\leqq i\leqq l(\alpha)-1$.

PROOF. (i) The first assertion is clear from the proof of [32; 8.2]. The second is a special case of (2.2.1) (i).

(ii) This is clear from (2.2.2).

(iii) This follows from the fact that δ induces an automorphism of (Σ, Π) and the classification of root systems.

(iv) Since $C(\alpha)$ is an $n(\alpha)$-th root of 1 and $\lambda_1(\delta)$ is a primitive m-th root of 1, there exists a t for which (2.2.6) holds. An element $\sum_{i=0}^{l(\alpha)-1} a_i X_{\delta^i\alpha}$ of $\mathfrak{g}_{(\alpha)}$ is contained in $\mathfrak{g}(j\lambda_1)$ $(j\in Z)$ if and only if

$$a_i c(\delta^i\alpha_i)=a_{i+1}\lambda_1(\delta^j)\,,\qquad i\in Z/l(\alpha)Z\,.$$

Solving this, we get (iii).

(2.2.8) REMARK. For a classification of finite order automorphisms of G up to the conjugacy in the automorphism group of G, see Kac [10 ; Ch. 8].

Let D, B and T be as in (2.2.1) (iv). Let Σ be the root system of G with respect to T, and Π the simple root system corresponding to B.

(2.2.9) LEMMA. *Let the notations be as above. We fix* $\lambda\in X(D)$ *and* $h\in H(\Sigma,\Pi)$ *(see* 2.1*), and assume that* h *is* D*-invariant. For* $j\in Z$*, we put*

$$\bar{g}(j)=\begin{cases} g((j/2)\lambda)\cap g(j)_h & \text{if } j/2\in Z\,; \\ \{0\} & \text{otherwise,}\end{cases} \tag{2.2.10}$$

and

$$\bar{g}=\bigoplus_{j\in Z}\bar{g}(j)\,.$$

(For the definition of $g(j)_h$*, see* (2.1.3).*)*

(i) $\bar{g}$ *is the Lie algebra of a connected reductive subgroup* $\bar{G}$ *of* G*, and* (2.2.10) *gives a* Z*-gradation of* $\bar{g}$*.*

(ii) *Let* $\bar{G}(0)=\bar{G}(0)_{\bar{h}}=G(0)\cap G(0)_h$*. Then this is a connected reductive subgroup of* G *with the Lie algebra* $\bar{g}(0)$*.*

PROOF. (i) To see that $\bar{g}$ is a Lie algebra with a Z-gradation (2.2.10) is easy. If $\mathrm{char}(K)=0$, one can prove the existence of a reductive $\bar{G}$ such that $\mathrm{Lie}\,\bar{G}=\bar{g}$ by observing the non-degeneracy of the restriction $\kappa(\cdot,\cdot)|\bar{g}$ of the Killing form $\kappa(\cdot,\cdot)$ on g. To get a proof which works in positive characteristics also, we argue as follows. For $\alpha\in\Sigma$, let (α) be the D-orbit through α and let

$$g_{(\alpha)}=\bigoplus_{\beta\in(\alpha)}g_\beta\,. \tag{2.2.11}$$

(Hence, in the case (T), $(\alpha)=\{\alpha\}$; in the case (C), (α) and $g_{(\alpha)}$ were already defined in (2.2.5).) Let

$$\bar{T}=T\cap G(0)$$

and

$$\bar{\Sigma}=\bar{\Sigma}_h=\{\bar{\alpha}\,;\ \alpha\in\Sigma,\ g_{(\alpha)}\cap\bar{g}(j)\neq\{0\}\ \text{for some}\ j\in Z\},$$

where $\bar{\alpha}$ is the restriction of $\alpha\in(\alpha)$ to $\bar{T}$. Note that $\bar{\alpha}\in X(\bar{T})$ does not depend on the choice of a representative α of (α), and that an integer j such that $g_{(\alpha)}\cap\bar{g}(j)\neq\{0\}$ is uniquely determined by (α). The subset $\bar{\Sigma}$ of $X(\bar{T})$ forms a root system ; in fact, the Weyl group $W(\bar{\Sigma})$ can naturally be identified with the subgroup of $W(\Sigma)$ generated by $\{w_{(\alpha)}\,;\ \bar{\alpha}\in\bar{\Sigma}\}$. Here we define $w_{(\alpha)} \in W(\Sigma)$ as follows. If $(\alpha)=\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ $(n=l(\alpha))$ satisfies (2.2.5) (iii a), then we put $w_{(\alpha)}=w_{\alpha_1}w_{\alpha_2}\cdots w_{\alpha_n}$, the product of reflections w_{α_i}. If (α) satisfies (2.2.5) (iii b), then we put $w_{(\alpha)}=w_{\alpha_1}w_{\alpha_2}w_{\alpha_1}=w_{\alpha_2}w_{\alpha_1}w_{\alpha_2}$, where $(\alpha)=\{\alpha_1, \alpha_2=\delta\alpha_1\}$. For $\bar{\alpha}\in\bar{\Sigma}$, we put

$$x_{\bar{\alpha}}(\xi)=\exp(\xi X_{(\alpha)}) \ (\in G), \qquad \xi\in K,$$

where $X_{(\alpha)}$ is a fixed non-zero element of $g_{(\alpha)}\cap\bar{g}(j)$ (see (2.2.7)). If (α) satisfies (2.2.5) (iii a), then there is no problem in doing this. If (α) satisfies (2.2.5) (iii b), then $\exp(\xi X_{(\alpha)})$ is equal to

$$1+\xi X_{(\alpha)}+\frac{\xi^2}{2}X_{(\alpha)}^2,$$

which makes sense since the characteristic p is odd or zero by our assumption (C) and (2.2.5) (ii). Now we define $\bar{G}$ to be the subgroup of G generated by $\bar{T}$ and $\{x_{\bar{\alpha}}(\xi)\,;\ \bar{\alpha}\in\bar{\Sigma},\ \xi\in K\}$. It is easy to see that $\bar{G}$ is a connected reductive group with a root system $\bar{\Sigma}$ and that $\bar{g}$ is the Lie algebra of $\bar{G}$.

(ii) As is shown in [32 ; 8.1], $G(0)$ is the group generated by a maximal torus $\bar{T}$ and the root subgroups

$$\{\exp\xi X_{(\alpha)}\,;\ \xi\in K\}, \qquad \alpha\in\Sigma,\ g_{(\alpha)}\cap g(0)\neq\{0\},$$

where $X_{(\alpha)}$ is a non-zero element of $g_{(\alpha)}\cap g(0)$. Hence, by comparing the Bruhat decomposition of G, $G(0)$ and $G(0)_h$ with respect to their Borel subgroups B, $B\cap G(0)$ and $B\cap G(0)_h$, we see that $G(0)\cap G(0)_h$ is the group generated by $\bar{T}$ and

$$\{\exp\xi X_{(\alpha)}\,;\ \xi\in K\}, \qquad \alpha\in\Sigma,\ g_{(\alpha)}\cap\bar{g}(0)\neq\{0\}.$$

Hence $G(0)\cap G(0)_h$ is connected, reductive and its Lie algebra is $\bar{g}(0)$.

Let the notations be as in (the proof of) Lemma (2.2.9). Let Σ^+ be the positive root system of Σ corresponding to Π. We put

$$\bar{\Sigma}^+=\{\bar{\alpha}\in\bar{\Sigma}\,;\ \alpha\in\Sigma^+\}.$$

Then $\bar{\Sigma}^+$ is a positive root system of $\bar{\Sigma}$. Let $\bar{\Pi}=\bar{\Pi}_h$ be the corresponding simple root system of $\bar{\Sigma}$. We define a weighted Dynkin diagram $\bar{h}\in H(\bar{\Sigma}_h, \bar{\Pi}_h)$ (see 2.1) by

$$\bar{g}_{\bar{\alpha}}\subset\bar{g}(\bar{h}(\bar{\alpha})), \qquad \bar{\alpha}\in\bar{\Pi}. \tag{2.2.12}$$

Evidently, the Z-gradation of $\bar{g}$ associated with $\bar{h}$ is just the one given by (2.2.10) ; hence we can write

$$\bar{g}(j)=\bar{g}(j)_{\bar{h}}, \qquad j\in Z.$$

(2.2.13) DEFINITION. In the situation of (2.2.9), we say that $h \in H(\Sigma, \Pi)$ is slim (with respect to λ) if the one parameter subgroup $S_h = \{s_h(\xi) ; \xi \in K^\times\}$ of $T \cap G(0)$ giving rise to the gradation (2.1.3), i. e., the one characterized by

$$s_h(\xi) \cdot X_\alpha = \xi^i X_\alpha, \qquad \alpha \in \Sigma,\ h(\alpha) = i,$$

is contained in the semisimple part of the reductive group $\bar{G}$.

Let $Y(T) = \mathrm{Hom}(K^\times, T)$ be the Z-module of one parameter subgroups of T. Since $Y(T) \underset{Z}{\otimes} R$ is dual to $X(T) \underset{Z}{\otimes} R$, it has a natural positive definite inner product. Put

$$C = \Big(\bigcap_{\bar{\alpha} \in \bar{\Sigma}} \mathrm{Ker}\, \bar{\alpha} \Big)^0 \subset T \cap G(0).$$

Then C is the maximal central torus of $\bar{G}$. The following lemma is obvious.

(2.2.14) LEMMA. *Let $\xi \to c_j(\xi)$ $(j = 1, 2, \cdots, l)$ be elements of $Y(T)$ generating C. Then the $s_h \in Y(T)$ defined in* (2.2.13) *is slim, if and only if it is orthogonal to $\{c_j ; 1 \leqq j \leqq l\}$.*

§ 3. Main results.

3.1. Statement of main results.

Let K, G, D and g be as in Introduction. We assume that $p = \mathrm{char}(K)$ is zero or a good prime [30 ; I, 4.3] for G. We fix a D-stable Borel subgroup B and a D-stable maximal torus T of G such that $B \supset T$ (see (2.2.1) (iv)). Let Σ be the root system of G with respect to T, and Π the simple system of Σ corresponding to B. Let $W(G(0))$ be the Weyl group of $G(0)$ with respect to $T \cap G(0)$. This can also be identified with the subgroup of $W(\Sigma)$ generated by $\{w_{\langle\alpha\rangle} ; \alpha \in \Sigma, g_{\langle\alpha\rangle} \cap g(0) \neq \{0\}\}$ (see the proof of (2.2.9) for the notations $w_{\langle\alpha\rangle}, g_{\langle\alpha\rangle}$).

(3.1.1) DEFINITION. Let λ be an element of $X(D)$. We define a subset $H(\Sigma, \Pi, \lambda)'_n$ of the set $H(\Sigma, \Pi)$ of weighted Dynkin diagrams (see 2.1) as follows. An element h of $H(\Sigma, \Pi)$ is contained in $H(\Sigma, \Pi, \lambda)'_n$ if and only if

(a) $h = k \circ w$ for some $k \in H(\Sigma, \Pi)_n$ (see 2.1) and some $w \in W(\Sigma)$;

(b) h is D-invariant ; in the case (C), this means that $h \circ \delta = h$ (see (2.2.2)) ; in the case (T), this condition is vacant ;

(c) h is slim (see (2.2.13) and (2.2.14)) with respect to λ ;

(d) $\bar{h}_+ \in H(\bar{\Sigma}_h, \bar{\Pi}_h)_n$, where $\bar{\Sigma}_h$, $\bar{\Pi}_h$ and $\bar{h}$ are as in (2.2.12), and $\bar{h}_+$ is defined from $\bar{h}$ as in (2.1.2).

Let $H(\Sigma, \Pi, \lambda)_n$ be a set of representatives of the set of $W(G(0))$-orbits in $H(\Sigma, \Pi, \lambda)'_n$. In the case (T), we can (and shall) take as $H(\Sigma, \Pi, \lambda)_n$ the set of elements of $H(\Sigma, \Pi, \lambda)'_n$ satisfying

(e) $$h(\alpha)\geqq 0 \quad \text{for any } \alpha\in\Pi(G(0)),$$

where

(3.1.2) $$\Pi(G(0))=\{\alpha\in\Pi\ ;\ g_\alpha\cap g(0)\neq\{0\}\}.$$

(3.1.3) LEMMA. *For $k\in H(\Sigma, \Pi)$, let $\Pi(0)_k=\{\alpha\in\Pi\ ;\ k(\alpha)=0\}$, and let $W(\Pi(0)_k)$ be the subgroup of $W(\Sigma)$ generated by reflections w_α, $\alpha\in\Pi(0)_k$. Then, for $k\in H(\Sigma, \Pi)_+$ and $w\in W(\Sigma)$, we have $k\circ w=k$ if and only if $w\in W(\Pi(0)_k)$.*

PROOF. The if-part is easy. For a proof of the only-if-part, it is enough to show that, for any $k\in H(\Sigma, \Pi)$, the group $\{w\in W(\Sigma)\ ;\ k\circ w=k\}$ is generated by reflections. But this follows from [30 ; II, 4.1] and [2 ; (11.12)].

Fix an arbitrary element h of $H(\Sigma, \Pi, \lambda)_n$. In (2.2.9), we defined the Z-graded Lie algebra

(3.1.4) $$\bar{g}=\bigoplus_{j\in Z}\bar{g}(j)$$

and the corresponding connected reductive subgroup $\bar{G}$ of G. Recall that $\bar{\Sigma}$ is the root system of $\bar{G}$ with respect to the maximal torus $\bar{T}=T\cap G(0)$, and that the gradation (3.1.4) is the one corresponding to the weighted Dynkin diagram $\bar{h}\in H(\bar{\Sigma}, \bar{\Pi})$ (see (2.2.12)). Hence, the condition (3.1.1) (d) for h and Theorem (2.1.7) (i) imply that there exists an element $N_{\bar{h}}$ of $\bar{g}(2)=\bar{g}(2)_{\bar{h}}$ such that $\bar{G}(0)_{\bar{h}}N_{\bar{h}}$ is dense in $\bar{g}(2)$. (Here we used the fact that p is good for $\bar{G}(0)$ also. This can be checked using (2.2.4) and the classification of semisimple algebraic groups.) By (2.1.11), $N_{\bar{h}}$ can be taken in an explicit way.

Now we can state the main result of this paper.

(3.1.5) THEOREM. *Let the notations be as in Introduction. We assume that $p=\mathrm{char}(K)$ is zero or a good prime for G. We fix $\lambda\in X(D)$. Let $H(\Sigma, \Pi, \lambda)_n$ be the set of weighted Dynkin diagrams h $(\in H(\Sigma, \Pi))$ defined in* (3.1.1).

(i) *For $h\in H(\Sigma, \Pi, \lambda)_n$, we take an element $N_{\bar{h}}\in\bar{g}(2)=g(\lambda)\cap g(2)_h$ such that $\bar{G}(0)_{\bar{h}}\cdot N_{\bar{h}}$ is dense in $\bar{g}(2)$. Then the correspondence $h\to o_h=G(0)\cdot N_{\bar{h}}$ is a bijection between the set $H(\Sigma, \Pi, \lambda)_n$ and the set of nilpotent $G(0)$-orbits in $g(\lambda)$. Moreover, $o_h\subset O_{h_+}$ in the notations of* (2.1.7) *and* (2.1.2).

(ii) $Z_{G(0)}(N_{\bar{h}})=Z_{\bar{G}(0)_{\bar{h}}}(N_{\bar{h}})\ Z_{G(0)\cap U_h}(N_{\bar{h}})$ *(semi-direct product), where* U_h *is as in* (2.1.7); $Z_{\bar{G}(0)_{\bar{h}}}(N_{\bar{h}})$ *is a reductive group whose global structure is known by* (2.1.9), *and* $Z_{G(0)\cap U_h}(N_{\bar{h}})$ *is the unipotent radical of* $Z_{G(0)}(N_{\bar{h}})$.

(iii) $\dim Z_{\bar{G}(0)_{\bar{h}}}(N_{\bar{h}})=\dim g(0)\cap g(0)_h-\dim g(\lambda)\cap g(2)_h$;

$$\dim Z_{G(0)\cap U_h}(N_{\bar{h}})=\sum_{j\geq 1}\dim g(0)\cap g(j)_h-\sum_{j\geq 3}\dim g(\lambda)\cap g(j)_h .$$

(iv) $\dim o_h=\sum_{j\leq -1}\dim g(0)\cap g(j)_h+\sum_{j\geq 2}\dim g(\lambda)\cap g(j)_h$.

(3.1.6) REMARK. (i) A primitive version of (3.1.5) was given in [13; (2.1.5)].

(ii) The second assertion of (ii) in Introduction follows from (3.1.5) (ii) and (2.1.10).

(iii) When D is of order 2, the nilpotent orbits in $g(\lambda)$ have been studied by Kostant-Rallis [17], Sekiguchi [26], [27], Ohta [21] and others. See also Richardson [22]. We have already mentioned, in Introduction, about a connection between (3.1.5) and a result in [27].

(iv) The classification of non-nilpotent orbits in $g(\lambda)$ can, in principle, be reduced to those of semisimple and nilpotent orbits by the Jordan decomposition. See [33]. The same can also be said about the determination of the structure of the stabilizer $Z_{G(0)}(A)$ of a non-nilpotent element A of $g(\lambda)$. A detailed study of semisimple orbits in $g(\lambda)$ is given in Vinberg [35]. Vinberg [34] also considered the classification of nilpotent orbits in $g(\lambda)$.

(v) It is not difficult to generalize (3.1.5) to the case when D is a general diagonalizable group acting morphically on G, but it seems that (3.1.5) is already sufficient for interesting applications. See also 3.3.

(vi) Let O be a nilpotent G-orbit in g, and let $\lambda\in X(D)$. If $D\cdot O\neq O$, then clearly $O\cap g(\lambda)$ is empty. If $D\cdot O=O$, the number of $G(0)$-orbits contained in $O\cap g(\lambda)$ is bounded by the number of elements of

$$(W^D\cap W(\Pi(0)_k))\backslash W^D/W(G(0)),$$

where W^D is the group of D-fixed elements of $W=W(\Sigma)$, $k\in H(\Sigma,\Pi)_n$ is the weighted Dynkin diagram associated with O, and $W(G(0))$ is the Weyl group of $G(0)$ with respect to $\bar{T}=T\cap G(0)$. This follows from (3.1.5) and (3.1.3).

(vii) The dimension formula in (3.1.5) (iv) does not seem to be very illuminating. When D is of order 2 and $\lambda\neq 0$, then it is known [17; Prop. 5] that

$$\dim G(0)\cdot A=(\dim G\cdot A)/2,\qquad A\in g(\lambda). \tag{3.1.7}$$

When D is a 1-dimensional torus, there also exists a formula comparable to (3.1.7). Let λ_1 be a generator of $X(D)\cong Z$, and let P be the parabolic subgroup of G with the Lie algebra $\bigoplus_{i\geq 0} g(i\lambda_1)$. Then we have

$$\dim P\cdot A=(\dim G\cdot A)/2\,,\qquad A\in g(\lambda_1)\,. \tag{3.1.8}$$

Let B be a Borel subgroup contained in P. Then the right hand side of (3.1.8) is equal to the dimension of the (equi-dimensional) variety $V(A)=(G\cdot A)\cap \mathrm{Lie}\, B$ (see Spaltenstein [29]). Hence, by (3.1.8), the closure of $P\cdot A$ ($A\in g(\lambda_1)$) in $V(A)$ is always an irreducible component of $V(A)$. For simplicity, we here prove (3.1.7) and (3.1.8) assuming that $p=0$ or $p\gg 0$. Then the Killing form $\kappa(\cdot,\cdot)$ on g is non-degenerate. Hence, as is well-known, we have

$$Z_g(A)=\{X\in g\,;\ \kappa(X,[g,A])=0\}\,. \tag{3.1.9}$$

Let M be an abelian group, and let

$$g=\bigoplus_{m\in M} g(m)\,,\qquad m\in M$$

be an M-gradation of g. By (3.1.9), it is easy to see that, if $A\in g(m)$, then

$$\dim[g(l),A]=\dim[g(-l-m),A]\,,\qquad l\in M\,. \tag{3.1.10}$$

The formula (3.1.7) follows from this by putting $M=X(D)=Z_2$, $m\neq 0$ and $l=0$. To prove (3.1.8), we put $M=X(D)=Z$ and $m=\lambda_1=1$, and sum up (3.1.10) for $l=0,1,2,\cdots$. Then, the resulting formula is:

$$\dim[\mathrm{Lie}\,P,A]=\dim\Big[\bigoplus_{i\leq -1} g(i),A\Big]\,.$$

Hence, using the decomposition $g=\mathrm{Lie}\,P\oplus\Big(\bigoplus_{i\leq -1} g(i)\Big)$, we get

$$\dim[g,A]=2\dim[\mathrm{Lie}\,P,A]\,,$$

which implies (3.1.8).

(viii) Let F_q be a finite field of q elements contained in K. Assume that G and D are given F_q-rational structures such that the D-action on G is also defined over F_q, and that B and T in (2.2.1) (iv) can be taken to be defined over F_q. We denote by σ the corresponding Frobenius map. Let $\lambda\in X(D)$ be a weight defined over F_q. A nilpotent orbit o_h ($h\in H(\Sigma,\Pi,\lambda)_n$) of $g(\lambda)$ is defined over F_q if and only if $h\circ\sigma=h$. (In particular, any nilpotent orbits are defined over F_q, if G is F_q-split.) Moreover, in this case, the orbital decomposition of $o_h(F_q)$ under $G(0)(F_q)$ and the structures of

$Z_{G(0)(F_q)}(N)$ for $N \in o_h(F_q)$ can be read off from the σ-action on $Z_{G(0)}(N)/Z_{G(0)}(N)^0$. See [30 ; I, 2]. In particular, we see that there exists a polynomial $N_\lambda(t) \in \boldsymbol{Q}[t]$ with coefficients independent of p such that $N_\lambda(q)$ is equal to the number of nilpotent elements of $g(\lambda)(F_q)$. When D is a torus, we have trivially

$$N_\lambda(q) = q^{\dim g(\lambda)}.$$

When D is a finite cyclic group, no general result seems to be known about the polynomials $N_\lambda(q)$, except for the case when λ is the 0-weight, in which case we have

$$N_0(q) = q^{\dim G(0) - \operatorname{rank} G(0)}$$

by Steinberg [32 ; 15.1]. See the final remark in 3.4.

3.2. Proof of Theorem (3.1.5), I (the case when $p=0$ or $p \gg 0$).

Let G be a connected semisimple group over K, and D a diagonalizable group over K acting morphically on G. We assume that $p = \operatorname{char}(K)$ is zero or large. For a nilpotent element N in $g(\lambda)$ ($\lambda \in X(D)$), let f_N be as in 2.1. We put $H = f_N\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $N' = f_N\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$. If we write $H = \sum_{\mu \in X(D)} H_\mu$ and $N' = \sum_{\mu \in X(D)} N'_\mu$ with H_μ, $N'_\mu \in g(\mu)$, we see that the linear map $f_{\tilde{N}} : sl_2(K) \to g$ defined by $f_{\tilde{N}}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = N$, $f_{\tilde{N}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = H_0$, $f_{\tilde{N}}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = N'_{-\lambda}$ is again a Lie algebra homomorphism. Hence H_0 ($\in g(0)$) is a characteristic of N (see 2.1). Such an H_0 will be called a normalized characteristic of N. Just as (2.1.5) was of fundamental importance for Dynkin's classification of nilpotent orbits in g, the following lemma plays an important role in the proof of Theorem (3.1.5).

(3.2.1) LEMMA (Vinberg [33]). (i) *Let N be a nilpotent element of $g(\lambda)$. Then two normalized characteristics of N are conjugate under $Z_{G(0)}(N)$.*

(ii) *Two nilpotent elements of $g(\lambda)$ are $G(0)$-conjugate if and only if their normalized characteristics are.*

When $D \cong Z_2$, this is proved in Kostant-Rallis [17 ; I.2]. The proof given there can easily be modified so that it works in the present more general situation.

(3.2.2) LEMMA. *Let D be as in* 3.1. *Let N be a fixed nilpotent element of $g(\lambda)$ ($\lambda \in X(D)$), and H a normalized characteristic of N. Let*

(3.2.3)
$$g=\bigoplus_{i\in Z} g(i)_H$$

be the Z-gradation of g defined by (2.1.6). *For $j\in Z$, we put*

$$\bar{g}(j)=\begin{cases} g((j/2)\lambda)\cap g(j)_H & \text{if } j/2\in Z\,; \\ \{0\} & \text{otherwise.}\end{cases}$$

Then

$$\bar{g}=\bigoplus_{j\in Z}\bar{g}(j)$$

is the Lie algebra of a connected reductive subgroup $\bar{G}$ of G. Moreover, the gradation $\{\bar{g}(j)\}_{j\in Z}$ of $\bar{g}$ coincides with the one defined using the characteristic $H\in\bar{g}(0)$ of $N\in\bar{g}(2)$. In particular, H is contained in the semisimple part of the reductive Lie algebra $\bar{g}$.

PROOF. The first assertion is a special case of (2.2.9) (i). The second and third assertions are obvious from our construction.

Proof of Theorem (3.1.5) in the case $p=0$ or $p\gg 0$. (i) Let o be a nilpotent $G(0)$-orbit in $g(\lambda)$, and $N=N_o$ an arbitrary element of o. Let $f_N: sl_2\to g$ be as in 2.1. We can assume that $H=f_N\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ is normalized. By replacing N and H with $g\cdot N$ and $g\cdot H$ for a suitable $g\in G(0)$, if necessary, we can further assume that $H\in \mathrm{Lie}(T\cap G(0))=\mathrm{Lie}\, T\cap g(0)$ by (2.2.5) (i). Then, by (3.2.1), H is uniquely determined from o exactly up to the transformations under the Weyl group $W(G(0))$ of $G(0)$ with respect to the maximal torus $T\cap G(0)$. We denote by h the weighted Dynkin diagram corresponding to the gradation (3.2.3). By our construction, we can write

(3.2.4)
$$h=k\circ w'$$

for some $w'\in W(\Sigma)$. Here $k=k_o\in H(\Sigma,\Pi)_n$ corresponds to the orbit $G\cdot N_o$ ((2.1.7) (ii)). Hence h satisfies (3.1.1) (a). That h satisfies (3.1.1) (b) follows from the fact $H\in g(0)$. The second and third assertions of (3.2.2) imply that h also satisfies (3.1.1) (d) and (3.1.1) (c) respectively. By (3.1.2) (i) and what we have said concerning the uniqueness of H ($\in\mathrm{Lie}\, T\cap g(0)$), we see that the orbit o determines the double coset $W(\Pi(0)_k)w'W(G(0))$ uniquely, and distinct orbits correspond to distinct double cosets. This means that o determines an element of $H(\Sigma,\Pi,\lambda)_n$ uniquely. In the case (T), one can choose B so that $W(G(0))$ is generated by $\{w_\alpha\,;\,\alpha\in\Pi(G(0))\}$. Hence, by [3; IV, 1, Ex. 3], the double coset $W(\Pi(0)_k)w'W(G(0))$ contains a unique $(\Pi(0)_k,\Pi(G(0)))$-reduced element w. If we put $h=k\circ w$, this satisfies (3.1.1) (a)-(e). Conversely, given an element h of $H(\Sigma, \Pi, \lambda)_n$, we write $h=k\circ w$. Since $k\in H(\Sigma, \Pi)_n$ and $h\circ\delta=h$, there exists a characteristic $H\in \mathrm{Lie}\, T\cap g(0) = \mathrm{Lie}\, T\cap \bar{g}(0)_{\bar{h}}$ of a nilpotent element of g such that $g(i)_H=g(i)_h$, $i\in Z$. Since h satisfies (3.1.1) (d), we can take a nilpotent element $N_{\bar{h}}\in \bar{g}(2)_{\bar{h}}\subset g(\lambda)$ such that $\bar{G}(0)_h\cdot N_{\bar{h}}$ is dense in $\bar{g}(2)_{\bar{h}}$. Then, by the second statement of (2.1.7) (i), there exists a characteristic $H'\in\bar{g}(0)$ of $N_{\bar{h}}$ such that the Z-gradation of $\bar{g}$ associated (see (2.1.6)) with H' and the one associated with $\bar{h}$ (or, equivalently, with H) are identical. This and (3.1.1) (c) imply that $H=H'$. Hence the G-orbit of $N_{\bar{h}}$ is equal to O_k in the notation of (2.1.7). Hence $o_h\subset O_k=O_{h_+}$. Now it is clear that the orbit $G(0)\cdot N_{\bar{h}}$ corresponds to the element h of $H(\Sigma, \Pi, \lambda)_n$ under the map defined above. This proves the part (i) of Theorem (3.1.5).

(ii) In (i), we have seen that $N_{\bar{h}}\in O_k$. By this and (2.1.7) (v), we have

$$\begin{aligned} Z_{G(0)}(N_{\bar{h}}) &= Z_G(N_{\bar{h}})\cap G(0) = (Z_{G(0)_h}(N_{\bar{h}})Z_{U_h}(N_{\bar{h}}))\cap G(0) \\ &= Z_{\bar{G}(0)}(N_{\bar{h}})Z_{G(0)\cap U_h}(N_{\bar{h}}) . \end{aligned}$$

Moreover, by (2.1.7) (v) (applied to $\bar{G}$), we see that $Z_{\bar{G}(0)}(N_{\bar{h}})$ is reductive. Hence we get (ii).

(iii), (iv) By (2.1.7) (v), $Z_{G(0)}(N_{\bar{h}})=Z_{G(0)\cap P_h}(N_{\bar{h}})$, and by (2.1.7) (iv), $(G(0)\cap P_h)\cdot N_{\bar{h}}$ is dense in $\bigoplus_{j\geq 2} g(\lambda)\cap g(j)_h$. Hence

$$\dim Z_{G(0)}(N_{\bar{h}})=\sum_{j\geq 0}\dim g(0)\cap g(j)_h-\sum_{j\geq 2}\dim g(\lambda)\cap g(j)_h .$$

Similarly, since $\bar{G}(0)\cdot N_{\bar{h}}$ is dense in $g(\lambda)\cap g(2)_h$, we have

$$\dim Z_{\bar{G}(0)}(N_{\bar{h}})=\dim g(0)\cap g(0)_h-\dim g(\lambda)\cap g(2)_h .$$

From these we get (iii) and (iv).

3.3. Proof of Theorem (3.1.5), II (the case when p is a good prime for G).

For a proof of (3.1.5) in the positive characteristic case, it is convenient to consider a situation slightly more general than that of (3.1.5). Let K and G be as in (3.1.5). Let D be a diagonalizable group over K acting morphically on G by algebraic group automorphisms. We assume:

(3.3.1) D stabilizes a pair (B, T) of a Borel subgroup B and a maximal torus T such that $B\supset T$.

For example, if D is a product of a torus and a finite cyclic group, then (3.3.1) is always satisfied. (This is not used in the sequel.) Returning to the general situation, let B and T be as in (3.3.1), and λ an element of the character module $X(D)$ of D. We can prove the statements in (2.2.9) in this more general situation by essentially the same arguments as in 2.2, and, using it, can define the set $H(\Sigma, \Pi, \lambda)_n$ as in 3.1. Then, for $h \in H(\Sigma, \Pi, \lambda)_n$, there exists an element $N_{\bar{h}}$ of $\bar{g}(2)_{\bar{h}} = g(2)_h \cap g(\lambda)$ such that $\bar{G}(0)_{\bar{h}} \cdot N_{\bar{h}}$ is dense in $\bar{g}(2)_{\bar{h}}$.

(3.3.2) THEOREM. *The statements in* (3.1.5) (i)-(iv) *are true in the present more general situation.*

If $p = \mathrm{char}(K)$ is zero or large, then one can get a proof of (3.3.2) by modifying slightly that of (3.1.5) in that case. We omit the details. Now we give a proof of (3.3.2) assuming that p is a good prime for G. We begin by proving the following lemma.

(3.3.3) LEMMA. *Let $\lambda \in X(D)$. Let k be an element of $H(\Sigma, \Pi)_n$, and O_k the corresponding nilpotent orbit in g (see* (2.1.7) (ii)).

(i) *Let $w \in W(\Sigma)$ and $h = k \circ w$ ($\in H(\Sigma, \Pi)$). Then $O_k \cap (g(\lambda) \cap g(2)_h)$ is an open, dense $\bar{G}(0)_{\bar{h}}$-orbit in $g(\lambda) \cap g(2)_h$, if it is non-empty.*

(ii) *Assume that $O_k \cap g(\lambda)$ is non-empty. Then the weighted Dynkin diagram k is D-invariant.*

(iii) *Let O_k be as in* (ii). *Let o be a $G(0)$-orbit contained in $O_k \cap g(\lambda)$. Then there exists an element $w = w_o$ of $W(\Sigma)$ such that*

(iii a) $o \cap g(2)_{k \circ w} \neq \emptyset$;

(iii b) $\delta \circ w(\alpha) = w \circ \delta(\alpha)$ *for any $\delta \in D$ and $\alpha \in \Sigma$.*

Moreover, the double coset $W(\Pi(0)_k) w_o W(G(0))$ is uniquely determined from o.

(iv) *Let o be a nilpotent $G(0)$-orbit in $g(\lambda)$. Let $k = k_o$ be an element of $H(\Sigma, \Pi)_n$ such that $o \subset O_k$, and $w = W(\Pi(0)_k) w_o W(G(0))$ be as in* (iii). *Then the correspondence $o \to (k_o, W(\Pi(0)_k) w_o W(G(0)))$ from the set of nilpotent $G(0)$-orbits in $g(\lambda)$ to the set $H(\Sigma, \Pi)_n \times W(\Pi(0)_k) \backslash W / W(G(0))$ is injective.*

PROOF. (i) We can assume that, for any nilpotent element N of g, we have $Z_g(N) = \mathrm{Lie}\, Z_G(N)$. (See [30 ; I, 5.3, 5.6].) Then, for any $N \in O_k \cap g(2)_h$, we have, by (2.1.7) (iii) (v),

$$Z_{g(0)_h}(N) = \mathrm{Lie}\, Z_{G(0)_h}(N) .$$

This and (2.1.7) (ii) (iii) imply that

$$[g(0)_h, N] = g(2)_h .$$

Hence, for any $N \in O_k \cap g(\lambda) \cap g(2)_h$, we have

$$[g(0) \cap g(0)_h, N] = g(\lambda) \cap g(2)_h .$$

Hence $\bar{G}(0)_{\bar{h}}\cdot N$ is open and dense in $g(\lambda)\cap g(2)_h$. This implies (i).

(ii) Take an element N of $O_k\cap g(\lambda)$. Since $\delta(N)=\lambda(\delta)N$ for any $\delta\in D$, and $\mu O_k=O_k$ for any $\mu\in K^\times$ by (2.1.7) (i) (iii), we have $D(O_k)=O_k$. On the other hand, by (2.2.4), the definition of $H(\Sigma,\Pi)_n$ and the existence of the graph automorphisms [31 ; § 10] in characteristic 0, we can easily show that $k\circ\delta\in H(\Sigma,\Pi)_n$ for any $\delta\in D$. Hence, if k is not D-invariant, we must have $D(O_k)\neq O_k$ by (2.1.7). This proves (ii).

(iii) Let N be an element of o. By (2.1.7) (i) (ii), there exists an element x of G such that

$$x^{-1}\cdot\Big(\bigoplus_{i\geqq 2} g(i)_k\Big)\ni N. \tag{3.3.4}$$

Then, by (2.1.7) (iv) (v), the parabolic subgroup $x^{-1}P_kx$ is uniquely determined from N. On the other hand, by Bruhat's lemma and the fact P_k normalizes $\bigoplus_{i\geqq 2} g(i)_k$, we can assume that

$$x=n_wu,$$

where $w\in W(\Sigma)=N_G(T)/T$ is $(\Pi(0)_k,\phi)$-reduced, n_w is a representative of w in $N_G(T)$ and u is an element of

$$\prod_\alpha\{\exp(\xi X_\alpha)\,;\ \xi\in K\},\qquad \alpha\in\Sigma^+\cap w^{-1}(\Sigma^-).$$

Since such a pair (w,u) is uniquely determined from the condition $u^{-1}n_w^{-1}P_kn_wu=x^{-1}P_kx$, it is uniquely determined from N. Applying $\delta\in D$ to (3.3.4) and using (ii), we get

$$\delta(u)^{-1}\delta(n_w)^{-1}\cdot\Big(\bigoplus_{i\geqq 2} g(i)_k\Big)\ni\lambda(\delta)N.$$

Hence, from the uniqueness of the pair (w,u) mentioned above, we have, for $\delta\in D$,

$$\delta(u)=u$$

and

$$\delta(n_w)=t_\delta n_w$$

for some $t_\delta\in T$. Hence we get

$$\bigoplus_{i\geqq 2} g(i)_{k\circ w}\ni u\cdot N\ (\in o) \tag{3.3.5}$$

for an element w of $W(\Sigma)$ satisfying (iii b). If w' is another element of $W(\Sigma)$ such that

$$o\cap g(2)_{k\circ w'}\neq\varnothing,$$

then there exist $N, N' \in O_k \cap g(2)_k$ such that

$$a \cdot (n_w^{-1} \cdot N) = n_{w'}^{-1} \cdot N'$$

for some $a \in G(0)$. Hence, by (2.1.7) (iv) (v), we have

$$n_{w'} a n_w^{-1} \in P_k .$$

Hence $P_k n_w G(0) = P_k n_{w'} G(0)$, which implies that

$$W(\Pi(0)_k) w W(G(0)) = W(\Pi(0)_k) w' W(G(0)) .$$

Hence o determines the double coset $W(\Pi(0)_k) w W(G(0))$ uniquely.

(iv) It is enough to prove the injectivity. But this follows from (i).

Proof of Theorem (3.3.2) in the case p is a good prime for G. For an element k of $H(\Sigma, \Pi)_n$, we denote by $H(\Sigma, \Pi, \lambda)_n(k)$ the set of elements h of $H(\Sigma, \Pi, \lambda)_n$ of the form $h = k \circ w$, $w \in W(\Sigma)$. Let O_k be the nilpotent orbit in g corresponding to k. We show

(3.3.6) $$O_k \cap g(\lambda) = \bigcup_h G(0) \cdot N_{\bar{h}} \text{ (disjoint)}, \ h \in H(\Sigma, \Pi, \lambda)_n(k)$$

and

(3.3.7) for $h \in H(\Sigma, \Pi, \lambda)_n(k)$, the statements (3.1.5) (ii)–(iv) are true in the situation of (3.3.2)

by induction on the closure inclusion relations among the nilpotent G-orbits $\{O_k ; k \in H(\sigma, \Pi)_n\}$. If O_k is minimal, i.e. $O_k = \{0\}$, then these are trivially true. For a general k, let h be an element of $H(\Sigma, \Pi, \lambda)_n(k)$. We claim that

(3.3.8) $$N_{\bar{h}} \in O_k .$$

In fact, by applying the induction assumption to the $D \times K^{\times}$-action on G giving rise to the $X(D \times K^{\times})$ $(\cong X(D) \oplus Z)$-gradation

$$g = \bigoplus_{(\lambda, i)} g((\lambda, i)_h) , \quad (\lambda, i)_h = (\lambda, i) \in X(D) \oplus Z ,$$

$$g((\lambda, i)_h) = g(\lambda) \cap g(i)_h$$

of g, we have

(3.3.9) $$O_{k'} \cap (g(\lambda) \cap g(2)_h) = \bigcup_l \bar{G}(0)_{\bar{h}} \cdot N_l , \ l \in H(\Sigma, \Pi, (\lambda, 2)_h)_n(k')$$

and

$$\begin{aligned}(3.3.10)\qquad \dim \bar{G}(0)_{\bar{h}}\cdot N_l = &\sum_{j\leq -1} \dim g(0)\cap g(0)_h \cap g(j)_l \\ &+\sum_{j\geq 2} \dim g(\lambda)\cap g(2)_h\cap g(j)_l, \\ &\qquad l\in H(\Sigma, \Pi, (\lambda, 2)_h)_n(k')\end{aligned}$$

if $\bar{O}_{k'}\subsetneqq \bar{O}_k$ (the bar denotes the closure of a given set). Let G^C be a connected semisimple algebraic group with the same root system Σ as G, and $g^C = \mathrm{Lie}\, G^C$. We denote by D^C a diagonalizable group over $\boldsymbol{C}$ such that $X(D^C)\cong X(D)$ as Z-modules. We assume that D^C acts on G^C in the 'same' way as D acts on G, i.e., for each $\alpha\in\Sigma$, the D-orbit $D(\alpha)=(\alpha)$ and the D^C-orbit $D^C(\alpha)$ are identical, and, for each $\lambda\in X(D)=X(D^C)$, $\dim g_{(\alpha)}\cap g(\lambda)$ and $\dim g^C_{(\alpha)}\cap g^C(\lambda)$ (which are either 0 or 1 by 2.2) are equal. The existence of such a pair (G^C, D^C) is clear from the analysis of D-actions given in 2.2. In this setting, it is evident that

$$\dim g(\lambda)\cap g(i)_h\cap g(j)_l = \dim g^C(\lambda)\cap g^C(i)_h \cap g^C(j)_l$$

for any $\lambda\in X(D)=X(D^C)$ and any $i, j\in Z$. Hence, by (3.3.10), we see that the dimensions of the nilpotent $\bar{G}(0)_{\bar{h}}$-orbits corresponding to l as above are independent of the characteristic p. Moreover, if $p=0$, we already know that (3.3.8) is true. This and the definition of $N_{\bar{h}}$ implies that the dimensions (3.3.10) are strictly smaller than $\dim g^C(\lambda)\cap g^C(2)_h = \dim g(\lambda)\cap g(2)_h$. Hence, by (3.3.9) and the denseness of $\bar{G}(0)_{\bar{h}}\cdot N_{\bar{h}}$ in $g(\lambda)\cap g(2)_h$, we get (3.3.8), or, equivalently, see that the right hand side of (3.3.6) is contained in the left hand side. The disjointness in (3.3.6) follows from (3.3.3) (iv). Once (3.3.8) is proved, then (3.3.7) follows from that and (2.1.7). See the proof of (3.1.5) (ii)-(iv) in the case $p=0$ or $p\gg 0$. It only remains to show that the right hand side of (3.3.6) contains the left hand side. Let o be a $G(0)$-orbit contained in $O_k\cap g(\lambda)$. Then, by (3.3.3) (iv), there exists an element w of $W(\Sigma)$ satisfying (3.3.3) (iii a)-(iii b). Let $f=k\circ w$. We must show

$$(3.3.11)\qquad f\in H(\Sigma, \Pi, \lambda)_n(k) \quad \mathrm{mod}\ W(G(0))\text{-action}.$$

Let G^C, g^C and D^C be as in the proof of (3.3.8). Then we see, from (3.3.3) (iv), that $g^C(\lambda)\cap g^C(2)_f$ is decomposed into finitely many $\bar{G}(0)_{\bar{f}}$-orbits, and hence, there exists an open, dense orbit a^C. Since (3.3.2) is true in characteristic 0, (3.3.11) is equivalent to

$$a^C\subset O^C_k,$$

where O^C_k is the nilpotent orbit in g^C corresponding to k. Hence, if (3.3.11) is not true, we must have

$$a^C = \bar{G}^C(0)_{\bar{f}} \cdot N_l{}^C$$

for some $l \in H(\Sigma, \Pi, (\lambda, 2)_f)_n(k')$ with $k' \neq k$ and an element N_l of $O_{k'} \cap g^C(\lambda) \cap g^C(2)_f \cap g^C(2)_l$. But, since we already have (3.3.7), we can show (see (3.3.10))

$$\dim \bar{G}(0)_{\bar{f}} \cdot N_l = \dim \bar{G}^C(0)_{\bar{f}} \cdot N_l{}^C .$$

Since the right hand side is equal to $\dim g^C(\lambda) \cap g^C(2)_f = \dim g(\lambda) \cap g(2)_f$, we see that $\bar{G}(0)_{\bar{f}} \cdot N_l$ is dense in $g(\lambda) \cap g(2)_f$. But, by our assumption on o and (3.3.3) (i), $o \cap g(\lambda) \cap g(2)_f$ (which is contained in O_k) is also open and dense in $g(\lambda) \cap g(2)_f$. This is absurd, since $N_l \in O_{k'}$ and $k' \neq k$. Hence (3.3.11) must be true, i.e., any $G(0)$-orbit contained in $O_k \cap g(\lambda)$ is contained in the right hand side of (3.3.6). Hence we get (3.3.6). This completes the proof of (3.3.2) and hence that of (3.1.5).

3.4. An example.

Here we consider the orbital decomposition of the 40-dimensional irreducible reduced prehomogeneous vector space, which is classified as type (11) in the table of Sato-Kimura [25; pp. 144–147]. See also [13; 3.3, 3.4]. In our notations in 2.1, this prehomogeneous vector space can be realized as the pair $(G(0)_h, g(2)_h)$, where G is a simple algebraic group of type E_8 and h is the weighted Dynkin diagram

$$\begin{matrix} 0002000 \\ 0 \end{matrix}$$

(see Rubenthaler [24]). By Theorem (3.1.5) (for the case when D is a 1-dimensional torus), we know that the orbital decomposition of $(G(0)_h, g(2)_h)$ is 'independent' of the characteristic p of the algebraically closed field K over which G is defined, provided that $p \neq 2, 3, 5$. Below we assume that p is positive and not equal to 2, 3 or 5. This enables us to consider the number of points over a finite field for a given orbit. As the reader will see, knowing such numbers is very helpful in actual calculations.

Let F_q be a finite subfield of q elements contained in K. Since the weighted Dynkin diagram h is contained in $H(\Sigma)_n$ (Σ = the root system of type E_8), the $G(0)_h$-orbit of maximal dimension 40 can be written as o_h in the notation of (3.1.5). By Mizuno [19; Lem. 70], $Z_{G(0)}(N_h) \cong S_5$ ($N_h \in o_h$) and the number of F_q-points of o_h is given by

$$|o_h(F_q)| = |G(0)_h(F_q)| = |GL_5(F_q)| \cdot |SL_4(F_q)|$$
$$= q^{40} - q^{39} + (\text{terms of lower degrees}).$$

Hence, using that the number of (nilpotent) elements of $g(2)_h(F_q)$ is q^{40}, we see that the number of $G(0)_h$-orbits of dimension $\underline{39}$ is just $\underline{1}$. Denote this orbit by o_{h_1} with $h_1 \in H(\Sigma, \Pi, 2)_n$. Noting that o_{h_1} is contained in the closure of o_h and taking the closure relations [19 ; p. 453] among nilpotent orbits in g into consideration, one can easily find out a candidate for h_1, namely $k \circ \mathrm{id.} \in H(\Sigma, \Pi, 2)_n$ with

$$k = \begin{pmatrix} 0010100 \\ 0 \end{pmatrix} \in H(\Sigma, \Pi)_n .$$

In fact, we have

$$\bar{k} = \begin{pmatrix} 002002 \\ 0 \end{pmatrix} \in H(\bar{\Sigma}_k, \bar{\Pi}_k)_n .$$

By [19 ; Lem. 21], and Theorem (3.1.5), we see that $Z_{G(0)}(N_{\bar{k}}) \cong S_3 \times GL_1(K)$. Hence we have indeed $h_1 = k$. Using [30 ; I, § 2], we also have

$$|o_{h_1}(F_q)| = |GL_5(F_q)| \cdot |SL_4(F_q)| (q-1)^{\ 1} .$$

Hence

$$|o_h(F_q)| + |o_{h_1}(F_q)| = q^{40} - \underline{2} q^{\underline{38}} + (\text{lower degrees}) .$$

This implies that there are exactly $\underline{2}$ $G(0)_h$-orbits of dimension $\underline{38}$. Using (3.1.5) and the closure relations mentioned above, one can show that they are $o_{h_{21}}$ and $o_{h_{22}}$ with

$$h_{21} = \begin{pmatrix} 1001010 \\ 0 \end{pmatrix} \circ w_{\alpha_5} = \begin{pmatrix} 101\text{-}1110 \\ 0 \end{pmatrix}$$

and

$$h_{22} = \begin{pmatrix} 0100010 \\ 1 \end{pmatrix} .$$

(For the numbering of simple roots, we follow [3 ; Table 7].) In fact, we have

$$\bar{h}_{21} = \begin{pmatrix} 20202 \\ 0 \end{pmatrix} \oplus (2) \in H(\bar{\Sigma}_{h_{21}}, \bar{\Pi}_{h_{21}})_n$$

and

$$\bar{h}_{22} = \begin{pmatrix} 202020 \\ 0 \end{pmatrix} \in H(\bar{\Sigma}_{h_{22}}, \bar{\Pi}_{h_{22}})_n .$$

Hence, by (3.1.5) and (2.1.9), we see that $Z_{G(0)}(N_{\bar{h}_{21}}) \cong (S_2 \times GL_1(K))\, U(1)$ ($U(1)$ = the unipotent radical of dimension 1), and $Z_{G(0)}(N_{\bar{h}_{22}}) \cong S_2 \ltimes (GL_1(K))^2$. Now we can see that

$$\begin{aligned} & |o_h(F_q)| + |o_{h_1}(F_q)| + |o_{h_{21}}(F_q)| + |o_{h_{22}}(F_q)| \\ & = q^{40} - \underline{2} q^{\underline{37}} + (\text{lower degrees}) . \end{aligned}$$

Hence, there are exactly $\underline{2}$ orbits of dimension $\underline{37}$. In this way, we go down one by one until we reach at the 0-dimensional (i.e., trivial) orbit. See [13; (2.1.8)] for further data concerning this example. In general, if we are given G, D and $\lambda \in X(D)$, then, after fixing a pair (B, T), we consider the set $H(\Sigma, \Pi)_n \times W(\Sigma)/W(G(0))$, which is finite. For each element (k, w) of this set, we check if $k \circ w$ is contained in $H(\Sigma, \Pi, \lambda)_n$ or not. Thus, after perhaps a lengthy but mechanical calculation, we can determine the set $H(\Sigma, \Pi, \lambda)_n$, or equivalently, can decompose the set of nilpotent elements of $\mathfrak{g}(\lambda)$ into $G(0)$-orbits. Moreover, by (3.1.5), we can get further information on each such nilpotent orbit. Of course, in actual calculations, one can appeal to various devices to shorten the above general procedure. For example, as we have seen in the above example, the knowledge of the polynomial $N_\lambda(q)$ (see (3.1.6) (viii)), if available beforehand, is very helpful.

References

[1] Alekseevsky, A. V., Component groups of centralizer for unipotent elements in semisimple algebraic groups (in Russian), Trudy Tbiliss. Mat. Inst. **62** (1979), 5-27.

[2] Borel, A., Linear Algebraic Groups, Benjamin, New York, 1969.

[3] Bourbaki, N., Groupes et Algèbres de Lie, Chap. IV, V, VI, Hermann, Paris, 1968.

[4] Carter, R. W., Finite Groups of Lie Type: conjugacy classes and complex characters, Wiley-Interscience, New York, 1985.

[5] Chang, B., The conjugacy classes of Chevalley groups of type (G_2), J. Algebra **9** (1968), 190-211.

[6] Dynkin, E. B., Semisimple subalgebras of semisimple Lie algebras, Am. Math. Soc. Transl. Ser. 2, **6** (1957), 111-244.

[7] Elashvili, A. G., The centralizers of nilpotent elements in the semisimple Lie algebras (in Russian), Trudy Tbiliss. Mat. Inst. **46** (1975), 109-132.

[8] Hirai, T., On semisimple Lie algebras over algebraically closed fields, Proc. Japan Acad. Ser. A Math. Sci. **60** (1984), 225-228.

[9] Kac, V. G., Some remarks on nilpotent orbits, J. Algebra **64** (1980), 190-213.

[10] Kac, V. G., Infinite Dimensional Lie Algebras, Birkhäuser, Boston-Basel-Stuttgart, 1983.

[11] Kawanaka, N., Fourier transforms of nilpotently supported invariant functions on a simple Lie algebra over a finite field, Invent. Math. **69** (1982), 411-435.

[12] Kawanaka, N., Generalized Gelfand-Graev representations and Ennola duality, algebraic groups and related topics, Advanced Studies in Pure Math. vol. 6, Kinokuniya, Tokyo and North-Holland, Amsterdam-New York-Oxford, 1985.

[13] Kawanaka, N., Generalized Gelfand-Graev representations of exceptional simple algebraic groups over a finite field I, Invent. Math. **84** (1986), 575-616.

[14] Kawanaka, N., Shintani lifting and Gelfand-Graev representations, to appear in Proc. Sympos. Pure Math. (Proc. AMS Summer Institute at Arcata, 1986)

[15] Kimura, T., Kasai, S. and O. Yasukura, A classification of the representations of reductive algebraic groups which admit only a finite number of orbits, Amer. J. Math. **108** (1986), 643-692.

[16] Kostant, B., The principal three-dimensional subgroup and the Betti numbers of

a complex simple Lie group, Amer. J. Math. **81** (1959), 973-1032.
[17] Kostant, B. and S. Rallis, Orbits and representations associated with symmetric spaces, Amer. J. Math. **93** (1971), 753-809.
[18] Mizuno, K., The conjugate classes of Chevalley groups of type E_6, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **24** (1977), 525-563.
[19] Mizuno, K., The conjugate classes of unipotent elements of the Chevalley groups E_7 and E_8, Tokyo J. Math. **3** (1980), 391-459.
[20] Muller, I., Rubenthaler, H. and G. Schiffmann, Structure des espaces préhomogènes associés a certaines algèbres de Lie graduées, Math. Ann. **274** (1986), 95-123.
[21] Ohta, T., On the singularities of the closures of nilpotent orbits in certain symmetric pairs, Tôhoku Math. J. 38 (1986), 441-468.
[22] Richardson, R. W., Orbits, invariants, and representations associated to involutions of reductive groups, Invent. Math. **66** (1982), 287-312.
[23] Richardson, R. W., Finiteness theorems for orbits of algebraic groups, Indag. Math. 88 (1985), 337-344.
[24] Rubenthaler, H., Espaces préhomogènes de type parabolique, Thèses, Publ. d'I. R. M. A., Univ. Strasbourg, 1982.
[25] Sato, M. and T. Kimura, A classification of irreducible prehomogeneous vector spaces and their relative invariants, Nagoya Math. J. **65** (1977), 1-155.
[26] Sekiguchi, J., The nilpotent subvariety of the vector space associated to a symmetric pair, Publ. Res. Inst. Math. Sci. **20** (1984), 155-212.
[27] Sekiguchi, J., Remarks on real nilpotent orbits of a symmetric pair, J. Math. Soc. Japan **39** (1987), 127-138.
[28] Shoji, T., The conjugacy classes of Chevalley groups of type (F_4) over finite fields of characteristic $p\neq 2$, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **21** (1974), 1-17.
[29] Spaltenstein, N., Classes unipotentes et Sous-groupes de Borel, Lecture Notes in Math., vol. 946, Springer, Berlin-Heidelberg-New York, 1982.
[30] Springer, T. A. and R. Steinberg, Conjugacy classes, in Borel et al. : Seminar on Algebraic Groups and Related Finite Groups, Lecture Notes in Math., vol. 131, Springer, Berlin-Heidelberg-New York, 1970.
[31] Steinberg, R., Lectures on Chevalley Groups, Yale University, 1967.
[32] Steinberg, R., Endomorphisms of linear algebraic groups, Mem. Amer. Math. Soc. **80** (1968).
[33] Vinberg, E. B., On the linear groups associated to periodic automorphisms of semisimple algebraic groups, Soviet Math. Dokl. **16** (1975), 406-409.
[34] Vinberg, E. B., On the classification of the nilpotent elements of graded Lie algebras, Soviet Math. Dokl. **16** (1975), 1517-1520.
[35] Vinberg, E. B., The Weyl group of a graded Lie algebra, Math. USSR-Izv. **10** (1976), 463-495.
[36] Wall, G. E., On the conjugacy classes in the unitary, symplectic and orthogonal groups, J. Austral. Math. Soc. 3 (1963), 1-62.

(Received April 9, 1987)

Department of Mathematics
Faculty of Science
Osaka University
Toyonaka, Osaka
560 Japan