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Shintani descent for exceptional groups over a finite field

Dedicated to Professor N. Iwahori on his sixtieth birthday

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Introduction

Let G be a connected reductive algebraic group defined over a finite field $\boldsymbol{F}_q$ of characteristic p, and $F: G\to G$ be the corresponding Frobenius map. For a positive integer m, let us denote by $G^{F^m}/\sim_F$ the set of F-twisted conjugacy classes in G^{F^m}. If $m=1$, it is just the set of conjugacy classes of G^F and we denote it simply by $G^F/\sim$. We have a canonical bijection called norm map, $N_{F^m/F}: G^F/\sim\to G^{F^m}/\sim_F$ by attaching to the class of $x=F^m(a)a^{-1}$ the class of $\hat{x}=a^{-1}F(a)$, where $x\in G^F$, $\hat{x}\in G^{F^m}$ and $a\in G$. Let $C(G^{F^m}/\sim_F)$ (resp. $C(G^F/\sim)$) be the space of $\bar{\boldsymbol{Q}}_l$-valued functions on the set $G^{F^m}/\sim_F$ (resp. $G^F/\sim$) and consider the induced map $N^*_{F^m/F}: C(G^{F^m}/\sim_F)\to C(G^F/\sim)$, which is an isomorphism of vector spaces. Following Digne-Michel [8], we shall call this map "Shintani descent" from G^{F^m} to G^F. However, notice that our definition of Shintani descent is slightly different from the original one given in [8], where the norm map is defined as the composition of the map $x\mapsto x^{-1}$ ($x\in G^{F^m}$) and our norm map $N_{F^m/F}$, and Shintani descent is defined as the transpose of this modified norm map.

We are now interested in describing the Shintani descent. In the case where $m=1$, $N^*_{F/F}$ (or rather its inverse map) coincides with the twisting operator in the sense of Asai [5], and the behaviour of $N^*_{F/F}$ has been studied extensively by him in a series of papers (see, e.g., [3], [4], [5]).

In contrast to the above situation, our main concern here lies in another extreme case, i.e., m is sufficiently divisible (we say that m is sufficiently divisible if m is a multiple of some sufficiently large fixed m_0 depending only on G^F). In [16], the author has described $N^*_{F^m/F}$ in connection with almost characters of G^F in the case of classical groups with connected center (cf. see the last paragraph of Introduction and § 4). In this paper, we shall show that the similar result holds also for exceptional groups of adjoint type under a mild restriction on p. In order to state our result more precisely, we shall prepare some notations. Let $\tilde{G}^{F^m}$ be the semidirect product of G^{F^m} with the cyclic group of order m with generator σ, where σ acts on G^{F^m} by $\sigma g\sigma^{-1}=F(g)$ ($g\in G^{F^m}$). For each representation $\tilde{\rho}$ of $\tilde{G}^{F^m}$, we denote by $[\tilde{\rho}]$ the restriction of the character of $\tilde{\rho}$ to $G^{F^m}\sigma$ $(\subset \tilde{G}^{F^m})$. If $m=1$, $[\tilde{\rho}]=[\rho]$ simply means the character of ρ. We may regard $[\tilde{\rho}]$ as an element of $C(G^{F^m}/\sim_F)$ under the natural bijection $G^{F^m}/\sim_F \cong G^{F^m}\sigma/\sim$ ($\sim$ in the last expression means the conjugation under $\tilde{G}^{F^m}$). Now, for each F-stable irreducible representation ρ of G^{F^m}, there exists an extension $\tilde{\rho}$ of ρ to $\tilde{G}^{F^m}$. Then $[\tilde{\rho}]$ is unique up to an m-th root of unity multiple for various extensions of ρ, and the set of all $[\tilde{\rho}]$ for F-stable irreducible representations ρ of G^{F^m} gives rise to a basis of $C(G^{F^m}/\sim_F)$.

According to Deligne-Lusztig [7], the set $\mathcal{E}(G^F)$ of isomorphism classes of irreducible representations of G^F over $\bar{\boldsymbol{Q}}_l$ is partitioned into a disjoint union of subsets $\mathcal{E}(G^F, \{s\})$, where $\{s\}$ runs over all the F-stable semisimple classes in the dual group G^* of G. Let $\{s\}$ be an F-stable class. We denote by $C^{(s)}(G^F/\sim)$ the subspace of $C(G^F/\sim)$ generated by all $[\rho]$ corresponding to $\rho \in \mathcal{E}(G^F, \{s\})$. We also consider the set $\mathcal{E}(G^{F^m}, \{s\})$ for each F^m-stable class $\{s\}$ in G. If $\{s\}$ is F-stable, F acts naturally on $\mathcal{E}(G^{F^m}, \{s\})$ and we denote by $\mathcal{E}(G^{F^m}, \{s\})^F$ its F-fixed point subset. For each F-stable class $\{s\}$, we denote by $C^{(s)}(G^{F^m}/\sim_F)$ the subspace of $C(G^{F^m}/\sim_F)$ generated by all $[\tilde{\rho}]$ corresponding to $\rho \in \mathcal{E}(G^{F^m}, \{s\})^F$.

Now our main result is Theorem 3.2, which asserts the following. Let G be an exceptional group of adjoint type. We assume $q \not\equiv -1 \pmod 3$ if G is of type E_8. Then for a sufficiently divisible m, the following hold: For each F-stable class $\{s\}$ in G^*, there exists a natural surjection from $X(W_s, \gamma)$ (see 3.1 for definition) onto $\mathcal{E}(G^{F^m}, \{s\})^F$ and we can attach to each $\rho \in \mathcal{E}(G^{F^m}, \{s\})^F$ an almost character R_{x_ρ} of G^F up to a root of unity multiple through this map, ($x_\rho \in X(W_s, \gamma)$ and $\rho \mapsto x_\rho$ is a cross section of this map). Then $N^*_{F^m/F}$ maps $C^{(s)}(G^{F^m}/\sim_F)$ onto $C^{(s)}(G^F/\sim)$ and

$$N^*_{F^m/F}([\mu_{\tilde{\rho}}\tilde{\rho}]) = R^\vee_{x_\rho} \tag{0.1}$$

for each $\rho \in \mathcal{E}(G^{F^m}, \{s\})^F$, where $\tilde{\rho}$ is an extension of ρ to $\tilde{G}^{F^m}$ and $\mu_{\tilde{\rho}}$ is a root of unity depending on the choice of $\tilde{\rho}$ and m. Here $R^\vee_{x_\rho}$ is a modified almost character obtained from R_{x_ρ} (see 3.1 for definition). (See Theorem 3.2 for precise statement, which involves some ambiguity and is not exactly the same as (0.1).) We notice that the similar class function as $R^\vee_{x_\rho}$ already appears in Asai [4], where he describes the twisting operators on the space of unipotent class functions in the case of exceptional groups, and our situation is parallel to him.

As in the case of classical groups [16], our theorem implies the decomposition of twisted induction $R^G_{L\subset P}(\pi)$ (cf. Cor. 3.27).

Our basic tool for the proof is, as in the case of classical groups, a kind of identity, called "Shintani descent identity", which connects via Shintani descent $N^*_{F^m/F}$, the trace evaluation with Frobenius action of the twisted induction on G^F and the twisted trace of the usual Harish-Chandra induction on G^{F^m}.

However, the information from this kind of identities is not enough to determine $N^*_{F^m/F}$ in the case of exceptional groups (in particular the case $G=E_8$, $s=1$) and we need some additional information. For this, we introduce an operator on the space $C(G^{F^m}/\sim)$ which is described by an easy formula and turns out to be compatible with the twisting operator on the space $C(G^F/\sim)$ via the norm map if m is large enough. Hence this operator plays the same role as twisting operators on G^F and is regarded as the "lift" of it to $C(G^{F^m}/\sim)$, which we call the "twisting operator on G^{F^m}".

In determining the twisting operators in the case of exceptional groups, Asai [4] uses the fact that the twisting operator stabilizes all uniform functions (which is easy to see for a good prime p and was proved in [5] in general) as an additional information. By making use of our operator together with the lifting theory by Kawanaka [12], we can prove a formula which is a counterpart to our situation of the above fact. Then many parts of his argument in [4] are applicable formally to our case. One more advantage of this operator is that it enables us to involve his result itself, not only his method, into our scheme. In fact, his result is essentially used in treating the most complicated family in E_8.

The paper is organized as follows: In § 1, we review some of known results and formulate Shintani descent identities in a more general form than [16]. § 2 is concerned with the property of twisting operators on $C(G^{F^m}/\sim_F)$. § 3 is devoted to the proof of our main theorem. In § 4, we take up again the case of classical groups. In fact, the argument used to the proof of Theorem 2.2 in [16] contains some gap. In particular, Lemma 2.17 in [16] is not enough to treat all the possible classes $\{s\}$. In this section, we generalize Lemma 2.17 so as to be valid for a general case. Thus we get Theorem 4.2, which is a revised form of Theorem 2.2 in [16]. The statement is the same for the case of type B_n or C_n and slightly weaker for the case of type D_n. In particular, if $G=CO^{\pm,\circ}_{2n}$, the same statement holds (cf. Remark 4.12).

The above mentioned gap in [16] was pointed out to the author by G. Lusztig and B. Srinivasan. The idea of improvement in § 4, in particular the use of cuspidal representations of Levi subgroups instead of maximal torus, was inspired from the discussions with F. Digne, J. Michel and B. Srinivasan. I am very grateful to them.

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The main result of this paper was announced in [17].

§ 1. Shintani descent identities

1.1. Let G be a connected reductive algebraic group defined over a finite field $\boldsymbol{F}_q$ as in Introduction. Throughout the paper, we assume that G has a connected center. We may, and shall, assume that G has a split $\boldsymbol{F}_p$-structure with Frobenius map F_0 such that $FF_0=F_0F$ and that a power of F_0 coincides with a power of F. Let us fix an F_0-stable Borel subgroup B of G and an F_0-stable maximal torus T contained in B. We further assume that B and T are F-stable. Let G^* be the dual group of G defined over $\boldsymbol{F}_q$, and T^* be a dual torus of T contained in G^*. Let $W=N_G(T)/T$ be the Weyl group of G with respect to T and $W^*=N_{G^*}(T^*)/T^*$ be the Weyl group of G^* with respect to T^*. W and W^* may be identified as in [14, 8.4] so that the action of F on W corresponds to the action of F^{-1} on W^*. For an F-stable class $\{s\}$ in G^*, we shall choose a representative s of $\{s\}$ in T^*. Since the center of G is connected, $W_s=\{w\in W^*|w(s)=s\}$ is the Weyl group of $Z_{G^*}(s)$ with respect to T^*. Put

$$Z_s=\{w\in W^*|F(s)=w(s)\}.$$

Then there exists a unique $w_0\in Z_s$ such that $\gamma=\gamma_s=w_0^{-1}F: W_s\rightarrow W_s$ leaves invariant the set of simple roots of $Z_{G^*}(s)$ determined naturally from the simple root system of G with respect to T and B. Since G has the connected center, we have $Z_s=w_0W_s$. Let $\tilde{W}_s=W_s\langle\gamma\rangle$ be the semidirect product of W_s with the cyclic group $\langle\gamma\rangle$ with generator γ. We denote by $(W_s)^\wedge_{\rm ex}$ the set of (isomorphism classes of) irreducible representations of W_s which is extendable to a representation of $\tilde{W}_s$.

We recall here the definition of Deligne-Lusztig's virtual representations of G^F. For each $w\in W$, put

$$X_w=\{gB\in G/B\mid g^{-1}F(g)\in BwB\}.$$

According to Lusztig [14], there exists $\theta_w\in(T^{wF})^\wedge$ for each $w\in Z_s$ (i.e., θ_w is nothing but $\bar{\lambda}_1$ in (2.1.2) in [14]), and one can associate to it a locally constant $\bar{\boldsymbol{Q}}_l$-sheaf $\mathcal{F}_{\dot{w},\theta_w}$ of rank 1 on X_w, where $\dot{w}$ is a representative of w in $N_G(T)^{F_0}$. Let $\bar{X}_w$ be the Zariski closure of X_w in G/B. Then we have an l-adic cohomology group $H_c^i(X_w,\mathcal{F}_{\dot{w},\theta_w})$ and a hypercohomology group $\boldsymbol{H}^i(\bar{X}_w,\mathcal{F}_{\dot{w},\theta_w})$ associated with the intersection cohomology complex on $\bar{X}_w$ obtained from $\mathcal{F}_{\dot{w},\theta_w}$. Both cohomologies have natural structures of G^F-modules. According to Lusztig [14, (3.7.1)], we shall define for each $E\in(W_s)^\wedge_{\rm ex}$ and its extension $\tilde{E}$ to $\tilde{W}_s$,

$$R_{\tilde{E}}=|W_s|^{-1}\sum_{\substack{y\in W_s\\ i\geq 0}}(-1)^i\,\mathrm{Tr}\,(\gamma y,\tilde{E})H_c^i(X_{w_0y},\mathcal{F}_{\dot{w}_0\dot{y},\theta_{w_0y}}), \tag{1.1.1}$$

which is an element of the Grothendieck group $\mathscr{R}(G^F)\otimes \boldsymbol{Q}$ tensored by $\boldsymbol{Q}$.

1.2. We fix $s\in T^*$. Let d be the smallest integer such that $F_0^d(s)=s$. Then for each $w\in Z_s$, X_w is F_0^d-stable and $\mathscr{F}_{\dot{w},\theta_w}$ has a natural $\boldsymbol{F}_{p^d}$-structure, which depends on the choice of a representative $\dot{w}$ of w. Hence the cohomologies $H_c^i(X_w,\mathscr{F}_{\dot{w},\theta_w})$ and $\boldsymbol{H}^i(\bar{X}_w,\mathscr{F}_{\dot{w},\theta_w})$ have structures of (G^F,F_0^d)-modules.

We now consider the Frobenius map F^m for a positive integer m. In the following, in order to distinguish the F^m-situation, we shall denote by $X_w^{(m)}, Z_s^{(m)},\cdots,$ objects corresponding to $X_w, Z_s,\cdots,$ defined using F^m instead of F. Let r_0 be the smallest integer such that F^{r_0} is a power of F_0^d and let $b=b^{(m)}$ be the smallest integer such that F^{br_0} is a power of F^m. Let $G^{F^m}\langle\phi\rangle$ be the semidirect product of G^{F^m} with the cyclic group of order b with generator ϕ, where ϕ acts on G^{F^m} by $\phi g\phi^{-1}=F^{r_0}(g)$. Then, by [14, Prop. 2.20], the following results are known. Each representation ρ in $\mathcal{E}(G^{F^m},\{s\})$ is F^{r_0}-stable, and to each ρ, one can associate an extension $\tilde{\rho}$ to $G^{F^m}\langle\phi\rangle$ and a root of unity $\lambda'_{\tilde{\rho}}\in\bar{\boldsymbol{Q}}_l^*$ satisfying the following properties. Eigenvalues of $(F^*)^{br_0}$ on $\boldsymbol{H}^i(\bar{X}_w^{(m)},\mathscr{F}_{\dot{w},\theta_w}^{(m)})$ have absolute values $\lambda_\rho q^{ibr_0/2}$, where $\lambda_\rho=(\lambda'_{\tilde{\rho}})^b$ is independent of the choice of the extension $\tilde{\rho}$ of ρ. Moreover ρ-isotypic subspace of $\boldsymbol{H}^i(\bar{X}_w^{(m)},\mathscr{F}_{\dot{w},\theta_w}^{(m)})$ is F^{r_0}-stable and has a (G^{F^m},F^{r_0})-stable filtration each of whose successive quotients is isomorphic to $\tilde{\rho}$ as $G^{F^m}\langle\phi\rangle$-modules (with ϕ acting as $\lambda'_{\tilde{\rho}}q^{ir_0/2}(F^*)^{-r_0}$).

1.3. We defined in Introduction the norm map $N_{F^m/F}$. For later use, we consider here norm maps in a more general setting. Let F' and F'' be two Frobenius maps of G such that $F'F''=F''F'$. We shall define a norm map

$$N_{F''/F'}: G^{F'}/\sim_{F''^{-1}} \longrightarrow G^{F''}/\sim_{F'}$$

by associating to the F''^{-1}-twisted class of x in $G^{F'}$ the F'-twisted class of $\hat{x}$ in $G^{F''}$, where $x\in G^{F'}$, $\hat{x}\in G^{F''}$ are related by $x=F''(a)a^{-1}$, $\hat{x}=a^{-1}F'(a)$ for $a\in G$. (Recall that x and y are F'-twisted conjugate if there exists z such that $y=z^{-1}xF'(z)$. Note $G^{F''}/\sim_{F'}$ means the set of F'-twisted classes in $G^{F''}$, and similarly for $G^{F'}/\sim_{F''^{-1}}$.)

Let r_0 be as in 1.2 and put $a=kr_0$ for a positive integer k. Then $s\in(T^*)^{F^a}$ and we can find $\theta=\theta^{(a)}\in\hat{T}^{F^a}$ corresponding to s via [7, 5.2]. Let W_θ be the stabilizer of θ in W. Then W_θ is isomorphic to W_s under $W\cong W^*$. We denote by $(W_\theta)\hat{}_{\mathrm{ex}}$ the set corresponding to $(W_s^{(m)})\hat{}_{\mathrm{ex}}$ with respect to $\tilde{W}_s^{(m)}=W_s\langle\gamma^{(m)}\rangle$.

Let us consider the induced representation $\mathrm{Ind}_{B^{F^a}}^{G^{F^a}}(\theta)$. It is decomposed as a $G^{F^a}\times H(q^a)$-module,

$$\mathrm{Ind}_{B^{F^a}}^{G^{F^a}}(\theta)\cong\bigoplus_{E\in W_s^{\wedge}}\rho_E^{(a)}\otimes E(q^a)\,,$$

where $H(q^a)$ is the Hecke algebra of W_θ and $E(q^a)$ is the irreducible representation of $H(q^a)$ corresponding to E, and $\rho_E^{(a)}$ is an irreducible constituent corresponding to E. Then as in [14, 3.6], for each $E\in(W_\theta)^{\wedge}_{\mathrm{ex}}$, $\rho_E^{(a)}$ can be extended canonically to a $G^{F^a}\langle F^m\rangle$-module, which we denote by $\tilde{\rho}_E^{(a)}$. Here $G^{F^a}\langle F^m\rangle$ is the semidirect product of G^{F^a} with the cyclic group $\langle F^m\rangle$ with generator F^m.

The following result is a direct consequence of Lusztig [14, Th. 3.8].

1.4. PROPOSITION. *Let m and $a=kr_0$ be as before. Assume $x\in G^{F^a}$ and $y\in G^{F^m}$ are corresponding each other via the norm map $N_{F^m/F^a}: G^{F^a}/\sim_{F^{-m}}\to G^{F^m}/\sim_{F^a}$. Then for each $E\in(W_\theta)^{\wedge}_{\mathrm{ex}}$,*

$$\sum_\rho\langle\rho, R_{\tilde{E}}^{(m)}\rangle_{G^{F^m}}(\lambda'_{\tilde{\rho}})^k\,\mathrm{Tr}\,((y\phi^k)^{-1},\tilde{\rho})=\mathrm{Tr}\,(x^{-1}F^m,\tilde{\rho}_E^{(a)})\,,$$

where ρ runs over all the elements in $\mathcal{E}(G^{F^m},\{s\})$.

PROOF. By [14, Prop. 2.20], for each $w=w_0z\in Z_s^{(m)}$,

$$\sum_i(-1)^i\,\mathrm{Tr}\,((yF^a)^*,\boldsymbol{H}^i(\bar{X}_w^{(m)},\mathcal{F}_{w,\theta_w}^{(m)}))$$
$$=\sum_i(-1)^i\sum_{\rho\in\mathcal{E}(G^{F^m},\{s\})}\langle\rho, M_{\dot{w}}^i\rangle_{G^{F^m}}(\lambda'_{\tilde{\rho}})^k q^{ik/2}\,\mathrm{Tr}\,((y\phi^k)^{-1},\tilde{\rho})\,,\tag{1.4.1}$$

where $M_{\dot{w}}^i$ is the (G^{F^m},F^a)-module $\boldsymbol{H}^i(\bar{X}_w^{(m)},\mathcal{F}_{\dot{w},\theta_w}^{(m)})$. Using the arguments in the proof of Theorem 3.8, in particular (3.8.1)$\sim$(3.8.3), of [14], we see that the left hand side of (1.4.1) is equal to

$$\sum_{v\in W_s}q^{(l(w_0z)-\tilde{l}(z))a/2}P_{v,z}(q^a)\sum_{E\in(W_\theta)^{\wedge}_{\mathrm{ex}}}\mathrm{Tr}\,(x^{-1}F^m,\tilde{\rho}_E^{(a)})\,\mathrm{Tr}\,(T_{\gamma v},\tilde{E}(q^a))\,,\tag{1.4.2}$$

where $T_{\gamma v}$ ($\gamma=\gamma^{(m)}, v\in W_s$) is a canonical basis in the extended Hecke algebra $\tilde{H}(q^a)$ associated with the extended Coxeter group $\tilde{W}_s^{(m)}$, and $\tilde{E}(q^a)$ is an extension of $E(q^a)$ to $\tilde{H}(q^a)$-module. $P_{v,z}$ are the Kazhdan-Lusztig polynomials with respect to W_s, and $l(w)$ (resp. $\tilde{l}(z)$) is the length function of W (resp. W_s).

On the other hand, using (3.8.16) in [14], the right hand side of (1.4.1) turns out to be

$$\sum_{\rho\in\mathcal{E}(G^{F^m},\{s\})}q^{(l(w_0z)-\tilde{l}(z))a/2}\sum_{E\in(W_\theta)^{\wedge}_{\mathrm{ex}}}\langle\rho, R_{\tilde{E}}^{(m)}\rangle_{G^{F^m}}$$
$$\times(\lambda'_{\tilde{\rho}})^k\,\mathrm{Tr}\,(\sum_{v\in W_s}P_{v,z}(q^a)T_{\gamma v},\tilde{E}(q^a))\,.\tag{1.4.3}$$

Now the proposition follows from the orthogonality relations of $\mathrm{Tr}\,(T_{\gamma v}, \tilde{E}(q^a))$ and of $P_{v,z}(q^a)$.

Several results can be obtained from this proposition. However, before doing it, we note an easy lemma.

1.5. LEMMA. *Let F' and F'' be two Frobenius maps of G as before. Let $\iota: G^{F''}/\sim_{F'} \to G^{F''}/\sim_{F'^{-1}}$ be a bijection defined by $x \mapsto x^{-1}$ $(x \in G^{F''})$ and similarly for $G^{F'}/\sim_{F''}$. Then the following diagram is commutative.*

$$\begin{array}{ccc} G^{F'}/\sim_{F''^{-1}} & \xrightarrow{N_{F''/F'}} & G^{F''}/\sim_{F'} \\ \iota \uparrow & & \downarrow \iota \\ G^{F'}/\sim_{F''} & \xleftarrow{N_{F'/F''}} & G^{F''}/\sim_{F'^{-1}} \end{array}$$

1.6. As in Introduction, for each representation $\tilde{\rho}$ of $\tilde{G}^{F^m} = G^{F^m}\langle\sigma\rangle$, we denote by $[\tilde{\rho}]$ the restriction of the character to $G^{F^m}\sigma$ and regard it as an element of $C(G^{F^m}/\sim_F)$. However, if there is no fear of confusion, sometimes we omit [ ] and express it as $\tilde{\rho}$.

Let us take an integer m large enough so that m is a multiple of r_0 and that all the eigenvalues of F^m on $\boldsymbol{H}^i(\bar{X}_w^{(1)}, \mathcal{F}_{w,\theta_w}^{(1)})$ are of the form $q^{im/2}$. Let $\theta = \theta^{(m)} \in (T^{F^m})^\wedge$ be the character corresponding to $s \in T^{F^m}$, and let $\rho_E^{(m)}$ be the irreducible constituent of $\mathrm{Ind}_{B^{F^m}}^{G^{F^m}}(\theta)$ corresponding to E, $\tilde{\rho}_E^{(m)}$ be its extension to $\tilde{G}^{F^m} = G^{F^m}\langle\sigma\rangle$ as in 1.3 with $a = m$. Then

1.7. COROLLARY. *Let m be sufficiently large as in* 1.6. *Then*

$$N^*_{F^m/F}(\tilde{\rho}_E^{(m)}) = R_E^{(1)}.$$

PROOF. Let $x \mapsto \hat{x}$ be in correspondence each other via the norm map $N_{F^m/F}: G^F/\sim \to G^{F^m}/\sim_F$ $(x \in G^F, \hat{x} \in G^{F^m})$. We shall apply Proposition 1.4 with $m=1$, $a=m$. Then taking Lemma 1.5 into account, we have

$$\sum_{\rho} \langle \rho, R_E^{(1)} \rangle_{G^F} \mathrm{Tr}\,(x, \rho) = \mathrm{Tr}\,(\hat{x}\sigma, \tilde{\rho}_E^{(m)}),$$

where ρ runs over all the elements in $\mathcal{E}(G^F, \{s\})$. This implies the corollary.

1.8. REMARK. This corollary, together with [14, Cor. 4.24], was used to show (2.9.1) in [16] although it was not explicit there.

1.9. We now consider the case where F is a power of F_0^d. Hence $s \in T^{*F}$ and $\gamma^{(m)}$ is trivial for any m. Moreover, all the elements in $\mathcal{E}(G^{F^m}, \{s\})$

are F-stable for any m. We fix an extension $\tilde{\rho}$ to $\tilde{G}^{F^m}$ for each ρ in $\mathcal{E}(G^{F^m},\{s\})$. According to 1.2, one can attach to each $\tilde{\rho}$, a root of unity $\lambda'_{\tilde{\rho}}$ such that $(\lambda'_{\tilde{\rho}})^m=\lambda_\rho$. Let us define a linear map

$$\tilde{\natural}: C^{(s)}(G^{F^m}/\sim_F) \longrightarrow C^{(s)}(G^{F^m}/\sim_F)$$

by $\tilde{\natural}[\tilde{\rho}]=\lambda'_{\tilde{\rho}}[\tilde{\rho}]$, and by extending linearly to the whole space. Put

$$\tilde{R}_E^{(m)}=\sum_\rho \langle \rho, R_E^{(m)}\rangle_{G^{F^m}}\tilde{\rho} \qquad (\in \mathcal{R}(\tilde{G}^{F^m})\otimes \boldsymbol{Q}), \tag{1.9.1}$$

where ρ runs over all the elements in $\mathcal{E}(G^{F^m},\{s\})$. Note the construction in 1.2 shows that $(\lambda'_{\tilde{\rho}})^{-1}[\tilde{\rho}]$ is independent of the choice of extensions $\tilde{\rho}$ of ρ. Hence $\tilde{\natural}^{-1}(\tilde{R}_E^{(m)})\in C^{(s)}(G^{F^m}/\sim_F)$ is independent of the choice of extensions $\tilde{\rho}$ for each ρ. Now we have the following.

1.10. COROLLARY. *Assume F is a power of F_0^d. Let m be an arbitrary positive integer. Then for each $E\in W_s^\wedge$,*

$$N^*_{F^m/F}\circ\tilde{\natural}^{-1}(\tilde{R}_E^{(m)})=\rho_E^{(1)}.$$

PROOF. Let $x\mapsto\hat{x}$ be in correspondence each other via the norm map $N_{F^m/F}$. Applying Proposition 1.4 (with $a=1$), we have

$$\sum_\rho \langle\rho, R_E^{(m)}\rangle_{G^{F^m}}\lambda'_{\tilde{\rho}}\operatorname{Tr}((\hat{x}\sigma)^{-1},\tilde{\rho})=\operatorname{Tr}(x^{-1},\rho_E^{(1)}),$$

where ρ runs over all the elements in $\mathcal{E}(G^{F^m},\{s\})$.

Let ρ^* be the dual representation of ρ. Then ρ belongs to $\mathcal{E}(G^{F^m},\{s\})$ if and only if ρ^* belongs to $\mathcal{E}(G^{F^m},\{s^{-1}\})$. If θ is the character of T^F corresponding to $s\in T^{*F}$, θ^{-1} corresponds to s^{-1}, and $\operatorname{Ind}_{B^F}^{G^F}(\theta^{-1})$ is the dual space to $\operatorname{Ind}_{B^F}^{G^F}(\theta)$. Both irreducible constituents are parametrized by $W_s^\wedge\cong W_{s^{-1}}^\wedge$ and $\rho_E^*=\rho_{E^*}$, where $E^*\in W_{s^{-1}}^\wedge$ is the character corresponding to $E\in W_s^\wedge$. Now, using the Poincaré duality pairing,

$$\boldsymbol{H}^i(\bar{X}_w^{(m)},\mathcal{F}_{\dot{w},\theta_w}^{(m)})\times\boldsymbol{H}^{2l(w)-i}(\bar{X}_w^{(m)},\mathcal{F}_{\dot{w},\theta_w^{-1}}^{(m)})\longrightarrow\bar{\boldsymbol{Q}}_l(-l(w)),$$

which is (G^{F^m},F)-equivariant, we see that the filtration of $\boldsymbol{H}^i(\bar{X}_w^{(m)},\mathcal{F}_{\dot{w},\theta_w}^{(m)})$ given in [14, Prop. 2.20] is compatible with that of the dual space $\boldsymbol{H}^{2l(w)-i}(\bar{X}_w^{(m)},\mathcal{F}_{\dot{w},\theta_w^{-1}}^{(m)})$ and that $\lambda_{\rho^*}=\lambda_\rho^{-1}$. Hence, if we choose an extension $\tilde{\rho}^*$ of ρ^* as the dual representation of $\tilde{\rho}$, we have $\lambda'_{\tilde{\rho}^*}=(\lambda'_{\tilde{\rho}})^{-1}$.

Moreover Theorem 3.8 and Lemma 6.11 in [14] combined with the Poincaré duality as above implies that $\langle\rho, R_E^{(m)}\rangle_{G^{F^m}}=\langle\rho^*, R_{E^*}^{(m)}\rangle_{G^{F^m}}$. Now the corollary follows easily from these facts.

1.11. We shall review here about the twisted induction. Let Σ be the set of roots of G with respect to T, Π be the set of simple roots with respect to B, and Σ^+ be the set of positive roots. For a subset J of Π, let $P_J=L_JU_J$ be the parabolic subgroup of G of type J containing B, where L_J is the Levi subgroup of P_J containing T and U_J is the unipotent radical of P_J. We denote by W_J the Weyl subgroup of W corresponding to L_J, which is generated by simple reflections belonging to J. Put $L=L_J$. For a $w\in W$ such that $Fw(J)=J$, choose a representative $\dot{w}$ in $N_G(T)^{F_0}$. Then $F\dot{w}: g\mapsto F(\dot{w}g\dot{w}^{-1})$ may be regarded as a Frobenius map of L with respect to some $\boldsymbol{F}_q$-structure commuting with F_0. Fix $w\in W$ as above and consider the variety

$$S=\{g\in G\,|\,g^{-1}F(g)\in F(\dot{w}U_J)\}/(U_J\cap F(\dot{w}U_J\dot{w}^{-1}))\,. \tag{1.11.1}$$

Then $G^F\times L^{F\dot{w}}$ acts on S naturally and we have an induced action of $G^F\times L^{F\dot{w}}$ on $H_c^i(S,\bar{\boldsymbol{Q}}_l)$. Following Lusztig [13], we associate a virtual G^F-module $R^G_{L(\dot{w})}(\pi)$ to an irreducible $L^{F\dot{w}}$-module π by

$$R^G_{L(\dot{w})}(\pi)=\sum_{i\geq 0}(-1)^i(H_c^i(S,\bar{\boldsymbol{Q}}_l)\otimes\pi)^{L^{F\dot{w}}}\,.$$

Hence, extending linearly, we get a homomorphism

$$R^G_{L(\dot{w})}:\mathcal{R}(L^{F\dot{w}})\longrightarrow\mathcal{R}(G^F)\,.$$

We denote also by $R^G_{L(\dot{w})}$ the corresponding linear map from $C(L^{F\dot{w}}/\sim)$ to $C(G^F/\sim)$. If we take an $F\dot{w}$-stable class $\{s\}$ in L^*, the class $\{s\}$ in G^* is F-stable and it is known that $R^G_{L(\dot{w})}$ induces a map from $C^{(s)}(L^{F\dot{w}}/\sim)$ to $C^{(s)}(G^F/\sim)$.

We now consider the twisted version of $R^G_{L(\dot{w})}$ by a Frobenius map as in [16, 1.8]. However, for later use in § 4, we consider here in a more general situation. Let F' and F'' be two commuting Frobenius maps as before and assume that F' (resp. F'') stabilizes T and B, respectively. Assume given $w\in W^{F''}$ and $J\subset\Pi$ such that $F'w(J)=J$ and that $F''(J)=J$. Take a representative $\dot{w}\in N_G(T)^{F''}$ of w. Then $F'\dot{w}$ may be regarded as a Frobenius map on $L=L_J$ commuting with another Frobenius map F'' on L. Let us consider the variety S as in (1.11.1) with respect to F' and $\dot{w}$. Now F'' acts naturally on S and induces an action $(F'')^*$ on $H_c^i(S)=H_c^i(S,\bar{\boldsymbol{Q}}_l)$. Let π be an irreducible representation of $L^{F'\dot{w}}$ stable by F''. Let ϕ'' be the restriction of F'' on $L^{F'\dot{w}}$ and consider an extension $\tilde{\pi}$ to $L^{F'\dot{w}}\langle\phi''\rangle$ of π. Then an endomorphism $(\tilde{F}'')^*=(F'')^*\otimes\phi''^{-1}$ on $H_c^i(S)\otimes\tilde{\pi}$ stabilizes the subspace $(H_c^i(S)\otimes\tilde{\pi})^{L^{F'\dot{w}}}$. Put for each $x\in G^{F'}$,

$$R^{(F'')}_{L(\dot{w})}(\tilde{\pi})(x)=\sum_{i\geq 0}(-1)^i\,\mathrm{Tr}\,((\tilde{F}'')^*(x^{-1})^*,H_c^i(S)\otimes\tilde{\pi})^{L^{F'\dot{w}}}\,. \tag{1.11.2}$$

This definition depends only on the restriction of $\tilde{\pi}$ to $L^{F'\dot{w}}\phi''^{-1}$ and the value depends only on F''^{-1}-twisted class of x. So, extending linearly, we can define a linear map

$$R_{L(\dot{w})}^{(F'')} : C(L^{F'\dot{w}}/\sim_{F''^{-1}}) \longrightarrow C(G^{F'}/\sim_{F''^{-1}}),$$

which also preserves the spaces corresponding to the class $\{s\}$ in L^* which is $F'\dot{w}$ and F''-stable.

1.12. We now define a linear map $a_{F'w}$ analogous to [16, 1.7]. Let π be an irreducible representation of $L^{F''}$ which is $F'\dot{w}$-stable. Let $\sigma'\dot{w}$ be the restriction of $F'\dot{w}$ on $L^{F''}$, and $\tilde{\pi}$ be an extension of π to the semidirect product $L^{F''}\langle\sigma'\dot{w}\rangle$. We denote by V the representation space of $\tilde{\pi}$. We lift π to a representation of $P^{F''}$ in a natural way. Let $\mathcal{P}$ be the space of all functions $f: G^{F''}\to V$ with $G^{F''}$-module structure by $(gf)(x)=f(xg)$, $g, x\in G^{F''}$, $f\in\mathcal{P}$. Let $\mathcal{P}_\pi$ be the subspace of $\mathcal{P}$ defined by

$$\mathcal{P}_\pi=\{f\in\mathcal{P}\mid f(pg)=\pi(p)f(g) \text{ for } p\in P^{F''},\ g\in G^{F''}\}.$$

Thus $\mathcal{P}_\pi$ is a $G^{F''}$-submodule of $\mathcal{P}$ realizing $\mathrm{Ind}_{P^{F''}}^{G^{F''}}(\pi)$. Since $F'w(J)=J$, we have $w(J)\subset\Pi$. Thus we can define a linear map $\tau_{\pi,\dot{w}}: \mathcal{P}\to\mathcal{P}$ by

$$\tau_{\pi,\dot{w}}(f)(x)=\frac{1}{|U_{wJ}^{F''}|}\sum_{y\in U_{wJ}^{F''}} f(\dot{w}^{-1}yx).$$

Define $F': \mathcal{P}\to\mathcal{P}$ by $F'(f)(x)=f(F'^{-1}(x))$ and $\tilde{\pi}(\sigma'\dot{w}): \mathcal{P}\to\mathcal{P}$ by $\tilde{\pi}(\sigma'\dot{w})(f)(x)=\tilde{\pi}(\sigma'\dot{w})(f(x))$, $(f\in\mathcal{P},\ x\in G^{F''})$. Then as in [16, 1.7], it is checked that a linear map $\tilde{\pi}(\sigma'\dot{w})F'\tau_{\pi,\dot{w}}: \mathcal{P}\to\mathcal{P}$ leaves $\mathcal{P}_\pi$ invariant. Now let us define a map $a_{F'w}: C(L^{F''}/\sim_{F'\dot{w}})\to C(G^{F''}/\sim_{F'})$ by

$$a_{F'w}([\tilde{\pi}])(\hat{x})=\mathrm{Tr}\,(\hat{x}\tilde{\pi}(\sigma'\dot{w})F'\tau_{\pi,\dot{w}}, \mathcal{P}_\pi) \qquad (\hat{x}\in G^{F''})$$

for each $[\tilde{\pi}]\in C(L^{F''}/\sim_{F'\dot{w}})$ and extending it linearly to the whole space. Note $a_{F'w}$ does not depend on the choice of a representative $\dot{w}$ of w. As in [16, Prop. 1.9], we have the following commutative diagram, which is called a Shintani descent identity. The proof is similar to [3], [8] and [14]. Note that a scalar multiplication $q^{md'}$ in (1.8.1) in [16] is unnecessary.

1.13. PROPOSITION (Shintani descent identity). *Let $F'w(J)=J$ and $F''(J)=J$. Then the following diagram is commutative.*

(1.13.1)
$$\begin{array}{ccc} C(G^{F''}/\sim_{F'}) & \xrightarrow{N^*_{F''/F'}} & C(G^{F'}/\sim_{F''^{-1}}) \\ a_{F'w}\uparrow & & \uparrow R_{L(\dot{w})}^{(F'')} \\ C(L^{F''}/\sim_{F'\dot{w}}) & \xrightarrow{N^*_{F''/F'\dot{w}}} & C(L^{F'\dot{w}}/\sim_{F''^{-1}}) \end{array}$$

Moreover, for a class $\{s\}$ in L^* which is stable by both of F'' and $F'\dot{w}$, $R_{L(\dot{w})}^{(F'')}$ (resp. $a_{F'w}$) maps $C^{(s)}(L^{F'\dot{w}}/\sim_{F''^{-1}})$ onto $C^{(s)}(G^{F'}/\sim_{F''^{-1}})$ (resp. $C^{(s)}(L^{F''}/\sim_{F'\dot{w}})$ onto $C^{(s)}(G^{F''}/\sim_{F'})$), respectively.

1.14. Returning to our previous setting, we consider the case where $F'=F$, $F''=F^m$ for a sufficiently large m. Hence Proposition 1.13 is reduced to Proposition 1.9 in [16]. We review here a more precise description of $a_{F'\dot{w}}$ in terms of Hecke algebras. Let L_K be the Levi subgroup corresponding to $K\subset\Pi$ and assume that there exists an irreducible cuspidal representation $\delta=\delta^{(m)}$ of $L_K^{F^m}$. Put

$$W_\delta=\{w\in W\mid wK=K, {}^w\delta\cong\delta\}.$$

By [14, 8.5], W_δ turns out to be a Coxeter group with the set of generators $S_\delta\subset W_\delta$. Let M be the F^m-stable subgroup of $N_G(L_K)$ generated by L_K and $w\in W_\delta$. Then δ can be extended to a representation of M^{F^m} by means of [10, 6.4], which we denote by $\tilde{\delta}$.

We also define for each $y\in W_\delta$, $T_y:\mathcal{P}_\delta\to\mathcal{P}_\delta$ by

$$T_y=\varepsilon_y^{(m)}(q_y)^{m/2}q^{l(y)m/2}\tilde{\delta}(\dot{y})\tau_{\delta,\dot{y}},$$

where $y\to\varepsilon_y^{(m)}=\pm1$ is a linear character of W_δ and $q_y=\Pi_s q^{\lambda(s)}$, s runs through the elements in a reduced expression of y in W_δ, and $\lambda:S_\delta\to\boldsymbol{Z}^+$ is a function which takes constant value under W_δ-conjugate. T_y is independent of the choice of a representative $\dot{y}$ of y. Then T_y ($y\in W_\delta$) gives rise to a basis of the Hecke algebra $H(q^m)\cong\mathrm{End}_{G^{F^m}}\mathrm{Ind}_{P_K^{F^m}}^{G^{F^m}}(\delta)$ over $\bar{\boldsymbol{Q}}_l$ with relations

$$\begin{cases} T_wT_{w'}=T_{ww'} & \text{if} \quad \tilde{l}(ww')=\tilde{l}(w)+\tilde{l}(w') \\ (T_s+1)(T_s-q^{m\lambda(s)})=0 & \text{for} \quad s\in S_\delta \end{cases}$$

where $\tilde{l}$ is the length function on W_δ with respect to S_δ. We shall define a set

$$Z_\delta=\{w\in W\mid Fw(K)=K, {}^{Fw}\delta\cong\delta\}.$$

Then W_δ acts on Z_δ from the right and there exists a unique $w_1\in W$ such that $Z_\delta=w_1W_\delta$ and that $\gamma_\delta=Fw_1: W_\delta\to W_\delta$ sends the set of positive roots with respect to S_δ determined naturally from Σ^+, to Σ^+ ([16, 1.10]).

$F\dot{w}_1$ can be regarded as a Frobenius map on M. We denote by $\sigma\dot{w}_1$ the restriction of $F\dot{w}_1$ on M^{F^m}. Then $\tilde{\delta}$ can be extended to a representation of $M^{F^m}\langle\sigma\dot{w}_1\rangle$, which we denote also by $\tilde{\delta}$.

More generally, we consider the Levi subgroup L_J ($J\subset\Pi$) and take $w\in W$ such that $Fw(J)=J$. Let us take m large enough so that $(F\dot{w})^m=$

F^m. Let $\pi=\pi^{(m)}$ be an irreducible representation of $L_J^{F^m}$ such that ${}^{F\dot{w}}\pi\cong\pi$. Then there exists a Levi subgroup L_K ($K\subset J\subset\Pi$) and an irreducible cuspidal representation δ of $L_K^{F^m}$ such that π is isomorphic to $\pi_{E'}$, where E' is an irreducible representation of $W'_\delta=W_\delta\cap W_J$, and $\pi_{E'}$ is an irreducible constituent corresponding to E' of the representation obtained by inducing up δ from $(P_K\cap L_J)^{F^m}$ to $L_J^{F^m}$. Since π is $F\dot{w}$-stable, there exists $w'\in W_J$ such that ${}^{F\dot{w}\dot{w}'}\delta\cong\delta$. Hence, $ww'\in Z_\delta$ and we can express ww' as $ww'=w_1y'$ for some $y'\in W_\delta$. If we define Z'_δ similar to Z_δ by replacing W by W_J, F by $F\dot{w}$, we see that $w'\in Z'_\delta$. Hence $w'=W'_1y''$, where $y''\in W'_\delta$ and w'_1 is the similar element as w_1 for Z'_δ. (We put $\gamma'_\delta=Fww'_1: W'_\delta\to W'_\delta$ the similar automorphism as γ_δ for W_δ.) Thus w is written as $w=w_1yw_1'^{-1}$ for some $y\in W_\delta$. Note $\gamma_\delta y: W_\delta\to W_\delta$ leaves W'_δ invariant and the restriction of $\gamma_\delta y$ on W'_δ coincides with γ'_δ. Let $\tilde{W}_\delta$ (resp. $\tilde{W}'_\delta$) be the semidirect product of W_δ with $\langle\gamma_\delta\rangle$ (resp. W'_δ with $\langle\gamma'_\delta\rangle$). The Hecke algebra $H(q^m)$ can be extended to an algebra $\tilde{H}(q^m)$ with basis T_w ($w\in\tilde{W}_\delta$) as in [14, 3.3]. Let $H'(q^m)$ be the subalgebra of $H(q^m)$ generated by T_z ($z\in W'_\delta$) and by $\tilde{H}'(q^m)$ the extended algebra corresponding to $\tilde{W}'_\delta$. As $E'\in(W'_\delta)^\wedge$ is γ'_δ-stable, we can find an extension $\tilde{E}'$ to $\tilde{W}'_\delta$-module. We denote by $\tilde{E}'(q^m)$ the corresponding $\tilde{H}'(q^m)$-module.

Let $(W_\delta)^\wedge_{\mathrm{ex}}$ be the set of isomorphism classes of irreducible W_δ-modules over $\boldsymbol{Q}$ which is extendable to a $\tilde{W}_\delta$-module over $\boldsymbol{Q}$. Take $E\in(W_\delta)^\wedge_{\mathrm{ex}}$ and let $\tilde{E}$ be an extension to a $\tilde{W}_\delta$-module over $\boldsymbol{Q}$. We denote by $E(q^m)$ the irreducible $H(q^m)$-module corresponding to E. Then corresponding to $\tilde{E}$, $E(q^m)$ can be extended to an $\tilde{H}(q^m)$-module, which we denote by $\tilde{E}(q^m)$. For these $\tilde{E}(q^m)$ and $\tilde{E}'(q^m)$, put

$$V^{(m)}_{\tilde{E}',\tilde{E}}=\mathrm{Hom}_{H'(q^m)}(\tilde{E}'(q^m),\tilde{E}(q^m))\,.$$

We shall define an endomorphism $\gamma^{(m)}_w: V^{(m)}_{\tilde{E}',\tilde{E}}\to V^{(m)}_{\tilde{E}',\tilde{E}}$ by $\gamma^{(m)}_w(f)=T_{\gamma y}\circ f\circ T_{\gamma'}^{-1}$, where $f\in V^{(m)}_{\tilde{E}',\tilde{E}}$, and $\gamma=\gamma_\delta$, $\gamma'=\gamma'_\delta$.

Let $\rho^{(m)}_E$ be an irreducible constituent of $\mathrm{Ind}_{P_K^{F^m}}^{G^{F^m}}(\delta)$ corresponding to $E\in(W_\delta)^\wedge_{\mathrm{ex}}$. Then by the same argument as in [14, 3.6], one can find an extension $\tilde{\rho}^{(m)}_E$ of $\rho^{(m)}_E$ to $\tilde{G}^{F^m}$, which is determined uniquely by the choice of an extension $\tilde{\delta}$ of δ and by the choice of $\tilde{E}$ of E.

Assume m is large enough so that

(1.14.1) $\qquad F^m$ is a power of F_0 and $\dot{w}_1F(\dot{w}_1)F^2(\dot{w}_1)\cdots F^{m-1}(\dot{w}_1)=1$.

Now the following Lemma is just the reformulation of Lemma 1.15 in [16].

1.15. LEMMA. *Let m be as in* (1.14.1). *For each Fw-stable irreducible representation $\pi^{(m)}_{E'}$ of $L_J^{F^m}$, let $w=w_1yw_1'^{-1}$ and $\gamma^{(m)}_w$ be as in* 1.13. *Then*

$$a_{Fw}([\tilde{\pi}_{E'}^{(m)}])=\varepsilon_y^{(m)}(q_y)^{-m/2}q^{-(l(w_1)+l(y)-l(w_1'))m/2}\sum_{E\in(W\delta)\hat{}_{\mathrm{ex}}}\mathrm{Tr}\,(\gamma_w^{(m)}, V_{\tilde{E}',\tilde{E}}^{(m)})[\tilde{\rho}_E^{(m)}],$$

where $y\mapsto\varepsilon_y^{(m)}$ *is a certain character of* W_δ.

§ 2. Twisting operators

As in Introduction, we shall define a "lift" of twisting operators to the space $C(G^{F^m}/\sim_F)$ and prove some of their properties. This provides us a more precise information about the map $N^*_{F^m/F}$, which seems to be difficult to obtain directly from Shintani descent identities as in (1.13.1).

2.1. We first recall the definition of the twisting operators due to Kawanaka [12]. For a positive integer r, a map $t_r=(t_1)^r: G^F/\sim\to G^F/\sim$ is defined by attaching to the conjugacy class of x in G^F the class of bxb^{-1} in G^F, where $b\in G$ is given by $x^r=b^{-1}F(b)$. t_r is a bijection on the set $G^F/\sim$ and the induced map $t_r^*: C(G^F/\sim)\to C(G^F/\sim)$ is called the twisting operator. (In fact, this definition agrees with the definition given in Asai [4, 3.2] only in the case F is of split type. However, in practice, we use this later in the split case only, and so, we adopt this definition here.) Let us fix an integer $m>0$ and consider $G^{F^m}/\sim_F$. We now define, for each integer $r>0$, a map $\tau_r=\tau_r^{(m)}: G^{F^m}/\sim_F\to G^{F^m}/\sim_F$ as follows: For $y\in G^{F^m}$, take $a\in G$ such that $y=a^{-1}F(a)$ and put $y'=F^{mr}(a^{-1})F(a)$. Then the assignment $y\mapsto y'$ induces a well-defined map τ_r on $G^{F^m}/\sim_F$. In fact, y' is independent of the choice of a, and we can check $y'\in G^{F^m}$ as follows. Put $x=F^m(a)a^{-1}$. Since $y\in G^{F^m}$, $x=N^{-1}_{F^m/F}(y)\in G^F$ and

$$(x^{-1})^r=x^{-1}F^m(x^{-1})\cdots F^{m(r-1)}(x^{-1})=aF^{mr}(a^{-1})\,.$$

Thus

$$y'=F^{mr}(a^{-1})F(a)=a^{-1}x^{-r}F(a)\,. \tag{2.1.1}$$

Hence

$$\begin{aligned}F^m(y')&=F^m(a^{-1})x^{-r}F^{m+1}(a)\\&=a^{-1}x^{-1}x^{\ r}xF(a)\qquad(\text{since } F^m(a)=xa,\ x\in G^F)\\&=y'\,.\end{aligned}$$

Clearly this correspondence is compatible with F-twisted conjugation.

Let

$$\tau_r^*: C(G^{F^m}/\sim_F)\longrightarrow C(G^{F^m}/\sim_F)$$

be the induced map. We shall call τ_r^* the twisting operator on $C(G^{F^m}/\sim_F)$. We note that τ_r is not necessarily bijective in general. However as the next lemma shows, τ_r turns out to be bijective for a sufficiently large m.

2.2. LEMMA. *Assume m is large enough so that, for each x in G^F, there exists $b \in G^{F^m}$ such that $x = F(b)b^{-1}$. Then the following diagram is commutative.*

$$\begin{tikzcd} G^{F^m}/\sim_F \arrow[r, "\tau_r"] & G^{F^m}/\sim_F \\ G^F/\sim \arrow[u, "N_{F^m/F}"] & G^F/\sim \arrow[l, "t_r"'] \arrow[u, "N_{F^m/F}"'] \end{tikzcd}$$

In particular, τ_r is a bijection.

PROOF. In the following, we shall often use a conventional expression such as $N_{F^m/F}(x), t_r(x), \cdots, (x \in G^F)$ to indicate the value at the class containing x under the map $N_{F^m/F}, t_r, \cdots$, by abuse of notation.

For $x \in G^F$, choose $b \in G^{F^m}$ such that $F(b)b^{-1} = x^r$. We can write also $x = F^m(a)a^{-1}$ for $a \in G$.
Put

$$\hat{x} = a^{-1}F(a) = N_{F^m/F}(x) \in G^{F^m}.$$

Then

$$\begin{aligned} t_r^{-1}(x) &= b^{-1}xb \\ &= F^m(b^{-1}a)(b^{-1}a)^{-1} \qquad (\text{since } b \in G^{F^m}). \end{aligned}$$

Hence

$$\begin{aligned} N_{F^m/F} \circ t_r^{-1}(x) &= (b^{-1}a)^{-1}F(b^{-1}a) \\ &= a^{-1}x^{-r}F(a) \\ &= F^{mr}(a^{-1})F(a) \qquad (\text{by } (2.1.1)) \\ &= \tau_r(\hat{x}) \\ &= \tau_r \circ N_{F^m/F}(x). \end{aligned}$$

This proves the lemma.

The following lemma gives an alternative description of τ_r and is used to apply the lifting theory.

2.3. LEMMA. *Assume m is large enough so that m is a multiple of $|G^F|$.*

(i) *There exists a natural bijection $\beta : G^{F^{mr-1}}/\sim_F \to G^{F^{mr-1}}/\sim_{F^m}$ such that for each F-twisted class C of $G^{F^{mr-1}}$, $C \cap G^F = \beta(C) \cap G^F \neq \emptyset$.*

(ii) *The following diagram is commutative.*

$$\begin{array}{ccccccc}
G^{F^m}/\sim_F & & \xrightarrow{\qquad\qquad\tau_r\qquad\qquad} & & G^{F^m}/\sim_F & = & G^{F^m}/\sim_{F^{-(mr-1)}} \\
\uparrow N_{F^m/F} & & & & & & \downarrow N_{F^{mr-1}/F^m} \\
G^F/\sim & \xleftarrow{t_1} & G^F/\sim & \xrightarrow{N_{F^{mr-1}/F}} & G^{F^{mr-1}}/\sim_F & \xrightarrow{\beta} & G^{F^{mr-1}}/\sim_{F^m}.
\end{array}$$

In particular, τ_r is a bijection.

PROOF. Let σ' be the restriction of F on $G^{F^{mr-1}}$. Since m is a multiple of $|G^F|$, the order of σ' is prime to $|G^F|$. Hence by a lemma of Glauberman [9] (see also Kawanaka [12, II, Lemma 2.1.6]), for each F-twisted class C in $G^{F^{mr-1}}$, $C\cap G^F$ is non-empty and consists of a single conjugacy class in G^F. Moreover $C\mapsto C\cap G^F$ gives a bijection $\beta_1: G^{F^{mr-1}}/\sim_F \to G^F/\sim$.

On the other hand, we note that the restriction of F^m on $G^{F^{mr-1}}$ is equal to $(\sigma')^m$ and the fixed point subgroup of $G^{F^{mr-1}}$ by $(\sigma')^m$ coincides with G^F. Since the order of $(\sigma')^m$ is prime to $|G^F|$, again we can apply [9] and we get a bijection $\beta_2: G^{F^{mr-1}}/\sim_{F^m}\to G^F/\sim$ satisfying the similar properties. Then the bijection $\beta=\beta_2^{-1}\beta_1: G^{F^{mr-1}}/\sim_F\to G^{F^{mr-1}}/\sim_{F^m}$ satisfies the assertion of (i).

Next we show (ii). Take $x\in G^F$ and put $x=F^m(a)a^{-1}$ for some $a\in G$. Put

$$\hat{x}=a^{-1}F(a)=N_{F^m/F}(x)\in G^{F^m}.$$

Then by definition, $\tau_r(\hat{x})=F^{mr}(a^{-1})F(a)$. If we put $b=F(a)$, then $\tau_r(\hat{x})=F^{mr-1}(b^{-1})b$. Hence

$$\begin{aligned}
N_{F^{mr-1}/F^m}\circ\tau_r(\hat{x}) &= bF^m(b^{-1}) \\
&= F(a)F^{m+1}(a^{-1}) \\
&= F(x^{-1}) \quad (\text{since } x=F^m(a)a^{-1}) \\
&= x^{-1} \qquad (\text{since } x\in G^F).
\end{aligned}$$

Since $x^{-1}\in G^F$, the F^m-twisted class containing x^{-1} in $G^{F^{mr-1}}$ is mapped by β to the F-twisted class containing x^{-1} in $G^{F^{mr-1}}$. Thus $\beta(x^{-1})=x^{-1}$. Now put $x^{-1}=c^{-1}F(c)$ for $c\in G$. Then

$$\begin{aligned}
N^{-1}_{F^{mr-1}/F}(x^{-1}) &= cx^{-1}F(x^{-1})\cdots F^{mr-2}(x^{-1})c^{-1} \\
&= cx^{-mr+1}c^{-1} \quad (\text{since } x^{-1}\in G^F) \\
&= cxc^{-1} \qquad (\text{since } x^m=1) \\
&= cF(c^{-1}).
\end{aligned}$$

Hence

$$t_1 \circ N^{-1}_{F^{mr-1}/F}(x^{-1}) = F(c^{-1})c = x\,.$$

This proves (ii).

2.4. REMARK. We assume that m satisfies both assumptions in Lemma 2.2 and Lemma 2.3, and consider the case $r=1$. Combining Lemma 2.2 and Lemma 2.3, we have the following commutative diagram.

$$\begin{tikzcd}
G^F/\sim \arrow[r, "N_{F^m/F}"] \arrow[d, "N_{F^{m-1}/F}"'] & G^{F^m}/\sim_F \arrow[r, equal] & G^{F^m}/\sim_{F^{-(m-1)}} \arrow[dl, "N_{F^{m-1}/F^m}"] \\
G^{F^{m-1}}/\sim_F \arrow[r, equal] & G^{F^{m-1}}/\sim_{F^m} &
\end{tikzcd}$$

(Compare this with Lemma 2.1.1 in Asai [5], where the similar formula is proved under a different assumption on m.)

2.5. Asai used Kawanaka's lifting theory [12] to obtain several properties concerning twisted operators. In our situation, the lifting theory again provides a powerful tool. However Glauberman's result [9] is already enough for our purpose. So we shall recall here his result. Let A be a finite group with an automorphism $\sigma: A \to A$. We assume that the order of σ is prime to $|A_\sigma|$, where A_σ is the fixed point subgroup of A by σ, (cf. Kawanaka [12, II, 3.1]). Let $A\langle\sigma\rangle$ be the semidirect product of A with the cyclic group generated by σ. The following result is a part of his more general theorem.

2.6. THEOREM (Glauberman [9], see also Isaacs [11, Th. 13.6]).

(i) *For each σ-twisted class C of A, $C \cap A_\sigma$ consists of a single conjugacy class. The map $C \mapsto C \cap A_\sigma$ gives a bijection $N_\sigma: A/\sim_\sigma \to A_\sigma/\sim$.*

(ii) *The lifting exists with respect to the map N_σ, i.e., the following holds: For each σ-stable irreducible representation ρ of A, let $\tilde{\rho}$ be an extension to $A\langle\sigma\rangle$. Then there exists an irreducible representation ρ_0 of A_σ such that*

$$\mathrm{Tr}\,(x\sigma, \tilde{\rho}) = \varepsilon\zeta_{\tilde{\rho}}\mathrm{Tr}\,(N_\sigma(\bar{x}), \rho_0)$$

for each $x \in A$, where $\bar{x}$ is the σ-twisted class in A containing x, $\varepsilon = \pm 1$ depends only on ρ, and $\zeta_{\tilde{\rho}}$ is an m-th root of unity depending on the extension $\tilde{\rho}$ of ρ. The correspondence $\rho \mapsto \rho_0$ gives a bijection $(\hat{A})_\sigma \simeq (A_\sigma)^\wedge$.

2.7. From now on until 2.11, we only consider the case where F is of split type and $s \in T^{*F}$. Hence r_0 in 1.2 is equal to 1 and γ acts trivially on W_s.

We assume m is large enough so that

$$(2.7.1)\qquad \begin{cases} \text{(i)} & \text{for each } x\in G^F, \text{ there exists } b\in G^{F^m} \text{ such that } x=F(b)b^{-1}, \\ \text{(ii)} & m \text{ is a multiple of } |G^F|, \\ \text{(iii)} & \lambda_\rho^m=1 \text{ for each } \rho\in\hat{G}^F. \end{cases}$$

Let $N_G: G^{F^{mr-1}}/\sim_{F^m}\to G^F/\sim$ be the map defined by $N_G=t_1\circ N_{F^{mr-1}/F}^{-1}\circ\beta^{-1}$, ($\beta$ as in Lemma 2.3). As was mentioned in the proof of Lemma 2.3, the fixed point subgroup of $G^{F^{mr-1}}$ by F^m coincides with G^F and the restriction of F^m on $G^{F^{mr-1}}$ has order prime to $|G^F|$ by our assumption (2.7.1). Now it is checked that $N_G=\iota\circ N_\sigma$, where N_σ is the map in Theorem 2.6 with σ the restriction of F^m to $G^{F^{mr-1}}$ and ι is as in Lemma 1.5. Thus, thanks to Theorem 2.6, the lifting exists in this case with respect to the map N_G.

Let $\theta_0\in\hat{T}^F$ and $\theta\in\hat{T}^{F^{mr-1}}$, $\theta: F$-stable, be the characters corresponding to $s\in T^*$ via [7, 5.2], respectively. Then $N^*_{F^{mr-1}/F}(\theta)=\theta_0$. Let $\mathcal{P}_{\theta_0}=\mathrm{Ind}_{B^F}^{G^F}(\theta_0)$ and let $\mathcal{P}_\theta=\mathrm{Ind}_{B^{F^{mr-1}}}^{G^{F^{mr-1}}}(\theta)$. F acts naturally on $\mathcal{P}_\theta$ and we can regard $\mathcal{P}_\theta$ as an $G^{F^{mr-1}}\langle(\sigma')^m\rangle$-module $\tilde{\mathcal{P}}_\theta$, $(\sigma'=F|_{G^{F^{mr-1}}})$. It is easy to check that

$$(2.7.2)\qquad N_G^*([\mathcal{P}_{\theta_0}])=[\tilde{\mathcal{P}}_\theta],$$

where $N_G^*: C(G^F/\sim)\to C(G^{F^{mr-1}}/\sim_{F^m})$ is the induced map of N_G. Since $W_{\theta_0}=W_\theta\cong W_s$, all the irreducible constituents of $\mathcal{P}_\theta$ are F^m-stable. Let $\tilde{\rho}$ be a canonical extension of ρ in $\mathcal{P}_\theta$ to $G^{F^{mr-1}}\langle(\sigma')^m\rangle$ as in [14, 3.6]. Then (2.7.2) implies that $\varepsilon=1$, $\zeta_{\tilde{\rho}}=1$ (the corresponding factors in Theorem 2.6 for N_G) for each ρ in $\mathcal{P}_\theta$ and that N_G^* gives a one-one correspondence between irreducible constituents of $\mathcal{P}_{\theta_0}$ and $\mathcal{P}_\theta$. Let $\rho_E^{(mr-1)}$ (resp. $\rho_E^{(1)}$) be the irreducible constituent of $\mathcal{P}_\theta$ (resp. $\mathcal{P}_{\theta_0}$) corresponding to $E\in W_\theta^\wedge\cong W_{\theta_0}^\wedge$, and $\tilde{\rho}_E^{(mr-1)}$ be the canonical extension of $\rho_E^{(mr-1)}$ to $G^{F^{mr-1}}\langle(\sigma')^m\rangle$ as before. We have the following proposition.

2.8. PROPOSITION. *Let m be as in* (2.7.1) *and we assume that F is of split type and $s\in T^{*F}$. Assume that*

$$(2.8.1)\qquad N_G^*(\rho_E^{(1)})=\tilde{\rho}_E^{(mr-1)}$$

for each $E\in W_s^\wedge$. Then for each $\tilde{R}_E^{(m)}$ in (1.9.1), *we have*

$$\tau_r^*\circ\tilde{\sharp}^{mr-1}(\tilde{R}_E^{(m)})=\tilde{\sharp}^{-1}(\tilde{R}_E^{(m)}).$$

PROOF. Take $y\in G^{F^m}$ and put $y_1=N_{F^{mr-1}/F^m}(y)$. By Lemma 1.5, $y^{-1}=N_{F^m/F^{mr-1}}(y_1^{-1})$. Now by applying Proposition 1.4 to the map $N_{F^m/F^{mr-1}}$

(with $a=mr-1$), we have

(2.8.2) $$\sum_{\rho\in\mathcal{E}(G^{F^m},\{s\})}\langle\rho, R_E^{(m)}\rangle_{G^{F^m}}(\lambda'_{\tilde\rho})^{mr-1}\,\mathrm{Tr}\,(y\sigma,\tilde\rho)=\mathrm{Tr}\,(y_1F^m,\tilde\rho_E^{(mr-1)})\,.$$

On the other hand, if we apply Proposition 1.4 to the map $N_{F^m/F}$,

(2.8.3) $$\sum_{\rho\in\mathcal{E}(G^{F^m},\{s\})}\langle\rho, R_E^{(m)}\rangle_{G^{F^m}}\lambda'_{\tilde\rho}\,\mathrm{Tr}\,((\hat{x}\sigma)^{-1},\tilde\rho)=\mathrm{Tr}\,(x^{-1},\rho_E^{(1)})\,,$$

where $x\in G^F$ and $\hat{x}=N_{F^m/F}(x)\in G^{F^m}$. Now replacing s by s^{-1} and passing to the dual representation as in the proof of Corollary 1.10, we have

(2.8.4) $$\sum_{\rho\in\mathcal{E}(G^{F^m},\{s\})}\langle\rho, R_E^{(m)}\rangle_{G^{F^m}}(\lambda'_{\tilde\rho})^{-1}\,\mathrm{Tr}\,(\hat{x}\sigma,\tilde\rho)=\mathrm{Tr}\,(x,\rho_E^{(1)})\,.$$

Put $y=\tau_r(\hat{x})$. By Lemma 2.3, $x=N_G(y_1)$. Then by (2.8.1),

(2.8.5) $$\mathrm{Tr}\,(x,\rho_E^{(1)})=\mathrm{Tr}\,(y_1F^m,\tilde\rho_E^{(mr-1)})\,.$$

Then, by (2.8.2), (2.8.4) and (2.8.5), we have

$$\tilde{\sharp}^{-1}(\tilde{R}_E^{(m)})(\hat{x}\sigma)=\tilde{\sharp}^{mr-1}(\tilde{R}_E^{(m)})(\tau_r(\hat{x}\sigma))\,.$$

This proves the proposition.

The following lemma is due to Asai [2].

2.9. LEMMA (Asai [2]). *Put $r=1$ and let $N_G=t_1\circ N_{F^{mr-1}/F}^{-1}$ be the map as before. We assume the same condition as in 2.8. Then for each $E\in W_s^\wedge$,*

$$t_1^*(\rho_E^{(1)})=\sharp^{-1}(R_E^{(1)})\,,$$

where $\sharp: C(G^F/\sim)\to C(G^F/\sim)$ is a linear map defined by $[\rho]\mapsto\lambda_\rho[\rho]$ for each $\rho\in\hat{G}^F$.

PROOF. Take $x\in G^F$ and put $\hat{x}=N_{F^{m-1}/F}(x)$. We shall apply Proposition 1.4 to the map $N_{F/F^{m-1}}$. Then, using Lemma 1.5, we have

$$\sum_{\rho\in\mathcal{E}(G^F,\{s\})}\langle\rho, R_E^{(1)}\rangle_{G^F}(\lambda_\rho)^{m-1}\,\mathrm{Tr}\,(x,\rho)=\mathrm{Tr}\,(\hat{x}F,\tilde\rho_E^{(m-1)})\,.$$

By our assumption, $(\lambda_\rho)^{m-1}=\lambda_\rho^{-1}$. Moreover, since $N_G=t_1\circ N_{F^{m-1}/F}^{-1}$, we have $N_G(\hat{x})=t_1(x)$. Thus condition (2.8.1) implies that

$$\mathrm{Tr}\,(\hat{x}F,\tilde\rho_E^{(m-1)})=\mathrm{Tr}\,(t_1(x),\rho_E^{(1)})=t_1^*(\rho_E^{(1)})(x)\,.$$

So, we have

$$\sharp^{-1}R_E^{(1)}=t_1^*(\rho_E^{(1)})\,.$$

2.10. COROLLARY. *Let the assumptions be as in* 2.8 (*with* $r=1$). *Then* *for each* $E\in W_s^\wedge$,

$$N^*_{F^m/F}\circ\tilde{\natural}^{m-1}(\tilde{R}_E^{(m)})=\natural^{-1}(R_E^{(1)}).$$

PROOF.

$$\begin{aligned} N^*_{F^m/F}\circ\tilde{\natural}^{m-1}(\tilde{R}_E^{(m)}) &= N^*_{F^m/F}\circ\tau_1^{*-1}(\tilde{\natural}^{-1}\tilde{R}_E^{(m)}) && \text{(by Prop. 2.8)}\\ &= t_1^*\circ N^*_{F^m/F}(\tilde{\natural}^{-1}\tilde{R}_E^{(m)}) && \text{(by Lemma 2.2)}\\ &= t_1^*(\rho_E^{(1)}) && \text{(by Cor. 1.10)}\\ &= \natural^{-1}R_E^{(1)}. && \text{(by Lemma 2.9)}. \end{aligned}$$

2.11. REMARKS. (i) In the case where G is almost simple and $s=1$, Asai [4, Lemma 3.2.4] has showed that (2.8.1) holds for $E\in W^\wedge$ whenever E is not an exceptional character of type E_7 or E_8. In the case of exceptional characters, it remains a possibility that N_G^* maps $\rho_{E_1}^{(1)}$ (resp. $\rho_{E_2}^{(1)}$) to $\tilde{\rho}_{E_2}^{(mr-1)}$ (resp. $\tilde{\rho}_{E_1}^{(mr-1)}$), respectively, where E_1 and E_2 are exceptional characters in W (of type E_7 or E_8). However, the above argument shows that, even in this case, Proposition 2.8 holds in a weaker form, i.e.,

$$\tau_r^*\tilde{\natural}^{mr-1}(\tilde{R}_{E_i}^{(m)})=\tilde{\natural}^{-1}(\tilde{R}_{E_j}^{(m)}), \qquad \{i,j\}=\{1,2\},$$

and so accordingly, Lemma 2.9 and Corollary 2.10 hold in a similar form as above.

(ii) Asai ([4]) uses the property that $R_E^{(1)}$ is t_1^*-invariant, to determine the twisting operators t_1^* in the case of exceptional groups, under the assumption that p is good. After that, in Theorem 2.4.1 (ii) in [5], he has proved this property without restriction on p. Then, using Corollary 1.7 and Lemma 2.2, we see that $\tilde{\rho}_E^{(m)}$ is fixed by τ_1^* if F is of split type. However, on the contrary, in the case where F is split and $s\in T^{*F}$, we can prove directly that τ_1^* fixes $\tilde{\rho}_E^{(m)}$. This, together with Corollary 1.7 and Lemma 2.2, gives an alternative proof of Asai's result, which is simpler in a sense that our proof proceeds only in a framework of principal series representations.

2.12. Before closing section 2, we consider here a certain compatibility of norm map $N_{F^m/F}$ and the restriction of scalars $R_{F_{q^n}/F_q}$. For later use in §4, we consider again two Frobenius maps F' and F'' on G as in 1.3. We assume that G is a product of r copies of G_1,

$$G=G_1\times\cdots\times G_r$$

such that

$$F'(G_i)=G_{i+1} \qquad (i\in \boldsymbol{Z}/r\boldsymbol{Z})\,.$$
$$F''(G_i)=G_i$$

Then F'^r stabilizes G_1 and the map

$$x_1 \longmapsto x=(x_1, F'(x_1), \cdots, F'^{(r-1)}(x_1))$$

gives an isomorphism of two groups, $G_1^{F'^r}$ and $G^{F'}$. Since this map is compatible with F''-action, we have a natural bijection

$$f' : G_1^{F'^r}/\sim_{F''^{-1}} \xrightarrow{\sim} G^{F'}/\sim_{F''^{-1}}\,.$$

Next, consider the injection $G_1^{F''}\hookrightarrow G^{F''}$ given by $x_1\mapsto(x_1, 1, \cdots, 1)$. This map induces a map

$$f'' : G_1^{F''}/\sim_{F'^r} \longrightarrow G^{F''}/\sim_{F'}\,.$$

In fact, take x_1 and y_1 in $G_1^{F''}$, which is F'^r-twisted conjugate in $G_1^{F''}$. So, there exists $z_1\in G_1^{F''}$ such that $y_1=z_1^{-1}x_1F'^r(z_1)$. Put $x=(x_1, 1, \cdots, 1)$, $y=(y_1, 1, \cdots, 1)$ and $z=(z_1, F'(z_1), \cdots, F'^{(r-1)}(z_1))$. Then $z\in G^{F''}$ and we have $y=z^{-1}xF'(z)$. Hence x and y are F'-twisted conjugate and so f'' is well-defined. Now, we have the following lemma.

2.13. LEMMA. *Let the notations be as in* 2.12.

(i) *The following diagram is commutative. In particular, f'' is a bijection.*

$$\begin{array}{ccc} G_1^{F'^r}/\sim_{F''^{-1}} & \xrightarrow{f'} & G^{F'}/\sim_{F''^{-1}} \\ N_{F''/F'^r}\big\downarrow & & \big\downarrow N_{F''/F'} \\ G_1^{F''}/\sim_{F'^r} & \xrightarrow{f''} & G^{F''}/\sim_{F'} \end{array}$$

(ii) *Let* $(f')^* : C(G^{F'}/\sim_{F''^{-1}})\to C(G_1^{F'^r}/\sim_{F''^{-1}})$ *and* $(f'')^* : C(G^{F''}/\sim_{F'})\to C(G_1^{F''}/\sim_{F'^r})$ *be the induced maps of f' and f'', respectively. Then for each F' and F''-stable class $\{s\}$ in G^*, there exists an F'^r and F''-stable class $\{s_1\}$ in G_1^* satisfying the followings.*

$$(2.13.1) \qquad (f')^* : C^{(s)}(G^{F'}/\sim_{F''^{-1}}) \xrightarrow{\sim} C^{(s_1)}(G_1^{F'^r}/\sim_{F''^{-1}})\,,$$

$$(2.13.2) \qquad (f'')^* : C^{(s)}(G^{F''}/\sim_{F'}) \xrightarrow{\sim} C^{(s_1)}(G_1^{F''}/\sim_{F'^r})\,.$$

Moreover, $(s_1, F'(s_1), \cdots, F'^{(r-1)}(s_1))$ *is conjugate to s in G^*.*

PROOF. First consider (i). It is enough to show the commutativity of the diagram. Take $x_1\in G_1^{F'^r}$ and put $x_1=F''(a_1)a_1^{-1}$ for $a_1\in G_1$. Then

$$N_{F''/F'^r}(x_1)=\hat{x}_1=a_1^{-1}F'^r(a_1)\,.$$

Now, if we put $a=(a_1, F'(a_1), \cdots, F'^{(r-1)}(a_1))\in G$, we have

$$a^{-1}F'(a)=(\hat{x}_1, 1, \cdots, 1)=f''(\hat{x}_1)\,.$$

Thus

$$N^{-1}_{F''/F'}\circ f''(\hat{x}_1)=F''(a)a^{-1}=(x_1, F'(x_1), \cdots, F'^{(r-1)}(X_1))=f'(x_1)\,.$$

This proves (i).

Next consider (ii). Choose $s'\in\{s\}$ in $(G^*)^{F'}$. Then via $(G^*)^{F'}\cong(G_1^*)^{F'^r}$, s' is written as $s'=(s_1, F'(s_1), \cdots, F'^{(r-1)}(s_1))$ for some $s_1\in(G_1^*)^{F'^r}$. Since $\{s\}$ is F''-stable, the class $\{s_1\}$ in G_1^* is also F''-stable. Then (2.13.1) is clear. (2.13.2) follows from the fact that the map $G^*\to G_1^*$ corresponding to the embedding $G_1\hookrightarrow G$ given by $x_1\mapsto(x_1, 1, \cdots, 1)$ is just the projection to the first factor, and that s' is mapped to s_1 under the above map $G^*\to G_1^*$.

2.14. We now consider the variant of Lemma 2.13, i.e., the case where F'' also permutes the factors. We only consider the case $r=2$. So, we consider two Frobenius maps F' and F'' of G as before. Here we assume further that $F'F''$ is also a Frobenius map on G. Let $G=G_1\times G_2$ be two copies of G_1 and assume that

$$\begin{aligned} F'(G_i)&=G_{i+1} \\ F''(G_i)&=G_{i+1} \end{aligned} \qquad (i\in \boldsymbol{Z}/2\boldsymbol{Z})\,.$$

Then $F'F''$ stabilizes G_1 and is regarded as a Frobenius map of G_1. Now, as before, $G_1^{F'^2}$ is isomorphic to $G^{F'}$ via $x_1\mapsto x=(x_1, F'(x_1))$, $(x_1\in G_1^{F'^2})$. Then it is easily checked that under this isomorphism, $(F'F'')^{-1}$-twisted conjugacy classes of $G_1^{F'^2}$ is mapped to the F''^{-1}-twisted conjugacy classes of $G^{F'}$. Hence we have a natural bijection

$$f' : G_1^{F'^2}/\sim_{(F'F'')^{-1}} \xrightarrow{\sim} G^{F'}/\sim_{F''^{-1}}\,.$$

Similarly, using $G_1^{F''^2}\simeq G^{F''}$, we have a natural bijection f'' which is defined as above replacing F' by F''. Now we have the following lemma.

2.15. LEMMA. *Let the notations be as in* 2.14.

(i) *The following diagram is commutative.*

$$\begin{array}{ccccc} G_1^{F'F''}/\sim_{F'^2} & \xleftarrow{N_{F'F''/F'^2}} & G_1^{F'^2}/\sim_{(F'F'')^{-1}} & \xrightarrow{f'} & G^{F'}/\sim_{F''^{-1}} \\ \| & & & & \downarrow{\scriptstyle N_{F''/F'}} \\ G_1^{F'F''}/\sim_{F''^{-2}} & \xrightarrow{N_{F''^2/F'F''}} & G_1^{F''^2}/\sim_{F'F''} & \xrightarrow{f''} & G^{F''}/\sim_{F'} \end{array}$$

(The identity on the left hand side follows from the fact that the restriction of F'^2 on $G^{F'F''}$ coincides with the restriction of F''^{-2} on $G^{F'F''}$.)

(ii) *Let $(f')^*: C(G^{F'}/\sim_{F''^{-1}}) \simeq C(G_1^{F'^2}/\sim_{(F'F'')^{-1}})$ be the induced map of f', and $(f'')^*$ be the induced map of f'' defined similarly as above replacing F' by F''. Then for each F' and F''-stable class $\{s\}$ in G^* there exists an F'^2 and $F'F''$-stable (hence F''^2-stable) class $\{s_1\}$ in G_1^* such that $(s_1, F'(s_1))$ and $(s_1, F''(s_1))$ are both conjugate to s in G^* and satisfies the following relations.*

$$(f')^*: C^{(s)}(G^{F'}/\sim_{F''^{-1}}) \xrightarrow{\sim} C^{(s_1)}(G_1^{F'^2}/\sim_{(F'F'')^{-1}}), \tag{2.15.1}$$

$$(f'')^*: C^{(s)}(G^{F''}/\sim_{F'^{-1}}) \xrightarrow{\sim} C^{(s_1)}(G_1^{F''^2}/\sim_{F'F''}). \tag{2.15.2}$$

PROOF. First consider (i). Let us take $\hat{x} \in G_1^{F'F''}$ and put

$$\hat{x} = \alpha^{-1}F'^2(\alpha) = F''^2(\beta)\beta^{-1}, \qquad \alpha, \beta \in G_1. \tag{2.15.3}$$

Then

$$y = N^{-1}_{F'F''/F'^2}(\hat{x}) = F'F''(\alpha)\alpha^{-1} \in G_1^{F'^2},$$

$$z = N_{F''^2/F'F''}(\hat{x}) = \beta^{-1}F'F''(\beta) \in G_1^{F''^2},$$

and

$$f'(y) = (y, F'(y)) \in G^{F'}$$

$$f''(z) = (z, F''(z)) \in G^{F''}.$$

It is enough to show that $N_{F''/F'}$ maps the class of $f'(y)$ to the class of $f''(z)$. Now put $c = (c_1, c_2) \in G$, where

$$c_1 = \alpha F''^2(\beta) = F'^2(\alpha)\beta \in G_1 \qquad \text{(by (2.15.3))}$$

$$c_2 = F'(\alpha)F''(\beta) \in G_2.$$

Then

$$\begin{aligned} F''(c)c^{-1} &= (F''(c_2), F''(c_1))(c_1^{-1}, c_2^{-1}) \\ &= (F''F'(\alpha)F''^2(\beta), F''F'^2(\alpha)F''(\beta)) \\ &\quad \times (F''^2(\beta^{-1})\alpha^{-1}, F''(\beta^{-1})F'(\alpha^{-1})) \\ &= (y, F'(y)) = f'(y). \end{aligned}$$

Hence $N_{F''/F'} \circ f'(y)$ is represented by $c^{-1}F'(c)$. On the other hand, the similar computation as above using (2.15.3) shows that

$$c^{-1}F'(c) = (z, F''(z)) = f''(z).$$

Hence we get the desired result.

Next consider (ii). Let $\{s\}$ be an F' and F''-stable class in G^*. We can find $s'\in\{s\}$ such that $s'\in G^{*F'}$. Since $G^{*F'}\cong(G_1^*)^{F'^2}$, we can find $s_1'\in(G_1^*)^{F'^2}$ such that $s'=(s_1', F'(s_1'))$. Since $\{s\}$ is F''-stable, the class $\{s_1'\}$ in G_1^* is $F'F''$-stable. For these representatives, (2.15.1) clearly holds. On the other hand, since $\{s_1'\}$ is also F''^2-stable, we can find $s_1''\in\{s_1'\}$ such that $s_1''\in(G_1^*)^{F''^2}$. Since $\{s_1'\}$ is $F''F'^{-1}$-stable, $(s_1', F'(s_1'))\sim(s_1', F''(s_1'))\sim(s_1'', F''(s_1''))$ (conjugate in G^*). Thus $s''=(s_1'', F''(s_1''))\in G^{*F''}$ is conjugate to s in G^*. Using these representatives we get (2.15.2). This proves (ii).

§ 3. The main result

3.1. Let G be as in 1.1 and let $\{s\}$ be an F-stable class in G^*. We shall choose a representative s in T^*. Then W_s, $\gamma: W_s\to W_s$ are defined as in 1.1. Now, according to [14, Chap. 4], two parameter sets $X(W_s,\gamma)$, $\bar{X}(W_s,\gamma)$ concerning with $\mathcal{E}(G^F,\{s\})$ and the pairing $\{\,,\}:\bar{X}(W_s,\gamma)\times X(W_s,\gamma)\to\bar{\boldsymbol{Q}}_l$ can be defined. $\bar{X}(W_s,\gamma)$ is a finite set and $X(W_s,\gamma)$ is an infinite set with a free M-action such that the orbit space of $X(W_s,\gamma)$ by M has the same cardinality as $\bar{X}(W_s,\gamma)$, where M is the group of all roots of unity in $\bar{\boldsymbol{Q}}_l^*$. In the fundamental case where γ acts trivially on W_s, these two sets are described as follows: $X(W_s,\gamma)\cong\bar{X}(W_s,\gamma)\times M$ and $\bar{X}(W_s,\gamma)$ is decomposed into a disjoint union of subsets called families, where each family $\mathcal{F}$ has the form $\mathcal{M}_\Gamma$ for a certain finite group Γ. Here the set $\mathcal{M}_\Gamma$ is defined for each finite group Γ as the set of pairs (y,σ) such that $y\in\Gamma$, $\sigma\in Z_\Gamma(y)^\wedge$ modulo the conjugation by Γ on it. Moreover, in this case, the pairing as above are essentially given by the pairing $\{\,,\}:\bar{X}(W_s,\gamma)\times\bar{X}(W_s,\gamma)\to\bar{\boldsymbol{Q}}_l$, which is defined for $(x,\sigma)\in\mathcal{F}$, $(y,\tau)\in\mathcal{F}'$ as follows.

$$\{(x,\sigma),(y,\tau)\}\tag{3.1.1}$$
$$=\begin{cases}\displaystyle\sum_{\substack{g\in\Gamma\\ xgyg^{-1}=gyg^{-1}x}}|Z_\Gamma(x)|^{-1}|Z_\Gamma(y)|^{-1}\operatorname{Tr}(g^{-1}x^{-1}g,\tau)\operatorname{Tr}(gyg^{-1},\sigma) & \text{if } \mathcal{F}=\mathcal{F}'\\[2ex] 0 & \text{if } \mathcal{F}\neq\mathcal{F}'\,.\end{cases}$$

We now define an involutive automorphism $x\mapsto x^*$ in $X(W_s,\gamma)$ as follows. In the case where γ is trivial, $x\mapsto x^*$ is defined on each $\mathcal{M}_\Gamma$ by associating to $x=(y,\sigma)\in\mathcal{M}_\Gamma$, $x^*=(y^{-1},\sigma)\in\mathcal{M}_\Gamma$. Along the process of defining $X(W_s,\gamma)$ for the general case as in 4.20~4.21 in [14], this operation $x\mapsto x^*$ is extended to the general case in a natural way.

According to the main result in Lusztig [14], the set $\mathcal{E}(G^F,\{s\})$ is parametrized by $\bar{X}(W_s,\gamma)$. We shall write as $\bar{x}_\rho$ an element in $\bar{X}(W_s,\gamma)$ corresponding to $\rho\in\mathcal{E}(G^F,\{s\})$. We say that ρ belongs to a family $\mathcal{F}$ if

$\bar{x}_\rho$ is in a family $\mathscr{F}$ in $\bar{X}(W_s,\gamma)$. Following [14], we shall define, for each $x\in X(W_s,\gamma)$, an almost character $R_x\in C^{(s)}(G^F/\sim)$ by the following formula,

$$R_x=(-1)^{l(w_0)}\sum_\rho\{\bar{x}_\rho, x\}\Delta(\bar{x}_\rho)[\rho], \tag{3.1.2}$$

where ρ runs over all the elements in $\mathcal{E}(G^F,\{s\})$. Here w_0 is as in 1.1, and $\Delta(\bar{x}_\rho)=\pm 1$ is the modification concerning exceptional characters of Weyl groups of type E_7 or E_8, and is defined as in [14, 4.21].

Now let us define the modified almost charcter $R_x^\check{}$ for each $x\in X(W_s,\gamma)$ by

$$R_x^\check{}=(-1)^{l(w_0)}\sum_\rho\{\bar{x}_\rho, x^*\}\Delta(\bar{x}_\rho)[\rho], \tag{3.1.3}$$

where the summation is the same as in (3.1.2).

Let c be the order of γ on W_s. Then R_x and $R_x^\check{}$ are determined uniquely by the M-orbit of x up to a c-th root of unity multiple. Note this modified almost character already appears in Asai [4] concerning the determination of twisting operators in the case of exceptional groups, and our situation is quite analogous to his.

Assume γ is trivial. Then there exists a natural embedding $W_s^\wedge\hookrightarrow\bar{X}(W_s,\gamma)$ according to [14, Chap. 4]. For each $E\in W_s^\wedge$, we shall write $\bar{x}_E$ the corresponding element in $\bar{X}(W_s,\gamma)$. Assume that the Coxeter diagram of W_s is connected. Then it is known by [14, Prop. 12.6], that $R_E=R_{\bar{x}_E}$ if E is not an exceptional character. While if W_s is of type E_7 (resp. E_8), let $\{E_1, E_2\}$ be two exceptional characters, (resp. $E_1=4096_z$, $E_2=4096_x$ or $E_1=4096'_z$, $E_2=4096'_x$, two of four exceptional characters). Then for $E\in\{E_1, E_2\}$,

$$R_E=R_{\bar{x}_{E_i}}\qquad i\in\{1,2\}.$$

The situation is similar for the case $W_s\cong E_7\times A_1$.

We say a family (4-element family) $\mathscr{F}\subset\bar{X}(W_s,\gamma)$ is exceptional if it contains an exceptional character $E\in W_s^\wedge$ (resp. $E\otimes E'\in W_s^\wedge$ with E exceptional) in the case where W_s is of type E_7 or E_8 (resp. of type $E_7\times A_1$).

We recall, as in 1.2, that a root of unity λ_ρ is associated to each $\rho\in\mathcal{E}(G^{F^m})$, provided that m is large enough so that F^m is a multiple of F_0 and that $s\in T^{*F^m}$.

We can now state our main theorem.

3.2. THEOREM. *Let G be an exceptional group of adjoint type. We assume $q\not\equiv -1$ (mod 3) if G is of type E_8. Then for any positive integer m divisible by some fixed number $m_0=m_0(G^F)$ depending only on G^F, the* *following hold.*

(i) $N^*_{F^m/F}$ *induces an isomorphism from* $C^{(s)}(G^{F^m}/\sim_F)$ *onto* $C^{(s)}(G^F/\sim)$.

(ii) *There exists a natural bijection between* $\mathcal{E}(G^F,\{s\})^F$ *and* $X(W_s,\gamma)/M$.

(iii) *Except the minor ambiguity below in* (a) *and* (b), $N^*_{F^m/F}$ *is described as follows. For each* $\rho\in\mathcal{E}(G^{F^m},\{s\})^F$, *let us take an extension* $\tilde{\rho}$ *of* ρ *to* $\tilde{G}^{F^m}$. *Let* x_ρ *be an element in* $X(W_s,\gamma)$ *which is mapped to* ρ *under the correspondence in* (ii). *Then we have*

$$N^*_{F^m/F}([\mu_{\tilde{\rho}}\tilde{\rho}])=R'_{x_\rho}, \tag{3.2.1}$$

where $\mu_{\tilde{\rho}}$ *is a root of unity (depending on an extension* $\tilde{\rho}$ *and on the choice of a representative* x_ρ *in its M-orbit) such that* $(\mu_{\tilde{\rho}})^m=\lambda_\rho^{-1}$.

(a) *Assume* W_s *is a Weyl group of type* E_7, E_8 *or* $E_7\times A_1$, *(hence* γ *is trivial). Let* $\mathcal{F}=\{\rho_1,\rho_2,\rho_3,\rho_4\}$ *be an exceptional family, where* ρ_1,ρ_2 *are in the principal series. Then* (3.2.1) *should be replaced by*

$$N^*_{F^m/F}([\mu_{\tilde{\rho}_i}\tilde{\rho}_i])=R'_{x_{\rho_j}}, \tag{3.2.2}$$

where $\{i,j\}$ *is equal to* $\{1,2\}$ *or* $\{3,4\}$.

(b) *Assume* (W_s,γ) *is of type* ${}^2E_6$. *Let* ρ_1,ρ_2 *be two cuspidal representations in* $\mathcal{E}(G^{F^m},\{s\})=\mathcal{E}(G^{F^m},\{s\})^F$. *If* $\rho\notin\{\rho_1,\rho_2\}$, *the same formula as in* (3.2.1) *holds. While if* $\rho\in\{\rho_1,\rho_2\}$, *we have*

$$N^*_{F^m/F}([\mu_{\tilde{\rho}}\tilde{\rho}])=R'_{x_{\rho_i}},\qquad i\in\{1,2\}. \tag{3.2.3}$$

3.3. REMARKS. (i) In the case where G is of type E_8 and $q\equiv -1 \pmod 3$, there exists $s\in T^*$ such that W_s is of type $E_6\times A_2$ and γ acts non-trivially on both factors. This case cannot be covered by our method.

(ii) The lower bound of m_0 cannot be determined explicitly (see 3.4 below and [14, 2.5]).

3.4. The remainder part of this section is devoted to the proof of Theorem 3.2.

In order to make the induction argument smoothly, we shall prove the theorem not only for adjoint (almost simple) groups but also for connected groups with connected center which arises as a Levi subgroup of exceptional groups.

Let $L=L_J$ be the proper Levi subgroup of a standard parabolic subgroup of G and $F\dot{w}$ be a Frobenius map on L (i.e., $Fw(J)=J$). We assume that the same statement as in Theorem 3.2 holds for $(L,F\dot{w})$. Let $\{s\}$ be an $F\dot{w}$-stable class in L^* and consider $\mathcal{E}(L^{F^m},\{s\})$ for sufficiently divisible $m>0$ (i.e., m is divisible by $m_0(L^{F\dot{w}})$ as in the theorem). We assume that $\mathcal{E}(L^{F^m},\{s\})^{F\dot{w}}$ contains a cuspidal representation δ. Let $\tilde{\delta}$ be an extension of δ to $\tilde{L}^{F^m}$. Then by our assumption, we can attach to $\tilde{\delta}$ a root of unity $\mu_{\tilde{\delta}}$ such that $N^*_{F^m/F\dot{w}}([\mu_{\tilde{\delta}}\tilde{\delta}])=R^{\check{}}_{x_\delta}$ is independent of m. Let ρ_E be the irreducible representation in $\mathcal{E}(G^{F^m},\{s\})^F$ corresponding to $E\in(W_\delta)^{\wedge}_{\text{ex}}$ and $\tilde{\rho}_E$ be its extension to $\tilde{G}^{F^m}$ as in 1.14. Then by the same argument as in [16, 2.5], but replacing (2.5.1) in [loc. cit.] by the formula

$$\operatorname{Tr}(T_{\gamma y},\tilde{E}(q^m))\in\boldsymbol{Q}[q^{m/2}],$$

which follows from Benson-Curtis [6], we can conclude that $N^*_{F^m/F}([\mu_{\tilde{\delta}}\tilde{\rho}_E])$ is independent of m when m runs through some (unknown) infinite set. Also we see that $\varepsilon^{(m)}_y$ $(y\in W_\delta)$ in 1.15 is independent of m for m as above.

3.5. Let us now define Euclidean spaces $\mathcal{U}^{(s)}(G,F)$ and $\mathcal{V}^{(s)}(G,F)$ associated with $X(W_s,\gamma)$ and $\bar{X}(W_s,\gamma)$, respectively, as follows. $\mathcal{U}^{(s)}(G,F)$ is a space over $\bar{\boldsymbol{Q}}_l$ with inner product $\langle\,,\rangle$ generated by f_x $(x\in X(W_s,\gamma))$ with relations:

$$f_{\zeta x}=\zeta f_x \qquad \text{for} \quad \zeta\in M$$

$$\langle f_x,f_y\rangle=\begin{cases}1 & \text{if} \quad x=y\\ 0 & \text{if} \quad y\notin Mx\end{cases}$$

$\mathcal{V}^{(s)}(G,F)$ is a space over $\bar{\boldsymbol{Q}}_l$ with inner product $\langle\,,\rangle$ and orthonormal basis $e_{\bar{x}}$, $(\bar{x}\in\bar{X}(W_s,\gamma))$.

Let $L=L_J$, $F\dot{w}$ and $\{s\}$ be as in 3.4. Thus $\mathcal{U}^{(s)}(L,F\dot{w})$ and $\mathcal{V}^{(s)}(L,F\dot{w})$ are defined similarly replacing (G,F) by $(L,F\dot{w})$ as above. Now consider the map $a_{Fw}:C^{(s)}(L^{F^m}/\sim_{F\dot{w}})\to C^{(s)}(G^{F^m}/\sim_F)$ as in 1.12. We denote by $C^{(s)}_1$ the subspace of $C^{(s)}(G^{F^m}/\sim_F)$ generated by the image of a_{Fw} for various $w\in W$ such that $Fw(J)=J$. Let $\mathcal{E}_1$ be the subset of $\mathcal{E}(G^{F^m},\{s\})^F$ consisting of ρ_E for $E\in(W_\delta)^{\wedge}_{\text{ex}}$ for cuspidal representations δ of various $L^{F^m}_K$ $(K\subset J)$. Then $[\tilde{\rho}]$ $(\rho\in\mathcal{E}_1)$ form a basis of $C^{(s)}_1$. Moreover, using the parametrization via Harish-Chandra induction, we see that there exists an M-stable subset X_1 of $X(W_s,\gamma)$ and a surjection $X_1\to\mathcal{E}_1$, $x\mapsto\rho_x$, which induces an isomorphism $\mathcal{U}^{(s)}_1\cong C^{(s)}_1$ via $f_x\leftrightarrow[\tilde{\rho}_x]$ $(x\in X_1)$, where $\mathcal{U}^{(s)}_1$ is the subspace of $\mathcal{U}^{(s)}(G,F)$ generated by f_x $(x\in X_1)$. We now identify $\mathcal{U}^{(s)}_1$ with $C^{(s)}_1$ via $f_x\leftrightarrow[\mu_{\tilde{\delta}}\tilde{\rho}_E]$, where $\mu_{\tilde{\delta}}$ and $\tilde{\rho}_E$ are as in 3.4 with respect to a cuspidal representation δ in $L^{F^m}_K$, and $\rho_x=\rho_E$ under the above map. Using the parametrization in 3.2, we may identify $\mathcal{U}^{(s)}(L^{F^m},F\dot{w})$ with $C^{(s)}(L^{F^m}/\sim_{F\dot{w}})$ via $f_{x_\rho}\leftrightarrow[\mu_{\tilde{\delta}}\tilde{\rho}]$ $(\rho\in\mathcal{E}(L^{F^m},\{s\})^{F\dot{w}})$, where $\rho\mapsto x_\rho$ is a cross section of the above map. Then, under these identifications, we get a map $a_{Fw}:\mathcal{U}^{(s)}(L,F\dot{w})\to\mathcal{U}^{(s)}_1$.

On the other hand, we shall identify $\mathcal{V}^{(s)}(G,F)$ (resp. $\mathcal{V}^{(s)}(L,F\dot{w})$ with $C^{(s)}(G^F/\sim)$ (resp. $C^{(s)}(L^{F\dot{w}}/\sim)$) via $e_{\bar{x}}\leftrightarrow\rho_{\bar{x}}$ using the parametrization of

$\mathcal{E}(G^F, \{s\}) \leftrightarrow \bar{X}(W_s, \gamma)$ and similarly for $L^{F\dot{w}}$. Hence, under these identifications, we get a map $R^{(m)}_{L(\dot{w})} : \mathcal{V}^{(s)}(L, F\dot{w}) \to \mathcal{V}^{(s)}(G, F)$. Thus Proposition 1.13 is translated to the following commutative diagram.

$$\begin{tikzcd}
\mathcal{V}^{(s)}(G, F) & \mathcal{U}^{(s)}_1 \arrow[l, "N^*_{F^m/F}"'] \\
\mathcal{V}^{(s)}(L, F\dot{w}) \arrow[u, "R^{(m)}_{L(\dot{w})}"] & \mathcal{U}^{(s)}(L, F\dot{w}) \arrow[l, "N^*_{F^m/F\dot{w}}"] \arrow[u, "a_{Fw}"']
\end{tikzcd} \tag{3.5.1}$$

Now the map $N^*_{F^m/F}$ is independent of m, for infinitely many m by 3.4, and the same is true for $N^*_{F^m/F\dot{w}}$. We denote by $\Delta = \Delta_G$ (resp. Δ_L) the map $N^*_{F^m/F}$ (resp. $N^*_{F^m/F\dot{w}}$) independent of m. We note that a_{Fw} is a linear map whose coefficients are (Laurent) polynomials in $q^{m/2}$ by Lemma 1.15 (notice that $\varepsilon_y = \varepsilon_y^{(m)}$ is constant for m as above by 3.4). On the other hand, by [16, Prop. 1.6], $R^{(m)}_{L(\dot{w})}$ is a linear map whose coefficients are polynomials in $q^{m/2}$. Thus one can specialize this diagram by $q^{m/2} \mapsto 1$. Note, thus obtained map from $\mathcal{V}^{(s)}(L, F\dot{w})$ to $\mathcal{V}^{(s)}(G, F)$ coincides with $R^G_{L(\dot{w})}$ in 1.11 under the above identifications. Summarizing the above arguments, we have

3.6. PROPOSITION (cf. [16, 2.7.2]). *Let $(L, F\dot{w})$ be as before and assume that the theorem holds for $(L, F\dot{w})$. Then for each Fw-stable class $\{s\}$ in L^*, the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{V}^{(s)}(G, F) & \mathcal{U}^{(s)}_1 \arrow[l, "\Delta_G"'] \\
\mathcal{V}^{(s)}(L, F\dot{w}) \arrow[u, "R^G_{(\dot{w})}"] & \mathcal{U}^{(s)}(L, F\dot{w}) \arrow[l, "\Delta_L"] \arrow[u, "a_w"']
\end{tikzcd}$$

The map a_w is given for each $x_{E'} = x_{\pi_{E'}}$ ($\pi_{E'}$ as in 1.13),

$$a_w(f_{x_{E'}}) = \varepsilon_y \sum_{E \in (W_{\delta})^\wedge_{\mathrm{ex}}} \mathrm{Tr}\,(\gamma_w, V_{\tilde{E}', \tilde{E}}) f_{x_E}, \tag{3.6.1}$$

where $w = w_1 y w_1'^{-1}$ is as in 1.14, and $\gamma_w : V_{\tilde{E}', \tilde{E}} \to V_{\tilde{E}', \tilde{E}}$ is the map obtained by specializing $q^{m/2} \mapsto 1$ from the map $\gamma_w^{(m)} : V^{(m)}_{\tilde{E}', \tilde{E}} \to V^{(m)}_{\tilde{E}', \tilde{E}}$ as in 1.14.

3.7. The following special case would be worth mentioning. Assume γ is trivial and L is F-stable. Thus we can consider the case $w=1$. We assume that L^{F^m} contains a cuspidal representation δ. Then $a_1([\tilde{\delta}])$ is nothing but the usual Harish-Chandra induction $\mathrm{Ind}^{G^{F^m}}_{P_J^{F^m}}(\delta)$. On the other hand, also $R^G_{(\dot{w})}$ coincides with the Harish-Chandra induction when w is equal to 1. Thus, we see that

$$\mathrm{Ind}^{G^F}_{P_J^F} \circ \Delta_L(\tilde{\delta}) = \Delta_G \circ \mathrm{Ind}^{G^{F^m}}_{P_J^{F^m}}(\tilde{\delta}). \tag{3.7.1}$$

3.8. Let $Z(G^*)$ be the center of G^*. Then for each $z\in Z(G^*)$, there corresponds an F-stable linear character ϕ of G^{F^m}. Now, for each $\rho\in\mathcal{E}(G^{F^m},\{s\})^F$, $\rho\otimes\phi\in\mathcal{E}(G^{F^m},\{sz\})^F$ and $\rho\to\rho\otimes\phi$ gives a bijection between $\mathcal{E}(G^{F^m},\{s\})^F$ and $\mathcal{E}(G^{F^m},\{sz\})^F$. If $\tilde{\rho}$ is an extension of ρ to $\tilde{G}^{F^m}$, $\tilde{\rho}\otimes\phi$ is an extension of $\rho\otimes\phi$, and we can associate a root of unity $\lambda'_{\tilde{\rho}}$, $\lambda'_{\tilde{\rho}\otimes\phi}$ to $\tilde{\rho}$, $\tilde{\rho}\otimes\phi$, respectively. Then, by Lemma 1.1.3 in Asai [5], we have

$$\lambda'_{\tilde{\rho}}=\lambda'_{\tilde{\rho}\otimes\phi}\,. \tag{3.8.1}$$

Now, thanks to Lemma 2.13 (applied with $F'=F$, $F''=F^m$) together with (3.8.1), Asai's argument ([1, §2], see also [14, 8.8]) can be applied to our case, and we see that the verification of the theorem (or rather its extended form in 3.4) is reduced to the case where G has connected center, G is simple modulo center and the derived group of G is simply connected, (except for the case of E_8, in which case, we assume G itself has trivial center). In [16], we have already verified the case of classical groups, and so we may assume that G is an exceptional simple group with connected center not of type E_8 or G is a simple group of type E_8, and that the theorem is verified for proper Levi subgroups.

3.9. We will prove the theorem separately for each $\mathcal{E}(G^{F^m},\{s\})$. We now show that the proof of the theorem is reduced to the special case where $Z_{G^*}(s)^*$ has the same semisimple rank as G. Suppose the semisimple rank of $Z_{G^*}(s)^*$ is less than that of G. Then there exists some Levi subgroup $(\neq G)$ containing $Z_{G^*}(s)^*$. Now, as in [14, 6.21], taking the conjugate of s in G^F if necessary, one can find a Levi subgroup $L=L_J\neq G$ with Frobenius map $F\dot{w}$ $(Fw(J)=J)$ such that $Z_{G^*}(s)$ is contained in L^* and that the class $\{s\}$ in L^* is Fw-stable. Note in this case W_s is contained in W_J and $X(W_s,\gamma)$, $\bar{X}(W_s,\gamma)$ for L coincide with those for G. Hence there is a natural identification $\mathcal{V}^{(s)}(L,F\dot{w})\cong\mathcal{V}^{(s)}(G,F)$ and a natural embedding $\mathcal{U}^{(s)}(L,F\dot{w})\cong\mathcal{U}_1^{(s)}\hookrightarrow\mathcal{U}^{(s)}(G,F)$. Using these identifications, it is checked that a_w is identity and that $R^G_{(\dot{w})}$ is a scalar multiplication $(-1)^{l(w)}$. Thus, as in [16, 2.8], by making use of Proposition 3.6, we can determine the map Δ_G on the subspace $\mathcal{U}_1^{(s)}$. In particular, Δ_G^{-1} is an isomorphism from $\mathcal{U}^{(s)}(G,F)$ to $\mathcal{U}_1^{(s)}$. It remains to show that $\mathcal{U}_1^{(s)}=\mathcal{U}^{(s)}(G,F)$. This is done in a final step of the proof of the theorem using the dimension argument as follows. Suppose we could show the theorem for each F-stable class $\{s\}$ in G^* such that $Z_{G^*}(s)$ is not contained in a proper Levi subgroup. Then, for such s, we see that $(N^*_{F^m/F})^{-1}$ induces an isomorphism from $C^{(s)}(G^F/\sim)$ onto $C^{(s)}(G^{F^m}/\sim_F)$ (or "into" is sufficient), and for other s, we have an isomorphism from $C^{(s)}(G^F/\sim)$ into $C^{(s)}(G^{F^m}/\sim_F)$. Since $(N^*_{F^m/F})^{-1}$ is an isomorphism from $C(G^F/\sim)$ onto $C(G^{F^m}/\sim_F)$, we see that the above restriction on the subspaces are all surjective. This gives the desired result.

3.10. We will discuss here some parametrization of $\mathcal{E}(G^F)$ briefly. The parametrization of $\mathcal{E}(G^F,\{s\})$ via $\bar{X}(W_s,\gamma)$ is characterized by the multiplicity of characters in various $R_E^{(1)}$ $(E\in(W_s)^{\wedge}_{\text{ex}})$, and the identification of this parametrization and the usual parametrization via Harish-Chandra induction has been done in the case of classical groups (for s as in 3.9) in [15, Prop. 6.6 and Prop. 5.3] and in the case of exceptional groups with s in the center of G^* in [14, 10.2]. We now consider the following case: $W_s\cong W_L\times W_r$, where W_L is a Weyl subgroup of W corresponding to a proper standard parabolic subgroup $P=LU_P$ of G and W_r is a Weyl group of type A_r $(r\geqq 1)$. We assume that γ acts trivially on W_r. By [14, Th. 8.6], we know already the structure of endomorphism algebras of induced representations from various cuspidal representations. Using this, we see easily that, for each $\pi\in\mathcal{E}(L^F,\{s\})$, irreducible constituent of $\mathrm{Ind}_{P^F}^{G^F}(\pi)$ is parametrized by the irreducible representation E of W_r, which we denote by $\rho_{\pi,E}$. All the irreducible representaions ρ in $\mathcal{E}(G^F,\{s\})$ are written uniquely as $\rho=\rho_{\pi,E}$ for some $\pi\in\mathcal{E}(L^F,\{s\})$ and $E\in W_r^{\wedge}$. Moreover since $s\in Z(L^*)$, there exists a linear character $\phi\in\hat{L}^F$ corresponding to s such that $\pi\to\phi\otimes\pi$ gives a bijection $\mathcal{E}(L^F,\{1\})\cong\mathcal{E}(L^F,\{s\})$.

On the other hand, $\bar{X}(W_s,\gamma)$ is decomposed as $\bar{X}(W_s,\gamma)\cong\bar{X}(W_L,\gamma_L)\times\bar{X}(W_r,1)$, where γ_L is a restriction of γ on W_L, and$\bar{X}(W_L,\gamma_L)\cong(L^F,\{1\})$, $\bar{X}(W_r,1)\cong W_r^{\wedge}$. Then the parametrization of ρ via $\bar{X}(W_s,\gamma)$ is compatible with the previous parametrization, i.e., if $\rho_{\bar{x}}=\rho_{\pi,E}$ with $x=(x_1,x_2)$ $x_1\in\bar{X}(W_L,\gamma_L)$, $x_2\in\bar{X}(W_r,1)$, then we have $\rho_{\bar{x}_1}=\pi$ and $\bar{x}_2$ is the element corresponding to E. This is shown by a similar method as in [15]. We omit the details.

Next consider the similar situation $W_s\cong W_L\times W_r$, but assume that γ is not necessarily trivial on W_r. We consider the set $\mathcal{E}(G^{F^m},\{s\})$ for a sufficiently large m. Since $\gamma^{(m)}$ is trivial on $W_s^{(m)}$, we have a parametrization of this set by the previous way. In particular, $\rho_{\pi,E}^{(m)}$ is F-stable if and only if $\pi=\pi^{(m)}\in\mathcal{E}(L^{F^m},\{s\})$ is F-stable. Since all the characters of W_r are γ-stable, we have a canonical bijection $X(W_s,\gamma)/M\cong X(W_L,\gamma_L)/M\times\bar{X}(W_r,1)$. We know already by induction assumption that $X(W_L,\gamma_L)/M\cong\mathcal{E}(L^{F^m},\{s\})^F$. Hence using the above bijection we have a natural bijection

$$\mathcal{E}(G^{F^m},\{s\})^F\cong X(W_s,\gamma)/M$$

given by $\rho_{\pi,E}^{(m)}\mapsto(x_\pi,x_E)$.

Finally consider the case where G is of type E_8 (resp. F_4) and W_s is of type D_8 (resp. B_4). The method employed in [15] can be also applied to this case, and we see that the parametrization is compatible with that from Harish-Chandra induction.

3.11. We now consider the case where F is of split type and $s\in Z(G^*)^F$. Thus γ is trivial and so we shall denote $\bar{X}(W_s, \gamma)$ by $X(W)$. As in 3.5, let $\mathcal{V}=\mathcal{V}^{(s)}(G, F)$ be an Euclidean space with orthonormal basis e_x $(x\in X(W))$. We have the pairing $\{ , \}: X(W)\times X(W)\to\bar{\boldsymbol{Q}}_l$ as in (3.1.1). Then the matrix $(\{x, y\})_{x,y\in X(W)}$ is a unitary and hermitian matrix. Hence the defintion of x^* in 3.1 implies that $\{x^*, y\}=\{y^*, x\}$ for $x, y\in X(W)$. For each $x\in X(W)$, let us define

$$R_x=\sum_{y\in X(W)}\{y, x\}\Delta(y)e_y\,,$$

$$R_x^{\vee}=R_{x^*}=\sum_{y\in X(W)}\{y, x^*\}\Delta(y)e_y\,,$$

where $\Delta(y)=\pm 1$ is the map defined similarly as in 3.2.

Let $\rho=\rho_x\in\mathcal{E}(G^F, \{s\})$ be an element corresponding to $x\in X(W)$. We denote by λ_x the root of unity λ_ρ associated to ρ (see 1.2), and we define a linear map $\sharp: \mathcal{V}\to\mathcal{V}$ by $e_e\mapsto\lambda_x e_x$ $(x\in X(W))$.

The following lemma is a special case of Asai's lemma [4].

3.12. LEMMA (Asai [4, Lemma 6.1.2]).

$$\sum_{y\in X(W)}\{x, y\}\lambda_y\{y, z\}=\lambda_x\{x, z^*\}\lambda_z\,.$$

Using this lemma, we can prove some properties of R_x and $R_x^{\vee}$.

3.13. LEMMA. (i) $R_{x_E}^{\vee}=R_{x_E}$.

(ii) *Assume x, x_E are not in the exceptional families. Then*

(a) $e_x=\sum_{y\in X(W)}\{x^*, y\}R_y^{\vee}$

(b) $\sharp^{-1}R_{x_E}=\sum_{y\in X(W)}\{x_E, y\}\lambda_y R_y^{\vee}$.

PROOF. By a direct observation of the table of the matrix $(\{x, y\})_{x,y\in X(W)}$ given in [14, 4.15], we can check that $x_E^*=x_E$ for $E\in W^{\wedge}$. Thus $R_{x_E}^{\vee}=R_{x_E^*}=R_{x_E}$ and (i) follows. (a) of (ii) follows from the fact that the square of the matrix is the identity matrix, and that $\{x^*, y\}=\{y^*, x\}$ for $x, y\in X(W)$. For (b) of (ii),

$$\begin{aligned}\sharp^{-1}R_{x_E}&=\sum_{y\in X(W)}\{y, x_E\}\lambda_y^{-1}e_y\\&=\sum_{z\in X(W)}\Big(\sum_{y\in X(W)}\{y, x_E\}\lambda_y^{-1}\{y^*, z\}\Big)R_z^{\vee}\qquad\text{(by (a))}\,.\end{aligned}$$

However, since $\lambda_y^{-1}=\lambda_{y^*}$ (cf. [14, Th. 11.2]), and $\{y, x_E\}=\{y, x_E^*\}=\{x_E, y^*\}$, we see that

$$\sum_{y\in X(W)}\{y, x_E\}\lambda_y^{-1}\{y^*, z\}=\sum_{y^*\in X(W)}\{x_E, y^*\}\lambda_{y^*}\{y^*, z\}$$
$$=\lambda_{x_E}\{x_E, z^*\}\lambda_z \qquad \text{(by Lemma 3.12)}\,.$$

Now $\lambda_{x_E}=1$ ([loc. cit.]), and $\{x_E, z^*\}=\{x_E, z\}$ since $\{x_E, z\}\in \boldsymbol{Q}$ for any $z\in X(W)$ (by [14, 4.14]). Thus the last expression is equal to $\{x_E, z\}\lambda_z$, which implies (b).

3.14. COROLLARY. *Let $\tilde{\Delta}: \mathcal{CV}\to\mathcal{CV}$ be the linear map defined by $e_x\to R_x^\vee$ $(x\in X(W))$. Then for each $E\in W^\wedge$, not of exceptional type, we have*

(i) $\tilde{\Delta}(e_{x_E})=R_{x_E}$
(ii) $\tilde{\Delta}(R_{x_E})=e_{x_E}$
(iii) $\sharp\tilde{\Delta}\sharp(R_{x_E})=R_{x_E}$.

PROOF. (i) is clear from Lemma 3.13 (i). Next, by definition,

$$\tilde{\Delta}(R_{x_E})=\sum_{y\in X(W)}\{y, x_E\}R_y^\vee\,.$$

However, since $\{y, x_E\}\in\boldsymbol{Q}$, $\{y, x_E\}=\{x_E, y\}=\{x_E^*, y\}$. Thus (ii) follows Lemma 3.13 (ii)-(a). Finally,

$$\tilde{\Delta}\sharp(R_{x_E})=\sum_{y\in X(W)}\{y, x_E\}\lambda_y R_y^\vee$$
$$=\sharp^{-1}R_{x_E} \qquad \text{(by Lemma 3.13 (ii)-(b))}.$$

Hence (iii) follows.

3.15. We preserve the preceding assumption. So, $s\in Z(G^*)^F$ and F is of split type. Then the set $\mathcal{E}(G^{F^m}, \{s\})$ is pointwise fixed by F and parametrized by $X(W)$. Let us identify $C^{(s)}(G^{F^m}/\sim_F)$ with the Euclidean space $\mathcal{CV}$ in 3.11 by associating $\lambda_{\tilde{\rho}}'^{-1}[\tilde{\rho}]$ to e_{x_ρ} ($x_\rho\in X(W)$ is the element corresponding to $\rho\in\mathcal{E}(G^{F^m}, \{s\})$). On the other hand, we have already identified $C^{(s)}(G^F/\sim)$ with $\mathcal{CV}$. We denote by $\Delta=\Delta_G: \mathcal{CV}\to\mathcal{CV}$ the linear map obtained from $N_{F^m/F}^*: C^{(s)}(G^{F^m}/\sim_F)\to C(G^F/\sim)$ under the above identifications. (In fact, Δ is the map sending e_{x_ρ} to the corresponding element of $N_{F^m/F}^*([\lambda_{\tilde{\rho}}'^{-1}\tilde{\rho}])$ and is defined only when its image lies in $\mathcal{CV}$.) Then the following proposition is an immediate consequence of Corollary 1.7, Corollary 1.10 and Corollary 2.10.

3.16. PROPOSITION. *Let the assumptions be as in* 3.15. *We assume $E\in W^\wedge$ is not of exceptional type. Then in each case below, Δ is well-defined and satisfies the following relations.*

(i) $\varDelta(e_{x_E})=R_{x_E}$
(ii) $\varDelta(R_{x_E})=e_{x_E}$
(iii) $\sharp\varDelta\sharp(R_{x_E})=R_{x_E}$.

3.17. We are now ready to verify the theorem in the case where $s\in Z(G^*)^F$ and F is of split type. The method employed by Asai [4] to determine the twisting operators on the space of unipotent class functions can be applied, in general, to our case once formulated appropriately, including the case where F is of non-split type. However, in sometimes (in particular, the case of type E_8) we need a different argument mainly because of the reason that the inverse map of $N^*_{F^m/F}$ is not so easily described as in the case of twisting operators.

We shall verify the theorem in the case where $s\in Z(G^*)$ and F is of split type. In this case, (ii) of the theorem is already clear. As is done in [4], we shall verify the statement (i) and (iii) for individual families $\mathcal{F}$ in $\mathcal{E}(G^{F^m},\{s\})^F=\mathcal{E}(G^{F^m},\{s\})$. If $\mathcal{F}$ consists of one element, (iii) is clear from Corollary 1.7. In the case where $\mathcal{F}$ is one of 4-element families (of non-exceptional type), 8-element families or 21-element family, the similar argument as in p. 2827–p. 2832 in [4] can be applied. In fact, under the identification given in 3.15, we have only to show that $\varDelta(e_x)=\tilde{\varDelta}(e_x)$ for $x\in\mathcal{F}$. Then the argument in [4] is translated to our case one after another. For example, the conditions (1), (2) and (3) in the case of 8-element family in [loc. cit.] corresponds to our conditions (i), (ii) and (iii) in Proposition 3.16. (Formally, $\sharp^{-1}t^*_{-1}\sharp^{-1}$ in [4] corresponds to our map $\varDelta$ by taking $r=m=-1$.) The condition (3) in the case of 21-element family corresponds to our condition (3.7.1).

We now consider the case where $\mathcal{F}$ is an exceptional (4-element) family. Let $\mathcal{F}=\{\rho_1^{(m)},\rho_2^{(m)},\rho_3^{(m)},\rho_4^{(m)}\}$, where $\rho_1^{(m)}$ and $\rho_2^{(m)}$ are principal series representations. Then, by making use of Corollary 1.7, Corollary 1.10 and Remarks 2.11 (i) instead of Proposition 3.16, the same argument as in [4] shows that

$$N^*_{F^m/F}(\lambda_i'^{-1}[\tilde{\rho}_i^{(m)}])=\begin{cases} R_i^{(1)} & \text{if } 1\leqq i\leqq 2 \\ -R_j^{(1)} \quad j\in\{3,4\}\,, & \text{if } 3\leqq i\leqq 4\,, \end{cases}$$

where $\lambda_i'=\lambda'_{\tilde{\rho}_i}$ and $R_i^{(1)}$ are almost characters corresponding to x_{ρ_i}. Hence by putting $\mu_i=\lambda_i'^{-1}$ (resp. $\mu_i=-\lambda_i'^{-1}$) for $i=1,2$ (resp. $i=3,4$), we get the required result.

3.18. Let us now consider the case where $\mathcal{F}$ is a 39-element family. The proof in this case will be done in 3.18–3.22. Let G be a simple group of type E_8 and $s=1$. We shall make a numbering $\{\rho_1,\cdots,\rho_{39}\}$ in 39 irreducible representations in such an order as listed in [14, Appendix]. Hence, ρ_i $(1 \leqq i \leqq 17)$ are in principal series. Let us denote by $x_i = x_{\rho_i}$ the corresponding elements in $\mathcal{M}_\Gamma \subset X(W)$ (where $\Gamma = \mathfrak{S}_5$) and by $e_i = e_{x_i}$ $(1 \leqq i \leqq 39)$ the corresponding vectors in $\mathcal{CV}$. We denote by $\lambda_i = \lambda_{x_i}$ a root of unity corresponding to ρ_i. As for the explicit description of $\mathcal{M}_\Gamma$ and the correspondence $\mathcal{F} \leftrightarrow \mathcal{M}_\Gamma$, we follow the notation in [14, Appendix]. Also, in the following proof, we use frequently the explicit value of the matrix of the pairing $\{\,,\}$ in $\mathcal{M}_\Gamma$, which we refer to the table in p. 112–p. 113 in [14].

It is enough to verify $\Delta = \tilde{\Delta}$ under our identifications in 3.15 to prove the theorem in this case. In the following proof, we use simultaneously Corollary 3.14 and Proposition 3.16. We refer to the conditions Corollary 3.14 (i) and Proposition 3.16 (i) as condition (i) and similarly for condition (ii) or (iii). Moreover, extending the map $E \mapsto x_E$ (resp. $E \mapsto R_{x_E}$) linearly, we can define x_E (resp. R_{x_E}) for a virtual representation E of W, and so we consider the conditions (i)$\sim$(iii) for this E.

Now, by condition (i), we know already

$$(3.18.1) \qquad \Delta = \tilde{\Delta} \quad \text{for } e_i, \qquad 1 \leqq i \leqq 17.$$

Let $E \in W^\wedge$ be the one corresponding to $(g_4, 1) \in \mathcal{M}_\Gamma$. Applying condition (ii) to this E, and using (3.18.1), we see that

$$(3.18.2) \qquad \Delta = \tilde{\Delta} \qquad \text{for } \frac{1}{4}e_{39} - \frac{1}{4}e_{37} - \frac{1}{2}e_{18}.$$

Applying condition (iii) to E by taking into account that $\lambda_{39} = \lambda_{37} = 1$, $\lambda_{18} = -1$, we have

$$(3.18.3) \qquad \Delta = \tilde{\Delta} \qquad \text{for } \frac{1}{4}e_{39} - \frac{1}{4}e_{37} + \frac{1}{2}e_{18}.$$

Thus

$$(3.18.4) \qquad \Delta = \tilde{\Delta} \qquad \text{for } e_{18},\ e_{39} - e_{37}.$$

Next consider $E_1 \leftrightarrow (g_2', 1)$, $E_2 \leftrightarrow (g_2', \varepsilon'')$, $E_3 \leftrightarrow (g_2', \varepsilon')$. Applying the condition (ii) as above to $E_1 + E_3$, we have

$$(3.18.5) \qquad \Delta = \tilde{\Delta} \qquad \text{for } \frac{1}{4}(e_{37} - e_{39}) - \frac{1}{2}e_{22}.$$

Hence by (3.18.4), we see that

$$(3.18.6) \qquad \Delta = \tilde{\Delta} \qquad \text{for } e_{22}.$$

Next by applying (ii) to E_1 and $E_2 - E_1$, we have

(3.18.7) $$\Delta=\tilde{\Delta} \quad \text{for} \quad e_{35}+e_{36}+e_{21}-e_{38}, \quad e_{39}+e_{35}+e_{36}.$$

Now $\lambda_{21}=-1$, $\lambda_{35}=\xi$, $\lambda_{36}=-\xi$, $\lambda_{38}=-1$ and $\lambda_{39}=1$, respectively, where ξ is a square root of -1. Hence by applying condition (iii) to E_1 and E_2-E_1, we have

(3.18.8) $$\Delta=\tilde{\Delta} \quad \text{for} \quad \xi e_{35}-\xi e_{36}-e_{21}+e_{38}, \quad e_{39}+\xi e_{35}-\xi e_{36}.$$

Thus, by (3.18.7) and (3.18.8) we have

(3.18.9) $$\Delta=\tilde{\Delta} \quad \text{for} \quad e_{35}, e_{36}, e_{39} \quad \text{and} \quad e_{21}-e_{38}.$$

Summing up the above arguments, we see that

(3.18.10) $$\Delta(e_i)=\tilde{\Delta}(e_i)=R_i \quad \text{for} \quad 1\leqq i\leqq 17, \quad \text{and} \quad \text{for} \quad i\in\{18, 22, 35, 36, 37, 39\}.$$

3.19. Until now, we did not use the result itself concerning the twisting operators in Asai [4], although the argument used here is quite analogous to his. However, at this step, we need to use his result in an essential way. For each positive integer r, let $t_r^*: C(G^F/\sim)\to C(G^F/\sim)$ be the twisting operator as in 2.1. It is known by [4] that t_r^* stabilizes the subspace $C^{(1)}(G^F/\sim)$. Under the identification $C^{(1)}(G^F/\sim)\cong \mathcal{V}$, we denote also by t_r^* the corresponding map on $\mathcal{V}$. Then by Asai's main theorem [4, Th. 6.2.1], t_r^* stabilizes the subspace of $\mathcal{V}$ generated by e_i $(1\leqq i\leqq 39)$, and

(3.19.1) $$t_r^*(R_i^\vee)=\lambda_i^r R_i^\vee \qquad (1\leqq i\leqq 39).$$

On the other hand, we have a twisting operator τ_r^* on $C(G^{F^m}/\sim_F)$ as in § 2. By abuse of notation, we denote by τ_r^* the corresponding map on $\mathcal{V}$ whenever it is well-defined. In order to proceed further, we need the following result.

(3.19.2) $$\Delta\natural^r R_{x_E}=\tilde{\Delta}\natural^r R_{x_E} \qquad \text{for} \quad \rho_E\in\mathcal{F}, \ r\geqq 0.$$

In fact,

$$\begin{aligned}
\Delta\natural^r R_{x_E} &= \Delta(\tau_r^*)^{-1}R_{x_E} && \text{(by Prop. 2.8)}\\
&= t_r^*\Delta R_{x_E} && \text{(by Lemma 2.2)}\\
&= t_r^*\tilde{\Delta} R_{x_E} && \text{(by condition (ii))}\\
&= t_r^*(\textstyle\sum_i a_i R_i^\vee) && \text{where} \quad R_{x_E}=\textstyle\sum_i a_i e_i, \ a_i\in\bar{\boldsymbol{Q}}_l
\end{aligned}$$

$$=\sum_i a_i\lambda_i^r R_i^\vee \qquad \text{(by (3.19.1))}$$
$$=\tilde{\varDelta}\sharp^r R_{x_E}\,.$$

Let us now take $E\in W^\wedge$ corresponding to $(g_5, 1)$. $R_{x_E}=R_{x_E}^\vee$ is written as

$$R_{x_E}=-\frac{1}{5}(e_{27}+e_{28}+e_{29}+e_{30})+\alpha$$

where α is a linear combination of e_i as in (3.8.10). Note that $\lambda_{27}=\zeta$, $\lambda_{28}=\zeta^2$, $\lambda_{29}=\zeta^3$, $\lambda_{30}=\zeta^4$, where ζ is a primitive fifth root of unity. Hence using (3.19.2), we see that

$$\text{(3.19.3)} \qquad \varDelta=\tilde{\varDelta} \quad \text{for} \quad \zeta^r e_{27}+\zeta^{2r}e_{28}+\zeta^{3r}e_{29}+\zeta^{4r}e_{30}\,.$$

By substituting $r=0,1,2$ and 3, we see that

$$\text{(3.19.4)} \qquad \varDelta=\tilde{\varDelta} \quad \text{for} \quad e_{27}, e_{28}, e_{29}, e_{30}\,.$$

3.20. We continue the proof. As in [4, p. 2833 (7)], it is checked, using (3.7.1), that $\varDelta\circ\tilde{\varDelta}^{-1}(e_{31}+e_{33})$ and $\varDelta\circ\tilde{\varDelta}^{-1}(e_{32}+e_{34})$ are well-defined and are "virtual characters" with support in $\mathscr{F}$, i.e., $\boldsymbol{Z}$-linear combinations of e_i $(1\leqq i\leqq 39)$. In the following, we shall show that

$$\text{(3.20.1)} \qquad \begin{aligned} &\varDelta\circ\tilde{\varDelta}^{-1}(e_{31}+e_{33})=e_{31}+e_{33} \\ &\varDelta\circ\tilde{\varDelta}^{-1}(e_{32}+e_{34})=e_{32}+e_{34}\,. \end{aligned}$$

We consider the case of $e_{31}+e_{33}$, and put $f=e_{31}+e_{33}$, $f'=\varDelta\circ\tilde{\varDelta}^{-1}(e_{31}+e_{33})$. Since $\varDelta$ and $\tilde{\varDelta}$ are isometries when they are defined, we have $\langle f', f'\rangle=2$. Moreover, the condition (i) implies that

$$\text{(3.20.2)} \qquad \langle f, R_i\rangle=\langle f', R_i\rangle \quad \text{for} \quad 1\leqq i\leqq 17\,.$$

By inspecting the explicit table of the matrix of the pairing, we can deduce easily from (3.20.2) that

$$\text{(3.20.3)} \qquad f'=e_i+e_j\,, \qquad i\in\{31, 32\}, j\in\{33, 34\}\,.$$

Take $E_1\leftrightarrow(1, \lambda^3)$, $E_2\leftrightarrow(g_2, 1)$. Applying condition (ii) to E_1-E_2, together with the previous results, we see that

$$\text{(3.20.4)} \quad \varDelta=\tilde{\varDelta} \quad \text{for} \quad h=\sharp\Big(e_{31}+e_{32}+e_{33}+e_{34}+e_{19}-e_{21}+\frac{1}{2}e_{38}-\frac{1}{2}e_{20}\Big)\,.$$

Thus

$$\text{(3.20.5)} \qquad \langle f, \tilde{\varDelta}(h)\rangle=\langle f', \tilde{\varDelta}(h)\rangle\,.$$

Now, $\lambda_{31}=-\theta$, $\lambda_{32}=-\theta^2$, $\lambda_{33}=\theta$, $\lambda_{34}=\theta^2$ and $\lambda_{19}=\lambda_{21}=\lambda_{33}=\lambda_{20}=-1$, where θ is a primitive cubic root of unity. Hence

$$\tilde{\Delta}(h)=-\theta R_{31}-\theta^2 R_{32}+\theta R_{33}+\theta^2 R_{31}-R_{19}+R_{21}-\frac{1}{2}R_{33}+\frac{1}{2}R_{20}\,.$$

We want to compute the inner product of $\tilde{\Delta}(h)$ with various e_i+e_j as in

Table 1.

	$(g_6,-\theta)$	$(g_6,-\theta^2)$	(g_6,θ)	(g_6,θ^2)	$(g_6,-1)$	(g_3,θ)	(g_3,θ^2)	$(g_3,\varepsilon\theta)$	$(g_3,\varepsilon\theta^2)$	$(g_2,-1)$	$(g_2,-\varepsilon)$	$(g_2,-r)$
$(g_6,-\theta)$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$
$(g_6,-\theta^2)$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$
(g_6,θ)	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$
(g_6,θ^2)	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$
$(g_6,-1)$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$
(g_3,θ)	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$
(g_3,θ^2)	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$
$(g_3,\varepsilon\theta)$	$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$
$(g_3,\varepsilon\theta^2)$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$
$(g_2,-1)$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{6}$
$(g_2,-\varepsilon)$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$
$(g_2,-r)$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$

(3.20.3). For this, we need a part of explicit entries of the matrix which is not included in the table in [14]. Later, we need also a calculation based on this part of the matrix. Thus, for convenience sake, we gave in Table 1, the part of the matrix which is needed for our computations. Now, using Table 1, we can compute $\langle e_i+e_j, \tilde{\varDelta}(h)\rangle$ for various e_i+e_j, $i\in\{31, 32\}$, $j\in\{32, 34\}$. Then we see that these values are mutually different, and so characterizes f. Hence, by (3.20.2), we see that

$$f'=f=e_{31}+e_{33}\,.$$

The case for $e_{32}+e_{34}$ is done similarly. So (3.20.1) is verified.

3.21. From (3.20.1), we see that

$$\varDelta=\tilde{\varDelta} \qquad \text{for} \quad \tilde{\varDelta}^{-1}(e_{31}+e_{33})\,. \tag{3.21.1}$$

Now, by using Lemma 3.13 (ii)-(a), we have

$$\tilde{\varDelta}^{-1}(e_i)=\sum_{1\le j\le 39}\{x_i^*, x_j\}e_j$$

for any e_i. Put

$$J=\{19, 20, 21, 23, 24, 25, 26, 31, 32, 33, 34, 38\}\,.$$

Then, by the previous result, $\varDelta(e_i)=\tilde{\varDelta}(e_i)$ for $i\in J$. Since $x_{31}^*=x_{31}$, (3.21.1) implies that

$$\varDelta=\tilde{\varDelta} \qquad \text{for} \quad \sum_{j\in J}(\{x_{31}, x_j\}+\{x_{33}, x_j\})e_j\,. \tag{3.21.2}$$

By the direct computation using Table 1, (3.21.2) implies

$$\varDelta=\tilde{\varDelta} \qquad \text{for} \quad -2e_{25}+e_{26}+2e_{23}-e_{24}\,. \tag{3.21.3}$$

Similarly, using $e_{32}+e_{34}$ in (3.20.1),

$$\varDelta=\tilde{\varDelta} \qquad \text{for} \quad e_{25}-2e_{26}-e_{23}+2e_{24}\,. \tag{3.21.4}$$

Take $E_1\leftrightarrow(1, 1)$, $E_2\leftrightarrow(1, \nu)$. Applying the conditions (ii) and (iii) to E_1-E_2, we have

$$\varDelta=\tilde{\varDelta} \qquad \text{for} \quad \begin{aligned}&e_{23}+e_{24}+e_{33}+e_{34}\,,\\ &\theta e_{23}+\theta^2e_{24}+\theta e_{33}+\theta^2e_{34}\,.\end{aligned} \tag{3.21.5}$$

Take $E_3\leftrightarrow(1, \lambda^3)$, $E_4\leftrightarrow(g_2, r)$. Applying the conditions (ii) and (iii) to E_3-E_4, we have

$$\varDelta=\tilde{\varDelta} \qquad \text{for} \quad \begin{aligned}&e_{25}+e_{26}+e_{23}+e_{24}\,,\\ &\theta e_{25}+\theta^2e_{26}+\theta e_{23}+\theta^2e_{24}\,.\end{aligned} \tag{3.21.6}$$

These 6 vectors in (3.21.3)$\sim$(3.21.6) are linearly independent and determine uniquely $e_{23}, e_{24}, \cdots, e_{34}$. Thus, we have

$$\Delta=\tilde{\Delta} \quad \text{for} \quad e_{23}, e_{24}, e_{25}, e_{26}, e_{33} \text{ and } e_{34}. \tag{3.21.7}$$

3.22. The remaining vectors are $e_{31}, e_{32}, e_{19}, e_{21}, e_{38}$ and e_{20}. Take $E_1 \leftrightarrow (1, 1)$. Applying (3.19.2) to $R_{x_{E_1}}$ with $r=0, 1, 2$ together with the previous result, we see that

$$\begin{aligned} \Delta=\tilde{\Delta} \quad \text{for} \quad & e_{31}+e_{32}+e_{19}+\frac{1}{2}e_{21}+\frac{1}{2}e_{38}+e_{20}, \\ & -\theta e_{31}-\theta^2 e_{32}-e_{19}-\frac{1}{2}e_{21}-\frac{1}{2}e_{33}-e_{20}, \\ & \theta^2 e_{31}+\theta e_{32}+e_{19}+\frac{1}{2}e_{21}+\frac{1}{2}e_{38}+e_{20}. \end{aligned} \tag{3.22.1}$$

Here we have used that $\lambda_{31}=-\theta$, $\lambda_{32}=-\theta^2$, $\lambda_{19}=-1$, $\lambda_{21}=-1$, $\lambda_{38}=-1$ and $\lambda_{20}=-1$.

On the other hand, take $E_2 \leftrightarrow (g_2', 1)$, $E_3 \leftrightarrow (g_2, \varepsilon)$, $E_4 \leftrightarrow (g_6, 1)$, $E_5 \leftrightarrow (1, \nu)$, $E_6 \leftrightarrow (1, \lambda')$. Then applying condition (ii) to E_2, E_3+E_4 and E_5+E_6, we have

$$\begin{aligned} \Delta=\tilde{\Delta} \quad \text{for} \quad & -e_{21}+e_{28} \\ & e_{19}+e_{21} \\ & e_{21}+e_{38}+2e_{20}. \end{aligned} \tag{3.22.2}$$

These 6 vectors in (3.22.1) and (3.22.2) are linearly independent and determine $e_{31}, \cdots, e_{20}$ uniquely. Thus, we have

$$\Delta=\tilde{\Delta} \quad \text{for} \quad e_{31}, e_{32}, e_{19}, e_{21}, e_{38} \text{ and } e_{20}. \tag{3.22.3}$$

This completes the proof for the 39-element family $\mathcal{F}$, and thus completes the proof in the case where $s \in Z(G^*)^F$ and F is of split type.

3.23. Next consider the case where (G, F) is of type ${}^3D_4$ and $s \in Z(G^*)^F$. Using (3.8.1), we may assume $s=1$. As in [4], we embed G as a Levi subgroup of a parabolic subgroup of a simple group of type E_6. So, we use a different notation here. Let G be an adjoint group of type E_6 with split Frobenius map F and L be the standard Levi subgroup of type D_4. Then there exists $w_0 \in W$ such that Fw_0 is a Frobenius map on L and that Fw_0 induces an automorphism γ of order 3 on the Dynkin diagram of L. $\bar{X}(W_L, \gamma)$ consists of four 1-element families and one 4-element family. They are all pointwise fixed by γ and $X(W_L, \gamma)=\bar{X}(W_L, \gamma)\times M$. Now $\mathcal{E}(L^{F^m}, \{1\})$ contains a unique cuspidal representation, which is necessarily Fw_0-stable, and others are all principal series representations for which the statement (iii) of the theorem is verified by Corollary 1.17. Thus, using the argument as in Lemma 2.17 in [16], (see also Remark 4.12, (ii)), we can conclude that the image of $C^{(1)}(G^{F^m}/\sim)$ under $N^*_{F^m/F}$ coincides with $C^{(1)}(G^F/\sim)$. Hence statements (i) and (ii) of the theorem are verified.

We now consider the 4-element family $\mathcal{F}$ in $\mathcal{E}(L^{F^m},\{1\})^{Fw_0}$. Let $\mathcal{F}=\{\rho_1^{(m)},\rho_2^{(m)},\rho_3^{(m)},\rho_4^{(m)}\}$ with $\rho_4^{(m)}$ cuspidal. Since $\rho_i^{(m)}$ $(1\leqq i\leqq 3)$ are in a principal series, by Corollary 1.7, we see that $N^*_{F^m/F}([\tilde{\rho}_i^{(m)}])=R^{(1)}_{\tilde{E}_i}$ for $1\leqq i\leqq 3$, where $E_i\in(W_L)^\wedge_{\text{ex}}$ are representations corresponding to ρ_i. Thus, by the orthogonality relations, we have

$$N^*_{F^m/F}([\tilde{\rho}_4^{(m)}])=cR_4 \tag{3.23.1}$$

for some $c\in\bar{\boldsymbol{Q}}_l^*$ of absolute value 1, $(R_4=R_x=R_x^\vee$ for $x=x_{\rho_4})$. We have to determine c. In the following, we shall denote $[\rho_i^{(1)}]$ (resp. $R^{(1)}_{x\rho_i}$, $[\tilde{\rho}_i^{(m)}]$) by e_i (resp. R_i, f_i) $(1\leqq i\leqq 4)$, respectively. As in [4], we shall make use of Proposition 3.6 with $w=w_0$. Let us consider e_1+e_4 in $\mathcal{V}^{(1)}(L,F\dot{w}_0)$. Since $e_1+e_4=R_1+R_4$, we have

$$\Delta_L^{-1}(e_1+e_4)=f_1+c^{-1}f_4.$$

Thus, by Proposition 3.6, we see that

(3.23.2) $\quad \Delta_G(a_{w_0}(f_1+c^{-1}f_4))$ is a $\boldsymbol{Z}$-linear combination of $e_x\in\mathcal{V}^{(1)}(G,F)$.

Let $\mathcal{F}_1$ be the family in E_6 corresponding to 30_p, i.e., $\mathcal{F}_1=\{[30_p],[15_q],[15_p],D_4[1]\}$ under the notation in [14, p. 363]. Taking the $\mathcal{F}_1$-part in (3.23.2), we see that

(3.23.3) $\quad \varepsilon R_{[30_p]}+c^{-1}\lambda' R_{D_4[1]}$ is a $\boldsymbol{Z}$-linear combination of e_x,

where $\varepsilon=\pm1$ and $\lambda'=\lambda'_\rho$ is the root of unity corresponding to $\rho=D_4[1]$, hence $(\lambda')^m=-1$. From this we see that

$$\begin{aligned}&\frac{1}{2}(\varepsilon+c^{-1}\lambda')\in\boldsymbol{Z}\\&\frac{1}{2}(\varepsilon-c^{-1}\lambda')\in\boldsymbol{Z}.\end{aligned} \tag{3.23.4}$$

Hence we see that $c^{-1}\lambda'\in\boldsymbol{Z}$. As $c^{-1}\lambda'$ has an absolute value 1, this implies that $c=\pm\lambda'$. Since $\lambda_4=-1$, (3.23.1) together with $\mu=c^{-1}=\pm\lambda'^{-1}$ satisfies the assertion of the theorem.

3.24. We now consider the case where (G,F) is of type ${}^2E_6$, and $s\in Z(G^*)^F$. It is clear that except two cuspidal representations $\rho_1^{(m)}$ and $\rho_2^{(m)}$, all the representations in $\mathcal{E}(G^{F^m},\{s\})$ are F-stable, while F may permute $\rho_1^{(m)}$ and $\rho_2^{(m)}$ in $\mathcal{E}(G^{F^m},\{s\})$. However the last possibility does not occur and the set $\mathcal{E}(G^{F^m},\{s\})$ is fixed pointwisely by F. In fact, we may assume, for each F-stable class $\{s\}$ such that $s\notin Z(G^*)$, that $\dim C^{(s)}(G^F/\sim)=\dim C^{(s)}(G^{F^m}/\sim_F)$. Moreover, for each $s\in Z(G^*)^F$, $\dim C^{(s)}(G^F/\sim)\geqq\dim C^{(s)}(G^{F^m}/\sim_F)=|\mathcal{E}(G^{F^m},\{s\})^F|$, and equality holds if and only if $\mathcal{E}(G^{F^m},\{s\})^F=\mathcal{E}(G^{F^m},\{s\})$. Since $\dim C(G^F/\sim)=\dim C(G^{F^m}/\sim_F)$, we see that the equality holds for each $s\in Z(G^*)^F$.

Now the determination of $N^*_{F^m/F}$ is done by entirely the same way as in Asai [4] under an appropriate formulation, by applying Proposition 3.6 to the cases $D_4\subset E_6$ and $E_6\subset E_7$. So we omit details of the proof.

3.25. We now consider the case where s is not in the center of G^*. If W_s is a product of various Coxeter groups of type A, each family consists of one element and it is enough to apply Corollary 1.7. In turn, if F is of split type and γ is trivial, then $s\in T^{*F}$ and at most 8-element family appears in $\bar{X}(W_s,\gamma)$. Thus the similar argument as in 3.17 can be applied as well to this case. Hence the remaining cases are the following three cases:

(i) G is of type E_7, W_s is of type $D_6\times A_1$ and γ acts non-trivially on the first factor,

(ii) G is of type E_8, W_s is of type $D_5\times A_3$ and γ acts non-trivially on both factors,

(iii) G is of type E_8, W_s is of type D_8 with non-trivial action of γ.

We only consider, in the following, the cases (ii) and (iii). The case (i) is done easily in a similar way as the case (ii).

The case (ii).

There exists $s\in T^*$ such that (W_s,γ) is as above only when $q\equiv 3 \pmod 4$, and in this case w_0 is given by $w_0=w^{D_5}w^{A_3}w^{E_8}$, where w^X is the longest element of the Weyl subgroup of type X. Let us now consider the standard Levi subgroup L of type D_5. We follow the notation in 3.10. In particular, $W_s\cong W_L\times W_3$, where W_3 is a Coxeter group of type A_3. Then for each $w\in W_3$, $F\dot{w}\dot{w}_0$ gives rise to a Frobenius map on L. Under the notation in 3.5, $\mathcal{V}_1^{(s)}=\mathcal{V}^{(s)}(G,F)$ and $\Delta_G:\mathcal{V}^{(s)}(G,F)\to\mathcal{C}\mathcal{V}^{(s)}(G,F)$ is defined. Thus, by the dimension argument, we can identify $C^{(s)}(G^{F^m}/\sim_F)$ with $\mathcal{V}^{(s)}(G,F)$ as in 3.5. So, to show 3.2 it is enough to determine the map Δ_G. Since $\mathcal{E}(G^{F^m},\{s\})$ consists of one-element families and 4-element families, we have only to consider the case of 4-element families. $\mathcal{E}(L^{F^m},\{s\})$ contains two 4-element families $\mathcal{F}$ and $\mathcal{F}'$. For each $E\in W_3^\wedge=(W_3)^\wedge_{\mathrm{ex}}$, there exists a 4-

element family in $\mathcal{E}(G^{F^m}, \{s\})$ associated with $\mathscr{F}$ (resp. $\mathscr{F}'$) which we denote by $\mathscr{F}_E$ (resp. $\mathscr{F}'_E$).

Let us denote by e_i $(1\leqq i\leqq 4)$ the vectors in $\mathcal{V}^{(s)}(L, F\dot{w}_0)$ and by f_i $(1\leqq i\leqq 4)$ the vectors in $\mathcal{U}^{(s)}(L, F\dot{w}_0)$ (by taking a representative) corresponding to the family $\mathscr{F}$. $\{e_i\}$, $\{f_i\}$ can be regarded also as vectors in $\mathcal{V}^{(s)}(L, F\dot{w}_0\dot{w})$, $\mathcal{U}^{(s)}(L, F\dot{w}_0\dot{w})$ for each $w\in W_3$, respectively. We denote by $e_{i,E}$ (resp. $f_{i,\tilde{E}}$) the vectors in $\mathcal{V}^{(s)}(G, F)$ (resp. $\mathcal{U}^{(s)}(G, F)$) corresponding to the family $\mathscr{F}_E$, where $\tilde{E}$ is a fixed extension of E. Also, we denote by $R_{i,\tilde{E}}$ the almost character corresponding to $f_{i,\tilde{E}}$ in $\mathcal{V}^{(s)}(G, F)$. The similar elements are defined with respect to the family $\mathscr{F}'$, which we denote by attaching primes, like $e'_i, f'_j, \cdots$. Since $f_{i,\tilde{E}}$ (resp. $f'_{i,\tilde{E}}$) for $1\leqq i\leqq 3$ are associated with principal series characters, we know already, by Corollary 1.7, that

$$\left.\begin{aligned} \Delta_G(f_{i,\tilde{E}}) &= R_{i,\tilde{E}} \\ \Delta_G(f'_{i,\tilde{E}}) &= R'_{i,\tilde{E}}\,. \end{aligned}\right. \qquad (1\leqq i\leqq 3)$$

Thus, to prove 3.2, it is enough to show that

$$\begin{aligned} \Delta_G(f_{4,\tilde{E}}) &= R_{4,\tilde{E}} \\ \Delta_G(f'_{4,\tilde{E}}) &= R'_{4,\tilde{E}}\,. \end{aligned} \tag{3.25.1}$$

We shall apply Proposition 3.6 to our case with $(L, F\dot{w}_0\dot{w})$ for each $w\in W_3$. Then (3.6.1) is written as

$$a_{w_0w}(f_i) = \sum_{E\in W_3^{\wedge}} \mathrm{Tr}\,(\gamma w, \tilde{E}) f_{i,\tilde{E}} \tag{3.25.2}$$

for each $w\in W_3$. We take a particular extension $\tilde{E}$ of E so that γ acts as a conjugation by w_3 (w_3 is the longest element in W_3). Hence $\mathrm{Tr}\,(\gamma w, \tilde{E}) = \mathrm{Tr}\,(w_3w, E)$. Put $\hat{f}_i = \Delta_L^{-1}(e_i)$, the Fourier transform of f_i. Then by (3.25.2),

$$a_{w_0w}(\hat{f}_i) = \sum_{E\in W_3^{\wedge}} \mathrm{Tr}\,(w_3w, E)\hat{f}_{i,\tilde{E}}\,, \tag{3.25.3}$$

where $\hat{f}_{i,\tilde{E}}$ is the Fourier transform of $f_{i,\tilde{E}}$ in $\mathcal{U}^{(s)}(G, F)$. By Proposition 3.6, $\Delta_G\circ a_{w_0w}(\hat{f}_i) = R^G_{(\dot{w}_0\dot{w})}(e_i)$ is a virtual character. Moreover Δ_G is an isometry. Using these facts together with the explicit information about the matrix of $\{\ ,\ \}$, we can determine $R_{(\dot{w}_0\dot{w})}(e_i)$ for each e_i $(1\leqq i\leqq 4)$ whenever $w\neq w_3$, i.e., we have

$$\Delta_G\circ a_{w_0w}(\hat{f}_i) = \sum_{E\in W_3^{\wedge}} \mathrm{Tr}\,(w_3w, E) e_{i,E} \qquad (1\leqq i\leqq 4),\ w\neq w_3\,. \tag{3.25.4}$$

Now put

$$
(3.25.5)\qquad
\begin{aligned}
f_w &= \sum_{E \in W_3^{\wedge}} \mathrm{Tr}\,(w_3 w, E) f_{4,\tilde{E}}\,, \\
\hat{e}_w &= \sum_{E \in W_3^{\wedge}} \mathrm{Tr}\,(w_3 w, E) R_{4,\tilde{E}}\,.
\end{aligned}
$$

Then by (3.25.4), we have

$$
(3.25.6)\qquad \varDelta_G(f_w) = \hat{e}_w \qquad \text{for} \quad w \neq w_3\,.
$$

Correspondingly, with respect to the family $\mathcal{F}'$,

$$
(3.25.7)\qquad \varDelta_G(f'_w) = \hat{e}'_w \qquad \text{for} \quad w \neq w_3\,.
$$

Using the orthogonality relations of characters in W_3, we see that (3.25.1) is reduced to showing the following two relations.

$$
(3.25.8)\qquad
\begin{aligned}
\varDelta_G(f_{w_3}) &= \hat{e}_{w_3} \\
\varDelta_G(f'_{w_3}) &= \hat{e}'_{w_3}\,.
\end{aligned}
$$

We shall show (3.25.8). We note that f_w and $f'_{w'}$ are orthogonal each other for any w and w' and that f_w (resp. f'_w) are mutually orthogonal for different conjugacy classes of $w_3 w$. Thus we can write

$$
(3.25.9)\qquad
\begin{aligned}
\varDelta_G(f_{w_3}) &= a\hat{e}_{w_3} + b\hat{e}'_{w_3} \\
\varDelta_G(f'_{w_3}) &= c\hat{e}_{w_3} + d\hat{e}'_{w_3}
\end{aligned}
$$

where $a, b, c, d \in \bar{\boldsymbol{Q}}_l$ such that

$$
(3.25.10)\qquad
\begin{cases}
|a|^2 + |b|^2 = 1 \\
|c|^2 + |d|^2 = 1 \\
a\bar{c} + b\bar{d} = 0\,.
\end{cases}
$$

As in the previous case, we shall compute $\varDelta_G(a_{w_0 w_3}(\hat{f}_4))$. Then by (3.25.9), we see easily that

$$
\varDelta_G(a_{w_0 w_3}(\hat{f}_4)) = \sum_{E \in W_3^{\wedge}} \dim E \cdot e_{i,E} + \alpha\,,
$$

where

$$
\alpha = \frac{1}{2}(1-a)\hat{e}_{w_3} - \frac{1}{2} b\hat{e}'_{w_3}\,.
$$

On the other hand, Proposition 3.6 implies that $\varDelta_G(a_{w_0 w_3}(\hat{f}_4)) = R_{(\dot{w}_0 \dot{w}_3)}(e_4)$ is a virtual character. Thus α is a virtual character. This implies that $1-a$, $b \in 4\boldsymbol{Z}$. Hence by (3.25.10) we see that $a=1$, $b=0$. Similar argument shows that $c=0$, $d=1$. This proves (3.25.8) and so proves 3.2 in the case (ii).

The case (iii).

This case is done using essentially the same method as in [3], [16]. However, the argument given in the proof of Lemma 2.10 in [16] cannot be applied directly to our case because of the lack of a Levi subgroup whose Weyl subgroup coincides with a Weyl subgroup of W_s of type $D_r \times A_{7-r}$ ($r \leqq 7$). For this reason, we need another argument to prove the lemma corresponding to Lemma 2.10. In fact, we have only to imitate the proof of Lemma 2.13 in [16] as follows. Under the notation in [3], we consider the map $I^+_{(v)}: V^-_r \to V^-_r$ (cf. [3, p. 577]). The proof of the lemma corresponding to Lemma 2.10 in [16] is reduced, under a suitable identifications, to showing that

$$\tilde{\delta}^-(f) = \delta^-(f) \qquad \text{for each } f \text{ in } I^+_{(v)}$$

with notations in [3, 2.8]. Now it is easily checked that each e^-_Λ appearing in the expression of $I^+_{(v)}(e^-_{(S,T)})$ belongs to mutually different families. Thus, if we apply Lemma 2.8.10 in [3] to $f = I^+_{(v)}(e^-_{(S,T)})$, we get $\tilde{\delta}^-(f) = \delta^-(f)$ by (II) in Lemma 2.8.10, since $a_i = 1$ for each i.

Other steps are done similarly, and we get the result. This proves the case (iii) and so completes the proof of the theorem.

3.26. Once Theorem 3.2 is established, Proposition 3.6 gives a decomposition of the twisted induction $R^G_{L(\dot{w})}$ in terms of a_w and $N^*_{F^m/F}$. More precisely, under the notation in 1.14, for each $E \in (W_\delta)^\wedge_{\mathrm{ex}}$, $E' \in (W'_\delta)^\wedge_{\mathrm{ex}}$, let $\tilde{E}, \tilde{E}'$ be their extensions to $\tilde{W}_\delta, \tilde{W}'_\delta$, respectively. Put $V_{\tilde{E}',\tilde{E}} = \mathrm{Hom}_{W'_\delta}(\tilde{E}', \tilde{E})$, and define an automorphism γ_w on $V_{\tilde{E}',\tilde{E}}$ by $\gamma_w(f) = (\gamma_\delta y) \circ f \circ \gamma'^{-1}_\delta$ for each $f \in V_{\tilde{E}',\tilde{E}}$, where $w = w_1 y w'^{-1}_1$. Then Proposition 3.6 is restated as follows:

3.27. COROLLARY. *Let $w \in W$ be such that $Fw(J) = J$. Let $L = L_J$ be the standard Levi subgroup and $\{s\}$ be an Fw-stable class in L^*. Then for a sufficiently divisible m, (in particular, $s \in T^{*F^m}$), the following diagram is commutative.*

$$\begin{array}{ccc} C^{(s)}(G^F/\sim) & \xleftarrow{N^*_{F^m/F}} & C^{(s)}(G^{F^m}/\sim_F) \\ {\scriptstyle R^G_{L(\dot{w})}} \uparrow & & \uparrow {\scriptstyle a_w} \\ C^{(s)}(L^{F\dot{w}}/\sim) & \xleftarrow{N^*_{F^m/F\dot{w}}} & C^{(s)}(L^{F^m}/\sim_{F\dot{w}}) \end{array}$$

Here a_w is a linear map defined for each $F\dot{w}$-stable representation $\pi_{E'} \in (L^{F^m})^\wedge$ by

$$a_w([\tilde{\pi}_{E'}]) = \varepsilon_y \sum_{E \in (W_\delta)^\wedge_{\mathrm{ex}}} \mathrm{Tr}\,(\gamma_w, V_{\tilde{E}',\tilde{E}})[\tilde{\rho}_E],$$

where $y\to\varepsilon_y=\pm 1$ *is a certain linear character of* W_δ, *and* $\pi_{E'}, \tilde{\rho}_E$ *are as in* 1.14.

§ 4. The case of classical groups

4.1. The similar result as in Theorem 3.2 for the case of classical groups with connected center has been treated in [16, Th. 2.2]. However the proof given in [16] contains some gap. In particular, Lemma 2.17 in [loc. cit.] can be applied only when $s\in T^{*F}$. Whereas, the case that $s\notin T^{*F}$ occurs, in general, for classical groups with connected center, and this case cannot be covered by the argument there. In this section, we shall fill the above gap, i.e., we shall prove a proposition which is a generalization of Lemma 2.17 and works also in a case where $s\notin T^{*F}$ in a slightly weaker form. Once this is done, we see that Theorem 2.2 in [16] holds with minor change by the argument there. (I am very grateful to G. Lusztig and B. Srinivasan for pointing out the above mentioned error in [16]. Also I wish to thank F. Digne, J. Michel and Srinivasan, from the discussions with whom the idea of using cuspidal representations of Levi subgroups instead of maximal torus was inspired.)

We now state the result in the case of classical groups.

4.2. THEOREM (cf. [16, Th. 2.2]). *Let* G *be a connected classical group with connected center and* $\{s\}$ *be an* F*-stable class in* G^*. *Then for a sufficiently divisible* m, *the following holds.*

(i) $N^*_{F^m/F}$ *induces an isomorphism from* $C^{(s)}(G^{F^m}/\sim_F)$ *onto* $C^{(s)}(G^F/\sim)$.

(ii) *There exists a natural bijection between* $\mathcal{E}(G^{F^m},\{s\})^F$ *and* $X(W_s,\gamma)/M$.

(iii) *For each* $\rho\in\mathcal{E}(G^{F^m},\{s\})^F$, *take an extension* $\tilde{\rho}$ *of* ρ *to* $\tilde{G}^{F^m}$. *Let* x_ρ *be an element in* $X(W_s,\gamma)$ *which is mapped to* ρ *under the correspondence* (ii). *Then we have*

$$N^*_{F^m/F}([\mu_{\tilde{\rho}}\tilde{\rho}])=R_{x_\rho},$$

where $\mu_{\tilde{\rho}}$ *is a constant of absolute value* 1. *If* G *is simple modulo center and of type* B_n *or* C_n, $\mu_{\tilde{\rho}}$ *is a root of unity such that* $(\mu_{\tilde{\rho}})^m=\lambda_\rho^{-1}$.

In view of the argument given in [16], to prove Theorem 4.2, it is enough to show the following proposition which is an extended form of Lemma 2.17 in [loc. cit.].

4.3. PROPOSITION (cf. [16, Lemma 2.17]). *Let* G *be a connected classical group (simple modulo center) with connected center. Let* $\{s\}$ *be an* F*-stable* *class in* G^* *such that* $Z_{G^*}(s)^*$ *has the same semisimple rank as* G *and that* $\mathcal{E}(G^{F^m},\{s\})^F$ *contains a cuspidal representation* ρ_0 *(which is unique in* $\mathcal{E}(G^{F^m},\{s\})$*). Assume that for each* $\rho\in\mathcal{E}(G^{F^m},\{s\})^F$ *except for* ρ_0*, the statement* (iii) *in Theorem* 4.2 *holds. Then* $N^*_{F^m/F}$ *preserves the spaces corresponding to* $\{s\}$ *and* (iii) *holds also for* ρ_0.

The remainder part of this section is devoted to the proof of Proposition 4.3. First we show the following lemma.

4.4. LEMMA. *Let* G *and* $\{s\}$ *be as in* 4.3. *Take* s *in* T^* *and let* $\gamma=w_0^{-1}F: W_s\to W_s$ *be the graph automorphism as in* 1.1. *Assume that* $w\neq 1$. *Then there exists an* F*-stable standard Levi subgroup* $L=L_J$ *such that*

(i) L *is of type* $A_1\times\cdots\times A_1$ *(r-times),*
(ii) $\dot{w}_0\in L^*$*, i.e., the class* $\{s\}$ *in* L^* *is* F*-stable,*
(iii) s *is regular semisimple in* L^*.

PROOF. We have the following four possibilities:

(i) G^*: type B_n, W_s: type $B_{m_1}\times D_{m_2}$ ($m_1\geqq 1$, $m_2\geqq 2$, $n=m_1+m_2$), and γ acts non-trivially on the second factor.

(ii) G^*: type C_n, W_s: type $C_m\times C_m$ ($m\geqq 1$, $n=2m$) and γ permutes two factors.

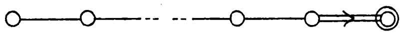
(iii) G^*: type D_n, W_s: type $D_{m_1}\times D_{m_2}$ ($m_1\geqq 2$, $m_2\geqq 2$, $n=m_1+m_2$), and γ preserves both factors.

(iv) G^*: type D_n, W_s: type $D_m\times D_m$ ($m\geqq 2$, $n=2m$) and γ permutes two factors.

In the following, we shall identify the Weyl group W_n of type B_n (or C_n) with the subgroup of the symmetric group of $2n$ letters $\{1,2,\cdots,n, n',\cdots,2',1'\}$ consisting of permutations which permute simultaneously $i\to j$, $i'\to j'$, or $i\to j'$, $i'\to j$. Then the Weyl group W'_n of type D_n is identified with the subgroup of W_n of index 2 generated by permutations $(1,2),\cdots,(n-1,n)$ and $(n-1,n')$, (here (i,j) denotes the permutation $i\leftrightarrow j$, $i'\leftrightarrow j'$). Then the simple root system of type B_n (resp. D_n) is determined naturally so that the corresponding set of simple reflections are $\{(1,2),(2,3),\cdots,(n-1,n),(n,n')\}$ (resp. $\{(1,2),(2,3),\cdots,(n-1,n),(n-1,n')\}$), respectively.

First consider the case (i). Then by replacing s by a suitable conjugate in W, we may assume that W_s is the following group; the first factor W_{m_1} is the Weyl group of type B_{m_1} corresponding to m_1 letters $\{1,2,\cdots,m_1\}$ and the second factor W'_{m_2} is the Weyl group of type D_{m_2} corresponding to m_2 letters $\{m_1+1, m_1+2,\cdots,n\}$. Taking the simple root system in a natural way, we see that $w_0=(n,-n)$ in this case. Now let $L^*=L_J^*$ be the standard

Levi subgroup corresponding to the circled dot in the Dynkin diagram below,

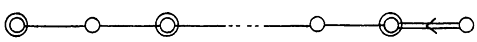
L_J^* :  type A_1.

Then it is clear that $\dot{w}_0 \in L^*$ and that $W_J \cap W_s = \{1\}$.

Next consider the case (ii). In this case, we may take $W_m \times W_m$ as W_s, the first factor (resp. the second factor) is the Weyl group of type C_m corresponding to m letters $\{1, 3, 5, \cdots, 2m-1\}$, (resp. $\{2, 4, 6, \cdots, 2m\}$). Then, with respect to a natural simple root system as before, we see that

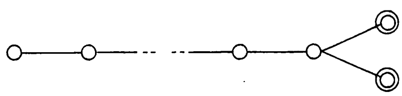
$$w_0 = (1, 2)(3, 4) \cdots (2m-1, 2m) .$$

Now let us take as $L^* = L_J^*$ the following type:

L_J^* :  type $A_1 \times \cdots \times A_1$
m times.

It is clear that $\dot{w}_0 \in L_J^*$ and that $W_J \cap W_s = \{1\}$.

Next consider the case (iii). We may take as $W_s \cong W'_{m_1} \times W'_{m_2}$ the following one, the first factor is the Weyl group of type D_{m_1} corresponding to m_1 letters $\{1, 2, \cdots, m_1-1, n-1\}$ and the second factor is the Weyl group of type D_{m_2} corresponding to m_2 letters $\{m_1, m_1+1, \cdots, n-2, n\}$. In this case, w_0 must stabilize both factors and act non-trivially on them. This implies, with respect to a natural simple root system, that $w_0 = (n-1, n)(n-1, n')$. In this case, we take as L_J^* the following one.

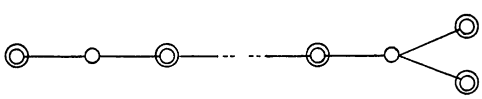
L_J^* :  type $A_1 \times A_1$.

Then L_J^* is F-stable (F: split or non-split) and $\dot{w}_0 \in L_J^*$, $W_J \cap W_s = 1$.

Finally consider the case (iv). In this case, we take as $W_s \cong W'_m \times W'_m$ the following one, the first factor (resp. the second factor) is the Weyl group of type D_m corresponding to m letters $\{1, 3, 5, \cdots, 2m-1\}$ (resp. $\{2, 4, 6, \cdots, 2m\}$. In this case, w_0 has order 2, and is given by

$$w_0 = (1, 2)(3, 4) \cdots (2m-1, 2m) .$$

Let us take L_J^* the following type,

L_J^* :  type $A_1 \times \cdots \times A_1$
$(m+1)$ *times*.

Then L_J^* is F-stable, (F: split or non-split) and $w_0 \in L_J^*$, $W_J \cap W_s = \{1\}$. Thus the lemma is proved.

4.5. Let L be an F-stable Levi subgroup of G as given in Lemma 4.4. We shall consider two Frobenius maps F', F'' on L as follows.

(4.5.1) $F'' = F$, and F' is a Frobenius map with respect to F_{q^m}-structure such that $F'F'' = F''F'$ and that some power of F' coincides with some power of F''.

Then we have the following crucial lemma.

4.6. LEMMA. *Let L be a Levi subgroup as in Lemma 4.4, and F' and F'' be two Frobenius maps as in 4.5. We assume m is sufficiently divisible. Then for each class $\{s\}$ in L^* which is both F' and F''-stable, $N^*_{F''/F'}$ induces an isomorphism*

$$N^*_{F''/F'} : C^{(s)}(L^{F''}/\sim_{F'}) \xrightarrow{\sim} C^{(s)}(L^{F'}/\sim_{F''^{-1}}) .$$

Moreover, in the case where s is regular semisimple, we denote by δ' (resp. δ'') the unique irreducible representation of $L^{F'}$ (resp. $L^{F''}$) belonging to $\mathcal{E}(L^{F'}, \{s\})^{F''}$ (resp. $\mathcal{E}(L^{F''}, \{s\})^{F'}$). Then there exists a suitable extension $\tilde{\delta}'$ of δ' (resp. $\tilde{\delta}''$ of δ'') to $L^{F'}\langle F''\rangle$ (resp. $L^{F''}\langle F'\rangle$) such that

$$N^*_{F''/F'}([\tilde{\delta}'']) = [\tilde{\delta}'] .$$

4.7. Assuming Lemma 4.6, we shall prove Proposition 4.3. The proof of Lemma 4.6 will be given in 4.10 and 4.11 after some preliminary.

If $w_0 = 1$, then $s \in T^{*F}$. So, Lemma 2.17 in [16] can be applied. Hence we assume that $w_0 \neq 1$. Let $L = L_J$ be the F-stable Levi subgroup associated with $\{s\}$ as in Lemma 4.6. Let $\delta_0 \in \hat{L}^F$ be a unique element in $\mathcal{E}(L^F, \{s\})$. Put

$$\mathcal{W} = \{w \in W^F \mid w(J) = J, {}^w\delta_0 \cong \delta_0\} .$$

For each $w \in \mathcal{W}$, we have the following commutative diagram by Proposition 1.13.

$$\begin{array}{ccc} C(G^F/\sim_{F^m}) & \xrightarrow{N^*_{F/F^m}} & C(G^{F^m}/\sim_{F^{-1}}) \\ {\scriptstyle a_{F^m w}} \uparrow & & \uparrow {\scriptstyle R^{(F)}_{L(\dot{w})}} \\ C(L^F/\sim_{F^m \dot{w}}) & \xrightarrow{N^*_{F/F^m \dot{w}}} & C(L^{F^m \dot{w}}/\sim_{F^{-1}}) . \end{array}$$

Let M be an F-stable subgroup of $N_G(L)$ generated by L and $\dot{w} \in N_G(T)$, a representative of $w \in \mathcal{W}$. Then δ_0 can be extended to a representation of

$M^F\langle F^m\rangle$ which we denote by $\tilde{\delta}_0$. Thus $\tilde{\delta}_0$ determines an element $[\tilde{\delta}_0]$ in $C^{(s)}(L^F/\sim_{F^m\dot{w}})$ for each $w\in\mathcal{W}$. Now, by Lemma 4.6, there exists $\delta_w\in\mathcal{E}(L^{F^m\dot{w}},\{s\})$, F-stable, such that $N^*_{F/F^m\dot{w}}([\tilde{\delta}_0])=[\tilde{\delta}_w]$ for some extension $\tilde{\delta}_w$ of δ_w. Hence the above diagram, together with Lemma 1.5, implies that

$$N^*_{F^m/F}\circ\iota^*\circ R^{(F)}_{L(\dot{w})}([\tilde{\delta}_w])=\iota^*\circ a_{F^m w}([\tilde{\delta}_0])\,. \tag{4.7.1}$$

Let us see the right hand side of (4.7.1) more precisely. We note that, since the class $\{s\}$ in L^* is regular semisimple, endomorphism algebra of $\mathrm{Ind}_{P_J^F}^{G^F}\delta_0$ is isomorphic to the endomorphism algebra of $\mathrm{Ind}_{P_K^F}^{G^F}\delta_0'$, where δ_0' is a cuspidal representation of a Levi subgroup L_K $(K\subset J)$ such that the inducing up of δ_0' to L_J coincides with δ_0. Hence we may assume that δ_0 is cuspidal. Now by Lusztig [14, Th. 8.6], it is known that the endomorphism algebra of $\mathrm{Ind}_{P_J^F}^{G^F}\delta_0$ is isomorphic to $H(\mathcal{W},q)$ the Hecke algebra associated with the Coxeter group $\mathcal{W}$ with some exponents. Thus, by making use of Lemma 1.15, we see that $a_{F^m w}([\tilde{\delta}_0])$ is expressed as

$$a_{F^m w}([\tilde{\delta}_0])=C_w(q)\sum_{E_1\in\mathcal{W}^\wedge}\mathrm{Tr}(T_w,E_1(q))[\rho_{E_1}]\,, \tag{4.7.2}$$

where $C_w(q)$ is an integral power of q and T_w is a standard basis of $H(\mathcal{W},q)$, and $E_1(q)$ is the irreducible representation of $H(\mathcal{W},q)$ corresponding to E_1.

Put $Y_w=a_{F^m w}([\tilde{\delta}_0])$. Then $Y_w\in C^{(s)}(G^F/\sim)$. Now, by passing to the dual representation, (4.7.1) implies that Y_w is contained in the image of $C^{(s)}(G^{F^m}/\sim_F)$ under the map $N^*_{F^m/F}$. We denote by V_1 the subspace of $C^{(s)}(G^F/\sim)$ generated by Y_w $(w\in\mathcal{W})$. We also define a subspace V of $C^{(s)}(G^F/\sim)$ as the one generated by R_{x_ρ} for each $\rho\in\mathcal{E}(G^{F^m},\{s\})^F$ such that $\rho\neq\rho_0$. Then V is a codimension one subspace of $C^{(s)}(G^F/\sim)$. Thus, to prove that $N^*_{F^m/F}$ preserves the spaces corresponding to $\{s\}$, it is enough to show that $V_1\subsetneqq V$. Suppose $V_1\subseteqq V$. Put $x_0=x_{\rho_0}\in X(W_s,\gamma)$. Now R_{x_0} is orthogonal to V, hence orthogonal to V_1. On the other hand, using the orthogonality relations of the characters of Hecke algebra $H(\mathcal{W},q)$, we see, by (4.7.2), that V_1 coincides with the subspace of $C^{(s)}(G^F/\sim)$ generated by ρ_{E_1}, $(E_1\in\mathcal{W}^\wedge)$. Thus R_{x_ρ} is orthogonal to each ρ_{E_1}. However, if we take the special representation E of W_s stable by γ associated with the family containing ρ_0, there exists $E_1\in\mathcal{W}^\wedge$ such that ρ_{E_1} coincides with ρ_{x_E}. Since R_{x_0} is not orthogonal to ρ_{x_E}, this is a contradiction. Thus we have showed that $N^*_{F^m/F}$ induces an isomorphism from $C^{(s)}(G^{F^m}/\sim)$ onto $C^{(s)}(G^F/\sim)$.

Now we can write as

$$N^*_{F^m/F}(\lambda'^{-1}_{\tilde{\rho}_0}[\tilde{\rho}_0])=\alpha_0 R_{x_0}\,,$$

by some $\alpha_0 \in \bar{\boldsymbol{Q}}_l$. Since $N^*_{F^m/F}$ is an isometry, we see that α_0 has absolute value 1. We want to determine α_0. Let r_0 be as in 1.2, hence $F^{r_0}(s)=s$ and F^{r_0} is a power of F_0. In this case $r_0=1, 2$ or 4.

We now consider the left hand side of (4.7.1). Note $R^{(F)}_{L(\dot{w})}([\tilde{\delta}_w])$ is given as in (1.11.2) with $F''=F$, $F'=F^m$. Put $M^i=(H^i_c(S, \bar{\boldsymbol{Q}}_l)\otimes\tilde{\pi})^{L^{F^m\dot{w}}}$. There exists an embedding $M^i \hookrightarrow H^j_c(X_{w'}, \mathcal{F}_{w',\theta_{w'}})$ as G^{F^m}-modules, for some j, w'. As in 1.2, F^{r_0} acts on $H^j_c(X_w, \mathcal{F}_{w',\theta_{w'}})$ naturally, and this embedding becomes F^{r_0}-equivariant, in a suitable sense, (see. (6.21.2) in [14]).

For each $\rho \in \mathcal{E}(G^{F^m}, \{s\})^F$, consider the ρ-isotypic subspace M^i_ρ of M^i. The eigenvalues of F^m on this subspace have the form $\lambda_\rho q^{jm/2}$ for some integer $j \leqq i$. For an eigenvalue $\mu=\lambda_\rho q^{jm/2}$, let $M^i_{\rho,\mu}$ be the generalized eigenspace of F^m on M^i_ρ, which is an G^{F^m}-submodule of M^i stable by F. There exists a filtration of $M^i_{\rho,\mu}$ by G^{F^m}-modules, stable by F, each of whose successive quotients is isomorphic to ρ as G^{F^m}-modules. If we define the action of σ on each quotient by $\sigma=\lambda'_{\tilde{\rho}} q^{j/2}(\tilde{F}^*)^{-1}$, (see 1.11 for $(\tilde{F})^*$), each successive quotient becomes a $\tilde{G}^{F^m}$-module. If we consider F^{r_0}-action instead of F, this filtration yields $G^{F^m}\langle\sigma^{r_0}\rangle$-modules. Now by making use of above embedding together with [14, Prop. 2.20], (see also Lemma 1.4 in [16]), we see that these $G^{F^m}\langle\sigma^{r_0}\rangle$-modules are mutually isomorphic.

We now restrict ourselves to the case where $r_0 \leqq 2$, (note if G is of type B_n or C_n, then $r_0 \leqq 2$). We fix an extension $\tilde{\rho}$ of ρ to $\tilde{G}^{F^m}$. Since $r_0 \leqq 2$, we have at most two possibilities for $\tilde{G}^{F^m}$-modules appearing in successive quotients, $\tilde{\rho}$ and $-\tilde{\rho}$, (which is obtained from $\tilde{\rho}$ by replacing σ-action by $-\sigma$). Thus, as in the proof of Lemma 2.17 in [16], we can write

$$R^{(F)}_{L(\dot{w})}([\tilde{\delta}_w]) = \sum_\rho \lambda'_{\tilde{\rho}} c_{\dot{w},\rho}[\tilde{\rho}]\,, \tag{4.7.3}$$

for some $c_{\dot{w},\rho} \in R$.

Now (4.7.1) implies that, by passing to the dual representation,

$$N^*_{F^m/F}\Big(\sum_\rho c_{\dot{w},\rho^*}\lambda'^{-1}_{\tilde{\rho}}[\tilde{\rho}]\Big) = C_w(q)\sum_{E_1}\mathrm{Tr}(T_w, E_1(q))[\rho_{E_1}] \tag{4.7.4}$$

for each $w \in \mathcal{W}$. On the other hand, since we know already that $N^*_{F^m/F}$ maps $[\tilde{\rho}]$ to R_{x_ρ} up to a scalar multiple for $\rho \neq \rho_0$, it follows from the argument of the former part that $c_{\dot{w},\rho_0^*} \neq 0$ for some $w \in \mathcal{W}$.

Now the right hand side of (4.7.4) is a $\boldsymbol{Q}$-linear combination of R_x $(x \in X(W_s, \gamma)/M)$. Hence we see that $\alpha_0 \in \boldsymbol{R}$. Since α_0 has absolute value 1, we get $\alpha_0 = \pm 1$. Hence, by replacing $\lambda'_{\tilde{\rho}_0}$ by $-\lambda'_{\tilde{\rho}_0}$ if necessary, we get the desired result. Thus Proposition 4.3 was proved (assuming Lemma 4.6).

4.8. Before starting the proof of Lemma 4.6, we need a general lemma. Let G be a connected reductive group with connected center endowed with two Frobenius maps F' and F'' as in (4.5.1) (replacing L by G). Let T be a maximal torus of G stable by both F' and F''. Let $N^*_{F''/F'}$ be the induced map of the norm map $N_{F''/F'}: T^{F'}/\sim_{F''-1}\to T^{F''}/\sim_{F'}$. Then

4.9. LEMMA. *Let the notations be as in 4.8. Then for each F'-stable character θ'' of $T^{F''}$, we have*

$$N^*_{F''/F'}([\theta''])=[\theta'],$$

where θ' is an F''-stable character of $T^{F'}$, ($[\theta']$ is the corresponding element in $C(T^{F'}/\sim_{F''-1})$ and similarly for θ''). Moreover $(T^{F'},\theta')$ and $(T^{F''},\theta'')$ are geometrically conjugate in G, i.e., θ' and θ'' correspond to the same class $\{s\}$ in G^*.

PROOF. Since T is abelian, $N_{F''/F'}$ may be regarded as a homomorphism from $T^{F'}$ onto the quotient of $T^{F''}$ by a subgroup consisting of $t^{-1}F'(t)$ $(t\in T^{F''})$. Thus, for each F'-stable $\theta''\in T^{F''}$, $N^*_{F''/F'}$ determines a character θ' of $\hat{T}^{F'}$ which is necessarily F''-stable. Hence the first statement follows.

Next consider the second statement. By (4.5.1) there exists a positive integer r such that $F''^{rm}=F'^{r}$. We denote by F^+ this common power of F' and F'' (i.e., $F^+=F'^{r}$), and consider two norm maps,

$$n'=N_{F'/F^+}\ :\ T^{F^+}/\sim_{F'-1}\longrightarrow T^{F'}/\sim\,,$$

$$n''=N_{F''/F^+}\ :\ T^{F^+}/\sim_{F''-1}\longrightarrow T^{F''}/\sim\,.$$

Then n' can be regarded as the usual norm map, and so lifts to a homomorphism $T^{F^+}\to T^{F'}$, and similarly for n''. Put $\tilde{\theta}'=n'^*(\theta')\in\hat{T}^{F^+}$ and $\tilde{\theta}''=n''^*(\theta'')\in\hat{T}^{F^+}$. It is enough to show that $\tilde{\theta}'=\tilde{\theta}''$. Take $\hat{t}\in T^{F^+}$ and choose $b,c\in T$ such that

$$\hat{t}=b^{-1}F'(b)=F''(c)c^{-1}.$$

Put $t=n'(\hat{t})\in T^{F'}$. Then, since n' is a homomorphism commuting with F'',

$$t=n'(F''(c)c^{-1})=F''(n'(c))n'(c)^{-1}.$$

Thus,

$$\begin{aligned}N_{F''/F'}(t)&=n'(c)^{-1}F'(n'(c))\\&=(cF'(c)\cdots F'^{(r-1)}(c))^{-1}F'(cF'(c)\cdots F'^{(r-1)}(c))\\&=c^{-1}F'^{r}(c)=n''(\hat{t}).\end{aligned}$$

Hence

$$\tilde{\theta}''(\hat{t})=\theta''(n''(\hat{t}))=\theta'(t)=\theta'(n'(\hat{t}))=\tilde{\theta}'(\hat{t}) .$$

This proves the lemma.

4.10. We shall prove Lemma 4.6. The proof is given in 4.10 and 4.11. In this section, we consider the case where F is of split type. By making use of the argument due to Asai ([1, § 2], see also [14, 8.8]), combined with Lemma 1.17 and (3.8.1), the proof is reduced to the case where $L=GL_2$. Now by changing the notation, put $G=GL_2$. We may write $F'=jF^m$, $F''=F$, where F is a Frobenius map of G with respect to an $\boldsymbol{F}_q$-structure, j is an automorphism of G of order 1 or 2, and m is a sufficiently divisible integer. For later use, we consider simultaneously the case where F is of non-split type. In particular, F is of split type or non-split type and F^m is of split type. Now, the case where j is identity is well known. So we may assume j is not identity.

We now see precisely the irreducible representations of G^F. Let us denote by $T\subset B$ a pair of maximal torus and a Borel subgroup of G, both stable by F and jF. Let $W=\{1, w\}$ be the Weyl group of G with respect to T. Then all the irreducible representations of G^F are given as follows;

$$\phi^{(1)}, \quad \phi^{(1)}\otimes\mathrm{St}_G^{(1)}, \quad R_{T_1}^{(1)}(\theta), \quad -R_{T_w}^{(1)}(\theta'),$$

where $\phi^{(1)}$ runs over all the linear characters of G^F, $\mathrm{St}_G^{(1)}$ is the Steinberg representation of G^F. $\theta\in\hat{T}^F$ (resp. $\theta'\in\hat{T}^{wF}$) runs over all the regular characters up to W-conjugate. Among them, jF^m-stable representations are those corresponding to jF^m-stable ϕ, and those corresponding to θ, θ' whose W-orbit is jF^m-stable. We also denote by $\phi^{(m)}, \cdots$, the similar representations of G^{jF^m}.

In view of Lemma 1.5, it is enough to determine $N^*_{jF^m/F}$. Now we see easily that under $N^*_{jF^m/F}$ a linear character $\phi^{(m)}$ of G^{jF^m}, stable by F, corresponds to a linear character $\phi^{(1)}$ of G^F, stable by jF^m, and that $\phi^{(1)}$ and $\phi^{(m)}$ belong to the same class $\{s\}$ in G^*. The same is true also for $\phi^{(1)}\otimes\mathrm{St}_G^{(1)}$ and $\phi^{(m)}\otimes\mathrm{St}_G^{(m)}$, ($\mathrm{St}_G^{(m)}$ is the Steinberg representation of G^{jF^m}). Hence $N^*_{jF^m/F}$ induces an isomorphism between $C^{(s)}(G^{jF^m}/\sim_F)$ and $C^{(s)}(G^F/\sim_{(jF^m)^{-1}})$ for such $\{s\}$.

Next consider the class $\{s\}$ corresponding to $R_{T_1}^{(1)}(\theta)$. Thus W-orbit of θ is jF^m-stable and so, θ is $w'jF^m$-stable for some $w'\in W$. Applying Proposition 1.13, we have the following commutative diagram, (note $w'jF^m=jF^mw'$).

$$\begin{tikzcd} C(G^F/\sim_{jF^m}) \arrow[r, "N^*_{F/jF^m}"] & C(G^{jF^m}/\sim_{F^{-1}}) \\ C(T^F/\sim_{w'jF^m}) \arrow[u, "a_{jF^m w'}"] \arrow[r, "N^*_{F/w'jF^m}"] & C(T^{w'jF^m}/\sim_{F^{-1}}) \arrow[u, "R^{(F)}_{T(\dot w')}"'] \end{tikzcd}$$

Suppose $\theta\in\hat{T}^F$ corresponding to $s\in T^*$. Then θ^{-1} corresponds to s^{-1}. Now $[\theta^{-1}]$ is an element of $C(T^F/\sim_{w'jF^m})$ and so, by Lemma 4.9, $N^*_{F/w'jF^m}([\theta^{-1}])=[\hat{\theta}^{-1}]$ for some $\hat{\theta}\in T^{w'jF^m}$, stable by F, and θ, $\hat{\theta}$ belong to the same class $\{s\}$. From the above diagram, we see that $N^*_{jF^m/F}$ induces an isomorphism between $C^{(s)}(G^{jF^m}/\sim_F)$ and $C^{(s)}(G^F/\sim_{(jF^m)^{-1}})$. Hence we see that $N^*_{jF^m/F}$ sends $\tilde{R}^{(m)}_{T_{w'}}(\hat{\theta})$ to $\tilde{R}^{(1)}_{T_1}(\theta)$ up to a scalar multiple α, where $\tilde{R}^{(1)}_{T_1}(\theta)$ (resp. $\tilde{R}^{(m)}_{T_{w'}}(\hat{\theta})$) is an extension of $R^{(1)}_{T_1}(\theta)$ (resp. $R^{(m)}_{T_{w'}}(\hat{\theta})$) to a $G^F\langle j\rangle$ module (resp. $G^{jF^m}\langle F\rangle$-module), respectively. Using the same argument as in 4.7, we see that $\alpha=\pm 1$ and so by taking another extension, if necessary, we conclude that $N^*_{jF^m/F}$ sends $\tilde{R}^{(m)}_{T_{w'}}(\hat{\theta})$ to $\tilde{R}^{(1)}_{T_1}(\theta)$. Thus, for the class $\{s\}$ corresponding to $R^{(1)}_{T_1}(\theta)$, the assertion of the lemma is verified.

Next consider the class $\{s\}$ corresponding to $-R^{(1)}_{T_w}(\theta')$. Thus, $\theta'\in\hat{T}^{wF}$ corresponds to s and W-orbit of θ' is jF^m-stable. First suppose θ' is jF^m-stable. Then, by applying Proposition 1.13, we have the following commutative diagram, (note $\dot{w}F=F\dot{w}$).

$$\begin{tikzcd} C(G^F/\sim_{(jF^m)^{-1}}) & C(G^{jF^m}/\sim_F) \arrow[l, "N^*_{jF^m/F}"'] \\ C(T^{wF}/\sim_{(jF^m)^{-1}}) \arrow[u, "R^{(jF^m)}_{T(\dot w)}"] & C(T^{jF^m}/\sim_{wF}) \arrow[u, "a_{Fw}"'] \arrow[l, "N^*_{jF^m/F}"'] \end{tikzcd}$$

Now θ' determines an element $[\theta']$ in $C(T^{wF}/\sim_{(jF^m)^{-1}})$. Hence by Lemma 4.9, $N^*_{jF^m/wF}([\hat{\theta}'])=[\theta']$ for some $\hat{\theta}'\in\hat{T}^{jF^m}$, stable by wF, and θ', $\hat{\theta}'$ belong to the same class $\{s\}$ in G^*. So, using the above diagram we see that $N^*_{jF^m/F}$ induces an isomorphism between $C^{(s)}(G^{jF^m}/\sim_F)$ and $C^{(s)}(G^F/\sim_{(jF^m)^{-1}})$, and by the same argument as before, we see that $N^*_{jF^m/F}$ sends $\tilde{R}^{(m)}_{T_1}(\hat{\theta}')$ to $-\tilde{R}^{(1)}_{T_w}(\theta')$ under suitable extensions.

Next suppose $\theta'\in\hat{T}^{wF}$ is wjF^m-stable. We shall realize $T\subset B$ and j as follows. T is the group of diagonal matrices, B is the group of upper triangular matrices, and j is the automorphism of order 2 given by $(x_{ij})\to\dot{w}^t(x_{ij})^{-1}\dot{w}^{-1}$. Since θ' is wjF^m-stable, we may assume θ' is wj-stable by taking m large enough. Thus $\theta'(t)=\theta'(t^{-1})$ for $t\in T^{wF}$ and we have $(\theta')^2=1$. Now suppose F is of split type. Then $T^{wF}\cong \boldsymbol{F}^*_{q^2}$ and there exists no regular character $\theta'\in\hat{T}^{wF}$ such that $(\theta')^2=1$. Next suppose that F is of non-split type. Then T^{wF} is a product of two cyclic groups of order $q+1$.

No such a regular character exists if $p=2$. While, up to W-conjugate, there exists exactly one regular character $\theta'\in\hat{T}^{wF}$ such that $(\theta')^2=1$ if $p\neq 2$. In this case, since we know already $N^*_{jF^m/F}$ preserves the spaces corresponding to the class $\{s'\}$ for each s' not conjugate to s, we see that the spaces corresponding to $\{s\}$ is also preserved by $N^*_{jF^m/F}$. It remains to show the second statement of the lemma. Let $\theta'_0\in\hat{T}^{wF}$ be a regular character stable by wjF^m, (so we assume $p\neq 2$). By Lemma 4.9, there exists $\hat{\theta}'_0\in\hat{T}^{wjF^m}$ such that $N^*_{wjF^m/wF}([\hat{\theta}'_0])=[\theta'_0]$, so $\hat{\theta}'_0$ is regular and wF-stable. Now, to prove the lemma, it is enough to show that

$$N^*_{jF^m/F}(\tilde{R}^{(m)}_{T_w}(\hat{\theta}'_0))=\tilde{R}^{(1)}_{T_w}(\theta'_0) \tag{4.10.1}$$

for suitable extensions. Let $\theta_0\in\hat{T}^F$ be such that the restriction of θ_0 to Z^F coincides with the restriction of θ'_0 to Z^F, where $Z\subset T$ is the center of G. From the previous argument, $(N^*_{jF^m/F})^{-1}(\tilde{R}^{(1)}_{T_1}(\theta_0))$ is a virtual character in $C(G^{jF^m}/\sim_F)$ and the same is true for $\mathrm{St}^{(1)}_G$. Thus the image of $\tilde{R}^{(1)}_{T_1}(\theta_0)\otimes\mathrm{St}^{(1)}_G$ by $(N^*_{jF^m/F})^{-1}$ is also a virtual character in $C(G^{jF^m}/\sim_F)$. However, it is easily checked that $R^{(1)}_{T_1}(\theta_0)\otimes\mathrm{St}^{(1)}_G$ contains $-R^{(1)}_{T_w}(\theta'_0)$ with multiplicity one. Since the character correspondence is already established for all other characters except $-R^{(1)}_{T_w}(\theta'_0)$, we see that the image of $-\tilde{R}^{(1)}_{T_w}(\theta'_0)$ by $(N^*_{jF^m/F})^{-1}$ is also a virtual character in $C^{(s)}(G^{jF^m}/\sim_F)$ and so (4.6.1) follows. Thus the lemma was proved in the case where F is of split type.

4.11. We continue the proof of Lemma 4.6 and now consider the case where F (Frobenius on L) is of non-split type. Hence G is of type D_n. Since the Dynkin diagram of L is of type $A_1\times\cdots\times A_1$, this case is also reduced to GL_2 by Asai's argument. However, since the action of F on the Dynkin diagram of L is non-trivial, we need an extra care for the process of reduction to GL_2. More precisely, using Asai's argument, the proof is reduced to the case $L=L_1\times L_2$, where L_1 (resp. L_2) is a product of some copies of GL_2 (resp. two copies of GL_2) stabilized by both of F' and F'', respectively. Moreover, F'' stabilizes each factor of L_1 and permutes two factors of L_2. To prove the lemma, we may consider L_1 and L_2 separately. In the case of L_1, by making use of Lemma 2.13, the proof is reduced to the case of GL_2 as in 4.10. However, note in this case, Frobenius map with respect to an $\boldsymbol{F}_q$-structure of GL_2 (F in the notation of 4.10) may be of non-split type. Anyway this case is done by the same method as in 4.10. In the case of L_2, if F' stabilizes each factor of L_2, again Lemma 2.13 can be applied and the previous argument works as well to this case. Thus we have only to consider the case where both of F' and F'' permutes two factors of L_2. Hence we can apply Lemma 2.15 to this case. In particular, the proof is reduced to showing the following.

(4.11.1) Let $G=GL_2$ and F be a Frobenius map on G (split or non-split), j be an automorphism of order 1 or 2 commuting with F. Consider the maps

$$\text{(i)}\quad N^*_{F^{m+1}/jF^{2m}} : C(G^{F^{m+1}}/\sim_{jF^{2m}}) \longrightarrow C(G^{jF^{2m}}/\sim_{F^{-(m+1)}})$$
$$\text{(ii)}\quad N^*_{F^2/F^{m+1}} : C(G^{F^2}/\sim_{F^{m+1}}) \longrightarrow C(G^{F^{m+1}}/\sim_{F^{-2}}).$$

Then for each class $\{s\}$ in G^* which is stable by F^{m+1} and jF^{2m} (resp. by F^2 and F^{m+1}) in the case (i) (resp. (ii)), the same statement holds as in Lemma 4.6.

We shall show (4.11.1). First we note that the class $\{s\}$ which is F^{m+1} and jF^{2m}-stable (resp. F^2 and F^{m+1}-stable) is in fact F-stable if we take m large enough. Now, among the arguments used in 4.10, the part concerning with the application of Proposition 1.13 works as well to this case. So we have only to consider the remaining case, i.e., under the notation in 4.10, the case concerning with $R^{(m+1)}_{T_w}(\theta')$ (resp. $R^{(2)}_{T_w}(\theta')$) where θ' is a regular character of $T^{wF^{m+1}}$ (resp. T^{wF^2}) which is wjF^{2m}-stable (resp. wF^{m+1}-stable) in the case (i) (resp. (ii)), respectively. First consider the case (i). Since the class $\{s\}$ is F-stable, θ' is F^2-stable. Thus θ' is wj-stable. If j is identity, this means that θ' is not regular and so is excluded. If j is non-identity, the situation is the same as in 4.10, and we see that θ' has order two. Thus by the similar argument as in 4.10, we get the result. Next consider the case (ii). As in the case (i), θ' is F^2-stable. Since θ' is wF^2-stable, θ' is w-stable and so θ' is not regular. Thus this case does not occur. Thus (4.11.1) was proved and so the case where F is of non-split type. This completes the proof of Lemma 4.6.

4.12. REMARKS. (i) If $r_0 \leqq 2$, the argument employed in the proof in Proposition 4.3 works as well, even in the case of type D_n, as in the case of type B_n or C_n. In fact, if $G=CO^{\pm,0}_{2n}$, $r_0 \leqq 2$ is shown as follows. In this case, there exists a natural homomorphism $\pi: G^* \to SO^{\pm}_{2n}$ whose kernel is a one dimensional central torus. If $s \in T^*$ is given as in Proposition 4.3 and is not stable by F, $\pi(s)$ is F-stable and has order 2. Then $F(s)=sz$, where z is the unique element of order 2 in $\operatorname{Ker}\pi$, and so F-stable. Hence s is F^2-stable and we see that $r_0=2$. Thus the same statement as in the case of type B_n or C_n holds for G in Theorem 4.2.

(ii) The argument in 4.7 can be applied also to the case where G is not necessarily of classical type. For example, consider the case (G, F) is ${}^3D_4$ and $s=1$. In this case, $\mathcal{E}(G^{F^m}, \{s\})^F$ contains only one cuspidal representations. Then using the argument in 4.7 (or rather, the argument in Lemma 2.17 in [16]) we can show, for a cuspidal representation ρ_0 of G^{F^m}, that $N^{*}_{F^m/F}([\tilde{\rho}_0])=\alpha_0 R_{x_0}$ for some $\alpha_0\in\bar{\boldsymbol{Q}}_l$ of absolute value 1. Since $r_0=3$, we cannot determine further this value α_0.

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