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The classification of involutions of simple algebraic groups

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Introduction.

The present note deals with the problem of classifying the involutions, i.e. automorphisms θ of order two, of a simple algebraic group G over an algebraically closed field k of characteristic not two. For $k=\boldsymbol{C}$ the problem is equivalent to that of classifying the real forms of simple complex Lie algebras, which was first solved by E. Cartan in 1914. Subsequently, S. Araki [1] introduced diagrams describing this classification. It was shown by A. G. Helminck [6], adapting Sugiura's simplification of Araki's work [9, appendix], that Araki's classification carries over to arbitrary k.

Here we shall discuss an approach which gives the classification in a fairly easy manner. In this approach no information about structure constants is required. We only need standard information about root systems, as contained in [4].

If G and θ are as above, we consider two kinds of θ-stable maximal tori of G, viz. the ones where the fixpoint set of θ has smallest resp. largest possible dimension. The description of θ by an Araki diagram is obtained using the root system of G with respect to a torus of the first kind. On the other hand, if θ is inner, it comes from an element of order two of a torus of the second kind. Such elements can be described in a well-known manner (see for example Iwahori's contribution in [2]), which leads to another description of inner involutions. Exploiting the interplay between the two descriptions we obtain the classification results for inner involutions. The other ones are then easy to handle. In the second description affine Weyl groups enter the picture. These are also used in the method of V. Kac to classify involutive automorphisms of simple Lie algebras, discussed in [5, Ch. X, § 5].

1. Reductive groups with involution, recollections.

k denotes an algebraically closed field of characteristic not two. Let G be a connected reductive linear algebraic group over k, provided with an involution θ (an automorphism of algebraic groups of order 2). We denote by K the fixed point group of θ, which is known to be reductive, too. We recall a number of known results, which can be found in [8] and [12]. The notations pertaining to the theory of algebraic groups will be those of [11].

1.1. (i) *There exists in* G *a maximal torus* T *and a Borel subgroup* B *containing* T *such that* $\theta T = T$, $\theta B = B$;

(ii) *Two maximal tori with the property of* (i) *are conjugate by an element of the identity component* K° *of* K.

For (i) see [13, § 7]. The conjugacy statement (ii) follows from the observation that $K^\circ \cap T$ is a maximal torus of K°, whose centralizer is T [8, § 5], using the conjugacy of maximal tori of K°. (A similar conjugacy statement for pairs (T, B) with the properties of (i) is not true.)

We call T a *fundamental torus* in G (relative to θ) and (T, B) a *fundamental pair*. It follows from 1.1(ii) that any fundamental torus is part of a fundamental pair.

A subtorus S of G is *split* (relative to θ) if $\theta x = x^{-1}$ for all $x \in S$. A parabolic subgroup P of G is *split* if P and θP are opposite, i.e. if their intersection is a Levi subgroup of both groups.

1.2. (i) *If* G *is not a torus then non-central split tori exist*;

(ii) *Two maximal split tori are conjugate by an element of* K°;

(iii) *Let* S *be a maximal split torus. There exists a minimal split parabolic subgroup* P *such that* $P \cap \theta P = Z_G(S)$, *the centralizer of* S;

(iv) *Two minimal split parabolic subgroups are conjugate by an element of* K°.

These results are due to Vust [14, § 1]. Notice that by (i) we have $P \neq G$, unless G is a torus.

Let S be a maximal split torus, T_1 a maximal torus containing S. It follows from 1.2(i) that $Z_G(S) \subset SK^\circ$, from which one infers, using 1.2(ii), that any two tori like T_1 are conjugate by an element of K°.

We fix a fundamental pair (T, B). Denote by X the character group of T, by R the root system of (G, T), by D the basis of R defined by B and by $W = N_G T/T$ the Weyl group. Then θ operates on R and stabilizes D. If I is a subset of D let R_I be the root system with basis I and P_I the corresponding parabolic subgroup containing B, i.e. its Levi subgroup L_I containing T has root system R_I. The Weyl group of (L_I, T), a subgroup of W, is denoted by W_I. The longest element of W_I (relative to I) is denoted by w_I. In particular, w_D is the longest element of $W = W_D$.

1.3. *There exist* $v \in G$ *and* $I \subset D$ *such that the following holds*:

(i) $n = v(\theta v)^{-1}$ *lies in* $N_G T$ *and represents the element* $w_I w_D$ *of* W;

(ii) *for all* $\alpha \in I$ *we have* $w_I w_D \theta \alpha = \alpha$;

(iii) $P = v^{-1} P_I v$ *is a minimal split parabolic subgroup*;

(iv) BvK° *is open in* G.

This is established in [12]. It follows from the uniqueness part of Bruhat's lemma [11, 10.2.12] that the double coset TvK is uniquely determined.

We draw a number of consequences of the preceding results. First notice that, with the notations of 1.3,

$$\theta(v^{-1}tv) = v^{-1}(w_I w_D \theta)(t)v \qquad (t \in T). \tag{1}$$

Put $T_1 = v^{-1}Tv$, $B_1 = v^{-1}Bv$. By (1), T_1 is a θ-stable maximal torus. We call (T_1, B_1) a *split pair*.

1.4. (i) *The maximal split torus* S *contained in* T_1 *is the identity component of the subgroup*

$$\{v^{-1}tv \mid t \in T, (w_I w_D \theta)(t) = t^{-1}\}.$$

(ii) *Let* T' *be a maximal torus of* G *containing a maximal split torus. If* B' *is a Borel subgroup containing* T' *then* (T', B') *is a split pair if and only if* B' *is contained in a minimal split parabolic subgroup*;

(iii) *Two split pairs are conjugate by an element of* K°.

Since T_1 is θ-stable, we have $T_1 \subset P \cap \theta P$, where P is as in 1.3. Now (i) follows from 1.3 and 1.2. To prove (ii) use 1.3(iii) and the conjugacy of minimal split parabolic subgroups. (iii) is proved similarly.

We introduce a realization of R in G [11, 11.2.4], i.e. a family of one-parameter subgroups $(x_\alpha)_{\alpha \in R}$ such that for all $\alpha \in R$

$$\begin{cases} t x_\alpha(\xi) t^{-1} = x_\alpha(\alpha(t)\xi) \quad (t \in T, \xi \in k), \\ x_\alpha(1) x_{-\alpha}(-1) x_\alpha(1) \in N_G T. \end{cases}$$

1.5. *The realization may be taken such that* $\theta(x_\alpha(\xi)) = n^{-1} x_{w_I w_D \theta \alpha}(\xi) n$ *for* $\pm\alpha \in D$, $\xi \in k$.

The notations are as in 1.3. First take $\alpha \in I$, so $w_I w_D \theta \alpha = \alpha$ (1.3(ii)). Let α_1 be the character of T_1 corresponding to α under the isomorphism $t \mapsto v^{-1}tv$ of T onto T_1. It follows from 1.4(i) that α_1 is trivial on the maximal split torus S. Hence $v^{-1}(\operatorname{Im} x_\alpha)v$ lies in the centralizer of S, which we know to be contained in $S.K^\circ$. We conclude that $v^{-1}(\operatorname{Im} x_\alpha)v \subset K^\circ$, which implies the statement of 1.5 for $\alpha \in I$. If $\alpha \in D - I$ then, using that w_I

sends positive roots (relative to D) outside R_I to positive roots, one sees that $w_I w_D \theta\alpha$ is negative, and in particular is different from α. We then can normalize the realization such that 1.5 also holds for $\alpha\in D-I$ (see [11, 11.2.1]). But then 1.5 holds also for $-\alpha\in D$, as follows from [loc. cit.].

Let G' be another connected reductive k-group, with an involution θ'. We assume given for (G', θ') data $T', B', R', \cdots$, similar to the ones introduced for (G, θ).

1.6. *Assume that there exists an isomorphism of algebraic tori* $f: T\xrightarrow{\sim}T'$, *which induces a bijection* $f^*: R'\rightarrow R$, *with* $f^*D=D', f^*I'=I$. *Then* f *extends to an isomorphism of algebraic groups* $f: G\xrightarrow{\sim}G'$ *with* $fB=B'$, *such that* $f\circ\theta=\theta'\circ f$. *If* f_1 *is another isomorphism* $G\xrightarrow{\sim}G'$ *with the same properties there is* $t\in T$ *such that* $f_1(x)=f(txt^{-1})$ $(x\in G)$ *and that* $t(\theta t)^{-1}$ *lies in the center of* G.

Introduce realizations $(x_\alpha)_{\alpha\in R}, (x_{\alpha'})_{\alpha'\in R'}$ for G resp G', with the properties of 1.3. Using the isomorphism theorem for reductive groups [11, 11.4.3] it is not hard to see that we can arrange matters such that

$$\begin{cases} f(n)=n' \\ f(x_{f^*\alpha'}(\xi))=x_{\alpha'}(\xi) \quad (\alpha'\in D', \xi\in k)\,. \end{cases}$$

Since f is determined by its values on T and the groups $\operatorname{Im} x_\alpha$ $(\alpha\in\pm D)$ [loc. cit., 10.1.11], we conclude from 1.5 that $f\circ\theta=\theta'\circ f$. The uniqueness statement is a direct consequence of the uniqueness part of the isomorphism theorem.

We give some additional results, which are fairly direct consequences of the preceding observations. Denote by $i=i_D$ the opposition involution of D, i.e. the permutation of D defined by

$$w_D\alpha=-i\alpha \qquad (\alpha\in D)\,.$$

Let $\tilde{\theta}$ be the permutation of D induced by θ. We put $\phi=i\tilde{\theta}$. Recall that the Weyl group W acts on T as a group of automorphisms.

1.7. (i) *ϕ stabilizes I and $\phi|_I=i_I$, the opposition involution of I;*
(ii) $\phi^2=1$.
(i) follows from (1). Since $w_D=w_{\tilde{\theta}D}=\theta w_D\theta^{-1}$ we have (ii).

Next we give characterizations of inner automorphisms of semi-simple groups.

1.8. *Let G be semi-simple. The following statements are equivalent*: (a) θ *is an inner automorphism,* (b) $\tilde{\theta}=1$, (c) $\operatorname{rank} G=\operatorname{rank} K$.

Let θ be an inner automorphism $\operatorname{Int} a$. Then a normalizes both B and T and hence lies in T (as is well-known). So θ fixes all elements of T, which implies that $\tilde{\theta}=1$. Conversely, if $\tilde{\theta}=1$ then θ fixes all elements of T (here we use the semi-simplicity of G). We can then find $a \in T$ such that $\theta x_\alpha(\xi)=a x_\alpha(\xi) a^{-1}$ for all $\alpha \in D$, $\xi \in k$. The same holds then for $\alpha \in -D$, which implies that $\theta=\operatorname{Int} a$. We have shown the equivalence of (a) and (b). The equivalence of (b) and (c) follows from the observation that $K^\circ \cap T$ is a maximal torus in K°.

By 1.2(ii) the dimension of a maximal split torus is an invariant of (G, θ), the *rank* of θ. We give several descriptions.

1.9. (i) $\operatorname{rank} \theta=\dim P_I-\dim K$;

(ii) $\operatorname{rank} \theta$ *equals the multiplicity of the eigenvalue* 1 *of the linear map* $\theta \otimes 1$ *in* $X \otimes_Z \boldsymbol{R}$;

(iii) *If G is semi-simple,* $\operatorname{rank} \theta$ *equals the number of ψ-orbits in* $D-I$.

(i) follows from the observation that $G=K.P$ and $K \cap P \cong L_I/S$ (see [14, § 1]). The assertions (ii) and (iii) follow from 1.4(i).

2. Classification, preliminaries.

2.1. In no. 1 we have associated to the involution θ of our reductive group the combinatorial data (D, I, ψ). These are described, after Araki, by a diagram (see [1]). Let $\mathscr{D}$ be the Dynkin diagram associated to D, a graph whose vertex set is D. We color black the vertices of I, and we indicate the action of ψ if it is nontrivial. The Araki diagram $\mathscr{D}_\theta$ of (G, θ) is the Dynkin diagram $\mathscr{D}$ provided with the extra data (I, ψ). If G is semi-simple and adjoint then $\mathscr{D}_\theta$ determines (G, θ) up to an inner isomorphism, as follows from 1.6.

One should keep in mind that the Dynkin diagram $\mathscr{D}$ is defined using a maximal torus T and a Borel group $B \supset T$. However, the Dynkin diagrams associated to two such pairs (T, B), (T_1, B_1) can be canonically identified.

In this section we shall only encounter split pairs. We fix one and denote it by (T, B) (in order to keep notations simple we do not write (T_1, B_1) as in no. 1). We view $\mathscr{D}$ as being defined using this split pair. The root system of (G, T) is again denoted by R. Recall that θ operates on T and hence on R. We denote the induced permutation of R by θ^*. In terms of (T, B) the data (I, ψ) can now be described as follows. We may assume that G is not a torus.

2.2. *If* $\alpha\in I$ *then* $\theta^*\alpha=\alpha$. *If* $\alpha\in D-I$ *then* $\theta^*\alpha+\psi\alpha$ *is a linear combination of roots in* I.

This is a direct consequence of (1).

We now forget for the moment the involution θ. Let (T, B) be any pair in G, and D the associated basis of the root system R of (G, T). Let I be a subset of D and ψ a permutation of D with $\psi^2=1$. Assume that ψ stabilizes I and that $\psi|_I=i_I$ (notations as before). We shall say that (I, ψ) is *admissible* (for D) if either (I, ψ) is obtained as above from an involution θ for which (T, B) is a split pair — in which case $I\neq D$, as a consequence of 1.2 — or if $I=D$, $\psi=i_D$. The problem of classification of the involutions of G leads to the problem of classifying the admissible pairs (I, ψ) (and is equivalent to it if G is adjoint).

We next mention some criteria for admissibility. These are essentially well-known, so we shall be brief (see [9, p. 101] or [6]). We assume G to be adjoint (as we may) and we take a realization $(x_\alpha)_{\alpha\in R}$. It follows from the isomorphism theorem [11, 11.4.3] that there exists a unique automorphism θ of G such that for $\alpha\in D$, $\xi\in k$

$$\theta(x_\alpha(\xi))=x_{-w_I\psi\alpha}(\xi)\,.$$

If $\alpha\in R$ is an arbitrary root there is $c_\alpha\in k^*$ such that

$$\theta(x_\alpha(\xi))=x_{-w_I\psi\alpha}(c_\alpha\xi)\,.$$

2.3. (I, ψ) *is admissible if and only if* $c_{-w_I\psi\alpha}=1$ *for* $\alpha\in D-I$.

If this condition is satisfied then $\theta^2x_\alpha(\xi)=x_\alpha(\xi)$ for all $\alpha\in D$ (notice that $w_I\psi\alpha=-\alpha$ for $\alpha\in I$), so that θ is an involution. It is then easy to check that (T, B) is a split pair for θ and that (I, ψ) is the pair defined by θ. We skip the rest of the proof of 2.3.

If J is a ψ-stable subset of $D-I$ we write $D_J=I\cup J$, $\psi_J=\psi|_{D_J}$.

2.4. (i) *If* (I, ψ) *is admissible for* D *then* (I, ψ_J) *is admissible for* D_J;

(ii) *If for each* ψ*-orbit* J *in* $D-I$ *we have that* (I, ψ_J) *is admissible for* D_J *then* (I, ψ) *is admissible for* D.

Let J be as above and denote by T_J the identity component of the intersection of the kernels of the characters α of T, for $\alpha\in I\cup J$. If the automorphism θ introduced above is an involution it induces an involution of the centralizer $Z_G(T_J)$, which has associated admissible pair (I, ϕ_J) for D_J. This establishes (i).

To prove (ii) one uses 2.3, which implies that admissibility of (I, ϕ) is characterized by a condition which involves only the ϕ-orbits in $D-I$.

In view of 1.9 (ii), the last result shows that the essential part of the classification of admissible pairs is the classification of those which lead to involutions θ with $\operatorname{rank}\theta=1$.

3. Classification, first step.

Let G and θ be as in no. 1 and first assume that $\operatorname{rank}\theta=1$. We also assume that G is semi-simple and adjoint. We fix a fundamental pair (T, B) and use the notations of no. 1 for the root system of (G, T). We assume given a realization (x_α) of R.

3.1. *Assume, moreover, that G is simple of* rank 2. *Then G is of type B_2, θ is an inner automorphism* $\operatorname{Int} a$, *with $a\in T$ such that $a^2=1$ and that $\alpha(a)=1$ for all long roots $\alpha\in R$.*

With the notations of 1.9 (i) we have $\dim P_I=\dim K+1$. There are three possible cases, where G is of type A_2, B_2, G_2, respectively. We shall only deal with the case of type B_2. The proof of the impossibility in the other cases is left to the reader.

In the case of B_2, θ must be inner, $\theta=\operatorname{Int} a$ with $a\in T$, $a^2=1$. Assume $\alpha(a)=-1$ for some long root α. We may assume α to be simple, by choosing B appropriately. Then one checks that the number of roots $\beta\in R$ with $\beta(a)=1$ equals 2, whence one concludes that $\dim K=4$. Hence $\dim P_I=5$, which is impossible.

We now assume that G is simple and that $\theta=\operatorname{Int} a$ is an inner automorphism, with $a\in T$, $a^2=1$. We denote the highest short root of R (relative to D) by α_s. So, if $R^\vee=\{\alpha^\vee \mid \alpha\in R\}$ is the dual root system, $\alpha_s^\vee$ is the highest root of $R^\vee$. The root α_s is also a dominant weight. If all roots of R have the same length we call them short. If we speak of long roots, it is understood that different root lengths occur.

3.2. *Assume* $\operatorname{rank}\theta=1$.

(i) *$\alpha(a)=1$ for all long roots $\alpha\in R$;*

(ii) *I is the set of roots γ of D which are orthogonal to α_s (i.e. such that $\alpha_s-\gamma$ is not a root);*

(iii) *If all roots of R have the same length and $D-I=\{\beta\}$ then β occurs with an odd coefficient in α_s.*

(i) follows from 3.1, using that any long root of R is contained in an irreducible subsystem of rank 2. Now let α be a short root with $\alpha(a)=-1$. Choosing B suitably we may assume that $\alpha=\alpha_s$. Let H be the subgroup of G generated by $\operatorname{Im} x_\alpha$ and $\operatorname{Im} x_{-\alpha}$. Then H is semi-simple of rank 1 and θ induces an involution of H. By 1.2(i), H contains a one-dimensional split torus S, which must be a maximal one for G, since $\operatorname{rank}\theta=1$. Within H, the torus S is conjugate with the torus $S_1=\operatorname{Im}\alpha^v$ (where α^v denotes the coroot defined by α, see [11, p. 190]), say $S=h^{-1}S_1h$.

Now $S_1\subset K$. It follows that $h(\theta h)^{-1}\in N_H(S_1)-S_1$. To the multiplicative one-parameter subgroup α^v one can associate in a familiar manner (see for example [11, p. 231]) a parabolic subgroup P_1. Its Lie algebra is spanned by the weight spaces.

$$V_n=\{X\in \operatorname{Lie}(G)\mid \operatorname{Ad}\alpha^v(\xi)X=\xi^n X\},$$

with $n\geqq 0$. Notice that P_1 is θ-stable.

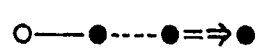
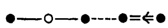
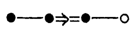
We conclude that $P=h^{-1}P_1h$ is a split parabolic subgroup of G containing $Z_G(S)$. If I is the set of $\gamma\in D$ orthogonal to $\alpha=\alpha_s$ then P_1 is a parabolic subgroup of type I. Hence P is of type I. This proves (ii).

Assume that all roots have the same length. Let $\alpha\in R$, $\alpha(a)=-1$. Again, we may assume that $\alpha=\alpha_s$. If $\gamma\in D$ is orthogonal to α_s then α_s and γ span a subsystem of type $A_1\times A_1$. If we had $\gamma(a)=-1$ we could construct a two-dimensional split torus, in the manner described above, which would violate the assumption $\operatorname{rank}\theta=1$. So $\gamma(a)=1$ for all $\gamma\in D$ orthogonal to α_s. Now write $\alpha_s=\sum_{\gamma\in D} m_\gamma\gamma$. Then

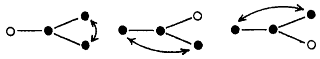
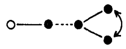
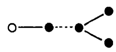
$$-1=\alpha_s(a)=\prod\gamma(a)^{m_\gamma}=\beta(a)^{m_\beta}.$$

Hence m_β must be odd, which establishes (iii).

3.3. Using the preceding results one obtains quickly a list of possible diagrams $\mathcal{D}_\theta$ (see no. 2), for the case that G is simple, θ is inner and $\operatorname{rank}\theta=1$. The α_s are readily found from the data in the tables of [4]. The cases D_n, E_6, E_7, E_8 are ruled out by using 3.2(iii) and the case G_2 by 3.1. We shall see that all diagrams of the list can be realized, i.e. that all corresponding (I,ϕ) are admissible. The list is as follows:

Type	Diagram
A_1	
A_l	
B_l	
C_l	
F_4	

There is a similar list for the case that θ is outer. Here it suffices already to use the conditions on (I, ψ) given in 1.7.

Type	Diagram
A_3	
D_4	
D_{2l} $(l \geqq 3)$	
D_{2l+1} $(l \geqq 2)$	

3.4. We now drop the assumption rank $\theta = 1$. Using 2.4 we can set up a list of possible diagrams $\mathcal{D}_\theta$ for the case that G is simple, as is done in [1]. The diagrams are given in the second columns of tables 1 and 2, for the inner resp. outer involutions.

If I is empty we say that θ is *quasi-split*. In that case a minimal split parabolic subgroup of G is a Borel group. If, moreover, rank θ = rank G then θ is *split*. It follows immediately that for given G there is a θ such that θ is split. Moreover this θ is unique up to conjugacy by an inner automorphism. In the tables we have indicated the split and quasi-split cases.

We shall next establish that all diagrams listed in tables 1 and 2 correspond to admissible pairs (I, ψ). For this we need well-known results about the classification of elements of order two in a simple group G, to be recalled in the next section.

4. Elements of order two in adjoint groups.

4.1. We assume G to be semi-simple and adjoint. The description of conjugacy classes of elements of order 2 in G is well-known. It is contained, for example, in Iwahori's paper in [2, p. 267–295], where the more complicated problem is discussed of describing the conjugacy classes of involutions in finite groups of Lie type (over a field of characteristic not two).

We fix a maximal torus T of G. The corresponding Weyl group $N_G T/T$ is denoted by W and the root system of (G, T) by R. If ${}_2T$ denotes the group of elements of order $\leqq 2$ of T there is a bijection of the set of conjugacy classes of involutions of G onto the set of W-orbits in ${}_2T - \{e\}$. Denote by X the character group of T and by $X^\vee$ its dual, which we identify with the group of one parameter subgroups of T. There is an isomorphism $X^\vee/2X^\vee \simeq {}_2T$, induced by the homomorphism $\lambda \mapsto \lambda(-1)$ of $X^\vee$

Table 1. Inner involutions.

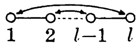
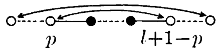
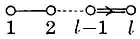
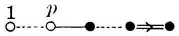
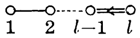
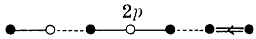
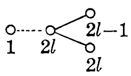
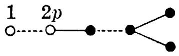
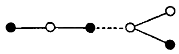
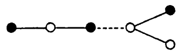
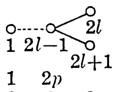
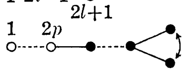
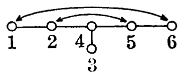
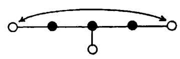
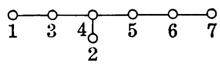
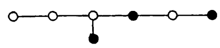
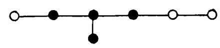
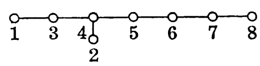
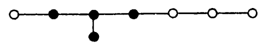
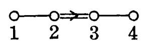
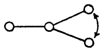
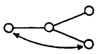
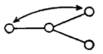
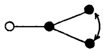
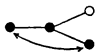
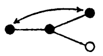
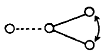
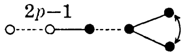
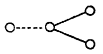
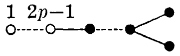
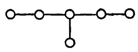
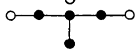
type	diagrams		matching
A_1		split	1
A_l $(l>1)$		quasi-split	$\left[\frac{l+1}{2}\right], \left[\frac{l+2}{2}\right]$
		$1\leqq p<\frac{1}{2}l$	$p,\ l+1-p$
B_l		split	$\left[\frac{l+1}{2}\right]$
		$1\leqq p\leqq l-1$	$\frac{l}{2}+(-1)^p\left(\left[\frac{p}{2}\right]-\frac{l}{2}\right)$
C_l $(l>2)$		split	l
		$1\leqq p\leqq \frac{1}{2}l$	$p,\ l-p$
D_{2l}		split	l
		$1\leqq p\leqq l-1$	p
			$2l-\frac{1}{2}(1+(-1)^l)$
			$2l-\frac{1}{2}(1-(-1)^l)$
D_{2l+1}		quasi-split	$l,\ l+1$
		$1\leqq p\leqq l-1$	p
			$2l,\ 2l+1$
E_6		quasi-split	2, 3, 5
			1, 6
E_7		split	2
			1, 6
			7
E_8		split	1
			8
F_4		split	1
			4
G_2		split	2

Table 2. Outer involutions.

type	diagram		matching
A_{2l}		split	
A_{2l+1}		split	$l+1$
A_{2l+1}			
D_4		quasi-split	1, 2
		quasi-split	2, 3
		quasi-split	2, 4
			
			
			
D_{2l} $(l>2)$		quasi-split	$l-1$, l
		$1\leqq p\leqq l-1$	$p-1$, $2l-p$ if $p>1$
D_{2l+1}		split	l
		$1\leqq p\leqq l$	$p-1$, $2l-p+1$ if $p>1$
E_6		split	3, 4
			

onto ${}_2T$. The isomorphism is W-equivariant. Hence there is a bijection of the set of W-orbits in ${}_2T$ onto the set of W-orbits in $X^v/2X^v$. Let W_a' be the semi-direct product of W and X^v, relative to the canonical action of W on X^v. Then W_a' operates on X^v by $(w, \lambda)(\mu)=w(\mu+2\lambda)$. It is clear that there is a bijection of the set of W-orbits in $X^v/2X^v$ onto the set of W_a'-orbits in X^v.

Since G is adjoint, X^v is isomorphic to the weight lattice P of the dual root system $R^v\subset X^v$. Let Q be the root lattice of R^v. The subgroup W_a generated by W and Q is a Coxeter group, viz. the affine Weyl group defined by W. It operates in the vector space $X^v\otimes_Z\boldsymbol{R}$ and has a fundamental domain in that vector space, described in [4, Ch. VI, § 2]. We recall a number of results established in that reference.

Assume G to be simple. Fix a basis D of R and denote by $\tilde{\alpha}=\sum_{\alpha\in D} n_\alpha\alpha$ the corresponding highest root. Put $D_a=D\cup\{-\tilde{\alpha}\}$, the vertex set of the affine Dynkin diagram. Let D_i $(i=1,2)$ be the set of $\alpha\in D$ with $m_\alpha=i$.

The group P/Q operates on D_a, and operates simply transitively on $D_1\cup\{-\tilde{\alpha}\}$.

Finally, let $(\alpha')_{\alpha\in D}$ be the basis of X^v dual to the basis D of X. The classification of conjugacy classes of elements of order two in the simple group G is now described in the following result.

4.2. (i) *Any element of order* 2 *in* G *is conjugate to an element* $\alpha'(-1)$ *of* T, *where* $\alpha\in D_1\cup D_2$;

(ii) *If* $\alpha, \beta\in D_1\cup D_2$ *then* $\alpha'(-1)$ *and* $\beta'(-1)$ *are conjugate in* G *if and only if we have one of the following situations*:

(a) $\alpha, \beta\in D_1$ *and there exists* $\gamma\in P/Q$ *such that* $\gamma\alpha=\gamma^{-1}\beta=-\tilde{\alpha}$,

(b) $\alpha, \beta\in D_2$ *and there exists* $\gamma\in P/Q$ *with* $\beta=\gamma\alpha$.

It is now an easy matter, using the data given in the tables of [4], to determine the number of conjugacy classes of elements of order two in the various simple cases. It turns out that in each case the number of classes equals the number of diagrams listed in table 1, for that case. It follows that the diagrams of table 1 do actually occur, so that the table gives the complete list of diagrams $\mathscr{D}_\theta$ for inner involutions θ.

One now checks that the diagrams of possible rank one outer involutions listed in 3.3 occur as diagrams (I, ψ_J) for a suitable diagram of table 1 (notation of 2.4). Using 2.4 we conclude that the diagrams listed in table 2 also occur. So tables 1 and 2 give the complete list of Araki diagrams of involutions of simple groups.

5. Matching two descriptions of inner automorphisms.

We keep the notations of no. 4. If a is an element of T of order two it defines one or more roots $\alpha\in D_1\cup D_2$, according to 4.2(i). On the other hand, the automorphism $\theta=\mathrm{Int}\, a$ determines a diagram listed in table 1. In the last column of table 1 we have listed for each diagram the corresponding roots α (if there are several they are related according to 4.2 (ii)). The column gives the numbers of the roots, the numbering being as in the tables of [4]. The numbering is indicated in the second column of table 1 (split and quasi-split cases).

In almost all cases the roots α can be found by computing in two ways the dimension of the centralizer K of a. In the first place, by 1.9(i),

$$\dim K=\dim P_I-\mathrm{rank}\,\theta,$$

showing that $\dim K$ is known if the diagram is given.

On the other hand we have the following result (going back to Borel-de Siebenthal [3]). We use the notations of 4.2. If $\alpha \in D_1 \cup D_2$ denote by H the connected centralizer of $\alpha'(-1)$. We denote by $\mathscr{D}_a$ the affine Dynkin diagram associated to $\mathscr{D}$.

5.1. (i) *If* $\alpha \in D_1$ *than* H *is isomorphic to the group* $L_{D-\{\alpha\}}$;

(ii) *If* $\alpha \in D_2$ *than* H *is a semi-simple group whose Dynkin diagram is obtained from* $\mathscr{D}_a$ *by removing the vertex* α *and all edges with endpoint* α.

This result permits us to compute $\dim H$ in all cases. A comparison of $\dim H$ and $\dim K$ then gives all entries in the last column of table 1 in a straightforward manner, except for the last two diagrams listed under D_{2l}.

In these two exceptional cases the comparison of dimensions only gives that $\alpha = \alpha_{2l}$ or α_{2l-1}. We can then make an explicit computation in $G_1 = SO_{4l}$, taking $G = G_1/\{\pm 1\}$. Let $V = k^{4l}$ and denote by $(\ ,\ )$ the bilinear symmetric form on V with ((e_i) denoting the canonical basis) $(e_i, e_{i+2l}) = 1$ for $1 \leqq i \leqq 2l$, $(e_i, e_j) = 0$ for all other pairs i, j. The group of nonsingular linear maps with respect to this basis of the form $x = \operatorname{diag}(x_1, \cdots, x_{2l}, x_1^{-1}, \cdots, x_{2l}^{-1})$ is a maximal torus of G_1. A basis D of the corresponding root system is given by the characters α_i $(1 \leqq i \leqq 2l)$ with $\alpha_i(x) = x_i x_{i+1}^{-1}$ $(1 \leqq i \leqq 2l-1)$, $\alpha_{2l}(x) = x_{2l-1} x_{2l}$. Let $a = \operatorname{diag}(i, \cdots, i, -i, \cdots, -i)$, where $i^2 = -1$ and let θ be the inner automorphism $\operatorname{Int} a$ of G_1. Then $a = \alpha'_{2l}(-1)$. To obtain a maximal split torus in G_1, relative to θ, one first constructs one in the case $l = 1$, e.g. by using that then G is isogeneous to $SL_2 \times SL_2$. One then obtains a maximal split torus in the general case by putting together l tori of the previous sort. It turns out that a maximal split torus is conjugate in G_1 with the l-dimensional diagonal torus of transformations of the form $\operatorname{diag}(x_1, x_1^{-1}, \cdots, x_l, x_l^{-1}, x_1, x_1^{-1}, \cdots, x_l, x_l^{-1})$. From this we see that $I = D - \{\alpha_{2l}\}$ if l is even and $I = D - \{\alpha_{2l-1}\}$ if l is odd, which leads to the result of the last column of table 1.

6. A remark on quasi-split inner automorphisms.

For each semi-simple G there is a unique class of quasi-split inner automorphisms. The next result gives an easy explicit description of this class. We assume G to be semi-simple. T is a maximal torus of G and D is a basis of the root system R of (G, T).

6.1. *Let* $a \in T$ *be such that* $\alpha(a) = -1$ *for all* $\alpha \in D$. *Then* $\operatorname{Int}(a)$ *is quasi-split. If the opposition involution* i_D *is trivial then* $\operatorname{Int}(a)$ *is split.*

Put $\operatorname{rank} G=l$, $\operatorname{rank} \operatorname{Int}(a)=r$. Let r_i be the number of roots in R of height $i>0$, relative to D. From a well-known description of the centralizer K of a [2, p. 201] we see that

$$\dim K=l+2\sum_{i \text{ even}} r_i .$$

Also, $\dim B=l+\sum_{i\geq 1} r_i$. It follows that

$$\dim B-(\dim K+r)=\sum_{i\geq 1}(-1)^{i-1}r_i-r . \tag{2}$$

By a result of Kostant [7, Cor. 8.7] we know that r_i-r_{i+1} equals the number of exponents of the Weyl group W which are equal to i. Hence the sum in the right-hand side of (2) equals the number of odd exponents. This is known to be equal to the number of i_D-orbits in D (see [10, 6.5]). We then conclude from 1.9 (iii) that the right-hand side of (2) is non-negative. Then 1.9 (i) shows that we must have $\dim K+r=\dim B$ and that the minimal split parabolic subgroups relative to $\operatorname{Int}(a)$ are Borel groups. This proves that $\operatorname{Int}(a)$ is quasi-split. The last assertion of 6.1 follows from 2.2.

Notice that if all exponents are odd, i.e. if -1 lies in the Weyl group W, we have that $\operatorname{Int} a$ is split.

7. Outer involutions.

Finally, we briefly discuss an analogue of the results of no. 5 for the case of outer involutions. We use the notations of no. 1. We assume G to be semi-simple. Fix a fundamental pair (T, B) and a realization $(x_\alpha)_{\alpha\in R}$. Assume that θ is outer. Then the permutation $\tilde{\theta}$ of the basis D is non-trivial. Put $E=\{\alpha\in D \mid \tilde{\theta}\alpha=\alpha\}$. There is a unique involution θ_0 of G such that (T, B) is a fundamental pair relative to θ_0 and that

$$\theta_0(x_\alpha(\xi))=x_{\tilde{\theta}\alpha}(\xi) \qquad (\alpha\in D, \xi\in k) .$$

It is readily seen that we may choose the realization such that $\theta=\theta_0\circ\operatorname{Int} a$, where $a\in T$ is such that $\alpha(a)=1$ for $a\in D-E$, $\alpha(a)=\pm 1$ for $\alpha\in E$.

If H is the semi-simple closed subgroup of G generated by the groups $\operatorname{Im} x_\alpha$ for $\alpha\in\pm E$ then θ induces an inner automorphism κ of H of order ≤ 2. If G is simple, an inspection of the possible cases (A_l, D_l, E_6) shows that H is either trivial or semi-simple of type A. By the previous results we know the possibilities for κ. We check that the number of those is equal to the number of corresponding diagrams in table 2. Hence these diagrams are completely determined by the conjugacy class of H in the group of inner automorphisms of H.

In the last column of table 2 we have given the numbers of the roots $\alpha \in E$ which define κ according to the last column of table 1, if κ is an involution. If κ is trivial there is no entry in the last column of table 2. (The numbering of roots used in that column is the numbering of the roots of D.)

The matching is again achieved by computing the dimension of K in two ways. To compute the dimension of the fixed point group of an automorphism $\theta_0 \circ \mathrm{Int}\, a$ one can make a direct check in the various cases. When G is of type A_l or E_6, there are at most two cases and it suffices to deal with the case that $a=1$, which is easy. When G is of type D_{2l} one can make a computation in SO_{2l}. The other computation of $\dim K$ uses 1.9 (i), as before. The identification of K is easily achieved. We omit the details.

As a byproduct we find the following analogue of 6.1 for outer automorphisms.

7.1. *Let $a \in T$ be such that $\alpha(a)=-1$ for $\alpha \in E$, $\alpha(a)=1$ for $\alpha \in D-E$. Then $\theta=\theta_0 \circ \mathrm{Int}\, a$ is quasi-split. If i_D is non-trivial then θ is split.*

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